1. A group $G$ action on a set $X$ is doubly transitive if it induces a transitive action on $S = \{(x, y) \in X \times X \mid x \neq y\}$ via $g(x, y) = (gx, gy)$. Let $X$ be a doubly transitive $G$-set. Show that $G_x$ is maximal for all $x \in X$: i.e., not properly contained in a proper subgroup of $G$.

2. Let $X$ be a transitive $G$-set, and consider the set $S$ of subsets $B$ of $X$ such that for every $g \in G$, either $B = gB$ or $B \cap gB = \emptyset$.
   a. Let $x \in X$. Show that the set of subgroups of $G$ containing $G_x$ is in one-to-one correspondence with the set \{ $B \in S \mid x \in B$ \}.
   b. Show that every $G_x$ for $x \in X$ is maximal if and only if $S$ consists of $X$ and the singleton subsets of $X$.

3. Let $G$ be a finite group with subgroups $H$ and $K$, and let $x \in G$. Show that
   \[ |HxK| = \frac{|H||K|}{|x^{-1}(Hx \cap K)|} \]

4. Show that the only proper subgroup of $S_n$ of index less than $n$ is $A_n$, unless $n = 4$.

5. Let $P$ be a Sylow $p$-subgroup of a finite group $G$. Show that $N_G(N_G(P)) = N_G(P)$.

6. Show that there are no simple groups of order 90, 112, or 120.

7. Show that any simple group of order $p^2qr$ where $p$, $q$, and $r$ are primes, is isomorphic to $A_5$. 