Assignment 3  
Math 210A, Fall 2018  
Due Wednesday, October 31 in class

1. Let $m, n, t \geq 2$ be integers. Show that there exists a group $G$ containing elements $x$ and $y$ in which the order of $x$ is $m$, the order of $y$ is $n$, and the order of $xy$ is $t$.

2. Let $p$ be an odd prime. Give a presentation of the group of upper triangular matrices in $GL_2(F_p)$ on three generators.

3. Find the order and exponent of $G = \text{Aut}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z})$ for a prime $p$.

4. Let $G$ be a finite group with $3 \nmid |G|$ such that $\phi: G \to G$ given by $\phi(g) = g^3$ for $g \in G$ is a homomorphism. Show that $G$ is abelian.

5. Let $G$ be a group, $H, H' \leq G$, and $K \trianglelefteq H, K' \trianglelefteq H'$. Show that

\[
\frac{(H \cap H')K}{(H \cap K')K} \cong \frac{(H \cap H')K'}{(K \cap H')K'}
\]

6. For $i \in \{1, 3, 5, 7\}$, we have homomorphisms $\varphi_i: \mathbb{Z}/2\mathbb{Z} \to (\mathbb{Z}/8\mathbb{Z})^\times$ sending 1 to $i$. Show that

\[
\mathbb{Z}/8\mathbb{Z} \rtimes_{\varphi_i} \mathbb{Z}/2\mathbb{Z} \cong \langle h, n \mid n^8 = h^2 = e, hn = n^ih \rangle,
\]

and show that all four of these groups are non-isomorphic.

7. Suppose that $G = N \rtimes H$ for some $N \trianglelefteq G$ and $H \leq G$. Let $K \leq G$ with $H \leq K$. Show that $K = N \rtimes (K \cap H)$.

8. Let $H$ be a cyclic group, and let $N$ be a group. Let $\phi, \psi: H \to \text{Aut}(N)$ be injective homomorphisms such that $\phi(H) = \psi(H)$. Show that $N \rtimes_{\phi} H \cong N \rtimes_{\psi} H$. 