Uniformization of Sierpiński carpets by square carpets

Dimitrios Ntalampakis
dimitrisnt@math.ucla.edu

Quasisymmetric uniformization

Uniformization is the problem of transforming a given metric space $X$ to a canonical space with a map that preserves the geometry. In the metric space setting we are interested in quasisymmetric maps.

A homeomorphism $f : X \to Y$ between metric spaces $X, Y$ is a quasisymmetry if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ called the distortion function such that for every triple $x, y, z \in X$:

$$\frac{d(x, y)}{d(x, z)} \leq \eta(t) \quad \text{and} \quad \frac{d(y, z)}{d(x, z)} \leq \eta(t).$$

A quasisymmetry quasi-preserves relative distances, instead of absolute distances.

Main Theorem

A square carpet is a carpet whose peripheral circles are squares, except possibly for $\partial \Omega$, which is a rectangle, and all their sides are parallel to the coordinate axes. The main result in [3] is the following:

Theorem. Let $S \subset \mathbb{C}$ be a Sierpiński carpet whose peripheral circles are uniformly relatively separated, uniform quasicircles. Furthermore, assume that $\text{Area}(S) = 0$. Then there exists a quasimmetry from $S$ onto a square carpet.

Why square carpets? Square carpets arise naturally as extremal domains for a minimizing problem.

Remark. If we remove the assumption of uniform relative separation, and weaken the assumption of uniform quasicircles to (e.g.) uniform John domains, then the uniformizing map is not quasimetric in general, but it is “quasiconformal” in a discrete sense.

Sierpiński carpets

Construction of a planar Sierpiński carpet $S$

Let $\Omega \subset \mathbb{C}$ be a Jordan region, and $Q_i \subset \Omega, i \in \mathbb{N}$ be Jordan regions such that:

1. $\cup_i Q_i = \Omega$ and $\cup_i \partial Q_i = \emptyset$
2. $\text{diam}(Q_i) = \delta$
3. $S := \Omega \setminus \bigcup_{i=1}^{\infty} Q_i$ has empty interior.

The peripheral circles of the carpet $S$ are homeomorphic to each other [3].

Geometric assumptions

A Jordan curve $C$ is a $k$-quasicircle if for every two points $x, y \in C$ there exists an arc $\gamma \subset C$ connecting $x$ and $y$ such that:

$$\text{diam}(\gamma) \leq k|y - y|.$$

Two continua $E, F$ are $\delta$-relatively separated if:

$$\frac{\text{dist}(E, F)}{\min\{\text{diam}(E), \text{diam}(F)\}} \geq \delta.$$}

The peripheral circles of a carpet $S$ are uniform quasicircles if they all are $k$-quasicircles for some $k > 0$. The peripheral circles are uniformly relatively separated if every pair of them is $\delta$-relatively separated for some $\delta > 0$.

Round carpets

Bonk [1] proved the following uniformization result for carpets:

Theorem. Let $S \subset \mathbb{C}$ be a Sierpiński carpet whose peripheral circles are uniformly relatively separated, uniform quasicircles. Then there exists a quasisymmetry from $S$ onto a round carpet.


References