Randomized projection algorithms for overdetermined linear systems

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Let $Ax = b$ be an overdetermined, standardized, full rank system of equations.
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From $A$ and $b$ we wish to recover unknown $x$. Assume $m \gg n$. 

*Goal*
The Kaczmarz method is an iterative method used to solve $Ax = b$.

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Method

Kaczmarz

1. Start with initial guess $x_0$
2. $x_{k+1} = x_k + (b[i] - \langle a_i, x_k \rangle) a_i$ where $i = (k \mod m) + 1$
3. Repeat (2)
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Theorem [Strohmer-Vershynin]: Consistent case $Ax = b$

1. Start with initial guess $x_0$

2. $x_{k+1} = x_k + (b_p - \langle a_p, x_k \rangle)a_p$ where $P(p = i) = \frac{\|a_i\|_2^2}{\|A\|_F^2} = 1/m$

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Randomized Kaczmarz (RK)

Theorem [Strohmer-Vershynin]

Let $R = m \| A^{-1} \|^2$ ($\| A^{-1} \| \overset{\text{def}}{=} \inf \{ M : M \| Ax \|_2 \geq \| x \|_2 \text{ for all } x \}$)

Then $\mathbb{E} \| x_k - x \|_2^2 \leq \left( 1 - \frac{1}{R} \right)^k \| x_0 - x \|_2^2$

Well conditioned $A \rightarrow$ Convergence in $\mathcal{O}(n)$ iterations $\rightarrow$ $\mathcal{O}(n^2)$ total runtime.

Better than $\mathcal{O}(mn^2)$ runtime for Gaussian elimination and empirically often faster than Conjugate Gradient.
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Well conditioned $A \rightarrow$ Convergence in $O(n)$ iterations $\rightarrow$ $O(n^2)$ total runtime.

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We now consider the system $Ax = b + e$. 
Randomized Kaczmarz (RK) with noise

**Theorem [N]**

Let $Ax = b + e$. Then

$$
\mathbb{E}\|x_k - x\|_2 \leq \left(1 - \frac{1}{R}\right)^{k/2} \|x_0 - x\|_2 + \sqrt{R}\|e\|_\infty
$$

This bound is sharp and attained in simple examples.
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- This bound is sharp and attained in simple examples.
Figure: Comparison between actual error (blue) and predicted threshold (pink). Scatter plot shows exponential convergence over several trials.
Recall $x_{k+1} = x_k + (b[i] - \langle a_i, x_k \rangle) a_i$.

Since these projections are orthogonal, the optimal projection is one that maximizes $\|x_{k+1} - x_k\|_2$.

What if we relax: $x_{k+1} = x_k + \gamma (b[i] - \langle a_i, x_k \rangle) a_i$.

Can we choose $\gamma$ optimally?

Idea: In each “iteration,” project once with relaxation optimally and then project normally.
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Two-subspace Kaczmarz

- Randomly select two rows, \( a_s \) and \( a_r \)
- Perform initial projection: \( y = x_k + \gamma (b[i] - \langle a_i, x_k \rangle) a_i \) with \( \gamma \) optimal
- Perform second projection: \( x_{k+1} = y + (b[i] - \langle a_i, y \rangle) a_i \)
- Repeat
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- Repeat
Geometrically, we choose $\gamma$ in such a way:
Two-subspace Kaczmarz

The optimal choice of $\gamma$ in a single iteration is

$$\gamma = \frac{-\langle a_r - \langle a_s, a_r \rangle a_s, x_k - x + (b_s - \langle x_k, a_s \rangle) a_s \rangle}{(b_r - \langle x_k, a_r \rangle) \| a_r - \langle a_s, a_r \rangle a_s \|^2_2}.$$ 

Two-Subspace Kaczmarz method

- Select two distinct rows of $A$ uniformly at random
- $\mu_k \leftarrow \langle a_r, a_s \rangle$
- $y_k \leftarrow x_{k-1} + (b_s - \langle x_{k-1}, a_s \rangle) a_s$
- $v_k \leftarrow \frac{a_r - \mu_k a_s}{\sqrt{1 - |\mu_k|^2}}$
- $\beta_k \leftarrow \frac{b_r - b_s \mu_k}{\sqrt{1 - |\mu_k|^2}}$
- $x_k \leftarrow y_k + (\beta_k - \langle y_k, v_k \rangle) v_k$
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### Two-Subspace Kaczmarz method

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- $\beta_k \leftarrow \frac{b_r - b_s \mu_k}{\sqrt{1 - |\mu_k|^2}}$
- $x_k \leftarrow y_k + (\beta_k - \langle y_k, v_k \rangle) v_k$
Figure: For coherent systems, the one-subspace randomized Kaczmarz algorithm (a) converges more slowly than the two-subspace Kaczmarz algorithm (b).
Define the coherence parameters:

\[ \Delta = \Delta(A) = \max_{j \neq k} |\langle a_j, a_k \rangle| \quad \text{and} \quad \delta = \delta(A) = \min_{j \neq k} |\langle a_j, a_k \rangle|. \] (1)

**Figure:** Randomized Kaczmarz (RK) versus two-subspace RK (2SRK). \( A \) has highly coherent rows with \( \delta = 0.992 \) and \( \Delta = 0.998 \).
Figure: Randomized Kaczmarz (RK) versus two-subspace RK (2SRK). $A$ has highly coherent rows with coherence parameters (a) $\delta = 0.837$ and $\Delta = 0.967$, (b) $\delta = 0.534$ and $\Delta = 0.904$, (c) $\delta = 0.018$ and $\Delta = 0.819$, and (d) $\delta = 0$ and $\Delta = 0.610$. 
Results

Recall the coherence parameters:

\[ \Delta = \Delta(A) = \max_{j \neq k} |\langle a_j, a_k \rangle| \quad \text{and} \quad \delta = \delta(A) = \min_{j \neq k} |\langle a_j, a_k \rangle|. \quad (2) \]

**Theorem [N-Ward]**

Let \( b = Ax + e \), then the two-subspace Kaczmarz method yields

\[
\mathbb{E}\| x - x_k \|_2 \leq \eta^{k/2} \| x - x_0 \|_2 + \frac{3}{1 - \sqrt{\eta}} \cdot \frac{\| e \|_\infty}{\sqrt{1 - \Delta^2}},
\]

where \( D = \min \left\{ \frac{\delta^2(1-\delta)}{1+\delta}, \frac{\Delta^2(1-\Delta)}{1+\Delta} \right\} \), \( R = m\| A^{-1} \|_2 \) denotes the scaled condition number, and \( \eta = \left(1 - \frac{1}{R}\right)^2 - \frac{D}{R} \).
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Remarks

1. When $\Delta = 1$ or $\delta = 0$ we recover the same convergence rate as provided for the standard Kaczmarz method since the two-subspace method utilizes two projections per iteration.

2. The bound presented in the theorem is a pessimistic bound. Even when $\Delta = 1$ or $\delta = 0$, the two-subspace method improves on the standard method if any rows of $A$ are highly correlated (but not equal).
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The parameter $D$

**Figure:** A plot of the improved convergence factor $D$ as a function of the coherence parameters $\delta$ and $\Delta \geq \delta$. 
Generalization to more than two rows?
Randomized Block Kaczmarz method

Given a partition of the rows, $T$:

- Select a block $\tau$ of the partition at random
- $x_k \leftarrow x_{k-1} + A_\tau^\dagger (b_\tau - A_\tau x_{k-1})$

The convergence rate heavily depends on the conditioning of the blocks $A_\tau \to$ need to control geometric properties of the partition.
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Row paving

An \((d, \alpha, \beta)\) row paving of a matrix \(A\) is a partition \(T = \{\tau_1, \ldots, \tau_d\}\) of the row indices that verifies

\[
\alpha \leq \lambda_{\text{min}}(A_\tau A_\tau^*) \quad \text{and} \quad \lambda_{\text{max}}(A_\tau A_\tau^*) \leq \beta \quad \text{for each} \ \tau \in T.
\]

Theorem [N-Tropp]

Suppose \(A\) admits an \((d, \alpha, \beta)\) row paving \(T\) and that \(b = Ax + e\). The convergence of the block Kaczmarz method satisfies

\[
\mathbb{E}\|x_k - x\|_2^2 \leq \left[1 - \frac{\sigma_{\text{min}}^2(A)}{\beta d}\right]^k \|x_0 - x\|_2^2 + \frac{\beta}{\alpha} \cdot \frac{\|e\|_2^2}{\sigma_{\text{min}}^2(A)}.
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(3)

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$$E\|x_k - x\|_2^2 \leq \left[1 - \frac{\sigma_{\min}^2(A)}{\beta d}\right]^k \|x_0 - x\|_2^2 + \frac{\beta}{\alpha} \cdot \frac{\|e\|_2^2}{\sigma_{\min}^2(A)}. \quad (3)$$
Good row pavings [Bougain-Tzafriri, Tropp]

For any $\delta \in (0, 1)$, $A$ admits a row paving with

$$d \leq C \cdot \delta^{-2} \|A\|^2 \log(1 + n)$$

and

$$1 - \delta \leq \alpha \leq \beta \leq 1 + \delta.$$

Theorem [N-Tropp]

Let $A$ have row paving above with $\delta = 1/2$. The block Kaczmarz method yields

$$\mathbb{E}\|x_k - x\|_2^2 \leq \left[1 - \frac{1}{C \kappa^2(A) \log(1 + n)}\right]^k \|x_0 - x\|_2^2 + \frac{3\|e\|_2^2}{\sigma_{\min}^2(A)}.$$
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Theorem [Bougain-Tzafriri, Tropp]

A random partition of the row indices with $m \geq \|A\|^2$ blocks is a row paving with upper bound $\beta \leq 6 \log(1 + n)$, with probability at least $1 - n^{-1}$.

Theorem [Bourgain-Tzafriri, Tropp]

Suppose that $A$ is incoherent. A random partition of the row indices into $m$ blocks where $m \geq C \cdot \delta^{-2} \|A\|^2 \log(1 + n)$ is a row paving of $A$ whose paving bounds satisfy $1 - \delta \leq \alpha \leq \beta \leq 1 + \delta$, with probability at least $1 - n^{-1}$. 
Figure: The matrix $A$ is a fixed $300 \times 100$ matrix consisting of 15 partial circulant blocks. Error $\|x_k - x\|_2$ per flop count.
Figure: The matrix $A$ is a fixed $300 \times 100$ matrix with rows drawn randomly from the unit sphere, with $d = 10$ blocks. Error $\|x_k - x\|_2$ over various computational resources.
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References: