Compactness

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There are two standard definitions of compactness, one using sequences and one using coverings by open sets. We shall prove that these two definitions are equivalent.

- We say that a metric space \((X, d)\) is **sequentially compact** if every sequence in \(X\) has a convergent subsequence. In other words, given a sequence \(\{x_n\} \subseteq X\), there exists a point \(x\) and a subsequence \(\{x_{n_k}\}\) such that \(x_{n_k} \to x\).

- We say that a metric space \((X, d)\) is **covering-compact** if every open cover of \(X\) has a finite subcover. In other words, if we have a collection \(\{U_\alpha\}_{\alpha \in I}\) of open sets such that \(X \subseteq \bigcup_{\alpha} U_\alpha\), then there exist finitely many open sets \(U_{\alpha_1}, \ldots, U_{\alpha_n}\) from our collection such that \(X \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}\).

**Theorem 1.** Let \(X\) be a metric space. The following are equivalent:

a. \(X\) is sequentially compact.

b. \(X\) is covering-compact.

The overall outline of the proof will be (a) \(\implies\) (b) then (b) \(\implies\) (a).

**Lemma 2.** Suppose \(X\) is sequentially compact and \(r > 0\). Then \(X\) can be covered by finitely many balls of radius \(r\). In other words, there exist \(x_1, \ldots, x_n\) such that \(X \subseteq \bigcup_{j=1}^{n} B(x_j, r)\).

**Proof.** Choose \(r > 0\). Suppose for contradiction that \(X\) cannot be covered by finitely many balls of radius \(r\). We construct a sequence \(\{x_n\}\) inductively as follows: For the base case, we let \(x_1\) be some point in \(X\). For the inductive step, assume that \(x_1, \ldots, x_n\) have been defined. We have assumed that \(X\) cannot be covered by \(B(x_1, r), \ldots, B(x_n, r)\). Therefore, we can choose \(x_{n+1} \in X \setminus \bigcup_{j=1}^{n} B(x_j, r)\).

Then \(\{x_n\}\) cannot have a convergent subsequence. Indeed, suppose for contradiction that \(\{x_{nk}\}\) is a subsequence converging to \(x\). Then there exists a \(K\) such that

\[
k \geq K \implies d(x_{nk}, x) < r/2.
\]

If \(k' > k \geq K\), then we have

\[
d(x_{nk}, x_{nk'}) \leq d(x_{nk}, x) + d(x_{nk'}, x) < r.
\]
On the other hand, our sequence was chosen so that \( x_{n_k} \not\in X \setminus \bigcup_{j=1}^{n_k-1} B(x_j, r) \). Hence, \( x_{n_k} \not\in B(x_{n_k}, r) \), which means that \( d(x_{n_k}, x) \geq r \). Thus, we obtain the contradiction \( d(x_{n_k}, x) < r \) and \( d(x_{n_k}, x) \geq r \).

**Lemma 3.** Suppose \( X \) is sequentially compact, and suppose that \( \{U_\alpha\} \) is an open cover. Then there exists an \( \delta > 0 \) such that every ball of radius \( \delta \) is contained in some open set \( U_\alpha \) from the open cover.

**Proof.** We proceed by contradiction. Suppose that there such not exist such an \( \delta \). In particular, the claim fails for \( \delta = 1/n \). Hence, for each \( n \), there exists a ball \( B(x_n, 1/n) \) of radius \( 1/n \) which is not contained in any \( U_\alpha \).

By sequential compactness, the sequence \( \{x_n\} \) has a subsequence \( \{x_{n_k}\} \) which converges to a point \( x \). The point \( x \) must be contained in a set \( U_\alpha \) from the open cover. Then because \( U_\alpha \) is open, there exists an \( r > 0 \) such that \( B(x, r) \subseteq U_\alpha \). Because \( x_{n_k} \rightarrow x \), we know that \( d(x_{n_k}, x) < r/2 \) for sufficiently large \( k \). Therefore, we can choose a large enough value of \( k \) that \( 1/n_k < r/2 \) and \( d(x_{n_k}, x) < r/2 \).

For this value of \( k \), we have \( B(x_{n_k}, 1/n_k) \subseteq B(x, r) \). Indeed, if \( y \in B(x_{n_k}, 1/n_k) \), then we have

\[
d(y, x) \leq d(y, x_{n_k}) + d(x_{n_k}, x) < \frac{1}{n_k} + \frac{r}{2} < \frac{r}{2} + \frac{r}{2} = r,
\]

hence \( y \in B(x, r) \). Therefore, we have

\[
B(x_{n_k}, 1/n_k) \subseteq B(x, r) \subseteq U_\alpha.
\]

This is a contradiction because we assumed that \( B(x_{n_k}, 1/n_k) \) is not contained in any \( U_\alpha \).

**Lemma 4** (a \( \Rightarrow \) b). If \( X \) is sequentially compact, then \( X \) is covering-compact.

**Proof.** Suppose that \( X \) is sequentially compact, and suppose that \( \{U_\alpha\} \) is an open covering of \( X \). By Lemma 3, there exists a \( \delta > 0 \) such that every ball of radius \( \delta \) is contained in one of the \( U_\alpha \)'s. By Lemma 2, \( X \) can be covered by finitely many balls \( B(x_1, \delta), \ldots, B(x_n, \delta) \). By our choice of \( \delta \), we have \( B(x_j, \delta) \subseteq U_{\alpha_j} \) for some index \( \alpha_j \). This implies that

\[
X \subseteq \bigcup_{j=1}^n B(x_j, \delta) \subseteq \bigcup_{j=1}^n U_{\alpha_j}.
\]

Therefore, \( \{U_{\alpha_j}\}_{j=1}^n \) is the desired finite subcover.

**Lemma 5.** Let \( \{x_n\} \) be a sequence and let \( x \in X \). Suppose that for every \( r > 0 \), \( B(x, r) \) contains infinitely many terms of the sequence. Then there is a subsequence \( \{x_{n_k}\} \) which converges to \( x \).

**Remark:** The phrases “infinitely many terms” and “finitely many terms” should be interpreted as follows: If \( m \neq n \), then \( x_m \) and \( x_n \) are considered distinct “terms” even if \( x_m = x_n \).
Proof. We construct the sequence inductively. Because \( B(x, 1) \) contains infinitely many terms of the sequence, we may choose \( n_1 \) such that \( x_{n_1} \in B(x, 1) \). Assume that \( n_k \) has been chosen. Then because \( B(x, 1/(k + 1)) \) contains infinitely many terms of the sequence, we may choose \( n_{k+1} > n_k \) such that \( x_{n_{k+1}} \in B(x, 1/(k+1)) \). Our subsequence \( \{x_{n_k}\} \) has been chosen so that \( d(x_{n_k}, x) < 1/k \). This implies that \( x_{n_k} \to x \).

**Lemma 6** (b \( \implies \) a). If \( X \) is covering-compact, then \( X \) is sequentially compact.

Proof. Assume that \( X \) is covering-compact. Suppose for the sake of contradiction that \( \{x_n\} \) is a sequence with no convergent subsequence. By the previous lemma, there cannot exist an \( x \) such that for every \( r > 0 \), \( B(x, r) \) contains infinitely many terms of the sequence. Therefore, it must be the case that for every \( x \) there exists an \( r_x > 0 \) such that \( B(x, r_x) \) contains only finitely many terms of the sequence.

The collection of balls \( \{B(x, r_x)\} \) is an open cover of \( X \) because every point \( x \in X \) is contained in the ball \( B(x, r_x) \). Therefore, there is a finite subcover \( \{B(x_j, r_{x_j})\}_{j=1}^n \). Now we have a contradiction: \( X \) is covered by finitely many balls and each ball contains finitely many terms of the sequence, which means that the sequence has only finitely many terms! This is absurd because a sequence has infinitely many terms by definition. \( \square \)