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Number 48

EXTENSION OF COMPACT OPERATORS

by

JORAM LINDENSTRAUSS

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EXTENSION OF COMPACT OPERATORS

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and Yale University

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CHAPTER I. INTRODUCTION

The starting point of the extension theory for operators is the classical Hahn-Banach theorem. This theorem may be formulated as follows.

Let X, Y and Z be Banach spaces with $Z \supset Y$. Let T be a bounded linear operator with a one-dimensional range from Y into X. Then there is a linear extension $\tilde{T}$ of T from Z into X with $\|\tilde{T}\| = \|T\|$.

There arises, naturally, the question whether similar results hold for more general operators T. It is well known that the answer to this question is, in general, negative. Not only that a norm preserving extension $\tilde{T}$ may fail to exist, but in general there does not exist even a bounded extension. Those spaces X for which the statement above remains valid if we drop the restriction that T has a one-dimensional range are called $\mathcal{P}_1$ spaces. More generally if a space X has one (and hence all) of the three equivalent properties stated below it is called a $\mathcal{P}_\lambda$ space ($\lambda \geq 1$, see Day [6, p. 94]).

(i) For every Banach space Z containing X there is a linear projection P from Z onto X with $\|P\| \leq \lambda$.

(ii) For every Banach space Z containing X and for every bounded linear operator T from X to a Banach space Y there is a linear extension $\tilde{T}$ of T from Z to Y with $\|\tilde{T}\| \leq \lambda \|T\|$.

(iii) For every bounded linear operator T from a Banach space Y to X and for every $Z \supset Y$ there is a linear extension $\tilde{T}$ of T from Z to X with $\|\tilde{T}\| \leq \lambda \|T\|$.

The $\mathcal{P}_1$ spaces were characterized by Nachbin [37], Goodner [11] and Kelley [25] who proved that X is a $\mathcal{P}_1$ space iff it is isometric to the space $C(K)$ of all the continuous functions on an extremally disconnected compact Hausdorff space K with the sup norm. (Iff means, as usual, if and only if. A topological space is called extremally disconnected if the closure of every open set is open.) The problem of the characterization of the $\mathcal{P}_\lambda$ spaces for $\lambda > 1$ is still open. In particular it is not known whether every $\mathcal{P}_\lambda$ space is isomorphic to a $\mathcal{P}_1$ space.

Our purpose in the present work is to study those Banach spaces X which have extension properties which are "between" the Hahn-Banach extension property (which is shared by all Banach spaces) and the extension properties (ii) and (iii) above which ensure the existence of an extension for <u>all</u> operators (from , resp. into X). We are interested in particular in extension properties for compact operators.

The discussion of the extension properties is divided into two parts. In the first part, which consists of Chapters II and III, we are concerned just with the question of the existence of a bounded or compact extension for certain compact operators. The results we obtain are rather incomplete and our main reason for including them here is that they form the framework in which we present the much more detailed theory of <u>norm preserving</u> or almost norm preserving (cf. the explanation of this notion below) extensions of compact operators. Chapters IV-VII are devoted to various aspects of the theory of norm preserving extensions.

We outline now briefly the contents of the various chapters. At the end of this chapter we give, besides the notations, also a list of the known results concerning $\mathcal{P}_\lambda$ spaces. In Chapter II we investigate the relation between various extension properties for compact or weakly compact operators in which the extension $\tilde{T}$ of the given operator T is assumed to satisfy the inequality $\|\tilde{T}\| \leq \lambda \|T\|$ for a certain λ which is independent of T (cf. Theorem 2.1). It is observed next (Theorem 2.2) that if all operators of a certain class can always be extended then there is a finite λ such that the extension $\tilde{T}$ can be chosen so that $\|\tilde{T}\| \leq$

$\lambda\|T\|$ (the same λ for all the operators T in the specific class). The proofs in Chapter II use rather standard compactness and embedding arguments. In Chapter III we introduce a class of spaces called $\mathcal{N}_\lambda$ spaces. These are Banach spaces which can be represented as the closure of the union of a directed set $\{B_\tau\}$ of finite-dimensional subspaces where each B_τ is a $\mathcal{P}_\lambda$ space (λ does not depend on τ). It is shown (Theorem 3.3) that these spaces have the extension properties treated in Chapter II. Conversely it is shown in Theorem 3.4 (under a certain assumption on X which is satisfied for example by every separable space with a basis) that if a Banach space X has extension properties which are even weaker (formally at least) than those considered in Chapter II then X is an $\mathcal{N}_\lambda$ space for some λ. All $C(K)$ spaces are $\mathcal{N}_\lambda$ spaces for every $\lambda > 1$.

Following Nachbin's study of $\mathcal{P}_1$ spaces [37] our main tool for investigating norm preserving extensions of compact operators is the use of intersection properties of cells. In Chapter IV the relation between some intersection properties are investigated. This chapter is combinatorial in nature and does not depend on Chapters II and III. Our main interest is in intersection properties which are important in the study of the extension of operators. However, a few theorems which may be of some independent interest are stated in a form which is stronger than actually needed in the subsequent chapters (for example Theorems 4.1 and 4.7).

In Chapter V the connection between extension and intersection properties is studied. Again, most of the results of Chapter V are used in Chapters VI and VII but some results, like the characterization of the Banach spaces whose cells have the finite intersection property (Theorem 5.9), are stated only since they follow rather easily from the discussion and are, perhaps, of some interest in themselves.

Chapters VI and VII contain the main results of the present work.

Chapter VI begins with a theorem (Theorem 6.1) which gives a long list of properties of a Banach space X, each of which is equivalent to the assumption that $X^* = L_1(\mu)$ for some measure μ. This theorem extends some previous results of Grothendieck [15]. In order to show the main direction of our discussion we state here some of the properties shown to be equivalent to the assumption that $X^* = L_1$. The six properties of a Banach space X which are stated here consist of three pairs. (a_1) and (a_2) are "from" extension properties, (b_1) and (b_2) are "into" extension properties and (c_1) and (c_2) are intersection properties. In each pair the second property is (only formally of course) weaker than the first.

(a_1) For every Banach space Y, every $Z \supset X$, and every compact operator T from X to Y there is a compact norm preserving extension $\tilde{T}$ of T from Z to Y.

(a_2) The same as (a_1) but with the further assumptions that $\dim Y = 3$ and $\dim Z/X = 1$.

(b_1) For any Banach spaces $Z \supset Y$, every $\varepsilon > 0$ and every compact operator T from Y to X there is a compact extension $\tilde{T}$ of T from Z to X with $\|\tilde{T}\| \leq (1+\varepsilon)\|T\|$.

(b_2) The same as (b_1) but with the further assumptions that $\dim Y = 3$ and $\dim Z = 4$.

(c_1) Every collection of mutually intersecting cells in X, whose set of centers is a compact subset of X, has a non empty intersection.

(c_2) Every collection of four mutually intersecting cells in X (all having radius 1) has a non empty intersection.

Properties (b_1) and (b_2) ensure the existence of what we call almost norm preserving extensions. In case the unit cell of X has at least one extreme point it is possible to get somewhat stronger results by introducing a partial order in X and investigating order properties which are equivalent to the usual decomposition property in partially ordered vector spaces.

In view of the results mentioned above the discussion in Chapter VI can be considered as the investigation of the properties of spaces X whose conjugates are abstract L spaces in the sense of Kakutani [23]. Theorem 6.6 gives a characterization of C(K) spaces in terms of the equivalent properties appearing in Theorem 6.1. A more general class of spaces than C(K) spaces which satisfy $X^* = L_1(\mu)$ is considered next. These spaces (called here G spaces) were introduced by Grothendieck. The chapter ends with a proof of the fact that a Banach space which is a $\mathcal{P}_{1+\varepsilon}$ space for every $\varepsilon > 0$ is already a $\mathcal{P}_1$ space (Theorem 6.10). The results of Chapters IV–VI solve some problems raised by Aronszajn and Panitchpakdi [2], Grothendieck [15], Grünbaum [17], Nachbin [37,38,39] and Semadeni. Several examples and counterexamples are given to illustrate the theorems and to show that some of the results are in a sense the best possible.

The question which spaces X have properties (b_1) or (b_2) with $\varepsilon = 0$ and related questions are the subject of Chapter VII. These questions turn out to be rather delicate and are closely related to the following problem. Given Banach spaces $Z \supset Y$, when does there exist a mapping ψ from Y^* into Z^* such that ψ is continuous (taking in Y^* and Z^* the norm topologies) and such that for every $y^* \in Y^*$ $\psi(y^*)$ is a norm preserving extension of y^* to Z. Theorem 7.3, Lemma 7.4 and the corollaries to Theorems 7.5 and 7.6 are results concerning this question. Another problem closely related to the existence of norm preserving extensions is the characterization of finite-dimensional spaces whose unit cells are polyhedra (cf. Theorem 7.7 and the corollary to Theorem 7.5). The extension theorems of Chapter VII deal not only with the extension of compact operators but also with the extension of certain isometries (cf. Theorems 7.5 and 7.8).

The problem of the lifting of compact operators, which is dual to the extension problem, is not discussed here. In [33] we gave some results concerning liftings which are the duals to some of the results in Chapters VI and VII. It turns out that the lifting problems are in many respects simpler than the corresponding extension problems which are discussed here.

Many unsolved problems are stated throughout the paper.

The present Memoir is a revised version of technical notes no. 28, 31 and 32 (the Hebrew University, Jerusalem, 1962) entitled extension of compact operators I, II and III. These notes in turn were based on the author's Ph.D. thesis prepared under the supervision of Professor A. Dvoretzky and Dr. B. Grünbaum of the Hebrew University. I wish to express my warm thanks to both for their valuable help and kind encouragement. I also wish to express my warm thanks to Professor S. Kakutani of Yale University for many helpful conversations concerning the subject of this paper.

The main results of chapters VI and VII were announced in [30] and [31]. Papers [32], [33], [34] and [35] are essentially results and examples which complement some parts of the present paper. The results of these papers are mentioned here in the proper context but only the results of Section 2 of [33] are reproduced here (in Chapter V).

Notations. We consider only Banach spaces over the real field R (R will denote also the one-dimensional space). The terminology and notions, from general topology and the theory of Banach spaces, used here are the standard ones. So is also the notation of special Banach spaces as

c_0, c, $m = \ell_\infty$, ℓ_p $(1 \le p \le \infty)$, $L_p(\mu)$ (where μ is a measure and $1 \le p \le \infty$) and $C(K)$, the Banach space of all bounded real-valued continuous functions on the topological space K with the sup norm. ℓ_p^n is the space of all n-tuples of real numbers $x = (x_1, x_2, \ldots, x_n)$ with $\|x\| = (\Sigma\, x_i^p)^{1/p}$ if $1 \le p < \infty$ and $= \max |x_i|$ if $p = \infty$. If s is a Banach space of (finite or infinite) sequences of real numbers and X_i are Banach spaces (whose number equals the number of the coordinates of the points of s) then

$$(X_1 \oplus X_2 \oplus \cdots \oplus X_i \oplus \cdots)_s$$

will denote the space of the sequences $x = (x_1, x_2, \cdots)$ with $x_i \in X_i$ and $(\|x_1\|, \|x_2\|, \cdots) \in s$. $\|x\|$ will be the norm of the latter sequence in s. $X \oplus Y$ will mean the direct sum of X and Y as a vector space in which the exact norm is not yet specified. If we write that $Z \supset Y$ we mean that Y is isometrically embedded in Z. If we consider X as a subspace of X^{**} we always assume that X is embedded canonically in X^{**}. The term operator will be used only for bounded linear operators. Let $Z \supset Y$ be Banach spaces and let T be an operator defined on Z. The restriction of T to Y is denoted by $T_{|Y}$. A similar notation will be used for restrictions of functionals and more general mappings.

In a normed space X we denote by $S_X(x_0, r)$ the cell $\{x;\ x \in X, \|x - x_0\| \le r\}$. If there can arise no confusion as to the space in which we take the cell we omit it from the notation and write simply $S(x_0, r)$. The unit cell of X, $S_X(0,1)$, is denoted also by S_X. The signs $\sim, \cap, \cup$ will be used to denote set theoretical operations while the signs $+$ and $-$ will be used for algebraic operations on sets in a vector space (thus for example $S_X(x_0, r) = x_0 + rS_X$). $\mathrm{Co}(A)$ denotes the convex hull of a set A in a vector space and $\bar{A}$ denotes the closure of a set A in a topological space.

In order to simplify the statement of the extension properties we shall assume (unless stated otherwise) that X is a fixed space (whose properties we investigate) while Z and Y are any Banach spaces satisfying the requirements (if any) imposed on them. T, T_0, $\tilde{T}$ etc. will denote operators. With this agreement the formulation of property (ii) above will be, for example:

Every T from X to Y has an extension $\tilde{T}$ from Z $(Z \supset X)$ to Y with $\|\tilde{T}\| \leq \lambda \|T\|$.

Let X be a Banach space. By $\dim X$ we understand the smallest cardinality $\mathfrak{m}$ such that X is the closed linear span of a set $\{x_\alpha\}$ consisting of $\mathfrak{m}$ elements.

A Banach space X has the metric approximation property (M.A.P.) if for every compact subset K of X and for every $\varepsilon > 0$ there is an operator T with a finite-dimensional range from X into itself such that $\|T\| = 1$ and $\|Tx-x\| \leq \varepsilon$ for every $x \in K$. This notion was introduced by Grothendieck [13]. Grothendieck has shown that the common Banach spaces have this property. It is an open question whether there exists a Banach space which does not have the M.A.P.

Some further notions will be defined in the subsequent chapters. The most important of them are: $\mathcal{N}_\lambda$ and $\mathcal{N}$ spaces (cf. the beginning of Chapter III), the various intersection properties (cf. the beginning of Chapter IV), G spaces (cf. Chapter VI before Lemma 6.7) and the notion of a continuous norm preserving extension (C.N.P.E.) map (cf. Chapter VII before Lemma 7.2).

<u>Preliminaries</u>. We shall list now some known results concerning $\mathcal{P}_\lambda$ spaces. These results and those mentioned already in the beginning of this chapter (i.e. the equivalence of (i), (ii) and (iii) and the characterization of $\mathcal{P}_1$ spaces) will be used freely in the sequel without

referring again to the literature.

Every Banach space can be embedded isometrically in a $\mathcal{P}_1$ space (e.g. in the space of the bounded real-valued functions on a set of a large enough cardinality). An immediate and well known consequence of this fact is

Lemma 1.1. Let T be an operator from a Banach space Y into a Banach space X and let $Z \supset Y$. Then there is a Banach space $V \supset X$ with $\dim V/X \leq \dim Z/Y$ such that T has a norm preserving extension $\tilde{T}$ from Z to V.

Let Z be a $\mathcal{P}_\lambda$ space and let X be a subspace of Z on which there is a projection with norm η, then X is a $\mathcal{P}_{\lambda\eta}$ space (cf. Day [6, p. 94]). Let X be a $\mathcal{P}_\lambda$ space then X^{**} is also a $\mathcal{P}_\lambda$ space (for $\lambda = 1$ this is a consequence of the characterization of $\mathcal{P}_1$ spaces. From this special case the general case follows immediately). Every infinite-dimensional $\mathcal{P}_\lambda$ space has a subspace isomorphic to c_0 (Pelczynski [41, 42]). This result implies some earlier results of Grothendieck [12], that no separable infinite-dimensional space is a $\mathcal{P}$ space (i.e. a $\mathcal{P}_\lambda$ space for some finite λ) and that there is no infinite-dimensional weakly sequentially complete (and in particular no infinite-dimensional reflexive) $\mathcal{P}$ space.

For every integer n there is a unique (up to isometry) n-dimensional $\mathcal{P}_1$ space. This is the space ℓ_∞^n whose unit cell is the n-dimensional cube. Every finite-dimensional space X is a $\mathcal{P}$ space. It is easily proved that if $\dim X = n$ then X is a $\mathcal{P}_n$ space and stronger results are also known (cf. Grünbaum [18]). The projection constants of some finite-dimensional spaces X (i.e. the inf of the λ such that X is a $\mathcal{P}_\lambda$ space) were computed by Grünbaum [18] and Rutowitz[44]. In general the projection constant tends to infinity with the dimension (cf. [29]).

The spaces of continuous functions which are $\mathcal{P}$ spaces were investigated recently by Amir [1] and Isbell and Semadeni [19]. The paper of Nachbin [38] gives a recent survey of various extension problems and contains an extensive bibliography.

CHAPTER II, GENERAL RESULTS

We begin this chapter by investigating the relation between the following eight properties

(1) X^{**} is a $\mathcal{P}_\lambda$ space.

(2) Every T from Y to X has an extension $\tilde{T}$ from Z $(Z \supset Y)$ to X^{**} with $\|\tilde{T}\| \leq \lambda \|T\|$.

(3) Every compact T from Y to X has (for every $\varepsilon > 0$) a compact extension $\tilde{T}$ from Z $(Z \supset Y)$ to X with $\|\tilde{T}\| \leq (\lambda+\varepsilon)\|T\|$.

(4) Every compact T from Y to X has (for every $\varepsilon > 0$) an extension $\tilde{T}$ from Z $(Z \supset Y,\ \dim Z/Y < \infty)$ to X with $\|\tilde{T}\| \leq (\lambda+\varepsilon)\|T\|$.

(5) Every T from X to a conjugate space Y has an extension $\tilde{T}$ from Z $(Z \supset X)$ to Y with $\|\tilde{T}\| \leq \lambda \|T\|$.

(6) Every compact T from X to Y has a compact extension $\tilde{T}$ from Z $(Z \supset X)$ to Y with $\|\tilde{T}\| \leq \lambda \|T\|$.

(7) Every weakly compact T from X to Y has a weakly compact extension $\tilde{T}$ from Z $(Z \supset X)$ to Y with $\|\tilde{T}\| \leq \lambda \|T\|$.

(8) Every compact T from X into itself has (for every $\varepsilon > 0$) an extension $\tilde{T}$ from Z $(Z \supset X,\ \dim Z/X < \infty)$ to X with $\|\tilde{T}\| \leq (\lambda+\varepsilon)\|T\|$.

Properties (2), (3) and (4) are "into" extension properties, (5), (6) and (7) are "from" extension properties and (8) is concerned with the extension of operators from X into itself.

<u>Theorem 2.1</u>. Let X be a Banach space. Then

$$(3) \Longrightarrow (4) \Longrightarrow (1) \Longleftrightarrow (2) \Longleftrightarrow (5) \begin{matrix} \nearrow (6) \searrow \\ \searrow (7) \nearrow \end{matrix} (8) .$$

For spaces X having the M.A.P. also $(8) \Longrightarrow (3)$, i.e. all the properties (1) - (8) are equivalent.

Proof. $(3) \Longrightarrow (4)$ is clear. In order to show that $(4) \Longrightarrow (1)$ we prove first that $(4) \Longrightarrow (9)$ where (9) is the following property

(9) The identity operator of X has an extension $\tilde{T}$ from Z $(Z \supset X)$ to X^{**} with $\|\tilde{T}\| \leq \lambda$.

Let $Z \supset X$, we define a partial order in the set of the pairs (B,ε) where B ranges over the finite-dimensional subspaces of Z and $1 \geq \varepsilon > 0$, by

$$(B_1,\varepsilon_1) \succ (B_2,\varepsilon_2) \Longleftrightarrow B_1 \supset B_2, \quad \varepsilon_1 \leq \varepsilon_2 .$$

From (4) it follows that for every pair (B,ε) there is an operator $T_{(B,\varepsilon)}$ from B to X satisfying:

a. $\|T_{(B,\varepsilon)}\| \leq \lambda + \varepsilon$.

b. The restriction of $T_{(B,\varepsilon)}$ to $B \cap X$ is the identity operator of that subspace.

For every r $S_{X^{**}}(0,r)$ is compact Hausdorff in the w^* topology hence, by Tychonoff's theorem, the same is true for

$$\prod = \prod_{z \in Z} S_{X^{**}}(0, \|z\|(\lambda+1))$$

in the usual product topology. For $t \in \prod$ we shall denote its z "co-ordinate" $(z \in Z)$ by $t(z)$. To every operator $T_{(B,\varepsilon)}$ (regarded as an operator from B to X^{**}) we assign a point $t_{(B,\varepsilon)}$ in $\prod$ by

$$t_{(B,\varepsilon)}(z) = \begin{cases} T_{(B,\varepsilon)}z & \text{if } z \in B \\ 0 & \text{if } z \notin B . \end{cases}$$

Let t be a limit point of the net $t_{(B,\varepsilon)}$. t has the following properties

(i) $t(x) = x$, $(x \in X)$.

(ii) $t(\alpha z_1+\beta z_2) = \alpha t(z_1) + \beta t(z_2)$, $(z_1, z_2 \in Z;\ \alpha,\beta$ scalars).

(iii) $\| t(z) \| \leq \lambda \| z \|$ $(z \in Z)$.

(i) follows from the fact that $t_{(B,\varepsilon)}(x) = x$ for $(B,\varepsilon) \succ ([x],1)$ ($[x]$ denotes the subspace spanned by x). (ii) and (iii) follow similarly. The operator $\tilde{T}$ from Z to X^{**} defined by $\tilde{T}z = t(z)$ has the required properties.

In the proof of (9) ==> (1) we shall use some well known facts concerning the duality in Banach spaces. We shall first list these facts. Let $U \supset V$ be Banach space, let $V^{\perp}$ be the annihilator of V in U^* and $V^{\perp\perp}$ the annihilator of $V^{\perp}$ in U^{**}. $V^{\perp\perp}$ is isometric to V^{**}. If T is an operator from U into itself with $TU \subset V$ then $T^*V^{\perp} = 0$ and $T^{**}U^{**} \subset V^{\perp\perp}$. If T is an operator from U into itself with $TV = 0$ then $T^*U^* \subset V^{\perp}$ and $T^{**}V^{\perp\perp} = 0$. In particular if Q is a projection in U with $V = Q^{-1}(0)$ then Q^* is a projection from U^* onto $V^{\perp}$. Let now $W \subset V \subset U$, then $W^{\perp\perp} \subset V^{\perp\perp} \subset U^{**}$ (the $\perp$ is taken with respect to U). For W we can take the $\perp$ also with respect to V. The two possible $W^{\perp\perp}$ thus obtained are connected by the fact that in the natural isometry from $V^{\perp\perp}$ to V^{**}, the subspace $W^{\perp\perp}$ (the $\perp$ with respect to U) is mapped onto $W^{\perp\perp}$ (the $\perp$ with respect to V).

We turn to the proof of (9) ==> (1). Let Z be a $\mathcal{P}_1$ space containing X^{**}, i.e. $Z \supset X^{**} \supset X$. From (9) follows the existence of an operator $\tilde{T}$ from Z into itself such that $\tilde{T}Z \subset X^{**}$, $\tilde{T}_{|X}$ is the identity operator of X and $\|\tilde{T}\| \leq \lambda$. Hence $\tilde{T}^{**}Z^{**} \subset X^{**\perp\perp}$ and the restriction of $\tilde{T}^{**}$ to $X^{\perp\perp}$ is the identity. Dixmier [7] has remarked that there is a projection Q with norm 1 from X^{***} onto X^*. Q is the restriction map - every functional on X^{**} is mapped to its restriction to X. Hence $Q^{-1}(0) = X^{\perp}$ (the $\perp$ here is taken with respect to the inclusion $X \subset X^{**}$), and therefore Q^* is a projection of X^{****}

onto $X^{\perp\perp}$. Using the isometry between X^{****} and $X^{**\perp\perp}$ we get a projection P with norm 1 from $X^{**\perp\perp}$ onto $X^{\perp\perp}$ (the $\perp$ are now with respect to the inclusion $X \subset Z$). It follows that $P\tilde{T}^{**}$ is a projection with norm $\leq \lambda$ from Z^{**} onto its subspace $X^{\perp\perp}$ and hence X^{**} is a $\mathcal{P}_\lambda$ space.

The implications (1) ==> (2) ==> (9) and (5) ==> (9) are clear. Hence we have proved already that (1) <==> (2) and that (5) ==> (1). We prove now that (9) (and hence (1)) implies (5).

Let Y be a conjugate space and let T be an operator from X into Y. By (9) there is an operator T_0 with norm $\leq \lambda$ from Z into X^{**} whose restriction to X is the identity. There is a projection with norm 1 (Q, say) from Y^{**} onto Y. $\tilde{T} = QT^{**}T_0$ is an extension of T from Z into Y with $\|\tilde{T}\| \leq \lambda \|T\|$.

The proofs of (9) ==> (6) and (9) ==> (7) are similar to that of (9) ==> (5). We do not need here the existence of a projection from Y^{**} onto Y since for weakly compact T (and in particular for compact T) T^{**} already maps X^{**} into the canonical image of Y in Y^{**} (see [8, pp. 482-483]). The implications (6) ==> (8) and (7) ==> (8) are immediate. Actually, each of (6) and (7) implies a stronger extension property for operators from X into itself (e.g. (8) with $\varepsilon=0$). (8) was stated in a weak version since already this version implies property (3) for spaces having the M.A.P.

Let us now assume that X has the M.A.P. and satisfies (8). Let $Z \supset Y$ be Banach spaces, let T be a compact operator from Y into X and let $\varepsilon > 0$. An easy and well known consequence of the assumption that X has the M.A.P. is that there exists an operator T_0 with a finite-dimensional range (B, say) from X into itself such that $\|T_0\| = 1$ and $\|T_0T-T\| \leq \varepsilon$. Let U be any Banach space satisfying $Z \supset U \supset Y$ and $\dim U/Y < \infty$. By Lemma 1.1 there exist a Banach space $V \supset X$ and an extension T_1 of T from U into V such that $\| T_1 \| = \| T \|$ and $\dim V/X = \dim U/Y$. By (8) there exists an extension $\tilde{T}_0$ of T_0 from V into X with $\| \tilde{T}_0 \| \leq \lambda + \varepsilon$. $\tilde{T}_U = T_0 \tilde{T}_0 T_1$ is an operator from U into B such that $\| \tilde{T}_{U|Y} - T \| \leq 2\varepsilon$ and $\| \tilde{T}_U \| \leq (\lambda+\varepsilon) \| T \|$. Since $\dim B < \infty$ we obtain by using a compactness argument similar to that used in the proof of (4) $\Longrightarrow$ (9) that there is an operator $\tilde{T}$ from Z into B satisfying $\| \tilde{T}_{|Y} - T \| \leq 2\varepsilon$ and $\| \tilde{T} \| \leq (\lambda+\varepsilon) \| T \|$. We have thus shown that X satisfies

(10) For every compact T from Y into X (and every $\varepsilon > 0$) there is a compact $\tilde{T}$ from Z $(Z \supset Y)$ into X such that $\| \tilde{T} \| \leq (\lambda+\varepsilon) \| T \|$ and $\| \tilde{T}_{|Y} - T \| \leq \varepsilon$.

To conclude the proof of the theorem we show that (10) $\Longrightarrow$ (3) (here we shall not use the assumption that X has the M.A.P.). Let $Z \supset Y$, $\varepsilon > 0$, and a compact T from Y into X be given. By (10) there exists a sequence $\{\tilde{T}_n\}_{n=1}^{\infty}$ of compact operators from Z into X satisfying

$$\| \tilde{T}_1 \| \leq (\lambda+\varepsilon) \| T \|, \qquad \| \tilde{T}_{1|Y} - T \| < \varepsilon/2 \qquad .$$

and for $n = 2,3,\cdots$

$$\| \tilde{T}_n \| \leq (\lambda+1) \| T-(\tilde{T}_1+\tilde{T}_2+\cdots+\tilde{T}_{n-1})_{|Y} \|$$

$$\| \tilde{T}_{n|Y} -(T-(\tilde{T}_1+\tilde{T}_2+\cdots+\tilde{T}_{n-1})_{|Y}) \| \leq \varepsilon/2^n \ .$$

For $n \geq 2$ we have in particular that $\| \tilde{T}_n \| \leq (\lambda+1)\varepsilon/2^{n-1}$ and hence the series $\sum_{n=1}^{\infty} \tilde{T}_n$ converges in the norm topology to a compact operator $\tilde{T}$ satisfying $\tilde{T}_{|Y} = T$ and

$$\| \tilde{T} \| \leq \| \tilde{T}_1 \| + \sum_{n=2}^{\infty} (\lambda+1)\varepsilon/2^{n-1} \leq (\lambda+\varepsilon) \| T \| + (\lambda+1)\varepsilon \ .$$

Hence X has property (3) and this concludes the proof of the theorem.

<u>Remark</u>. For $\lambda = 1$ some of the equivalencies of Theorem 2.1 were

obtained by Grothendieck [13, 15] using his theory of tensor products. For that case (i.e. $\lambda=1$) the theorem can be strengthened considerably. We shall discuss this case in detail in Chapter VI.

<u>Corollary 1</u>. Let X be a Banach space satisfying (1) and let Y be a direct summand of a conjugate space. Then every operator T from X into Y has an extension $\tilde{T}$ from Z $(Z \supset X)$ into Y.

<u>Proof</u>. Let Y be a complemented subspace of $U = V^*$. By (5) T can be extended to an operator $\tilde{T}_0$ from Z into U. $\tilde{T} = P\tilde{T}_0$, where P is a projection from U onto Y, has the required properties.

<u>Remark</u>. A Banach space Y is a direct summand of a conjugate space if and only if it is a direct summand of Y^{**}. Indeed, let $U = V^*$ be a space containing Y and let P be a projection from U onto Y. Then, canonically

$$U^{**} \begin{array}{l} \supset Y^{**} \\ \supset U \end{array} \supset Y \, .$$

Let Q be a projection with norm 1 from U^{**} onto U. Then $PQ_{|Y^{**}}$ is a projection from Y^{**} onto Y and $\| PQ_{|Y^{**}} \| \leq \| P \|$.

<u>Corollary</u> 2. Suppose that $X^{**} \supset Z \supset X$ and that Z^{**} is a $\mathcal{P}_\lambda$ space. Then also X^{**} is a $\mathcal{P}_\lambda$ space.

<u>Proof</u>. We proceed as in the proof of (9) ==> (1). Let P be a projection with norm 1 from X^{****} onto $X^{\perp\perp}$. Since $X^{****} \supset Z^{\perp\perp} \supset X^{\perp\perp}$, $P_{|Z^{\perp\perp}}$ is a projection from a $\mathcal{P}_\lambda$ space on a space isometric to X^{**}.

<u>Corollary 3</u>. Let X have property (1) and let B be a finite-dimensional subspace of X on which there is a projection with norm η. Then B is a $\mathcal{P}_{\lambda\eta}$ space.

<u>Proof</u>. Use (6) with $B = Y$, T a projection from X onto B and Z a $\mathcal{P}_1$ space containing X.

Corollary 4. Let X have property (1) and let K be the closure (in the w^* topology) of the extreme points of the unit cell of X^*. Then any weakly compact T from X to Y can be represented as

$$Tx = \int_K \langle x,x^*\rangle \, d\mu(x^*)$$

where μ is a measure on K with values in Y and $\langle x,x^*\rangle = x^*(x)$. The image under T of a w convergent sequence is convergent in the norm topology. If $Y = X$ (i.e. if T maps X into itself) then T^2 is compact.

Proof. K is compact Hausdorff and the mapping $x \longrightarrow \langle x,x^*\rangle$ is an isometry of X into $C(K)$. By (7) T can be extended to $C(K)$. The corollary follows now from known results about weakly compact operators defined on $C(K)$ spaces (see [8, Chapter 6, Section 7]).

In the extension properties listed before Theorem 2.1 it was always required that the norm of the extension $\tilde{T}$ could be estimated by an inequality of the form $\|\tilde{T}\| \le \eta \|T\|$. However it is easily seen that in some cases the mere existence of an extension implies the possibility of such an estimation.

Theorem 2.2.(a) Let X be a Banach space such that every compact T from Y to X has a compact extension from Z $(Z \supset Y)$ to X. Then there is a constant η so that for every such Y, Z and T there is a compact extension $\tilde{T}$ with $\|\tilde{T}\| \le \eta \|T\|$.

(b) Let X be a Banach space such that every compact T from X to Y has a compact extension from Z $(Z \supset X)$ to Y. Then there exists a constant η so that for every such Y, Z and T there is a compact extension $\tilde{T}$ with $\|\tilde{T}\| \le \eta \|T\|$.

Proof. (a) Suppose no such η exists. Then for every n there are spaces $Z_n \supset Y_n$ and a compact operator T_n from Y_n to X with

$\| T_n \| = 1$ such that any compact extension $\tilde{T}_n$ of T_n, from Z_n to X satisfies $\| \tilde{T}_n \| \geq n^3$. Let $Y = (Y_1 \oplus \cdots \oplus Y_n \oplus \cdots)_{c_0}$ and let T be the compact operator from Y to X defined by

$$T(y_1, \cdots, y_n, \cdots) = \sum_{n=1}^{\infty} \frac{T_n y_n}{n^2} .$$

Let $\tilde{T}$ be a compact extension of T from $(Z_1 \oplus \cdots \oplus Z_n \oplus \cdots)_{c_0}$ to X. The restriction of $n^2 \tilde{T}$ to Z_n (i.e. to the sequences $(0, \ldots, 0, z_n, 0, \ldots))$ is an extension of T_n. This contradicts our assumption (for $n > \| \tilde{T} \|$).

(b) It is enough to prove that if Z is a (fixed) $\mathcal{P}_1$ space containing X, then there is an η such that every compact T from X to Y has a compact extension $\tilde{T}$ from Z to Y with $\| \tilde{T} \| \leq \eta \| T \|$. Keeping this in mind the proof proceeds now in the same manner as in part (a).

<u>Remark</u>. (a) and (b) here correspond, respectively, to the extension properties (3) and (6). Similar results corresponding to the extension properties (2), (5) and (7) can be proved in the same manner. Obviously no similar results can be obtained for (4) and (8).

As remarked in the beginning of the introduction the "from" extension property for the class of all operators (property (ii) there) is equivalent to the "into" extension property (property (iii) there). We shall now investigate how far this symmetry between "from" and "into" extension properties carries over to the case of compact or weakly compact operators.

For spaces X having the M.A.P. it was shown in Theorem 2.1 that a "from" extension property for compact operators (property (6)) is equivalent to the "into" extension property (3). The formulation of these properties is, however, not completely symmetric. While in the "from" extension property the extension $\tilde{T}$ satisfies $\| \tilde{T} \| \leq \lambda \| T \|$ the "into" property only assures the existence, for every given $\varepsilon > 0$, of a $\tilde{T}$ with $\|\tilde{T}\| \leq (\lambda+\varepsilon)\|T\|$. In Chapter VII it will be shown that (6) does not imply (3) with $\varepsilon = 0$ even if $\dim Z/X < \infty$.

For weakly compact operators the symmetry breaks down completely. Even if X is a $\mathscr{P}_\lambda$ space it does not have an "into" property corresponding to the "from" property (7) for weakly compact operators. The reason for this phenomenon lies in the special properties of weakly compact operators appearing in Corollary 4 to Theorem 2.1.

Let T be the formal identity operator from ℓ_2 to c_0, i.e. T maps the sequence $(x_1,x_2,\ldots)$ in ℓ_2 to the same sequence in c_0. ℓ_2 is reflexive and hence T is weakly compact. Let Z be any space containing ℓ_2 such that Z^{**} is a $\mathscr{P}$ space (e.g. $Z = C(0,1)$) and let X be any space containing c_0 (in particular X can be any infinite-dimensional $C(K)$ space). T does not have a weakly compact extension from Z to X since the sequence $\{Te_i\}_{i=1}^\infty$ does not converge in the norm topology while $\{e_i\}_{i=1}^\infty$ is w convergent to 0 ($\{e_i\}_{i=1}^\infty$ denotes the natural basis of ℓ_2). Moreover, if X is not a $\mathscr{P}$ space (but still X^{**} may be a $\mathscr{P}$ space) it is possible that T will not have even a bounded extension. This is the case for example if $Z = m$ and $X = c_0$, since any operator from m to a separable space is necessarily weakly compact (Grothendieck [12]).

We conjecture that only finite-dimensional spaces X have the following extension property:

Every weakly compact operator from Y to X has a weakly compact extension from Z $(Z \supset Y)$ to X.

From (4) ==> (1) of Theorem 2.1 and the proof of Theorem 2.2(a) it follows immediately that if X has this extension property then X^{**} is a $\mathscr{P}$ space. Further, as we have seen above, such an X cannot contain a subspace isomorphic to c_0 (the same remark shows that X has no infinite-dimensional reflexive subspace). Hence our conjecture will be proved if it can be shown that every infinite-dimensional space X whose second conjugate is a $\mathcal{P}$ space has a subspace isomorphic to c_0.

We pass now to the problem of extending compact operators between general spaces, when we allow the enlargement of the range space. The question is the following: suppose that the operator T appearing in the statement of Lemma 1.1 is compact [weakly compact], can $\tilde{T}$ be chosen to be also compact [weakly compact]? For the case of weakly compact T the answer is negative even if we discard any restriction on $\|\tilde{T}\|$. This follows from the preceding discussion. In Chapter VII we shall see that for compact T the answer is also in general negative if we require that $\|\tilde{T}\| = \|T\|$. If however we allow even an arbitrarily small increase of the norm the situation is different.

Theorem 2.3. Let T be a compact operator from a Banach space Y into a Banach space X. Then there exists a Banach space $V \supset X$ such that

(i) V/X is separable

(ii) For every $\varepsilon > 0$ and every $Z \supset Y$ there is a compact extension $\tilde{T}$ of T from Z to V with $\|\tilde{T}\| \leq \|T\| + \varepsilon$.

Proof. It is convenient (though not necessary) to use some results which will be proved in the next chapter. Let U be a $\mathcal{P}_1$ space containing X. The subspace $\overline{T(Y)}$ of X is separable, hence (Lemma 3.2) there exists a separable $\mathcal{N}_1$ space U_0 with $T(Y) \subset U_0 \subset U$. The subspace V of U spanned by X and U_0 has the required property (Theorem 3.3).

CHAPTER III. THE $\mathcal{N}_\lambda$ SPACES

We begin with the definition of the $\mathcal{N}_\lambda$ spaces.

Definition. A Banach space X is called an $\mathcal{N}_\lambda$ space if there exists a set $\{B_\tau\}$ of finite-dimensional subspaces of X, directed by inclusion, such that $X = \overline{\bigcup_\tau B_\tau}$ and such that every B_τ is a $\mathcal{P}_\lambda$ space. A Banach space is called an $\mathcal{N}$ space if it is an $\mathcal{N}_\lambda$ space for some λ.

The simplest example of an infinite-dimensional $\mathcal{N}_1$ space is the space c_0. Indeed let $\{e_i\}_{i=1}^\infty$ be the natural basis of this space and let B_n be the subspace spanned by $\{e_i\}_{i=1}^n$. Then clearly $B_n \subset B_{n+1}$, $c_0 = \overline{\bigcup_n B_n}$ and every B_n is a $\mathcal{P}_1$ space. The spaces $C(K)$ with K a totally disconnected compact Hausdorff space, are also simple examples of $\mathcal{N}_1$ spaces. In particular every $\mathcal{P}_1$ space is an $\mathcal{N}_1$ space.

Before beginning to investigate the properties of $\mathcal{N}_\lambda$ spaces it should be remarked that an $\mathcal{N}_\lambda$ space $X = \overline{\bigcup_\tau B_\tau}$ is not fully determined by the spaces B_τ -- it also depends on the nature of the embedding of B_{τ_1} in B_{τ_2} (for $\tau_1 \prec \tau_2$). For example, both c_0 and $C(0,1)$ can be represented as $\overline{\bigcup_n B_n}$ where $B_{n+1} \supset B_n$ and B_n is the (unique up to isometry) n-dimensional $\mathcal{P}_1$ space. That c_0 has such a representation was shown above; we shall prove that this holds also for $C(0,1)$. Let $\{f_i\}_{i=1}^\infty$ be the Schauder basis [6, p. 69] of $C(0,1)$. For every $f \in C(0,1)$ the sum of the first n terms of the expansion of f with respect to $\{f_i\}_{i=1}^\infty$ ($P_n(f)$, say) is a function whose graph is a polygon with vertices belonging to the graph of f. Hence $\|P_n(f)\| \leq \|f\|$, and thus P_n is a projection with norm 1 from $C(0,1)$ onto the subspace B_n spanned by $\{f_i\}_{i=1}^n$. By Corollary 3 to Theorem 2.1 B_n is a $\mathcal{P}_1$ space. From this the above assertion concerning $C(0,1)$ follows.

An $\mathcal{N}_\lambda$ space is defined as the closure of a union of $\mathcal{P}_\lambda$ spaces. In the following lemma it is shown that we can dispense with the closure provided λ is replaced by $\lambda + \varepsilon$ with $\varepsilon > 0$.

<u>Lemma 3.1</u>. Let X be an $\mathcal{N}_\lambda$ space and let $\lambda' > \lambda$. Then there exists a set $\{B_{\tau'}\}$ of finite-dimensional subspaces of X, directed by

inclusion, so that $X = \bigcup B_{\tau'}$ and that every $B_{\tau'}$ is a $\mathcal{P}_{\lambda'}$ space.

<u>Proof</u>. The statement of the lemma is clearly equivalent to the following: every finite set $\{x_i\}_{i=1}^n$ of points of X is contained in a finite-dimensional $\mathcal{P}_{\lambda'}$ space. Without loss of generality we may assume that the $\{x_i\}_{i=1}^n$ are linearly independent. Hence there is a constant M such that for every choice of $\{\gamma_i\}_{i=1}^n$, $\sum_{i=1}^n |\gamma_i| \leq M \| \sum_{i=1}^n \gamma_i x_i \|$. Let $\varepsilon > 0$. Since X is an $\mathcal{N}_\lambda$ space there exists a finite-dimensional subspace B of X which is a $\mathcal{P}_\lambda$ space and which contains points $\{y_i\}_{i=1}^n$ satisfying $\| y_i - x_i \| \leq \varepsilon$ $(i=1,\ldots,n)$. If ε is sufficiently small the $\{y_i\}_{i=1}^n$ will also be linearly independent. Denote the subspace spanned by the $\{y_i\}_{i=1}^n$ by C and let P be a projection from X onto C with $\|P\| \leq n$. We now choose points $\{z_j\}_{j=1}^m$ with $Pz_j = 0$ $(j=1,\ldots,m)$ such that the $\{y_i\}_{i=1}^n$ and $\{z_j\}_{j=1}^m$ form together a basis of B. Thus every $b \in B$ admits a unique representation of the form

$$b = \sum_{j=1}^m \alpha_j z_j + \sum_{i=1}^n \beta_i y_i .$$

By the choice of the z_j we have that $Pb = \sum_{i=1}^n \beta_i y_i$. By our assumption on the y_i

$$\| \sum_{i=1}^n \beta_i y_i - \sum_{i=1}^n \beta_i x_i \| \leq \varepsilon \sum_{i=1}^n |\beta_i| \leq \varepsilon M \| \sum_{i=1}^n \beta_i x_i \| .$$

Hence, if $\varepsilon M < 1$,

$$\| \sum_{i=1}^n \beta_i y_i - \sum_{i=1}^n \beta_i x_i \| \leq \frac{M\varepsilon}{1-M\varepsilon} \| \sum_{i=1}^n \beta_i y_i \| \leq \frac{Mn\varepsilon}{1-M\varepsilon} \|b\| ,$$

and therefore

$$(1 - \frac{Mn\varepsilon}{1-\varepsilon M}) \|b\| \leq \| \sum_{j=1}^m \alpha_j z_j + \sum_{i=1}^n \beta_i x_i \| \leq (1 + \frac{Mn\varepsilon}{1-\varepsilon M}) \|b\| .$$

Thus if ε is sufficiently small B will be "almost" isometric to the subspace $\tilde{B}$ of X spanned by $\{z_j\}_{j=1}^m$ and $\{x_i\}_{i=1}^n$ (M and n

depend only on $\{x_i\}_{i=1}^n$ while B and $\tilde{B}$ depend also on ε). In particular ε may be chosen so small that $\tilde{B}$ will be a $\mathcal{P}_{\lambda'}$ space.

Remark. Lemma 3.1 does not hold with $\lambda = \lambda'$. As a counter example we take the space $C(0,1)$. We already know that it is an $\mathcal{N}_1$ space. Let B be a finite-dimensional subspace of $C(0,1)$ whose unit cell is not a polyhedron. Since the unit cell of a finite-dimensional $\mathcal{P}_1$ space is a polyhedron such a space cannot contain B as a subspace. Hence $C(0,1)$ cannot be represented as the union of a set of $\mathcal{P}_1$ spaces directed by inclusion. In this connection we would like to remark that in Chapter VII it will be shown that every finite-dimensional subspace of an $\mathcal{N}_1$ space X (and even of a space which is an $\mathcal{N}_{1+\varepsilon}$ space for every $\varepsilon > 0$) whose unit cell is a polyhedron is contained in a finite-dimensional $\mathcal{P}_1$ subspace of X. In particular, every $\mathcal{N}_1$ space all whose finite-dimensional subspaces have a polyhedron as their unit cell (for example c_0) can be represented as a union of a directed set of finite-dimensional $\mathcal{P}_1$ spaces (for details see Theorem 7.9).

Lemma 3.2. (a) Let X be an $\mathcal{N}_\lambda$ space and Y a separable subspace of X. Then there exists a separable $\mathcal{N}_\lambda$ space Z with $Y \subset Z \subset X$.

(b) Let X be a Banach space such that for every separable subspace Y of X there exists an $\mathcal{N}$ space Z with $Y \subset Z \subset X$. Then X is an $\mathcal{N}$ space. If for every such Y there exists a Z (with $Y \subset Z \subset X$) which is an $\mathcal{N}_\lambda$ space (λ does not depend on Y) then X is an $\mathcal{N}_{\lambda'}$ space for every $\lambda' > \lambda$.

Proof. (a) Let $X = \overline{\bigcup_\tau B_\tau}$ where the B_τ form a set, directed by inclusion, of subspaces of X which are $\mathcal{P}_\lambda$ spaces. Let $\{y_i\}_{i=1}^\infty$ be a dense sequence in Y. For every i let $\{x_i^n\}_{n=1}^\infty$ be a sequence such that $\| x_i^n - y_i \| \longrightarrow 0$ as $n \longrightarrow \infty$ and $x_i^n \in \bigcup_\tau B_\tau$ (i.e. for every i and n there is a τ_i^n with $x_i^n \in B_{\tau_i^n}$). Since the B_τ are directed by inclusion there exists a sequence τ_n such that

$$B_{\tau_{n+1}} \supset \bigcup_{i,m\leq n+1} B_{\tau_i^m} \cup B_{\tau_n}$$

$(B_{\tau_1} = B_{\tau_1^1})$. The subspace $Z = \overline{\bigcup_n B_{\tau_n}}$ has the required properties.

(b) Suppose that X is not an $\mathcal{N}$ space. Then for every n there is a finite-dimensional subspace B_n of X such that no finite-dimensional subspace of X containing B_n is a $\mathcal{P}_n$ space. By Lemma 3.1 $\bigcup_n B_n$ is not contained in an $\mathcal{N}$ space and this contradicts our assumption. The second part of assertion (b) also follows immediately from Lemma 3.1.

Lemma 3.2 shows that some problems concerning $\mathcal{N}$ spaces can be reduced to the separable case. In this respect $\mathcal{N}$ spaces differ from $\mathcal{P}$ spaces, which are either finite-dimensional or non-separable.

We show now that the $\mathcal{N}_\lambda$ spaces have the extension properties which were listed in the beginning of Chapter II. By Theorem 2.1 it is sufficient to show that the $\mathcal{N}_\lambda$ spaces have property (3). This is done in

Theorem 3.3. Let X be an $\mathcal{N}_\lambda$ space. Then for every $Z \supset Y$, every $\varepsilon > 0$, and every compact T from Y into X there is a compact extension $\tilde{T}$ of T from Z into X with $\|\tilde{T}\| \leq (\lambda+\varepsilon)\|T\|$.

Proof. By the proof of Theorem 2.1 it is enough to show that every $\mathcal{N}_\lambda$ space has property (10) (this property was defined at the end of the proof of Theorem 2.1). Let $Z \supset Y$, an $\varepsilon > 0$ and a compact $T \neq 0$ from Y into X be given. Let $K = T(S_Y)$ and let $\delta > 0$. By the compactness of T there exists a finite set $\{x_i\}_{i=1}^n$ in X such that $K \subset \bigcup_{i=1}^n S_X(x_i,\delta)$. By Lemma 3.1 there is a finite-dimensional subspace B of X such that $\{x_i\}_{i=1}^n \subset B$ and B is a $\mathcal{P}_{\lambda+\delta}$ space. Let P be a projection from X

onto B with $\|P\| \leq \lambda+\delta$. For every $y \in Y$ with $\|y\| \leq 1$ there is an $x \in B$ with $\|Ty-x\| \leq \delta$. Hence

$$\|PTy-Ty\| \leq \|PTy-Px\| + \|x-Ty\| \leq \delta(\lambda+1+\delta) \ .$$

Since B is a $\mathscr{P}_{\lambda+\delta}$ space there is an extension $\tilde{T}$ of PT from Z into B satisfying

$$\|\tilde{T}\| \leq (\lambda+\delta)\|PT\| \leq (\lambda+\delta)(\|T\| + (\lambda+1+\delta)\delta) \ .$$

Thus if δ is small enough we get that $\|\tilde{T}\| \leq (\lambda+\varepsilon)\|T\|$ and $\|\tilde{T}_{|Y}-T\| \leq \varepsilon$, and this concludes the proof.

We pass to the question of the validity of the converse of Theorem 3.3. We shall first introduce the following notion.

Definition. Let η be a scalar ≥ 1. A Banach space X has the η projection approximation property (η - P.A.P.) if $X = \overline{\bigcup_\tau B_\tau}$ where $\{B_\tau\}$ is a set, directed by inclusion, of finite-dimensional subspaces of X such that for every τ there exists a projection P_τ from X onto B_τ with $\|P_\tau\| \leq \eta$.

Examples. Evidently every $\mathscr{N}_\lambda$ space has the λ - P.A.P. As well known, every separable Banach space with a basis has the η - P.A.P. for some η. The $L_p(\mu)$ spaces $(1 \leq p \leq \infty$, μ an arbitrary measure) have the 1 - P.A.P. This follows from the fact that for every decomposition of the measure space Ω into n disjoint sets $\{\Omega_i\}_{i=1}^n$ there is a projection with norm 1 from $L_p(\mu)$ onto the subspace spanned by the characteristic functions of the Ω_i (for $p < \infty$ only of those Ω_i with $\mu(\Omega_i) < \infty$). We do not know whether there exists a space which does not have the η-P.A.P. for any η.

Let X have the η- P.A.P. and let $X = \overline{\bigcup_\tau B_\tau}$ be the representation of X ensured by this property. Suppose X^{**} is a $\mathscr{P}_\lambda$ space. Then it follows immediately (see Corollary 3 to Theorem 2.1.) that every B_τ is a

$\mathcal{P}_{\lambda\eta}$ space and hence X is an $\mathcal{N}_{\lambda\eta}$ space. In the next theorem we shall show that for spaces having the η - P.A.P. much weaker extension properties than those listed before Theorem 2.1 or those appearing in Theorem 2.2 imply that X is an $\mathcal{N}$ space.

Theorem 3.4. Let X be a Banach space which has the η - P.A.P. for some η but which is not an $\mathcal{N}$ space. Then

(a) There exists a compact operator T from X into itself which does not have even a bounded extension from some Z (containing X) to X^{**}.

(b) There exists a compact operator T from X into a separable reflexive space Y which does not have even a bounded extension from some Z (containing X) to Y.

As Z one may take in (a) and (b) any space containing X such that Z^{**} is a $\mathcal{P}$ space.

Proof. (a) Let $X = \overline{\bigcup_\tau B_\tau}$ be the representation of X ensured by the η - P.A.P., and let P_τ be a projection from X onto B_τ with $\| P_\tau \| \leq \eta$. For every τ_0 $X = \overline{\bigcup_{\tau > \tau_0} B_\tau}$ and hence, since X is not an $\mathcal{N}$ space, there exists a sequence τ_n with $\tau_{n+1} \succ \tau_n$ and $\lambda_{\tau_n} = P(B_{\tau_n}) \longrightarrow \infty$. ($P(Y)$ denotes the projection constant of the space Y, that is $\inf \{\lambda;\ Y \text{ is a } \mathcal{P}_\lambda \text{ space}\}$).

We shall choose now, inductively, a sequence of integers n_k and two sequences of positive numbers α_k and β_k satisfying (among other requirements),

$$(3.1) \qquad \beta_i > \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \quad 1 \leq i \leq j .$$

We take first $n_1 = 1$, $\alpha_1 = 1$ and $\beta_1 = 2$. Suppose we have already chosen α_i, β_i and n_i for $i \leq k$ so that (3.1) is satisfied for $1 \leq j \leq k$. The operator

$$Q_k = \alpha_1 P_{\tau_{n_1}} + \cdots + \alpha_k P_{\tau_{n_k}}$$

maps X into its subspace $B_{\tau_{n_k}}$ and hence has only a finite number of eigenvalues. We choose $\beta_{k+1} > 0$ so that $Q_k + aI$ (I is the identity operator of X) has an inverse for $0 < a \leq \beta_{k+1}$. We next choose $\alpha_{k+1} > 0$ so that (3.1) holds for $i \leq j \leq k+1$, i.e. α_{k+1} must satisfy the inequality

$$0 < \alpha_{k+1} < \min\,(\beta_{k+1};\ \beta_i - \sum_{h=i}^{k} \alpha_h,\ 0 < i \leq k)\ .$$

Let

$$(3.2) \qquad \gamma_{k+1} = \sup_{\alpha_{k+1} \leq a \leq \beta_{k+1}} \|(Q_k + aI)^{-1}\|\ .$$

That $\gamma_{k+1} < \infty$ follows from the choice of β_{k+1}. Finally we choose n_{k+1} so that

$$(3.3) \qquad \lambda_{\tau_{n_{k+1}}} = P(B_{\tau_{n_{k+1}}}) \geq k\,\gamma_{k+1}\ .$$

In order to simplify the notation we put

$$\lambda_k = \lambda_{\tau_{n_k}}, \quad B_k = B_{\tau_{n_k}}, \quad P_k = P_{\tau_{n_k}}\ .$$

Let $T = \sum_{i=1}^{\infty} \alpha_i P_i$. The series converges absolutely since $\|P_i\| \leq \eta$, $\alpha_i > 0$ and $\sum_{i=1}^{\infty} \alpha_i \leq \beta_1 = 2$. Hence T is a compact operator from X into itself. Suppose there exists a bounded extension $\tilde{T}$ of T from Z to X^{**} ($Z \supset X$ and Z^{**} is a $\mathcal{P}_\lambda$ space). P_i^{**} is a projection from X^{**} onto B_i and hence $\tilde{T}_i = P_i^{**}\tilde{T}$ maps Z into B_i ($i=1,2,\ldots$). Since $B_{k+1} \subset B_i$ for $i > k+1$ we have

$$\tilde{T}_{k+1}|_{B_{k+1}} = (\alpha_1 P_1 + \cdots + \alpha_k P_k + (\alpha_{k+1} + \cdots)I)|_{B_{k+1}}\ ,$$

or

$$\tilde{T}_{k+1}|_{B_{k+1}} = (Q_k + \delta_k I)|_{B_{k+1}}$$

where $\delta_k = \alpha_{k+1} + \alpha_{k+2} + \cdots$ satisfies $\alpha_{k+1} < \delta_k \leq \beta_{k+1}$ (see (3.1)).

Let $S_k = (Q_k + \delta_k I)^{-1}$. By (3.2) $\| S_k \| \leq \gamma_{k+1}$. Since $(Q_k+\delta_k I)x \in B_{k+1}$ if and only if $x \in B_{k+1}$ it follows that $S_k\tilde{T}_{k+1}$ is a projection from Z onto B_{k+1}. $\| S_k\tilde{T}_{k+1}\| \leq \eta\gamma_{k+1}\|\tilde{T}\|$ and therefore (Corollary 3 to Theorem 2.1) $P(B_{k+1}) \leq \lambda\eta\gamma_{k+1}\|\tilde{T}\|$ and this contradicts (3.3) for $k > \lambda\eta\|\tilde{T}\|$.

(b) From the assumptions on X it follows that there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of finite-dimensional subspaces of X such that $P(B_n) \geq n^2$ and such that there exists a projection P_n with norm $\leq \eta$ from X onto B_n $(n = 1,2,\ldots)$. The operator T from X into $(B_1 \oplus B_2 \oplus \cdots)_{\ell_2}$ defined by

$$Tx = (P_1x, P_2x/2, \ldots, P_nx/n, \ldots)$$

has the required properties. We omit the easy details.

We end this chapter with some remarks and open problems concerning $\mathcal{N}$ spaces. First some words on the relation between $\mathcal{N}$ and $\mathcal{P}$ spaces. By Theorems 2.1 and 3.3 an $\mathcal{N}$ space is a $\mathcal{P}$ space if and only if it is complemented in a conjugate space. It is easily seen that if Y is a complemented subspace of an $\mathcal{N}$ space and if Y has the η - P.A.P. for some η then also Y is an $\mathcal{N}$ space. Actually, by using an argument similar to that used in the proof of Lemma 3.2 it can be seen that for the validity of the statement in the preceding sentence, it is enough to assume that every separable subspace of Y has the η - P.A.P. for some η (which may depend on the subspace). In particular every $\mathcal{P}$ space which has the η - P.A.P. for some η (or all whose separable subspaces have the η- P.A.P.) is an $\mathcal{N}$ space.

The question of a suitable functional representation of $\mathcal{N}$ spaces remains open and seems to be difficult. In the beginning of the chapter we remarked that some C(K) spaces are $\mathcal{N}_1$ spaces. It is easy to show that every C(K) space is an $\mathcal{N}_{1+\varepsilon}$ space for every $\varepsilon > 0$ (we do not know whether every $C(K)$ space is an $\mathcal{N}_1$ space). Indeed, let $\{f_i\}_{i=1}^n \subset C(K)$ and an $\varepsilon > 0$ be given. Then there is a partition of the unity $\{\varphi_j\}_{j=1}^m$ such that the distance of each f_i from the subspace (B, say) spanned by the $\{\varphi_j\}_{j=1}^m$ is less than ε. (By a partition of the unity we mean here a finite set $\{\varphi_j\}_{j=1}^m \subset C(K)$ such that $\varphi_j \geq 0$, $\|\varphi_j\| = 1$ for every j and $\sum_j \varphi_j(k) = 1$ for every k.) B is isometric to ℓ_∞^m and hence it is a $\mathcal{P}_1$ space. Combining these remarks with the proof of Lemma 3.1 we get immediately that $C(K)$ is an $\mathcal{N}_\lambda$ space for every $\lambda > 1$. Hence a Banach space which is isomorphic to a $C(K)$ space is an $\mathcal{N}$ space. We do not know whether conversely, every $\mathcal{N}$ space is isomorphic to a $C(K)$ space. From Theorems 2.1 and 3.3 and from known results concerning $\mathcal{P}$ spaces it follows that the common Banach spaces which are not isomorphic to $C(K)$ spaces are also not $\mathcal{N}$ spaces.

The question whether a Banach space is isomorphic to a $C(K)$ space if (and only if) it is an $\mathcal{N}$ space seems to be of interest not only from the point of view of the extension properties which we study here. An affirmative answer to this question would contribute much to the knowledge of the structure of $C(K)$ spaces. There are some easily established facts concerning $\mathcal{N}$ spaces for which it seems to be an open question whether they hold also for spaces isomorphic to $C(K)$ spaces. One such fact is Lemma 3.2 part (b), another is the following result. Let $X \supset Y$ be Banach spaces such that Y and X/Y are $\mathcal{N}$ spaces. Then also X is an $\mathcal{N}$ space. (Indeed, it can be shown by first proving a similar result for $\mathcal{P}$ spaces and then using Lemma 3.1 that if Y is an $\mathcal{N}_\lambda$ space and if X/Y is an $\mathcal{N}_\eta$ space then X is an $\mathcal{N}_\rho$ space for every $\rho > \lambda + \eta + \lambda\eta$.)

CHAPTER IV. INTERSECTION PROPERTIES OF CELLS

In this chapter we shall first discuss the relation between certain intersection properties of cells and then some other geometrical properties, which are closely related with a certain intersection property, will be investigated. The results obtained in the present chapter will be used in Chapters V, VI and VII for the study of problems concerning extension of operators. However, in the present chapter extension properties are not considered at all.

We define now the main intersection properties which will concern us in this chapter.

A normed space X has the n, k intersection property $(n,k.I.P.)$, where n and k are integers with $n > k \geq 2$, if for every collection of n cells in X such that any k of them have a non void intersection, there is a point common to all the n cells.

A normed space X has the finite k intersection property (F.k.I.P.) if it has the $n,k.I.P.$ for every $n > k$.

A normed space X has the restricted n,k intersection property $(R.n,k.I.P.)$, where $n > k \geq 2$, if for every collection of n cells in X with a common radius such that any k of them have a non void intersection, there is a point common to all the n cells.

Similarly we define the R.F.k.I.P.

The definitions of the intersection properties remain meaningful for general metric spaces. However we shall study here these properties only in normed (linear) spaces.

We begin with a theorem showing that for Banach spaces the $n,k.I.P$ already implies the $F.k.I.P$ if n is sufficiently large.

<u>Theorem 4.1</u>. Let n and k be integers such that $k \geq 2$ and

$$n > \frac{4k - 5 + \sqrt{1 + 8(k-1)^2}}{2} \tag{4.1}$$

Then, for Banach spaces, the n,k.I.P implies the F.k.I.P.

Proof. It is clearly enough to show that for a Banach space X and for n satisfying (4.1) the n,k.I.P. implies the $(n+1)$, k.I.P. Let $\{S_i\}_{i=1}^{n+1}$ be $n+1$ cells in X such that any k of them intersect. Let x be an arbitrary point in X and denote

$$\theta = \max_{1 \le i_1 < i_2 < \cdots < i_{k-1} \le n+1} d(x, \bigcap_{j=1}^{k-1} S_{i_j})$$

($d(x,K)$ denotes the distance of the point x from the set K). Let ε be a positive number. From the definition of θ it follows that if we add the cell $S = S(x,\theta+\varepsilon)$ to the given $n+1$ cells we shall still have a collection of cells in which any k intersect. Consider the set $A = \{1,2,\ldots,n+1\}$. Let Ω denote the set of all the subsets α of A consisting of $n-1$ numbers. The number of elements of Ω is $n(n+1)/2$. For every $\alpha \in \Omega$ there is a $y_\alpha \in X$ such that

$$y_\alpha \in \bigcap_{i \in \alpha} S_i \cap S .$$

Let

$$y = \frac{2}{n(n+1)} \cdot \sum_{\alpha \in \Omega} y_\alpha .$$

Since $y_\alpha \in S$ for every α the same is true for y i.e.

$$\|y - x\| \le \theta + \varepsilon . \tag{4.2}$$

Let now $\{i_j\}_{j=1}^{k-1}$ be $k-1$ integers satisfying $1 \le i_1 < i_2 \cdots < i_{k-1} \le n+1$. We shall estimate $d(y, \bigcap_{j=1}^{k-1} S_{i_j})$. The number of the α such that $\{i_j\}_{j=1}^{k-1} \subset \alpha$ is equal to $(n-k+2)(n-k+1)/2$. Hence, since $\bigcap_{j=1}^{k-1} S_{i_j}$ is convex, we have

$$d(y, \bigcap_{j=1}^{k-1} S_{i_j}) \le \frac{2}{(n+1)n} \sum_{\alpha\in\Omega} d(y_\alpha, \bigcap_{j=1}^{k-1} S_{i_j})$$

$$= \frac{2}{(n+1)n} \cdot \sum_{\alpha \not\supset \{i_j\}_{j=1}^{k-1}} d(y_\alpha, \bigcap_{j=1}^{k-1} S_{i_j})$$

$$\le \frac{2}{(n+1)n} \cdot \sum_{\alpha \not\supset \{i_j\}_{j=1}^{k-1}} (d(x, \bigcap_{j=1}^{k-1} S_{i_j}) + \| y_\alpha - x \|)$$

$$\le \frac{2}{(n+1)n} \left(\frac{(n+1)n}{2} - \frac{(n-k+2)(n-k+1)}{2} \right)(2\theta+\varepsilon) = (\theta + \tfrac{\varepsilon}{2})\, f(n,k) ,$$

where $f(n,k) = 2((2k-2)n-k^2+3k-2)/n(n+1)$. For n satisfying (4.1) it is easily seen that $f(n,k) < 1$. Hence if c satisfies $f(n,k) < c < 1$ and if ε is taken small enough we have

$$c\theta \ge \max_{1\le i_1<i_2\cdots<i_{k-1}\le n+1} d(y, \bigcap_{j=1}^{k-1} S_{i_j}) . \tag{4.3}$$

From (4.2) and (4.3) it follows that there exists a sequence z_m (with $z_0 = x$) satisfying

$$\| z_{m+1} - z_m \| \le 2c^m\theta$$

$$c^m\theta \ge \max_{1\le i_1<i_2\cdots<i_{k-1}\le n+1} d(z_m, \bigcap_{j=1}^{k-1} S_{i_j}) .$$

The sequence z_m is a Cauchy sequence and since X is complete it converges to a point z. $d(z, \bigcap_{j=1}^{k-1} S_{i_j}) = 0$ for every $\{i_j\}_{j=1}^{k-1}$, i.e. $z \in \bigcap_{i=1}^{n+1} S_i$, and this concludes the proof of the theorem.

Remark. For k=2 we obtain that 4,2.I.P $\Longrightarrow$ F.2.I.P. As we shall see later the 3,2.I.P. is a weaker property and does not imply the F.2.I.P. Hence for k=2 Theorem 4.1 gives the best possible n. We do not know whether this is the case also for $k \ge 3$. For the special case k=2 and X finite-dimensional the result of Theorem 4.1 was obtained, by different methods, by Hanner [20]. In Chapter VI we shall give an alternative proof for the case k=2 which is valid also for non complete normed spaces (assuming, however, that the unit cell has at least one extreme point). The case $k=2$ of Theorem 4.1 solves problem 1 in the paper of Aronszajn and Panitchpakdi [2].

From now on our interest will be focused on the n,k.I.P. with $k=2$ i.e. on the 3,2.I.P. and the 4,2.I.P.

Lemma 4.2. Suppose the Banach space X has (for some integer $n \geq 3$) the following property:

Every collection of n cells $\{S(x_i, r_i)\}_{i=1}^n$ in X, such that every two of them intersect, has for every $\varepsilon > 0$ a point $x = x_\varepsilon \in X$ satisfying

$$\| x - x_i \| \leq r_i + \varepsilon \ , \ i = 1,2,\dots,n \ .$$

Then X has the n,2.I.P.

Proof. Let $\{S(x_i, r_i)\}_{i=1}^n$ be a collection of n cells in X such that every two of them intersect (i.e. $\| x_i - x_j \| \leq r_i + r_j$ for every i and j). We have to prove that there is a point common to all the n cells. Let $\varepsilon > 0$, and let x be a point satisfying $\| x - x_i \| \leq r_i + \varepsilon$, $i = 1,2,\dots,n$. The cell $S(x, \varepsilon)$ intersects any of the n given cells. Hence, for every i, $1 \leq i \leq n$, and every $\delta > 0$, there is a point $y_i = y_i(\delta)$ satisfying

$$\| y_i - x \| \leq \varepsilon + \delta; \quad \| y_i - x_j \| \leq r_j + \delta \ , \quad i \neq j \ .$$

Let

$$y = \frac{1}{n} \cdot \sum_{i=1}^{n} y_i \ .$$

Then $\| y - x \| \leq \varepsilon + \delta$ and

$$\| y - x_j \| \leq \frac{1}{n} \Big(\sum_{i \neq j} \| y_i - x_j \| + \| y_j - x_j \| \Big)$$

$$\leq \frac{1}{n} \big((n-1)(r_j + \delta) + \| y_j - x \| + \| x - x_j \| \big) \leq r_j + \delta + 2\varepsilon/n.$$

Since $n \geq 3$ we obtain for $\delta \leq \varepsilon/6$ that

$$(4.4) \qquad \| y - x \| \leq 2\varepsilon \ ; \quad \| y - x_j \| \leq r_j + 5\varepsilon/6, \quad j = 1,\dots,n \ .$$

From (4.4) it follows that there exists a sequence z_m (with $z_0 = x$) satisfying

$$\| z_{m+1} - z_m \| \leq 2(\tfrac{5}{6})^m \varepsilon \ ; \quad \| z_{m+1} - x_j \| \leq r_j + (\tfrac{5}{6})^m \varepsilon, \quad j=1,\ldots,n \ .$$

Let $z = \lim_{m \to \infty} z_m$, then $z \in \bigcap_{j=1}^{n} S(x_j, r_j)$ and this concludes the proof of the lemma.

Remark. Aronszajn and Panitchpakdi [2] proved (Theorem 4 of section 3 in their paper) that for general complete metric spaces the property appearing in the formulation of the lemma implies the $(n-1)$, 2.I.P. They raised the problem (problem 4 in their paper) whether it also implies the n, 2.I.P. Lemma 4.2. solves this problem for Banach spaces.

The following corollary is an immediate consequence of Lemma 4.2.

Corollary. Let X be a normed space having the n, 2.I.P. (for some $n \geq 3$). Then the completion of X also has the n, 2.I.P.

It should be remarked that the converse statement is false. In Chapter VI we shall give an example of a normed space which does not have the 4,2.I.P., while its completion has this property.

Lemma 4.2. will be used in the next two theorems.

Theorem 4.3. Let X be a Banach space. Then for every $n \geq 3$ the R.n,2.I.P implies the n,2.I.P.

Proof. Suppose X has the R.n,2.I.P. Let $\{S(x_i, r_i)\}_{i=1}^{n}$ be a collection of n cells in X such that every two of them intersect. From Lemma 4.2 it follows that it is enough to show that for every $\varepsilon > 0$ there is a point $x \in X$ satisfying $\| x - x_i \| \leq r_i + \varepsilon$ for every i. Suppose that there is an $\varepsilon > 0$ for which no such x exists and let r be a number satisfying $r > r_i$ $(1 \leq i \leq n)$. We shall construct n cells $S(y_i, r)$ in X such that

$$S(y_i, r) \supset S(x_i, r_i) \ , \quad i = 1, \ldots, n$$

and

(4.5) $$\bigcap_{i=1}^{n} S(y_i,\ r) = \emptyset\ .$$

This will contradict our assumption that X has the R.n,2.I.P.

The y_i will be chosen inductively. Suppose that we have already chosen y_i for $i \leq j$ ($< n$) such that

(4.6) $$S(y_i,\ r) \supset S(x_i,\ r_i)\ , \qquad i \leq j$$

and

(4.7) $$\bigcap_{i=1}^{j} S(y_i,\ r) \cap \bigcap_{i=j+1}^{n} S(x_i,\ r_i + \varepsilon) = \emptyset\ .$$

We shall choose y_{j+1} (j may also be equal to 0, we adhere to the usual convention that an empty intersection is the whole space).

Let
$$K = \bigcap_{i=1}^{j} S(y_i,\ r) \cap \bigcap_{i=j+2}^{n} S(x_i,\ r_i + \varepsilon)\ .$$

K is a closed convex set whose intersection with the cell $S(x_{j+1},\ r_{j+1}+\varepsilon)$ is empty. Hence there exists a functional $f \in X^*$ such that $\|f\| \leq 1/(r_{j+1} + \varepsilon)$ and such that for every $x \in K$ we have $f(x - x_{j+1}) \geq 1$. Let $z \in X$ be a point satisfying $\|z\| = 1$ and $f(z) \leq -\|f\| + \delta$, where δ is a positive number which will be fixed below. We claim that if δ is small enough and if we set $y_{j+1} = x_{j+1} + (r - r_{j+1})z$ then (4.6) and (4.7) (with j replaced in both by $j + 1$) will be satisfied. Let $x \in S(x_{j+1},\ r_{j+1})$ then,

$$\|x - y_{j+1}\| \leq \|x - x_{j+1}\| + r - r_{j+1} \leq r$$

and hence $S(y_{j+1},\ r) \supset S(x_{j+1},\ r_{j+1})$. Let now $x \in S(y_{j+1},\ r)$, then

$$r\|f\| \geq f(x - y_{j+1}) = f(x - x_{j+1}) - (r - r_{j+1})f(z)$$

or

$$f(x - x_{j+1}) \leq r\|f\| + (r - r_{j+1})(-\|f\| + \delta) \leq$$

$$\leq \frac{r_{j+1}}{r_{j+1}+\varepsilon} + \delta(r - r_{j+1}) \ .$$

Thus if δ is small enough, $f(x - x_{j+1}) < 1$ for every $x \in S(y_{j+1}, r)$ and hence $K \cap S(y_{j+1}, r) = \emptyset$, which is (4.7) for $j + 1$.

Substituting $j = n$ in (4.7) we obtain (4.5) and this concludes the proof of the theorem.

Remark. Theorem 4.3 was proved by Hanner [20] for finite-dimensional spaces. The basic idea of our proof is taken from the proof of Hanner. The difference between the infinite-dimensional case and the finite-dimensional one (i.e. the case treated by Hanner) is that for finite-dimensional spaces stronger separation theorems for convex sets are available. This fact necessitated the use of Lemma 4.2 in the proof given here (for finite-dimensional spaces Lemma 4.2 follows, of course, immediately from the local compactness).

Theorem 4.4. Let X be a Banach space.

(a) If X has the F.2.I.P. then for every separable subspace Y of X there exists a separable space Z having the F.2.I.P. with $Y \subset Z \subset X$.

(b) If for every 3-dimensional subspace Y of X there exists a space Z having the F.2.I.P. with $Y \subset Z \subset X$, then X has the F.2.I.P.

Proof. (a) Let $\{y_i^1\}_{i=1}^{\infty}$ be a dense sequence in Y. Let Ω_1 be the set of all the collections of a finite number of cells with rational radii and with centers taken from the set $\{y_i^1\}_{i=1}^{\infty}$, such that any two cells in a collection intersect. Ω_1 is denumerable. For every collection α in Ω_1 we choose a point $x_\alpha \in X$ belonging to the intersection of all the cells in α. Let Y_2 be the subspace of X spanned by Y and the points x_α, $\alpha \in \Omega_1$. Y_2 is separable. Proceeding similarly we obtain an increasing sequence Y_m of separable subspaces of X and a dense sequence $\{y_i^m\}_{i=1}^\infty$ in Y_m such that for every collection of a finite number of mutually intersecting cells with rational radii and with centers taken from the sequence $\{y_i^m\}_{i=1}^\infty$ there is a point in Y_{m+1} common to all the cells in the collection. Let $Z = \overline{\bigcup_m Y_m}$. Z is separable and $Y \subset Z \subset X$. We shall show that Z has the F.2.I.P. Let $\{S(z_i, r_i)\}_{i=1}^n$ be n mutually intersecting cells in Z and let $\varepsilon > 0$. There exists an m and $j_1, \ldots, j_n$ such that

$$\| y_{j_i}^m - z_i \| \le \varepsilon/2 \ , \quad i = 1,2,\ldots,n \ .$$

Let $\{R_i\}_{i=1}^n$ be rational numbers satisfying

$$\varepsilon/2 < R_i - r_i < \varepsilon \ , \quad i = 1,2,\ldots,n \ .$$

Any two of the n cells $S(y_{j_i}^m, R_i)$ intersect and hence there is a point $z \in Y_{m+1} \subset Z$ common to all these cells. In particular $\| z - z_i \| \le r_i + \frac{3}{2}\varepsilon$ for every i. The desired result follows by using Lemma 4.2.

(b) From Theorem 4.1 it follows that it is enough to prove that for every four cells in X such that any two of them intersect there is a point common to all the cells. This property is invariant with respect to translations, hence we can assume that the center of one of the four cells is the origin. The assertion is now immediate.

Remark. It is clear that a result similar to Theorem 4.4 holds also for the 3,2.I.P. (for this property we can even replace 3 by 2 in (b)). It is likely that also Theorem 4.4 (b) itself holds when 3 is replaced by 2. We shall prove in chapter VI that this is indeed the case if we assume that the unit cell of X has at least one extreme point.

The next result shows that the F.2.I.P. implies a (formally) stronger intersection property in which the set of centers of the cells is assumed to be compact instead of finite. In the proof of the theorem we shall use some methods which were also used by Aronszajn and Panitchpakdi in [2].

Theorem 4.5. Let X be a Banach space having the F.2.I.P. Let $\{S(x_\alpha, r_\alpha)\}_{\alpha\in A}$ be a collection of cells in X such that any two of them intersect and such that the set of the centers $\{x_\alpha\}_{\alpha\in A}$ is conditionally compact. Then

$$\bigcap_{\alpha\in A} S(x_\alpha,\ r_\alpha) \neq \emptyset \ .$$

Proof. Since a compact set is separable there exists a sequence $\{\alpha_j\}_{j=1}^\infty \subset A$ such that

$$\overline{\{x_{\alpha_j}\}_{j=1}^\infty} = \overline{\{x_\alpha\}_{\alpha\in A}} \ .$$

We define inductively a sequence of numbers R_j by

$$R_1 = \sup_{\alpha\in A} (\| x_{\alpha_1} - x_\alpha\| - r_\alpha)$$

$$R_j = \max (\sup_{\alpha\in A} \| x_{\alpha_j} - x_\alpha\| - r_\alpha \ ; \ \ \| x_{\alpha_j} - x_{\alpha_k}\| - R_k, \ \ 1 \le k < j), j=2,3,\ldots$$

Since $r_\alpha + r_\beta \geq \| x_\alpha - x_\beta\|$ for every $\alpha,\beta \in A$ it follows immediately that $R_j \le r_{\alpha_j}$ for every j. If for some j $R_j \le 0$ then $x_{\alpha_j} \in \bigcap_\alpha S(x_\alpha,\ r_\alpha)$ and there is nothing to prove. Thus we can assume that $R_j > 0$ for every j. The R_j are such that if we replace the cells $S(x_{\alpha_j},\ r_{\alpha_j})$ in the given collection $\{S(x_\alpha,\ r_\alpha)\}_{\alpha\in A}$ by the cells $S(x_{\alpha_j},\ R_j)$ we shall still have a collection of mutually intersecting cells, but if we now replace any $S(x_{\alpha_j},\ R_j)$ by a cell $S(x_{\alpha_j},\ R)$ with $R < R_j$ the collection will no longer have this property. We can therefore assume without loss of generality that $R_j = r_{\alpha_j}$ for every j and hence that for every j and ε there is a $\beta_{\varepsilon,j} \in A$ with

$$r_{\alpha_j} + r_{\beta_{\varepsilon,j}} \le \| x_{\alpha_j} - x_{\beta_{\varepsilon,j}}\| + \varepsilon \ .$$

Let now ε be a positive number. From the compactness of $\overline{\{x_\alpha\}_{\alpha\in A}}$ it follows that there is an $n = n(\varepsilon)$ such that

$$\{x_\alpha\}_{\alpha \in A} \subset \bigcup_{j=1}^{n} S(x_{\alpha_j}, \varepsilon) .$$

X has the F.2.I.P. hence there is a point $y \in X$ satisfying

$$\| y - x_{\alpha_j} \| \leq r_{\alpha_j} , \quad 1 \leq j \leq n .$$

Let $\alpha \in A$ and let $\| x_{\alpha_j} - x_\alpha \| \leq \varepsilon$. We have

$$r_{\beta_{\varepsilon,j}} + r_\alpha \geq \| x_{\beta_{\varepsilon,j}} - x_\alpha \|$$

$$\geq \| x_{\beta_{\varepsilon,j}} - x_{\alpha_j} \| - \varepsilon \geq r_{\alpha_j} + r_{\beta_{\varepsilon,j}} - 2\varepsilon ,$$

i.e. $r_\alpha \geq r_{\alpha_j} - 2\,\varepsilon$. Hence for every $\alpha \in A$

$$\| y - x_\alpha \| \leq \| y - x_{\alpha_j} \| + \| x_{\alpha_j} - x_\alpha \| \leq r_\alpha + 3\varepsilon .$$

Before proceeding in the proof we remark that y may be chosen so that it belongs not only to $\bigcap_{i=1}^{n} S(x_{\alpha_j}, r_{\alpha_j})$ but also to any given finite number of cells which intersect each other and all the cells in the given collection.

Let ε_k be a sequence of positive numbers tending to 0 and let $n_k = n(\varepsilon_k)$ be the integer corresponding to ε_k in the above argument. Let

$$z_1 \in \bigcap_{j=1}^{n_1} S(x_{\alpha_j}, r_{\alpha_j}) .$$

The cell $\sigma_1 = S(z_1, 3\varepsilon_1)$ intersects any cell of the given collection. We continue inductively and choose a sequence z_k satisfying

$$z_k \in \bigcap_{j=1}^{n_k} S(x_{\alpha_j}, r_{\alpha_j}) \cap \bigcap_{j=1}^{k-1} \sigma_j$$

where $\sigma_j = S(z_j, 3\varepsilon_j)$. In particular $\| z_k - z_h \| \leq 3\varepsilon_k$ for $h > k$ and hence the sequence z_k converges to a point $z \in X$. Letting k tend to ∞ in $\| z_k - x_\alpha \| \leq 3\varepsilon_k + r_\alpha$ $(\alpha \in A)$ we obtain

$$z \in \bigcap_{\alpha \in A} S(x_\alpha, r_\alpha) ,$$

and this concludes the proof of the theorem.

Remarks. The theorem and its proof are valid in general complete metric spaces.

The requirement imposed in Theorem 4.5 on the set of the centers cannot, in general, be weakened. It is clear (from the result of Nachbin [37] cited after Lemma 5.3) that if all the centers belong to a subspace of X which is a $\mathcal{P}_1$ space (e.g. a one-dimensional space) then $\bigcap_{\alpha \in A} S(x_\alpha, r_\alpha) \neq \emptyset$ without any further assumption on the centers. However (see [32] and also Chapter VII) even if all the centers belong to a two-dimensional subspace of X, $\bigcap_{\alpha \in A} S(x_\alpha, r_\alpha)$ may be empty (we assume of course that X has the F.2.I.P. and that any two cells in the collection intersect). This shows that in the theorem compactness cannot be replaced by finite-dimensionality. Neither can it be replaced by weak compactness. For example let $\{e_n\}_{n=1}^{\infty}$ be the usual basis of the space c_0, and let $S_n = S(e_n, \frac{1}{2})$. c_0 has the F.2.I.P., the sequence e_n converges weakly to 0, $S_n \cap S_m \neq \emptyset$ for every n and m but $\bigcap_{n=1}^{\infty} S_n = \emptyset$.

Spaces having the 4,2.I.P. will be studied in detail in Chapter VI. It will be shown there, in particular, that a Banach space X has the 4,2.I.P. if and only if X^* is (isometric to) an $L_1(\mu)$ space for some measure μ. Our purpose now is to study some properties of spaces having the 3,2.I.P. In [20] Hanner gave a geometrical characterization of the unit cells of finite-dimensional spaces having the 3,2.I.P. The methods of Hanner do not seem to apply to the infinite-dimensional case, and thus our methods are different from those he uses.

We first give some methods for obtaining spaces having the 3,2.I.P.

Theorem 4.6. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of Banach spaces having the 3,2.I.P., then

(a) $(X_1 \oplus X_2 \oplus \cdots)_{c_0}$

(b) $(X_1 \oplus X_2 \oplus \cdots)_m$

(c) $(X_1 \oplus X_2 \oplus \cdots)_{\ell_1}$

all have the 3,2.I.P.

Proof. (a) Let $x^i = (x_1^i, x_2^i, \ldots)$ be three sequences with $x_n^i \in X_n$ and $\|x_n^i\| \xrightarrow[n\to\infty]{} 0$. Let r^i be three positive numbers satisfying $\|x_n^i - x_n^j\| \le r^i + r^j$ $(i, j = 1,2,3;\ n = 1,2,\ldots)$. Since X_n has the 3,2.I.P for every n, there exists an $x_n \in X_n$ with $\|x_n^i - x_n\| \le r^i$, $i = 1,2,3$. Let n_0 be an integer such that $n > n_0$ implies $\|x_n^i\| \le \min(r^1, r^2, r^3)$ for every i, and put

$$x = (x_1, x_2, \ldots, x_{n_0}, 0, 0, \ldots) .$$

Then $\|x - x^i\| \le r^i$ for every i and this proves that the space (a) has the 3,2.I.P. The proof for (b) is similar and even simpler.

We turn to the space (c). Let three mutually intersecting cells be given. We can assume without loss of generality that (at least) two pairs of the cells have no interior points in common (otherwise we replace the cells by cells with smaller radii). Furthermore we may assume that the cell which belongs to both pairs has the origin as center. Denote the centers of the other two cells by x and y and the radii of the cells by r^i, $i = 1,2,3$. We have

$$x = (x_1, x_2, \ldots); \quad \|x\| = \sum_{n=1}^{\infty} \|x_n\| = r^1 + r^2 ,$$

$$y = (y_1, y_2, \ldots); \quad \|y\| = \sum_{n=1}^{\infty} \|y_n\| = r^1 + r^3 ,$$

$$\|x - y\| = \sum_{n=1}^{\infty} \|x_n - y_n\| \le r^2 + r^3 .$$

It follows that there are $\lambda_n \ge 1$ $(n = 1,2,\ldots)$ such that

$$\sum_{n=1}^{\infty} \lambda_n \cdot \|x_n - y_n\| = r^2 + r^3; \quad \lambda_n \cdot \|x_n - y_n\| \le \|x_n\| + \|y_n\| .$$

Put

$$\alpha_n = \frac{\|x_n\| + \|y_n\| - \lambda_n\|x_n - y_n\|}{2},$$

$$\beta_n = \frac{\|x_n\| - \|y_n\| + \lambda_n\|x_n - y_n\|}{2},$$

$$\gamma_n = \frac{-\|x_n\| + \|y_n\| + \lambda_n\|x_n - y_n\|}{2}.$$

For every n we have

$$\alpha_n, \beta_n, \gamma_n \geq 0,$$

$$\|x_n\| = \alpha_n + \beta_n, \quad \|y_n\| = \alpha_n + \gamma_n, \quad \|x_n - y_n\| \leq \beta_n + \gamma_n.$$

Since X_n has the 3,2.I.P. there is a $z_n \in X_n$ satisfying

$$\|z_n\| \leq \alpha_n, \quad \|z_n - x_n\| \leq \beta_n, \quad \|z_n - y_n\| \leq \gamma_n.$$

Let $z = (z_1, z_2, \ldots)$, then

$$\|z\| \leq \sum_{n=1}^{\infty} \alpha_n = \frac{(r^1 + r^2) + (r^1 + r^3) - (r^2 + r^3)}{2} = r^1.$$

Similarly

$$\|z - x\| \leq \sum_{n=1}^{\infty} \beta_n = r^2, \quad \|z - y\| \leq \sum_{n=1}^{\infty} \gamma_n = r^3,$$

and this concludes the proof of the theorem.

Remarks. (i) The theorem holds also if we take only a finite number of summands or, on the other hand, a non-countable number of them.

(ii) If the X_n have the 4,2.I.P. (or any other intersection property defined in this chapter) the same is true for the direct sums (a) and (b). This is not, however, the case for (c). For example if X_n is, for every n, the one-dimensional space the direct sum (c) will be the space ℓ_1 which does not have the 4,2.I.P.

Corollary 1. Every $L_1(\mu)$ space, and more generally every $L_1(\mu, X)$ space (i.e. the space consisting of all X-valued Bochner integrable functions with the usual norm) with X having the 3,2.I.P., has the

3,2.I.P.

Proof. It is enough to prove that a dense subspace of $L_1(\mu, X)$ has the 3,2.I.P. (Corollary to Lemma 4.2). We shall prove this for the subspace consisting of the simple functions (= finite sums of functions of the form $x\varphi$ with $x \in X$ and φ a characteristic function of a set with finite measure). Let f_1, f_2 and f_3 be three simple functions. Then there is a subspace of $L_1(\mu, X)$ which contains the f_i and which is isometric to $(X \oplus X \oplus \cdots \oplus X)_{\ell_1^n}$ for some finite n. The desired result follows now from Theorem 4.6 (c).

Corollary 2. Let X be a Banach space having the n,2.I.P. $(n \geq 3)$, and let K be a compact Hausdorff topological space. Then the Banach space $C(K, X)$ consisting of all the continuous functions from K to X (with the usual norm) has the n,2.I.P.

Proof. Let $\{\varphi_i\}_{i=1}^n$ be continuous real valued functions on K satisfying $\varphi_i \geq 0$, $\|\varphi_i\| = 1$ $(i = 1,2,\ldots,n)$ and $\sum_{i=1}^n \varphi_i(k) = 1$ $(k \in K)$. Let B be the subspace of $C(K, X)$ consisting of functions of the form $\Sigma_i\, x_i\varphi_i$ with $x_i \in X$. Then B is isometric to $(X \oplus \cdots \oplus X)_{\ell_\infty^n}$ and hence has the n,2.I.P. For every finite set $\{f_i\}_{i=1}^m$ in $C(K, X)$ and every $\varepsilon > 0$ there is a subspace B of the type described above such that the distance of each of the f_i from B is less than ε. This proves the corollary (use Lemma 4.2).

We shall now characterize spaces having an intersection property which is weaker than the 3,2.I.P. This characterization shows the connection between intersection properties and order properties and it will be the starting point for the discussion in Chapter VI of the decomposition property (in partially ordered vector spaces). In Chapter VI we shall consider also non complete normed spaces and therefore we shall not assume here that the space is complete.

Theorem 4.7. Let X be a normed space, then statements (1) - (6) are equivalent and imply (7).

(1) Let $\{S_i\}_{i=1}^3$ be three mutually intersecting cells in X such that $S_1 \cap S_2$ is a single point e. Then $e \in S_3$.

(2) The same as (1) for three cells with a common radius.

(3) Let e be an extreme point of S_X and let a partial order be defined in X by $x \geq 0 \Longleftrightarrow x = \lambda(e + u)$, $\lambda \geq 0$, $\|u\| \leq 1$. Then $\|x\| \leq 1 \Longleftrightarrow -e \leq x \leq e$.

(4) Let e be an extreme point of S_X. Then X is isometric to a subspace of some $C(K)$ (K compact Hausdorff), in such a manner that e corresponds to the function identically equal to 1.

(5) Let e be an extreme point of S_X and x^* an extreme point of the unit cell of X^*. Then $|x^*(e)| = 1$.

(6) Let e be an extreme point of S_X and let $x \in X$ with $\|x\| = 1$. Then at least one of the two segments joining x with e and $-e$ is contained in the boundary of S_X.

(7) Let $e_1 \neq e_2$ be two extreme points of S_X. Then $\|e_1 - e_2\| = 2$.

Proof. (1) => (2) is clear. We shall show first that (2) => (3).

It is clear that the order defined in (3) is compatible with the linear structure of X. Since e is an extreme point of S_X $x \geq 0$ and $x \leq 0$ imply that $x = 0$. That $\|x\| \leq 1$ implies $-e \leq x \leq e$ is also obvious (for all these remarks we do not use the fact that X satisfies (2)). We assume now that X satisfies (2) and that $x \in X$ satisfies $-e \leq x \leq e$, i.e.

$$(4.8) \qquad x = -\lambda_1(e + u_1) + e, \qquad x = \lambda_2(e + u_2) - e$$

with $\lambda_1, \lambda_2 \geq 0$ and $\|u_1\|$, $\|u_2\| \leq 1$. Since for every $\lambda, \mu > 0$ and u

with $\|u\| \leq 1$ we have

$$\lambda(e+u) = (\lambda+\mu)(e + \frac{\lambda u-\mu e}{\lambda+\mu}) , \quad \| \frac{\lambda u-\mu e}{\lambda+\mu} \| \leq 1 ,$$

we may replace λ_1 and λ_2 in (4.8) by any larger number. Hence without loss of generality we may assume that $\lambda_1 = \lambda_2 > 2$ (we denote the common value of the λ_i by λ). Consider the cells

$$S_1 = S(x+(\lambda-1)e,\lambda-1), \quad S_2 = S(x-(\lambda-1)e,\lambda-1)$$

$$S_3 = S(-\frac{\lambda-2}{\|x\|}x, \lambda-1) .$$

Clearly all three cells have a common radius. All points of S_1 are $\geq x$ while all those of S_2 are $\leq x$. Since x belongs to both we have that $S_1 \cap S_2$ consists of the single point x. $-u_1 \in S_1 \cap S_3$. Indeed,

$$\| u_1 + x + (\lambda-1)e\| = \| -(\lambda-1)u_1\| \leq \lambda-1$$

$$\| u_1 - ((\lambda-2)x)/\|x\| \| \leq 1 + \lambda - 2 = \lambda - 1.$$

Similarly $u_2 \in S_2 \cap S_3$. By (2) it follows that $x \in S_3$. That is

$$\|x\| + \lambda - 2 = \| x + ((\lambda-2)x)/\|x\| \| \leq \lambda - 1.$$

Hence $\|x\| \leq 1$ and this concludes the proof of (2) => (3). The proof of (3) => (1): Let S_1, S_2, S_3 and e be as in (1). We have to show that $e \in S_3$. Without loss of generality we may assume that $S_1 = S_X = S(0,1)$, and hence $S_2 = S((1+\lambda)e,\lambda)$ with some $\lambda > 0$. e is an extreme point of S_1. Indeed, suppose that there is a $u \neq 0$ with $\|e \pm u\| = 1$. Since $(1+\lambda)e - (e+\eta u) = \lambda(e-\eta u/\lambda)$ it follows that $e + \eta u \in S_1 \cap S_2$ if $|\eta| \leq \min(1,\lambda)$ and this contradicts our assumptions. Let $\geq$ be the (partial) order defined in X as in (3) corresponding to the extreme point e, and let $S_3 = S(x,\eta)$. $S_1 \cap S_3 \neq \emptyset$ implies that $x \leq (1+\eta)e$. $S_2 \cap S_3 \neq \emptyset$ implies that $x \geq (1+\lambda)e - (\lambda+\eta)e$. Hence $-\eta e \leq x-e \leq \eta e$ i.e. $\|x-e\| \leq \eta$

or $e \in S_3$.

(3) => (4). Let X be ordered as in (3). Then if $x_n \geq 0$ and $x_n \longrightarrow x_0$ also $x_0 \geq 0$. Indeed, let $2k > \sup_n \|x_n\|$; then all the x_n belong to the cell $S(ke,k) = \{x;\ 0 \leq x \leq 2ke\}$, and therefore also x_0 belongs to this cell. In view of this remark (3) => (4) is exactly a representation theorem of Kadison ([21], Theorem 2.1).

(4) => (5) follows from the Hahn Banach theorem and the well known fact that any extreme point x^* of the unit cell of $C(K)^*$ has the form $x^*(f) = \pm f(k)$ $(k \in K,\ f \in C(K))$.

(5) => (6). From the Krein-Milman theorem it follows that there exists an extreme point x^* of the unit cell of X^* satisfying $x^*(x) = 1$. Since $x^*(e) = \theta$ with $|\theta| = 1$ we have $x^*(\lambda x + (1-\lambda)\theta e) = 1$ for every λ. Hence the segment joining x with θe is contained in the boundary of S_X.

(6) => (3). As in the proof of (2) => (3) we have only to prove that $-e \leq x \leq e \Rightarrow \|x\| \leq 1$. Suppose that $\|x\| > 1$ and put $y = x/\|x\|$. By (6) we may assume that the segment joining y with e belongs to the boundary of S_X (otherwise replace e by $-e$). Hence the intersection of S_X with the segment joining x with e is the point e alone and this contradicts the assumption that $x \leq e$.

(5) => (7). By the Krein-Milman theorem there is an extreme point x^* of the unit cell of X^* satisfying $x^*(e_1) \neq x^*(e_2)$. Since $|x^*(e_1)| = |x^*(e_2)| = 1$ we have $|x^*(e_1-e_2)| = 2$ and hence $\|e_1-e_2\| = 2$.

Remarks. (i) Many of the implications proved here are not essentially new. Nachbin [37] proved that (1) => (3) and our proof here of (2) => (3) is a modification of the argument used by Nachbin. As mentioned already in the proof, the implication (3) => (4) (and some related results) appear in Kadison [21]. A result closely related to the equivalence of (3) and (5) was proved by Fullerton ([9], Theorem 4.1). Concerning this theorem of Fullerton we remark that one step of its proof is not justified, since it is not true that a maximal convex subset of the boundary of the unit cell of a conjugate space is w^* closed. We do not know whether the theorem itself is true.

(ii) Our result that the 3,2.I.P implies property (5) of Theorem 4.7 reduces, in the finite-dimensional case, to the following result of Hanner [20]:

Let X be an n-dimensional space having the 3,2.I.P., then S_X is affinely equivalent to the convex hull of some of the vertices of an n-dimensional cube.

Indeed (we follow the reasoning of Hanner), let $\{x_i^*\}_{i=1}^n$ be n linearly independent extreme points of S_{X^*}. It follows from (5) that the extreme points of S_X are vertices of the parallelepiped $|x_i^*(x)| \leq 1$, $i = 1,\ldots,n$.

(iii) The properties appearing in Theorem 4.7 are strictly weaker than the 3,2.I.P., even if we restrict ourselves to spaces X in which the unit cells have enough extreme points. Hanner [20, Remark 3.6] gave an example of a 5-dimensional space which does not have the 3,2.I.P. though it satisfies (5) of Theorem 4.7.

We conclude this chapter with a result concerning the connection between the 3,2.I.P. and the notion of CL spaces. This latter notion was introduced by Fullerton [9] and its definition is as follows:

A Banach space X is called a CL space if for every maximal convex subset F of the boundary of the unit cell S_X of X, $S_X = \mathrm{Co}(F \cup -F)$.

It is clear that this notion is closely connected with statement (5) of Theorem 4.7. Fullerton ([9] Theorem 4.1.) proved that every CL space has property (3) of Theorem 4.7. For finite-dimensional spaces the converse also holds. In particular a finite-dimensional space having the 3,2.I.P is a CL space. Hanner has shown that if a finite-dimensional space X has the 3,2.I.P the same is true for X^*. Hence if X is a finite-dimensional space having the 3,2.I.P both X and X^* are CL spaces. We prove now a result which reduces in the finite-dimensional case to the latter statement.

Theorem 4.8. Let $X = Y^*$ be a Banach space having the 3,2.I.P. Then

(a) for every maximal convex subset F of the boundary of S_X, $S_X = \overline{\text{Co}(F \cup -F)}$ (the closure in the w^* topology).

(b) For every maximal convex subset F of the boundary of S_Y, $S_Y = \overline{\text{Co}(F \cup -F)}$ (the closure in the norm topology).

Proof. (a) is a consequence of the Krein-Milman Theorem and (1) $\Rightarrow$ (5) of Theorem 4.7.

(b) Let F be a maximal convex subset of the boundary of S_Y. From the separation theorems for convex sets it follows that there is an $x \in X$ satisfying $x(y) = \|x\| = 1$ for every $y \in F$. Let $x_0 \in X$ satisfy $\|x_0\| = 1$, $\|x - x_0\| = \lambda < 2$, and suppose that there is a $y_0 \in Y$ with $x_0(y_0) = \|y_0\| = 1$. Consider the following three cells

$$S(x_0,\lambda/2), \quad S(x,\lambda/2), \quad S(0,1-\lambda/2) .$$

Clearly any two of them intersect and hence there is a point $u \in X$ common to all the three, i.e.

$$\|u\| \leq 1 - \frac{\lambda}{2}, \quad \|u-x\| \leq \frac{\lambda}{2}, \quad \|u - x_0\| \leq \frac{\lambda}{2} .$$

We have

$$1 = x_0(y_0) = (x_0 - u)(y_0) + u(y_0) \leq \|x_0 - u\| + \|u\| \leq 1 ,$$

and hence $u(y_0) = \|u\|$. Similarly, from $x(y) = 1 = \|y\|$ it follows that $u(y) = \|u\|$. Hence y_0 and F are included in the convex set of all the $y \in Y$ satisfying $\|y\| = 1$ and $u(y) = \|u\|$. From the maximality of F we infer that $y_0 \in F$. We have thus proved that

$$\|x_0\| = \|y_0\| = x_0(y_0) = 1, \quad \|x - x_0\| < 2 \Rightarrow y_0 \in F.$$

Let us denote $\overline{\mathrm{Co}(F \cup -F)}$ (the norm closure) by $\tilde{S}$. Clearly $\tilde{S} \subset S_Y$. We have to show that also $S_Y \subset \tilde{S}$. Suppose this were false. By the separation theorems it would follow that there exists a $u \in X$ with $\|u\| = 1$ and

$$\alpha = \sup_{y \in \tilde{S}} |u(y)| < 1 .$$

We show first that this u satisfies $\|u - x\| = 2$. We use the theorem of Bishop and Phelps [4] stating that in a conjugate space Y^* those functionals which attain their supremum on S_Y are dense in the norm topology. Suppose that $\|u - x\| < 2$. From the theorem of Bishop and Phelps we infer that there exist $x_0 \in X$ and $y_0 \in Y$ satisfying

$$x_0(y_0) = \|x_0\| = \|y_0\| = 1, \quad \|x_0 - u\| < \frac{1-\alpha}{2}, \quad \|x_0 - x\| < 2.$$

As we have shown above $y_0 \in F$ and hence

$$1 = x_0(y_0) \leq \sup_{y \in F} x_0(y) \leq \sup_{y \in F} u(y) + \|x_0 - u\| \leq \alpha + \frac{1-\alpha}{2} = \frac{1+\alpha}{2} .$$

This is a contradiction and thus $\|u - x\| = 2$. There is an $x^* \in X^*$ satisfying $2 = x^*(u - x) = \|x^*\| \, \|u - x\|$, i.e. $\|x^*\| = x^*(u) = -x^*(x)$. For every $\lambda > 0$ we have therefore $\lambda+1 = \|\lambda u - x\| \, (= x^*(\lambda u - x))$. We have thus shown that the equation $\|\lambda v - x\| = \lambda+1$ holds for every λ and v satisfying $\|v\| = 1$, $\sup_{y \in F} |v(y)| < 1$ and $\lambda > 0$. In particular, $\|w - x\| = \|w\| + \|x\|$ for every w in the open cone $\|w/\|w\| - u\| < 1 - \alpha$.

By using again the theorem of Bishop and Phelps it follows that there exist $x_0 \in X$ and $y_0 \subset Y$ satisfying

$$\| y_0\| = 1, \quad (x - x_0)(y_0) = \| x - x_0\| = \| x\| + \| x_0\| ,$$
$$\| x_0 - u\| < \frac{1-\alpha}{3} .$$

In particular $x(y_0) = \| x\| = 1$ and hence (by the maximality of F) $y_0 \in F$. We have

$$\frac{2+\alpha}{3} \le \| x_0\| = |x_0(y_0)| \le \sup_{y\in F} |x_0(y)|$$
$$\le \sup_{y\in F} |u(y)| + \| u - x_0\| \le \frac{1+2\alpha}{3} ,$$

and this is the desired contradiction.

CHAPTER V. THE CONNECTION BETWEEN INTERSECTION AND EXTENSION PROPERTIES

The connection between intersection and extension properties is based on the following, well known, three lemmas (cf. for example [37], [2] and [17]).

<u>Lemma 5.1</u>. Let $Y \supset X$ be Banach spaces and assume that there is a projection with norm 1 from Y onto X. Let $\{S_Y(x_\alpha, r_\alpha)\}$ be a collection of cells in Y whose centers belong to X. Then

$$\bigcap_\alpha S_Y(x_\alpha, r_\alpha) \neq \emptyset \Rightarrow \bigcap_\alpha S_X(x_\alpha, r_\alpha) \neq \emptyset .$$

<u>Proof</u>. Obvious.

<u>Corollary</u>. Let $Y \supset X$ be Banach spaces such that there is a projection with norm 1 from Y onto X. If Y has the n,k.I.P. (or the R.n,k.I.P.) for some $n > k$ then the same is true for X.

<u>Lemma 5.2</u>. Let T be an operator from a Banach space X into a Banach space Y. Let $Z \supset X$ be a Banach space with $\dim Z/X = 1$ and let

$z \in Z \sim X$. Then T has a norm preserving extension to an operator from Z into Y if and only if $\bigcap_{x \in X} S_Y(Tx, \|T\| \, \|z-x\|) \neq \emptyset$.

Proof. If $\tilde{T}$ is a norm preserving extension of T then $\|\tilde{T}z - Tx\| \leq \|T\| \, \|z-x\|$ for every $x \in X$. Conversely if $\|u - Tx\| \leq \|T\| \, \|z-x\|$ for every $x \in X$ then $\tilde{T}(x+\lambda z) = Tx + \lambda u$ is a norm preserving extension of T. Indeed, for $\lambda \neq 0$ we have

$$\|Tx + \lambda u\| = |\lambda| \, \|T(x/\lambda)+u\|$$

$$\leq |\lambda| \, \|T\| \, \|x/\lambda + z\| = \|T\| \, \|x + \lambda z\| .$$

Lemma 5.3. Let X be a Banach space and let $\{S_X(x_\alpha, r_\alpha)\}$ be a collection of mutually intersecting cells in it. Then there is a Banach space $Z \supset X$ with $\dim Z/X = 1$ such that $\bigcap_\alpha S_Z(x_\alpha, r_\alpha) \neq \emptyset$.

Proof. The original proof of Nachbin [37] was rather long. We give here two simple proofs. The second, which has the advantage that it shows what S_Z is, is due to Grünbaum [17].

First proof. It is obvious that for every index set I every collection of mutually intersecting cells in $m(I)$ (= the Banach space of all bounded real-valued functions on I with the sup norm) has a non empty intersection. Embed X isometrically in some $m(I)$. Then there is a $z \in m(I)$ such that $\|z - x_\alpha\| \leq r_\alpha$ for every α. The subspace Z of $m(I)$ spanned by X and z has the required properties. (If $z \in X$ then we can take as Z any Banach space satisfying $\dim Z/X = 1$.)

Second proof. If $\inf_\alpha r_\alpha = 0$ then it follows easily from the completeness of X that $\bigcap_\alpha S_X(x_\alpha, r_\alpha) \neq \emptyset$, and hence any $Z \supset X$ with $\dim Z/X = 1$ will have the required property. Assume now that $\inf_\alpha r_\alpha > 0$. In the vector space $X \oplus R$ let K be the set consisting of all the points of the form $z_\alpha = (x_\alpha/r_\alpha, 1/r_\alpha)$. Let S_Z be the closed convex hull of

$S_X \cup K \cup (-K)$ (in Z we take the product topology of the norm topology in X and the usual topology in R). It is easily checked that $S_Z \cap X = S_X$ and hence if we introduce in Z a norm whose unit cell is S_Z this norm will coincide on X with the given norm there (X is identified with the subspace of Z consisting of the points $(x,0)$). $(0,-1) \in \bigcap_\alpha S_Z(x_\alpha, r_\alpha)$ and hence Z has the required property.

Before passing to our applications of the preceding lemmas we would like to mention some elegant (though straightforward) applications of them which appear in the literature. One consequence of Lemmas 5.1, 5.2 and 5.3 is the result of Nachbin [37] that a Banach space is a $\mathcal{P}_1$ space if and only if every collection of mutually intersecting cells in X has a common point. Another consequence of Lemmas 5.1 and 5.2 (together with Helly's theorem in the plane and Kakutani's well known characterization of a Hilbert space) is the following observation, due to Comfort and Gordon [5]:

A Banach space is isometric to a Hilbert space if and only if for every collection of three cells $\{S(x_i,r_i)\}_{i=1}^3$ in X for which $\bigcap_{i=1}^3 S(x_i,r_i) \neq \emptyset$ also $\bigcap_{i=1}^3 S(x_i,r_i) \cap L \neq \emptyset$ where L is the plane determined by the x_i. (In case the x_i lie on a line the condition is satisfied in every Banach space X and for every plane L containing the x_i).

The importance of the connection between intersection and extension properties goes far beyond the possibility of just getting elegant characterizations like those described above. This connection can be applied to establish relations between certain extension properties by proving the corresponding results for intersection properties and conversely. We hope the results proved here and in the next chapters will illustrate this point. Another use of the connection between extension and intersection properties is in obtaining a functional representation for Banach spaces having certain extension properties. Nachbin, for example, obtained the functional representation of $\mathcal{P}_1$ spaces X as C(K) spaces with extremally disconnected K by using his result that in such X every collection of mutually intersecting cells has a common point. (He assumed that S_X has at least one extreme point and applied (1) $\Rightarrow$ (3) of Theorem 4.7.) In the next chapter we shall give further applications of this kind to the connection between extension and intersection properties.

Unfortunately, the method of using intersection properties for studying extension properties applies here only to the study of immediate extensions, that is to extending operators defined on a space Y to operators defined on Z where $Z \supset Y$ and $\dim Z/Y = 1$. It is possible to use some more complicated intersection properties for studying nonimmediate extensions. However, generalizations of this kind seem to contribute only little to the problem discussed here. (Such generalizations are useful for considering some extension problems concerning non linear mappings, and we shall discuss these generalizations elsewhere.) In some problems the possibility of finding always an immediate extension implies easily the possibility of extending operators defined on Y to operators defined on $Z \supset Y$ without any restriction on Z/Y. This is the case, for example, in the proof of the Hahn-Banach theorem or in the results of Nachbin and Comfort and Gordon mentioned above. In general the passage from extensions to Z with $\dim Z/Y = 1$ to extensions to an arbitrary $Z \supset Y$ is not easy if at all possible. In [34] we give some results and counterexamples concerning this question. Most of the unsolved problems mentioned in this chapter are nothing but the question whether it is possible to discard the requirement $\dim Z/Y = 1$ in the statement of certain extension properties without changing the properties themselves.

We now give some applications of Lemmas 5.1, 5.2 and 5.3 to the characterization of certain "into" extension properties.

Theorem 5.4. Let X be a Banach space. The following three statements are equivalent.

(a) X has the F.2.I.P.

(b) Every compact operator T from Y to X has (for every $\varepsilon > 0$) and extension $\tilde{T}$ from $Z(Z \supset Y$ with $\dim Z/Y = 1)$ to X with $\|\tilde{T}\| \leq (1+\varepsilon)\|T\|$.

(c) Every operator T from Y to X with a range of dimension ≤ 3 has (for every $\varepsilon > 0$) an extension $\tilde{T}$ from $Z(Z \supset Y$ with $\dim Z/Y = 1)$ to X with $\|\tilde{T}\| \leq (1+\varepsilon)\|T\|$.

Proof. (a) $\Rightarrow$ (b). Let T (with $\|T\| = 1$), Y, Z and ε be given, and let $z \in Z \sim Y$ with $\|z\| = 1$. In order to show the existence of an extension $\tilde{T}$ of T from Z into X with $\|\tilde{T}\| \leq 1 + \varepsilon$ we have to show the existence of a point $u \in X$ satisfying

(5.1) $\quad \|u-Ty\| \leq (1+\varepsilon)\|z-y\| \quad$ for every $y \in Y$

(cf. the proof of Lemma 5.2). Let M be a positive number and consider the collection of cells $\{S_X(Ty, \|y-z\|)\}_{\|y\| \leq M}$. Any two of these cells intersect since

$$\|Ty_1-Ty_2\| \leq \|y_1-y_2\| \leq \|y_1-z\| + \|y_2-z\| .$$

By the compactness of T and Theorem 4.5 there exists a point $u_M \in X$ satisfying

$$\|Ty-u_M\| \leq \|y-z\|, \quad \text{for every } y \in Y \text{ with } \|y\| \leq M.$$

Taking in particular $y = 0$ we obtain that $\|u_M\| \leq \|z\| = 1$. Let $y \in Y$ be a point with $\|y\| \geq M$. Then

$$\|Ty-u_M\| \leq \|Ty\| + \|u_M\| \leq \|y\| + 1$$

and

$$\|y - z\| \geq \|y\| - \|z\| = \|y\| - 1 ,$$

hence

$$\frac{\|Ty-u_M\|}{\|y-z\|} \leq \frac{\|y\|+1}{\|y\|-1} \leq \frac{M+1}{M-1} .$$

Therefore, for large enough M the point u_M will satisfy (5.1) and this shows that (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a). It is enough to show that if X satisfies (c) then for every collection of four mutually intersecting cells $\{S(x_i,r_i)\}_{i=1}^4$ in X and for every $\varepsilon > 0$ there is a point $x \in X$ satisfying $\|x-x_i\| \leq r_i(1+\varepsilon)$ for every i (cf. Theorem 4.1 and Lemma 4.2). Without loss of generality we may assume that $x_1 = 0$ (otherwise we translate the cells) and hence there is a 3-dimensional subspace B of X containing $\{x_i\}_{i=1}^4$. By Lemma 5.3 there exists a space $C \supset B$ with $\dim C/B = 1$ and a point $z \in C$ satisfying $\|z-x_i\| \leq r_i$, $i = 1,2,3,4$. Let $\tilde{T}$ be an operator from C into X whose restriction to B is the identity and for which $\|\tilde{T}\| \leq 1+\varepsilon$. $x = \tilde{T}z$ satisfies $(1 \leq i \leq 4)$

$$\|x-x_i\| = \|\tilde{T}(z-x_i)\| \leq (1+\varepsilon)\|z-x_i\| \leq (1+\varepsilon)r_i$$

as required.

Remarks. In Chapter VII we shall show that (a) does not imply (b) or even (c) with $\varepsilon = 0$ (cf. also [32]). The use of Theorem 4.5 in the proof of (a) $\Rightarrow$ (b) can be avoided by slightly modifying the argument given here.

The proof of the following two results is similar to (and even simpler than) the proof of Theorem 5.4.

Corollary 1. Let X be a Banach space. The following two statements are equivalent

(d) For every finite collection of mutually intersecting cells $\{S(x_i,r_i)\}$ in X whose centers form a 2-dimensional subspace of X, and for every $\varepsilon > 0$, $\bigcap_i S(x_i, r_i+\varepsilon) \neq \emptyset$.

(e) Every operator T from Y into X with a 2-dimensional range has, for every $\varepsilon > 0$, an extension $\tilde{T}$ from Z ($Z \supset Y$ with $\dim Z/Y = 1$) into X with $\|\tilde{T}\| \leq (1+\varepsilon)\|T\|$.

Remark. We conjecture that (d) and (e) are also equivalent to statements (a), (b) and (c) in Theorem 5.4. In Chapter VI we shall prove that this is indeed the case if we assume that S_X has at least one extreme point.

Corollary 2. Let X be a Banach space and let $\mathfrak{m}$ be a (finite or infinite) cardinal number. The following two statements are equivalent.

(i) Every collection of mutually intersecting cells in X whose centers span a subspace of dimension $\leq \mathfrak{m}$ has a non empty intersection.

(ii) Every operator T from Y (with Y of dimension $\leq \mathfrak{m}$) into X has a norm preserving extension from Z ($Z \supset Y$, $\dim Z/Y = 1$) into X.

Remark. In the next chapter we shall show that spaces having the F.2.I.P. also have the property obtained from statement (b) of Theorem 5.4 (and hence also from statement (c)) if we discard any requirement on Z/Y. Concerning the properties appearing in Corollary 2 it is not hard to see that if we assume the generalized continuum hypothesis then property (ii) (for an infinite cardinal $\mathfrak{m}$) implies the property obtained from it by discarding any requirement on Z/Y (cf. [32]). For finite $\mathfrak{m}$ we were able to show that the same is true in some special cases, for instance for $C(K)$ spaces (cf. Chapter VII and [32]).

Theorem 5.5. Let X be a Banach space. The following three statements are equivalent

(i) X has the F.2.I.P.

(ii) Every operator (not necessarily compact) T from Y into X has an extension $\tilde{T}$ from Z ($Z \supset Y$ with $\dim Z/Y = 1$ and the unit cell S_Z of Z is the convex hull of S_Y and a finite set of points $\{\pm z_i\}_{i=1}^n$) into X with $\|\tilde{T}\| = \|T\|$.

(iii) Every operator T from Y into X with a range of dimension ≤ 3 has an extension $\tilde{T}$ from Z ($Z \supset Y$ with $\dim Z/Y = 1$ and $S_Z = \mathrm{Co}(S_Y \cup \{\pm z_i\}_{i=1}^4)$) into X with $\|\tilde{T}\| = \|T\|$.

<u>Proof</u>. (i) $\Rightarrow$ (ii). Let Z, Y and T with $\|T\| = 1$ be given, and let $z \in Z \sim Y$. Then $z_i = \lambda_i z + y_i$, $i=1,\dots,n$, where $y_i \in Y$ and the λ_i are scalars. We may assume without loss of generality that $\lambda_i \neq 0$ for every i. $\|\tilde{T}\|$ will be equal to 1 if $\|\tilde{T}z_i\| \leq 1$ for every i, i.e. if $u = \tilde{T}z$ is chosen so that $\|u + Ty_i/\lambda_i\| \leq 1/|\lambda_i|$.

$$\begin{aligned} \|Ty_i/\lambda_i - Ty_j/\lambda_j\| &\leq \|y_i/\lambda_i - y_j/\lambda_j\| \\ &= \|z_i/\lambda_i - z_j/\lambda_j\| \leq 1/|\lambda_i| + 1/|\lambda_j| , \end{aligned}$$

and since X has the F.2.I.P. such a choice of u is possible.

(ii) $\Rightarrow$ (iii) is clear and (iii) $\Rightarrow$ (i) follows as in Theorem 5.4 if we use the second proof of Lemma 5.3.

For the 3,2.I.P. we obtain similarly

<u>Corollary</u>. A Banach space X has the 3,2.I.P. if and only if for every $Z \supset Y$ with $\dim Z/Y = 1$ and $S_Z = \mathrm{Co}(S_Y \cup \{\pm z_i\}_{i=1}^3)$, every operator from Y to X has a norm preserving extension from Z into X.

<u>Remark</u>. If $Z \supset Y$ and $S_Z = \mathrm{Co}(S_Y \cup \{\pm z_i\}_{i=1}^2)$ then it is easily seen that there is a projection of norm 1 from Z onto Y. Hence for every Banach space X and every operator from Y into X there is a norm preserving extension from Z to X.

We pass now to "from" extension properties. We shall use the following result of Klee [26].

Lemma 5.6. Let X be a Banach space and let $\{C_i\}_{i=1}^{n+1}$ be convex open sets in X such that $\bigcap_{i=1}^{n+1} C_i = \emptyset$. Then there is a closed subspace V of X with $\dim X/V \leq n$ so that no translate of V meets all the C_i.

Corollary. Let $\{S_X(x_i,r_i)\}_{i=1}^n$ be n cells in X such that $\bigcap_{i=1}^n S_X(x_i,r_i+\varepsilon) = \emptyset$ for some $\varepsilon > 0$. Then there is a quotient space Y of X with $\dim Y \leq n-1$ such that $\bigcap_{i=1}^n S_Y(Tx_i,r_i) = \emptyset$ where T denotes the quotient map from X onto Y.

Proof. Let V be a closed subspace of X with $\dim X/V \leq n-1$ such that no translate of V meets all the cells $S_X(x_i,r_i+\varepsilon/2)$, $i = 1,2,\ldots,n$ (apply the lemma to corresponding open cells with slightly bigger radii). Let $Y = X/V$ and let T be the quotient map. Then for every $x \in X$ and positive r and δ

$$TS_X(x,r) \subset S_Y(Tx,r) \subset TS_X(x,r+\delta) \ .$$

By the choice of V $\bigcap_{i=1}^n TS_X(x_i,r_i+\varepsilon/2) = \emptyset$ and hence $\bigcap_{i=1}^n S_Y(Tx_i,r_i) = \emptyset$.

Theorem 5.7. Let X be a Banach space and let $n \geq 3$. The following statements are equivalent

(i) X has the $n,2$.I.P.

(ii) Every operator from X to Y (with Y having a dimension $\leq n-1$) has a norm preserving extension $\tilde{T}$ from Z ($Z \supset X$ with $\dim Z/X = 1$) into Y.

Proof. (i) $\Rightarrow$ (ii). Let Y, Z and T be given with $\|T\| = 1$ and let $z \in Z \sim X$. By Lemma 5.2 we have only to show that $\bigcap_{x \in X} S_Y(Tx, \|x-z\|) \neq \emptyset$. By Helly's theorem it is enough to show that, for every $\{x_i\}_{i=1}^n$ in X, $\bigcap_{i=1}^n S_Y(Tx_i, \|z-x_i\|)$ is not empty. But since X has the $n,2$.I.P. and the cells $S_X(x_i, \|x_i-z\|)$ are mutually intersecting we get that $\bigcap_{i=1}^{n} S_X(x_i, \|x_i - z\|) \neq \emptyset$. This concludes the proof.

(ii) $\Rightarrow$ (i) follows from Lemma 4.2, Lemma 5.3 and the corollary to Lemma 5.6. We omit the details (cf. [33]).

Remark. In the next chapter we shall show that for $n \geq 4$ we can discard in (ii) the requirement that $\dim Z/X = 1$ and still get a property which is equivalent to (i). In [33] we have proved that the most important class of spaces which have the 3,2.I.P. but not the 4,2.I.P, that is the $L_1(\mu)$ spaces, have the property obtained from (ii) (for $n = 3$) by discarding any requirement on Z/X.

Lemma 5.8. Let X be a Banach space and let $\{S_X(x_i,r_i)\}_{i=1}^{n}$ be a finite number of cells in X. Then

$$\bigcap_{i=1}^{n} S_{X^{**}}(x_i,r_i) \neq \emptyset \iff \bigcap_{i=1}^{n} S_X(x_i,r_i+\varepsilon) \neq \emptyset \text{ for every } \varepsilon > 0.$$

Proof. If $\bigcap_{i=1}^{n} S_X(x_i,r_i+\varepsilon) \neq \emptyset$ for every $\varepsilon > 0$ then clearly $\bigcap_{i=1}^{n} S_{X^{**}}(x_i,r_i+\varepsilon) \neq \emptyset$ for every $\varepsilon > 0$ and hence by the w^* compactness of cells in X^{**} also $\bigcap_{i=1}^{n} S_{X^{**}}(x_i,r_i) \neq \emptyset$. Conversely, suppose that for some $\varepsilon > 0$ $\bigcap_{i=1}^{n} S_X(x_i,r_i+\varepsilon) = \emptyset$. Then by the corollary to Lemma 5.6 there is a finite-dimensional quotient space Y of X such that $\bigcap_{i=1}^{n} S_Y(Tx_i,r_i) = \emptyset$ where T is the quotient map from X onto Y. Suppose there is an x^{**} which belongs to all the cells $S_{X^{**}}(x_i,r_i)$, $i = 1,\dots,n$. Then $\|T^{**}x^{**} - Tx_i\| \leq r_i$ for every i and this is a contradiction. Hence $\bigcap_{i=1}^{n} S_{X^{**}}(x_i,r_i) = \emptyset$ and this establishes the lemma.

Corollary. Let X^{**} have the n,2.I.P. for some $n \geq 3$. Then also X has the n,2.I.P.

We say that the cells in a Banach space X have the finite intersection property if for every collection $\{S_\alpha\}_{\alpha \in A}$ of cells in X

such that $\bigcap_{i=1}^{n} S_{\alpha_i} \neq \emptyset$ for every finite subset $\{\alpha_i\}_{i=1}^{n}$ of A also $\bigcap_{\alpha \in A} S_\alpha \neq \emptyset$.

Theorem 5.9. Let X be a Banach space. The cells in X have the finite intersection property if and only if for every Y with $X^{**} \supset Y \supset X$ and $\dim Y/X \leq 1$ there is a projection of norm 1 from Y onto X.

Proof. Let $\{S_X(x_\alpha, r_\alpha)\}_{\alpha \in A}$ be a collection of cells in X such that any finite subcollection has a non void intersection. Then clearly $\bigcap_{\alpha \in A} S_{X^{**}}(x_\alpha, r_\alpha) \neq \emptyset$. Let x^{**} be a point in this intersection and let Y be the subspace of X^{**} spanned by X and x^{**}. Suppose that there is a projection P of norm 1 from Y onto X; then Px^{**} belongs to all the cells $S_X(x_\alpha, r_\alpha)$. Conversely, suppose that the cells of X have tne finite intersection property and let $X \subset Y \subset X^{**}$ with $\dim Y/X = 1$. Let $x^{**} \in Y \sim X$. By Lemma 5.2 it is enough to show that $\bigcap_{x \in X} S_X(x, \|x^{**}-x\|) \neq \emptyset$. By Lemma 5.8 we have, for every $\varepsilon > 0$ and every finite set $\{x_i\}_{i=1}^{n}$ in X, that $\bigcap_{i=1}^{n} S_X(x_i, \|x^{**}-x_i\| + \varepsilon) \neq \emptyset$. Hence, since the cells in X have the finite intersection property,

$$\bigcap_{x \in X} S_X(x, \|x^{**}-x\|) = \bigcap_{\varepsilon > 0} \bigcap_{x \in X} S_X(x, \|x^{**}-x\| + \varepsilon) \neq \emptyset$$

and this concludes the proof of the theorem.

Remarks. 1. It is well known that if $X = L_1(\mu)$ there is always a projection of norm 1 from X^{**} onto X. Since the unit cell of $L_1(0,1)$ has no extreme points we thus get an example of a Banach space whose cells have the finite intersection property though they have no extreme points. This solves a problem of Nachbin ([39], cf. also [38 problem 1]).

2. We do not know whether for every Banach space X whose cells have the finite intersection property there is a projection of norm 1 from X^{**} onto X.

To end this chapter we would like to give an example which shows that the corollary to Lemma 5.6 as well as Lemma 5.8 do not hold if ε is allowed to be 0. This example shows also the role of Lemma 4.2 in the discussions in this chapter and in the preceding one.

Example. There is a Banach space Z and three cells in it such that $\bigcap_{i=1}^{3} S_Z(z_i, r_i+\varepsilon) \neq \emptyset$ for every $\varepsilon > 0$ but $\bigcap_{i=1}^{3} S_Z(z_i, r_i) = \emptyset$.

Proof. Introduce in $Y = R \oplus m$ the following norm

$$\| y \| = \max \left(|\lambda|, \sup_n |x_n| + \left(\sum_{n=1}^{\infty} (x_n/n)^2 \right)^{1/2} \right)$$

where $y = (\lambda, x)$ with $\lambda \in R$ and $x = (x_1, x_2, \dots) \in m$. Let Z be the subspace of Y consisting of all the vectors (λ, x) such that $\lim_{n\to\infty} x_{2n}$ and $\lim_{n\to\infty} x_{2n+1}$ exist and satisfy

$$2\lambda = 3 \lim_{n\to\infty} x_{2n+1} - \lim_{n\to\infty} x_{2n} .$$

Take $z_1 = (0,0)$ and $z_2 = (0,x)$ where $x = (1,0,0,\dots)$. These points belong to Z, satisfy $\| z_1 - z_2 \| = 2$ and as easily seen $z_0 = (z_1+z_2)/2$ is the only point in Z for which $\| z_0 - z_1 \| = \| z_0 - z_2 \| = 1$. However, if $u_n = (1/2, x_n/2)$ where $x_n = (1,0,0,\dots,0,1,1,\dots)$ (in the places from 2 to n we have 0 otherwise 1) then $\| u_n - z_1 \| \to 1$ and $\| u_n - z_2 \| \to 1$ as $n \to \infty$. Let k_0 be an even integer such that

$$3/8 + \left(\sum_{k>k_0} 1/k^2 \right)^{1/2} < 7/16 ,$$

and put $z_3 = (1/2, v)$ where $v = (1/2, 0, 0, \dots, 0, 3/8, 1/8, 3/8, \dots)$ (the first coordinate is $1/2$, the coordinates from 2 to k_0 are 0 and the rest are alternately $3/8$ and $1/8$). Then $z_3 \in Z$, $\| z_3 - u_n \| < 7/16$ for $n > k_0$ while $\| z_3 - z_0 \| = 1/2$. Hence

$$S(z_1,1) \cap S(z_2,1) \cap S(z_3, 7/16) = \emptyset$$

while for every $\varepsilon > 0$

$$S(z_1,1+\varepsilon)\cap S(z_2,1+\varepsilon)\cap S(z_3,7/16) \neq \emptyset .$$

(Actually even $S(z_1,1+\varepsilon)\cap S(z_2,1)\cap S(z_3,7/16) \neq \emptyset$.)

CHAPTER VI. THE EXTENSION THEOREM FOR $\lambda = 1$ AND ITS APPLICATIONS

We begin this chapter with the statement of a theorem which is, in a sense, the central result of this paper. This theorem is in part the special case $\lambda = 1$ of Theorem 2.1. However the information available in this case is much more precise and complete than in the general case which was treated in Chapter II. Theorem 6.1 summarizes many results which were proved in the preceding chapters but it contains also some new assertions which will be proved here. The equivalences (1) <==> (2) <==> (3) <==> (5) are due to Grothendieck [14,15]. We recall the convention made in the introduction that in the statement of the extension properties Y and Z are arbitrary Banach spaces satisfying the requirements (if any) imposed on them.

Theorem 6.1. Let X be a Banach space. The following statements are equivalent.

(1) X^{**} is a $\mathcal{P}_1$ space.

(2) X^* is an $L_1(\mu)$ space (for some measure μ).

(3) Every compact operator T from Y to X has (for every $\varepsilon > 0$) a compact extension $\tilde{T}$ from Z $(Z \supset Y)$ to X with $\|\tilde{T}\| \leq (1+\varepsilon)\|T\|$.

(4) Every operator T from Y $(\dim Y \leq 3)$ to X has (for every $\varepsilon > 0$) an extension $\tilde{T}$ from Z $(Z \supset Y$ and $\dim Z/Y = 1)$ to X with $\|\tilde{T}\| \leq (1+\varepsilon)\|T\|$.

(5) Every operator T from Y to X has an extension $\tilde{T}$ from Z $(Z \supset Y)$ to X^{**} with $\|\tilde{T}\| = \|T\|$.

(6) Every operator T from Y to X has an extension $\tilde{T}$ from

Z $(Z \supset Y)$ to X with $\|\tilde{T}\| = \|T\|$ provided that S_Z is the convex hull of S_Y and a finite set of points.

(7) Every operator T from X to a conjugate space Y has an extension $\tilde{T}$ from Z $(Z \supset X)$ to Y with $\|\tilde{T}\| = \|T\|$.

(8) Every compact operator T from X to Y has a compact extension $\tilde{T}$ from Z $(Z \supset X)$ to Y with $\|\tilde{T}\| = \|T\|$.

(9) Every weakly compact operator T from X to Y has a weakly compact extension $\tilde{T}$ from Z $(Z \supset X)$ to Y with $\|\tilde{T}\| = \|T\|$.

(10) Every operator T from X to Y $(\dim Y \leq 3)$ has an extension $\tilde{T}$ from Z $(Z \supset X$ and $\dim Z/X = 1)$ to Y with $\|\tilde{T}\| = \|T\|$.

(11) X has the M.A.P. and every compact operator T from X into itself has an extension $\tilde{T}$ from Z $(Z \supset X$ with $\dim Z/X = 1)$ to X with $\|\tilde{T}\| = \|T\|$.

(12) X has the R.4,2.I.P.

(13) Every collection of mutually intersecting cells $\{S(x_\alpha, r_\alpha)\}$ in X such that the set of centers $\{x_\alpha\}$ is conditionally (norm) compact, has a non empty intersection.

If S_X has at least one extreme point the following statements are also equivalent to the preceding ones.

(14) X is isometric to a subspace X_1 of some $C(K)$ (K compact Hausdorff) having the following properties:

(a) 1_K, the function identically equal to 1 on K, belongs to X_1.

(b) The decomposition property. $f,g,h \in X_1$, $f,g,h \geq 0$ and $f + g \geq h \Longrightarrow$ there are $f_0, g_0 \in X_1$ such that $f \geq f_0 \geq 0$, $g \geq g_0 \geq 0$ and $f_0 + g_0 = h$.

(15) Every collection of four mutually intersecting cells $\{S(x_i, r_i)\}_{i=1}^4$ in X such that the $\{x_i\}_{i=1}^4$ span a 2-dimensional subspace of X has a non empty intersection.

(16) Every operator T from Y ($\dim Y = 2$) to X has (for every $\varepsilon > 0$) an extension $\tilde{T}$ from Z ($Z \supset Y$ and $\dim Z = 3$) to X with $\|\tilde{T}\| \leq (1+\varepsilon)\|T\|$.

<u>Proof</u>. The equivalence of (1) - (13) (except (2)) follows from Theorems 2.1, 4.1, 4.3, 4.5, 5.4, 5.5, 5.7 and the following remarks.

(i) By Theorem 5.4 (13) implies (3) for spaces Z with $\dim Z/Y = 1$ and hence for Z with $\dim Z/Y < \infty$. Therefore by (4) => (1) of Theorem 2.1 we get here the implication (13) => (1).

(ii) It is easily seen that (6) is equivalent to the property obtained from it by adding the requirement that $\dim Z/Y = 1$. Hence by Theorem 5.5 property (6) is equivalent to the F.2.I.P.

(iii) (11) => (12). Let $\{S(x_i,r_i)\}_{i=1}^n$ be a finite collection of mutually intersecting cells in X and let $\varepsilon > 0$. By the M.A.P. there is a compact T from X into itself such that $\|T\| = 1$ and $\|Tx_i - x_i\| \leq \varepsilon$ for every i. By Lemma 5.3 there is a space $Z \supset X$ with $\dim Z/X = 1$ and a point $z \in Z$ such that $\|z-x_i\| \leq r_i$, $i=1,\ldots,n$. Let $\tilde{T}$ be a norm preserving extension of T from Z into X. $\|\tilde{T}z-x_i\| \leq r_i + \varepsilon$ for every i and (12) follows now by Lemma 4.2. It should be remarked that unlike the cases treated in (i) and (ii) it does not seem to be immediate that (11) implies that X has a similar property for all Z with $\dim Z/X < \infty$.

(iv) A Banach space X which has the F.2.I.P. also has the M.A.P. Indeed, let B be a finite-dimensional subspace of X and let $\varepsilon > 0$. By approximating B by a space B_1 whose unit cell is a polyhedron and embedding B_1 in an ℓ_∞^n space for some n it follows that there is a finite-dimensional space $U \supset B$ which is a $\mathcal{P}_{1+\varepsilon}$ space. By Theorem 5.4 the identity operator from B into X has an extension T_0 from U into X with $\|T_0\| \leq 1 + \varepsilon$. Since U is a $\mathcal{P}_{1+\varepsilon}$ space there is an operator T_1 from X into U such that $T_{1|B}$ is the identity and $\|T_1\| \leq 1 + \varepsilon$. The operator $T = T_0 T_1$ from X into itself satisfies $\|T\| \leq (1+\varepsilon)^2$ and its restriction to B is the identity. Hence X has the M.A.P. It follows now (use Theorem 5.7) that also (10) implies the M.A.P. Since clearly (8) => (10) and (9) => (10) it follows from Theorem 2.1 that any one of the properties (1) - (10) implies the M.A.P.

The equivalence of (1) and (2) (in which the non obvious implication is (1) => (2)) was proved by Grothendieck [15].

We turn now to spaces X for which S_X has at least one extreme point, and show that for such X (14) is equivalent to (1) - (13). It is easy to prove that (14) => (12). We prefer, however, to prove that (14) => (2) and thus to obtain a new proof to the equivalence of (1) and (2) (valid of course only for spaces X in which S_X has an extreme point). Let X_1 satisfy (a) and (b) of (14). We order X_1^* by

$$x^* \geq 0 \iff x^*(f) \geq 0 \quad \text{for every} \quad f \geq 0 \quad \text{in} \quad X_1.$$

Clearly $x^* \geq 0$ iff $x^*(1_K) = \|x^*\|$. F. Riesz [43] has shown that (b) implies that in the order defined above X_1^* is a lattice (cf. also Kadison [21, Lemma 5.1] or Day [6, p. 98] and the references there). It is immediate that $x^*, y^* \geq 0 \implies \|x^*+y^*\| = \|x^*\| + \|y^*\| \ (= x^*(1_K) + y^*(1_K))$ and that $x^* \wedge y^* = 0$ ($\wedge$ is the lattice operation) implies $\|x^*+y^*\| = \|x^*-y^*\|$. Hence X_1^* is an L space in the terminology of Kakutani [23]. (14) ==> (2) follows now from the representation theorem of Kakutani [23].

We prove now that (12) ==> (14). From (2) ==> (4) of Theorem 4.7 it follows that X is isometric to a subspace X_1 of some $C(K)$ satisfying (a) of (14). We shall show that if X_1 has the R.4,3.I.P. (and in particular if it has the R.4,2.I.P.) then it also has the decomposition property. Let $f,g,h \in X_1$ $f,g,h \geq 0$ and $f+g \geq h$. Without loss of generality we may assume that $\|f\|, \|g\|, \|h\| < 1$. Consider the four cells

$$S_1 = S(1_K,1) , \qquad S_2 = S(f-1_K,1) ,$$

$$S_3 = S(h-1_K,1) , \qquad S_4 = S(1_K+h-g,1) .$$

Every three of them intersect. Indeed $0 \in S_1 \cap S_2 \cap S_3$ ($0 \in S_1$ is clear, $0 \in S_2$ follows from $0 \leq f \leq 1_K$, $0 \in S_3$ follows from $0 \leq h \leq 1_K$) and similarly $h \in S_1 \cap S_3 \cap S_4$, $h-g \in S_2 \cap S_3 \cap S_4$ and $f \in S_1 \cap S_2 \cap S_4$. Hence by the R.4,3.I.P. there is an $f_0 \in S_1 \cap S_2 \cap S_3 \cap S_4$. Put $g_0 = h-f_0$. We have $f_0 \in S_1 \implies f_0 \geq 0$, $f_0 \in S_2 \implies f \geq f_0$, $f_0 \in S_3 \implies g_0 \geq 0$, $f_0 \in S_4 \implies g \geq g_0$ and this proves (b) of (14).

It should be mentioned that as K in (14) we may take the w^* closure of the positive extreme points of S_{X^*}. (X is ordered by taking any extreme point e of S_X and letting $x \geq 0 \iff x = \lambda(e+u)$, $\lambda \geq 0$ and $\|u\| \leq 1$. X^* is ordered as in the proof of (14) $\implies$ (2).) The mapping of X into $C(K)$ is then the canonical one $x(k) = k(x)$, $k \in K$, $x \in X$ (cf. Kadison [21, Theorem 5.2]). It is possible, and will be convenient sometimes, to take as K the set of positive extreme points itself (this K, however, will not in general be compact).

The equivalence of (15) and (16) with (1) - (14), under the assumption that S_X has an extreme point, will be shown later on in this chapter (cf. Lemma 6.5).

We give some corollaries of Theorem 6.1.

Corollary 1. Let X be an $\mathcal{N}_\lambda$ space for every $\lambda > 1$. Then X satisfies (1) - (13) of Theorem 6.1.

Proof. Use Theorem 3.3.

We do not know whether the converse of Corollary 1 is true. At the end of Chapter III we showed that every $C(K)$ space is an $\mathcal{N}_\lambda$ space for every $\lambda > 1$. A similar argument (however technically more complicated) can be used to show that certain other special classes of spaces which satisfy (1) - (13) (for example $C_\sigma(K)$ spaces, cf. Day [6, p. 89] for their definition) are $\mathcal{N}_\lambda$ spaces for every $\lambda > 1$. In this connection see also Corollary 3 to Theorem 7.9.

<u>Corollary 2</u>. Let X satisfy (1) - (13) of Theorem 6.1 and let Y be a separable subspace of X. Then there is a separable subspace Z of X containing Y which satisfies (1) - (13).

<u>Proof</u>. Use Theorem 4.4.

<u>Corollary 3</u>. Let X be a conjugate space satisfying (1) - (16) of Theorem 6.1. Then X is a $\mathcal{P}_1$ space.

<u>Proof</u>. There exists a projection with norm 1 from X^{**} onto X (Dixmier [7]). Or, alternatively, the fact that every collection of mutually intersecting cells in X has a common point follows from the F.2.I.P. and the w^* compactness of the cells.

<u>Corollary 4</u>. Let X be a conjugate CL space (in the sense of Fullerton [9], cf. also the discussion preceding Theorem 4.8). Then X is a C(K) space <==> X has the R.4,3.I.P.

<u>Proof</u>. ==> is clear. <== follows from Corollary 3, from the proof of (12) ==> (14) in Theorem 6.1 and from the fact that CL spaces have the properties appearing in Theorem 4.7.

Another property characterizing C(K) spaces among the conjugate CL spaces was given by Fullerton [9].

<u>Corollary 5</u>. Let X satisfy (1) - (13) of Theorem 6.1 and let F be a maximal convex subset of the boundary of S_X. Then S_X is the (norm) closed convex hull of $F \cup -F$.

<u>Proof</u>. Use Theorem 4.8(b) and Corollary 1 to Theorem 4.6.

We shall now study some questions related to the decomposition property (property (14) (b) of Theorem 6.1).

Lemma 6.2. Let X be a linear space of functions on a set K. X has the decomposition property iff it satisfies

$(^+)$ For every $n + m$ functions in X $\{f_i\}_{i=1}^n$, $\{g_j\}_{j=1}^m$, such that $f_i \leq g_j$, $i = 1,\ldots,n$, $j = 1,\ldots,m$ there is an $\tilde{h} \in X$ satisfying $f_i \leq \tilde{h} \leq g_j$, $i = 1,\ldots,n$, $j = 1,\ldots,m$.

Proof. We observe first that the decomposition property is equivalent to $(^+)$ for $m = n = 2$. Indeed, put $f_1 = 0$, $f_2 = h - g$, $g_1 = f$, $g_2 = h$; then the requirements of $(^+)$ on $\{f_i\}_{i=1}^2$, $\{g_j\}_{j=1}^2$ are satisfied iff $f,g,h \geq 0$ and $f+g \geq h$. The fact that $(^+)$ with $n = m = 2$ implies that $(^+)$ holds for every n and m follows easily by induction (first on m for $n = 2$ and then on n for a fixed m).

This simple lemma enables us to give a new proof to the fact that the R.4,2.I.P. implies the F.2.I.P.

Theorem 6.3. Let X be a normed space whose unit cell has at least one extreme point. X has the F.2.I.P. if it has the following two properties:

(i) The R.4,3.I.P.

(ii) Let $\{S_i\}_{i=1}^3$ be a collection of 3 mutually intersecting cells in X with a common radius and such that $S_1 \cap S_2$ is a single point e. Then $e \in S_3$.

Proof. From Theorem 4.7 and the proof of Theorem 6.1 ((12) $\Longrightarrow$ (14)) it follows that X is (isometric to) a subspace of some $C(K)$ containing the function 1_K and satisfying $(^+)$ of Lemma 6.2. Let $\{S(x_i,r_i)\}_{i=1}^n$ be n mutually intersecting cells in X. Put $f_i = x_i - r_i 1_K$, $g_i = x_i + r_i 1_K$, $i = 1,\ldots,n$. Then $f_i \leq g_j$, $i,j = 1,\ldots,n$ and hence by $(^+)$ there is an $\tilde{h} \in X$ such that $f_i \leq \tilde{h} \leq g_i$

(i.e. $\|\tilde{h} - x_i\| \leq r_i$) for every i.

From the proof of Theorem 6.3 it follows that a (not necessarily closed) subspace of $C(K)$ containing 1_K has the F.2.I.P. iff it has the decomposition property. We shall now prove that for closed subspaces the decomposition property is equivalent to its special case where $f + g = 1_K$.

Lemma 6.4. Let X be a closed subspace of some $C(K)$ containing the function 1_K. Then X has the F.2.I.P. iff it has the following property

$(^{++})$ $\quad f,g,h \in X,\ \ f,g,h \geq 0,\ \ 1_K = f + g \geq h \implies$ there are $f_0, g_0 \in X$ satisfying $0 \leq f_0 \leq f,\ 0 \leq g_0 \leq g,\ \ f_0 + g_0 = h$.

Proof. By extending the argument of Riesz [43] we shall prove that $(^{++})$ implies that X^* is a lattice in its natural order. Then (as in the proof of (14) $\implies$ (2) in Theorem 6.1) it will follows that X^* is an L space and this will conclude the proof of the lemma.

Let $x^* \in X^*$. For every $f \in X$, $f \geq 0$, we define

$$x^{*+}(f) = \sup_{0 \leq h \leq f} x^*(h) .$$

It is clear that

$$(6.1) \qquad x^{*+}(\lambda f) = x^{*+}(f)\ ,\ \lambda \geq 0,\ \ f \geq 0,$$

$$(6.2) \qquad x^{*+}(f) + x^{*+}(g) \leq x^{*+}(f + g),\quad f,g \geq 0.$$

We intend to show that (6.2) is actually an equality, i.e. that

$$(6.3) \qquad x^{*+}(f) + x^{*+}(g) = x^{*+}(f + g),\quad f,g \geq 0.$$

If $f + g = 1_K$ then by $(^{++})$ every $h \leq f + g$ is of the form $f_0 + g_0$, with $0 \leq f_0 \leq f$ and $0 \leq g_0 \leq g$, and hence in this case (6.3) is clear. In order to show that (6.3) holds for every positive f and g we may assume (by (6.1)) that $f,g \leq 1_K$.

We have

$$x^{*+}(f) + x^{*+}(1_K - f) = x^{*+}(1_K) ,$$
$$x^{*+}(g) + x^{*+}(1_K - g) = x^{*+}(1_K) ,$$
$$x^{*+}(f+g) + x^{*+}(2.1_K - f - g) = 2x^{*+}(1_K) .$$

These equalities, (6.2) and the inequality

$$x^{*+}(1_K - f) + x^{*+}(1_K - g) \le x^{*+}(2.1_K - f - g)$$

imply (6.3). From (6.3) it follows that x^{*+} can be extended to a continuous linear functional on X by defining $x^{*+}(f) = x^{*+}(f_1) - x^{*+}(f_2)$, where $f = f_1 - f_2$ and $f_1, f_2 \ge 0$. It is now clear that X^* becomes a lattice if we define $x^* \vee y^* = y^* + (x^* - y^*)^+$.

Every algebra of functions (with the usual pointwise multiplication) which contains 1_K satisfies $(^{++})$. Indeed, if $f + g = 1_K \ge h \ge 0$ then $f_0 = fh$ and $g_0 = gh$ have the required properties. We remark that the same is true if the multiplication in the algebra is not the usual (pointwise) one, provided it is distributive and satisfies $1_K.f = f.1_K = f$ and $f \ge 0$, $g \ge 0 \Longrightarrow fg \ge 0$. (Commutativity and associativity are not required; these are exactly the same requirements as those appearing in Kadison [21, Section 3].) We shall now give an example of an algebra of continuous functions containing 1_K which does not have the decomposition property. If we take in this algebra the sup norm we obtain an example of a normed space which does not have the F.2.I.P. though it satisfies $(^{++})$ and its completion has the F.2.I.P. (it follows, in particular, that Lemma 6.4 does not hold for non closed subspaces X).

<u>Example</u>. Let $r_1(x)$ and $s_1(x)$ be two continuous functions on $[0,\frac{1}{2}]$ such that $x, r_1(x), s_1(x)$ are algebraically independent and $r_1(x) \le x \le s_1(x)$ $(0 \le x \le \frac{1}{2})$, $r_1(\frac{1}{2}) = s_1(\frac{1}{2}) = \frac{1}{2}$. Similarly let $r_2(x)$ and $s_2(x)$ be two continuous functions on $[\frac{1}{2},1]$ such that $x, r_2(x)$, $s_2(x)$ are algebraically independent and $r_2(x) \le x \le s_2(x)$ $(\frac{1}{2} \le x \le 1)$, $r_2(\frac{1}{2}) = s_2(\frac{1}{2}) = \frac{1}{2}$. We define four continuous functions on

$[0,1]$ as follows:

$$f_1(x) = \begin{cases} r_1(x) & 0 \leq x \leq \frac{1}{2} \\ x & \frac{1}{2} \leq x \leq 1 \end{cases}, \qquad f_2(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ r_2(x) & \frac{1}{2} \leq x \leq 1 \end{cases},$$

$$g_1(x) = \begin{cases} s_1(x) & 0 \leq x \leq \frac{1}{2} \\ x & \frac{1}{2} \leq x \leq 1 \end{cases}, \qquad g_2(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ s_2(x) & \frac{1}{2} \leq x \leq 1 \end{cases}.$$

Let A be the subalgebra of $C(0,1)$ generated by $1_{[0,1]}$, f_1, f_2, g_1, g_2. The function x does not belong to A. Indeed, suppose that

$$x = \Sigma\, a_{i_1,i_2,j_1,j_2}\ f_1(x)^{i_1}\, f_2(x)^{i_2}\, g_1(x)^{j_1}\, g_2(x)^{j_2}\ .$$

For $0 \leq x \leq \frac{1}{2}$ we obtain

$$0 = x - \Sigma\, a_{i_1,i_2,j_1,j_2}\ r_1(x)^{i_1}\, s_1(x)^{j_1}\, x^{i_2+j_2}\ .$$

Since x, r_1 and s_1 are algebraically independent the homogeneous part of degree 1 is 0 i.e.

$$x = a_{1,o,o,o}\, r_1(x) + a_{o,o,1,o}\, s_1(x) + (a_{o,1,o,o} + a_{o,o,o,1})x$$

or $a_{1,o,o,o} = a_{o,o,1,o} = 0$, $a_{o,1,o,o} + a_{o,o,o,1} = 1$. Similarly by taking $\frac{1}{2} \leq x \leq 1$ we obtain $a_{o,1,o,o} = a_{o,o,o,1} = 0$ and this is a contradiction. We have shown that $x \notin A$. But $x = \max(f_1,f_2) = \min(g_1,g_2)$ and hence A does not have the decomposition property (Lemma 6.2).

We are now ready to prove that each of the properties (15) and (16) of Theorem 6.1 is equivalent to (1) - (13) if S_X has at least one extreme point. By Corollary 1 to Theorem 5.4 it is enough to prove the following

Lemma 6.5. Let X be a Banach space such that S_X has at least one extreme point and such that X has the following property

$(15)_0$ For every collection of four mutually intersecting cells $\{S(x_i,r_i)\}_{i=1}^4$ in X, such that the $\{x_i\}_{i=1}^4$ span a 2-dimensional subspace of X, and for every $\varepsilon > 0$ $\bigcap_{i=1}^{4} S(x_i, r_i+\varepsilon) \neq \emptyset$.

Then X^* is an $L_1(\mu)$ space.

Proof. $(15)_0$ implies the 3,2.I.P. (cf. Lemma 4.2) and hence by Theorem 4.7 X is isometric to a subspace of some $C(K)$ containing 1_K. We take in X the order induced on it by $C(K)$ and define for every x^* in X^* and every $f \geq 0$ in X

$$(6.4) \qquad x^{*+}(f) = \lim_{\varepsilon \to 0} \sup_{0 \leq h \leq f+\varepsilon 1_K} x^*(h) .$$

It is clear that the limit always exists. If $f \geq \delta \cdot 1_K$ for some $\delta > 0$ then $f \leq f+\varepsilon \cdot 1_K \leq (1+\varepsilon/\delta)f$ and it follows that $x^{*+}(f) = \sup_{0 \leq h \leq f} x^*(h)$. It is easily seen that (6.1) and (6.2) hold and that

$$(6.5) \qquad \lim_{\varepsilon \to 0} x^{*+}(f+\varepsilon g) = x^{*+}(f) \quad , \quad f,g \geq 0 .$$

As in the proof of Lemma 6.4 the present lemma will be proved once (6.3) is established. We prove first the following special case of (6.3)

$$(6.6) \qquad x^{*+}(1_K+f) + x^{*+}(f) = x^{*+}(1_K+2f), \quad 0 \leq f \leq 1_K .$$

Let h satisfy $0 \leq h \leq 1_K + 2f$ and consider the following four cells

$$S(1_K,1) , \qquad S(f-1_K,1) ,$$
$$S(h-2\cdot 1_K,2) , \qquad S(h-f,1) .$$

It is easily checked that these cells are mutually intersecting. If we translate all the centers of the cells by -1_K we get the four points 0, $f-2\cdot 1_K$, $h-3\cdot 1_K$ and $h-f-1_K = (h-3\cdot 1_K) - (f-2\cdot 1_K)$, which lie in a 2-dimensional subspace of X. Hence by $(15)_0$ there is for every $\varepsilon > 0$ an element u_ε in X such that

$$\| 1_K-u_\varepsilon \| \leq 1 + \varepsilon , \qquad \| 1_K-f+u_\varepsilon \| \leq 1 + \varepsilon ,$$
$$\| 2\cdot 1_K-h+u_\varepsilon \| \leq 2 + \varepsilon , \qquad \| h-u_\varepsilon-f \| \leq 1 + \varepsilon ,$$

and thus

$$-\varepsilon\cdot 1_K \leq u_\varepsilon \leq f + \varepsilon\cdot 1_K \ , \quad -\varepsilon\cdot 1_K \leq h-u_\varepsilon \leq f + (1+\varepsilon)\cdot 1_K \ .$$

Hence

$$x^{*+}(f+2\varepsilon\cdot 1_K) + x^{*+}(f+(1+2\varepsilon)\cdot 1_K)$$
$$\geq x^*(u_\varepsilon+\varepsilon\cdot 1_K) + x^*(h-u_\varepsilon+\varepsilon\cdot 1_K) = x^*(h) + 2\varepsilon x^*(1_K) \ .$$

Since $\varepsilon > 0$ was arbitrary and h had only to satisfy $0 \leq h \leq 1_K + 2f$ we get by (6.5) that

$$x^{*+}(1_K+f) + x^{*+}(f) \geq x^{*+}(1_K+2f) \ ,$$

and this together with (6.2) prove (6.6). By (6.6) we have for every integer $n \geq 0$ and $0 \leq f \leq 1_K$

$$x^{*+}(1_K+f) = (2^{-1}+2^{-2}+\dots+2^{-n})x^{*+}(f) + x^{*+}(1_K+2^{-n}f) \ ,$$

and hence, by (6.5),

(6.7) $$x^{*+}(1_K+f) = x^{*+}(1_K) + x^{*+}(f) \qquad 0 \leq f \leq 1_K \ .$$

Let now $g \in X$ with $0 \leq g \leq 1_K$, let $x^* \in X^*$ and let $n > 1$ be an integer. We take an $h \in X$ such that $0 \leq h \leq 1_K$ and

(6.8) $$|x^{*+}(1_K) - x^*(h)| \leq 1/n^2 \ .$$

Consider the four cells

$$S((2n^2-1)\cdot 1_K, \ 2n^2-1) \ , \qquad S(-n\cdot 1_K+g,n) \ ,$$
$$S(-n\cdot 1_K+(n+1)h, \ n) \ , \qquad S((n-1)\cdot 1_K + \tfrac{n-1}{n+1}\, g+h, \ n) \ .$$

It is easily checked that the cells are mutually intersecting and that if we translate the centers by $-(2n^2-1)\cdot 1_K$ we get four points which lie in a 2-dimensional subspace of X. Hence by $(15)_0$ we get that for every $\varepsilon > 0$ there is an element u_ε in X satisfying

$$-\varepsilon\cdot 1_K \leq u_\varepsilon \leq g + \varepsilon\cdot 1_K \ ,$$
$$-\varepsilon\cdot 1_K \leq (n+1)h-u_\varepsilon \leq nh - \tfrac{n-1}{n+1}\, g + (1+\varepsilon)\cdot 1_K \leq (n+1+\varepsilon)\cdot 1_K - \tfrac{n-1}{n+1}\, g \ .$$

Hence

$$\varepsilon x^*(1_K) + x^*(u_\varepsilon) \leq x^{*+}(g + 2\varepsilon\cdot 1_K)$$

and

$$(n+1)x^*(h) - x^*(u_\varepsilon) + \varepsilon x^*(1_K) \leq x^{*+}((n+1+2\varepsilon)\cdot 1_K - \tfrac{n-1}{n+1}\, g\ ) .$$

It follows from these inequalities, (6.7) and (6.8) that

$$\begin{aligned} & nx^{*+}(1_K) + x^{*+}((1+2\varepsilon)\cdot 1_K - \tfrac{n-1}{n+1}\, g) \\ = {} & x^{*+}((n+1+2\varepsilon)\cdot 1_K - \tfrac{n-1}{n+1}\, g) \geq (n+1)x^*(h) - x^{*+}(g+2\varepsilon\cdot 1_K) + 2x^*(\varepsilon\cdot 1_K) \\ \geq {} & (n+1)x^{*+}(1_K) - x^{*+}(g+2\varepsilon\cdot 1_K) - (n+1)/n^2 - 2\varepsilon\|x^*\| . \end{aligned}$$

Hence

$$\begin{aligned} & x^{*+}((1+2\varepsilon)\cdot 1_K - \tfrac{n-1}{n+1}\, g) + x^{*+}(g+2\varepsilon\cdot 1_K) \\ & \qquad \geq x^{*+}(1_K) - (n+1)/n^2 - 2\varepsilon\|x^*\| . \end{aligned}$$

Letting $\varepsilon \longrightarrow 0$ and $n \longrightarrow \infty$ we obtain that $x^{*+}(1_K - g) + x^{*+}(g) \geq x^{*+}(1_K)$ and from this (6.3) follows (cf. Lemma 6.4). This concludes the proof of Lemma 6.5 and thus of Theorem 6.1.

Before we leave the discussion of the decomposition property we would like to give some examples of (in general not closed) subspaces of $C(K)$ which have the decomposition property. Every subspace of $C(K)$ which is a lattice in the usual order for functions (but not necessarily a sublattice of $C(K)$) has the decomposition property (such spaces were considered by Geba and Semadeni [10]). Another category of examples (cf. [43]) are the spaces of continuous rational functions on $[0,1]$, real analytic functions on $[0,1]$, and functions having derivatives up to order n everywhere in $[0,1]$ (for some integer n). For all these spaces the decomposition property is established by taking (given $f,g,h \geq 0$, $f+g \geq h$) ,

$$f_0(x) = \begin{cases} \dfrac{f(x)h(x)}{f(x)+g(x)} & f(x)+g(x) \neq 0 \\ 0 & f(x)=g(x)=0 \end{cases} , \quad g_0(x) = \begin{cases} \dfrac{g(x)h(x)}{f(x)+g(x)} & f(x)+g(x) \neq 0 \\ 0 & f(x)=g(x)=0 \end{cases} .$$

We omit the straightforward verification that if f,g and h have n derivatives the same is true for f_0 and g_0. Less immediate is the fact that also the space of all the polynomials (with the norm $\|f\| = \max_{0\le x\le 1}|f(x)|$) has the F.2.I.P. (the proof of this fact was shown to us by M. Perles). Let f_i and g_i, $i = 1,2$ be four polynomials satisfying $f_i(x) \le g_j(x)$, $i,j = 1,2$, $0 \le x \le 1$. Excluding the trivial case where at least two of the polynomials coincide, there is only a finite (perhaps empty) set of points x in $[0,1]$ for which $\max_{i=1,2} f_i(x) = \min_{i=1,2} g_i(x)$. We denote this set by $\{x_p\}_{p=1}^n$ and assume for the moment that it does not contain 0 or 1. Let x_p be one such point then

$$f_i(x_p) = g_j(x_p) \tag{6.9}$$

for some i and j. Since $g_j - f_i$ has a minimum at the point $x = x_p$ there is an integer m such that

$$f_i^{(\nu)}(x_p) = g_j^{(\nu)}(x_p),\ \nu < 2m,\quad f_i^{(2m)}(x_p) < g_j^{(2m)}(x_p)\ .$$

We call this m the order of contact of the pair (f_i,g_j) at x_p. There may be more than one pair of i and j for which (6.9) is satisfied. The maximal of the (at most four) orders of contact thus assigned to x_p will be denoted by m_p. Let $r(x)$ be a polynomial which satisfies for every p and for every pair (f_i,g_j) whose order of contact at x_p is m_p

$$r^{(\nu)}(x_p) = f_i^{(\nu)}(x_p) = g_j^{(\nu)}(x_p)\ ,\ \nu < 2m_p$$

and

$$f_i^{(2m_p)}(x_p) < r^{(2m_p)}(x_p) < g_j^{(2m_p)}(x_p)\ .$$

Let $s(x)$ be the non negative polynomial

$$s(x) = \prod_{p=1}^{n}(x - x_p)^{2m_p}\ .$$

Put now $F_i(x) = (f_i(x) - r(x))/s(x)$, $G_i(x) = (g_i(x) - r(x))/s(x)$,

$i = 1,2$. The F_i or G_i need not be finite at $x = x_p (1 \leq p \leq n)$. However, the following facts are evident

(i) $F_i(x) < G_j(x)$, $i,j = 1,2$, $x \neq x_p$, $(1 \leq p \leq n)$.

(ii) $F_i(x_p)$ $(1 \leq p \leq n)$ is either $-\infty$ or finite and negative ($F_i(x_p)$ is defined by $\lim_{x \to x_p} F_i(x)$). $G_i(x_p)$ is either $+\infty$ or finite and positive.

(iii) For every p at least one of the $F_i(x_p)$ and one of the $G_i(x_p)$ is finite.

Hence, $F(x) = \max(F_1(x), F_2(x))$ and $G(x) = \min(G_1(x), G_2(x))$ are finite and continuous functions which satisfy $G(x) > F(x)$ for every x in $[0,1]$. By the Weierstrass approximation theorem there is a polynomial $H(x)$ for which $G(x) > H(x) > F(x)$, $0 \leq x \leq 1$. The polynomial $h(x) = H(x)s(x) + r(x)$ satisfies $f_i(x) \leq h(x) \leq g_i(x)$, $i = 1,2$, $0 \leq x \leq 1$.

Suppose now that $x_p = 0$ for some p (the case $x_p = 1$ is similar). We cannot claim in this case that $g_j - f_i$ has a zero of an even order at $x = 0$. But since x^ν is non negative in $[0,1]$ for every ν, we may take as a factor in $s(x)$ (corresponding to $x_p = 0$) also x^ν with an uneven ν. With this slight modification the proof proceeds as in the case treated above.

The $C(K)$ spaces have properties (1) - (13) of Theorem 6.1. In fact these properties "almost" characterize $C(K)$ spaces. We have

Theorem 6.6. A Banach space X is isometric to a $C(K)$ space (K compact Hausdorff) iff it has the following properties:

(i) X has one (and hence all) of the properties (1) - (13) of Theorem 1.1.

or (i_0) X is an $\mathcal{N}_\lambda$ space for every $\lambda > 1$.

(ii) S_X has at least one extreme point.

(iii) The set of extreme points of S_{X^*} is w^* closed.

Proof. That a $C(K)$ space satisfies (i), (ii) and (iii) is well known, and that it satisfies (i_0) was proved in Chapter III. We only have to prove the converse. $(i_0) \Rightarrow (i)$ by Corollary 1 to Theorem 6.1. From (i) and (ii) it follows that X satisfies (14) of Theorem 6.1 (we identify X with X_1). As remarked at the end of the proof of Theorem 6.1 we may assume that K is the w^* closure of the set of positive extreme points of S_{X^*}. The set of positive extreme points of S_{X^*} is the intersection of the set of extreme points with the w^* closed hyperplane $x^*(1_K) = 1$. Hence, by (iii), every point of K is an extreme point of S_{X^*}. Kadison [21, the proof of Theorem 4.1] has shown that if X is a separating subspace of $C(K)$ containing 1_K and if the functional φ_k corresponding to a point $k \in K$ $(\varphi_k(f) = f(k))$, is an extreme point of S_{X^*} then for every $h \in C(K)$

$$\sup\{f(k);\ f \leq h,\ f \in X\} = h(k) = \inf\{g(k);\ g \geq h,\ g \in X\}\ . \tag{6.10}$$

Hence in our case (6.10) holds for every $k \in K$.

Let $h \in C(K)$ and let $\varepsilon > 0$. For every $k \in K$ there are functions $f_k,\ g_k \in X$ satisfying

$$f_k \leq h \leq g_k, \quad f_k(k) + \varepsilon > h(k) > g_k(k) - \varepsilon\ .$$

Put $G_k = \{p \in K;\ f_k(p) + \varepsilon > h(p) > g_k(p) - \varepsilon\}$. G_k is an open set which contains k. By the compactness of K there are $\{k_i\}_{i=1}^n$ such that $K = \bigcup_{i=1}^n G_{k_i}$. By Lemma 6.2 there is an $\tilde{h} \in X$ which satisfies

$$f_{k_i} \leq \tilde{h} \leq g_{k_i}\ , \qquad i = 1,2,\dots,n\ .$$

It is clear from the construction that $\|\tilde{h} - h\| \leq 2\varepsilon$. Hence X is dense in $C(K)$ and since it is complete it coincides with $C(K)$.

No one of the properties (i), (ii) and (iii) is implied by the other two. Clearly (ii) and (iii) do not imply (i). The subspace of $C(0,1)$ consisting of all the functions for which $f(0) + f(1) = 0$ satisfies (i).

(By using the Schauder basis of $C(0,1)$ as in chapter III it can be even proved that it is an $\mathcal{N}_1$ space.) This space satisfies also (iii), since the set of extreme points of the unit cell of its conjugate (with the w^* topology) is homeomorphic to the circle, but it does not satisfy (ii).

Let X be the space of all the sequences $x = (x_1, x_2, \ldots)$ for which $\lim x_i = (x_1 + x_2)/2$, with $\|x\| = \max_i |x_i|$. X satisfies (ii) (the sequence $(1,1,\ldots)$ is an extreme point of S_X), but not (iii) since the functionals φ_i ($\varphi_i(x) = x_i$, $i = 1,2,\ldots$) are extreme points of S_{X^*} and converge in the w^* topology to the functional $(\varphi_1 + \varphi_2)/2$ which is not extremal. We shall show that X is an $\mathcal{N}_1$ space (and hence it satisfies (i)). X has the following basis

$$e_1 = (1,0,\tfrac{1}{2},\tfrac{1}{2},\ldots)$$

$$e_2 = (0,1,\tfrac{1}{2},\tfrac{1}{2},\ldots)$$

$$e_3 = (0,0,1,0,\ldots)$$

and in general for $j \geq 3$

$$e_j = (0,0,0,\ldots,0,1,0\ldots)$$

(all the coordinates are zero except the j-th which is 1). Let B_k be the subspace of X spanned by $\{e_j\}_{j=1}^k$. In B_k we take the following basis

$$e_1' = e_1 - \tfrac{1}{2}e_3 - \ldots - \tfrac{1}{2}e_k = (1,0,0,\ldots,0,\tfrac{1}{2},\tfrac{1}{2},\ldots)$$

$$e_2' = e_2 - \tfrac{1}{2}e_3 - \ldots - \tfrac{1}{2}e_k = (0,1,0,\ldots,0,\tfrac{1}{2},\tfrac{1}{2},\ldots)$$

$$e_j' = e_j \qquad 3 \leq j \leq k .$$

For every $\{\lambda_j\}_{j=1}^k$ we have

$$\Big\| \sum_{j=1}^k \lambda_j e_j' \Big\| = \max\Big(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_k|, \Big|\frac{\lambda_1+\lambda_2}{2}\Big|\Big) = \max_{1\leq j\leq k} |\lambda_j| .$$

Thus B_k is a $\mathcal{P}_1$ space and hence $X = \overline{\bigcup_{k=1}^\infty B_k}$ is an $\mathcal{N}_1$ space.

This example solves a problem raised by Nachbin [37], [38]. The same problem was mentioned also by Aronszajn and Panitchpakdi [2] in connection with their characterization of $C(K)$ spaces having the following intersection property: Let $\mathfrak{m}$ be a cardinal number. A normed space X has the $\mathfrak{m}$.2.I.P. if for every collection of $\mathfrak{m}$ mutually intersecting cells in X there is a point common to all the cells. We remark that for every cardinal number $\mathfrak{m}$ there is a Banach space X which has the $\mathfrak{m}$.2.I.P. and whose unit cell has an extreme point but which does not satisfy (iii) of Theorem 6.6. Indeed, let Ω be a set of cardinality $2^{\mathfrak{m}}$ and let ω_1 and ω_2 be two elements of Ω. The space of all the bounded functions on Ω satisfying $f(\omega) = \frac{f(\omega_1)+f(\omega_2)}{2}$ except for a set of ω whose cardinality is $\leq \mathfrak{m}$, with the sup norm, has the properties described above.

Grothendieck [15] conjectured that a Banach space X satisfies (1) in Theorem 6.1 iff X is isometric to a subspace of some $C(K)$ consisting of all the functions satisfying a set Ω of conditions of the form

$$(6.11) \qquad f(k_\alpha^1) = \lambda_\alpha f(k_\alpha^2), \quad k_\alpha^1,\ k_\alpha^2 \in K, \quad \lambda_\alpha \text{ a scalar}, \quad \alpha \in \Omega.$$

We shall call a Banach space which admits such a functional representation a <u>G space</u>. Every M space in the sense of Kakutani [22] and every $C_\sigma(K)$ space (cf. Day [6, p. 89]) is a G space.

<u>Lemma 6.7</u>. Let X be a subspace of $C(K)$ consisting of all the functions satisfying a set of conditions of the form (6.11). Then

$$(6.12) \qquad \{f_i\}_{i=1}^n \subset X \implies g = \max_i f_i + \min_i f_i \in X.$$

(the max and min are the usual pointwise ones).

<u>Proof</u>. Clearly $g \in C(K)$. We have to show that if all the f_i satisfy (6.11) the same is true for g. Suppose first that $\lambda_\alpha \geq 0$, and let $f_{i_0}(k_\alpha^2) = \max_i f_i(k_\alpha^2)$. By (6.11) also $f_{i_0}(k_\alpha^1) = \max_i f_i(k_\alpha^1)$ and hence

also $\max_i f_i$ satisfies (6.11). The same holds for $\min_i f_i$ and hence for g. Suppose now that $\lambda_\alpha < 0$ and let $f_{i_0}(k_\alpha^2) = \max_i f_i(k_\alpha^2)$, $f_{i_1}(k_\alpha^2) = \min_i f_i(k_\alpha^2)$. By (6.11) $f_{i_1}(k_\alpha^1) = \max_i f_i(k_\alpha^1)$ and $f_{i_0}(k_\alpha^1) = \min_i f_i(k_\alpha^1)$ and hence also in this case g satisfies (6.11).

Lemma 6.8. Let X be a subspace of $C(K)$ such that $1_K \in X$ and

$$\{f_i\}_{i=1}^3 \subset X \Longrightarrow \max_i f_i + \min_i f_i \in X.$$

Then X is a sublattice of $C(K)$.

Proof. Let $f \in X$ and put

$$g = \max(1_K, -1_K, f) + \min(1_K, -1_K, f).$$

$g \in X$ and we have

$$f(k) - g(k) = \begin{cases} 1 & \text{if} \quad f(k) \geq 1 \\ f(k) & \text{if} \quad |f(k)| \leq 1 \\ -1 & \text{if} \quad f(k) \leq -1 . \end{cases}$$

In particular, if $f \geq 0$ then $\min(1_K, f) \in X$. By translating and multiplying by a scalar we infer that for every $f \in X$ also $\min(0, f) \in X$ and hence X is a sublattice of $C(K)$.

Theorem 6.9. Let X be a G space. Then

(a) X satisfies (1) - (13) of Theorem 6.1.

(b) If S_X has an extreme point then X is isometric to a $C(K)$ space.

Proof. We assume, as we may, that X is a subspace of some $C(K)$ consisting of all the functions which satisfy (6.11) (and not only isometric to such a subspace).

(a) We show that X satisfies (12) of Theorem 6.1. Let $\{S(f_i, 1)\}_{i=1}^4$ be four mutually intersecting cells in X, i.e. $\|f_i - f_j\| \leq 2$, $i,j = 1,2,3,4$. We have $f_i - 1_K \leq f_j + 1_K$ for every i and j and hence

$$1_K + \min_j f_j = \min_j(1_K + f_j) \geq f_i - 1_K \qquad i = 1,2,3,4.$$

$$-1_K + \max_j f_j = \max_j(-1_K + f_j) \geq f_i - 1_K \qquad i = 1,2,3,4.$$

Adding, we obtain

$$g = (\max_j f_j + \min_j f_j)/2 \geq f_i - 1_K \qquad i = 1,2,3,4.$$

Similarly $g \leq f_i + 1_K$ for every i and thus

$$g \in \bigcap_{i=1}^{4} S(f_i, 1) .$$

(b) We first extend the functions of X to functions on $K \cup -K$ (the disjoint union of K and a set $-K$ homeomorphic to K by the mapping $k \longrightarrow -k$), by defining $f(-k) = -f(k)$, $k \in K$. These are also conditions of the form (6.11). Hence X is (isometric to) a subspace of $C(K \cup -K)$ satisfying (6.12), and further every extreme point of S_{X^*} is of the form φ_p, $p \in K \cup -K$ $(\varphi_p(f) = f(p))$. Let e be an extreme point of S_X and order X by the relation $x \geq 0 \iff x = \lambda(e + u)$, $\lambda \geq 0$, $\|u\| \leq 1$. From part (a) and Theorem 6.1 (see the remark at the end of its proof) it follows that X is isometric to a subspace $\tilde{X}$ of $C(K_0)$ containing 1_{K_0}, where K_0 is the set of positive extreme points of S_{X^*} and the mapping from X onto $\tilde{X}$ is the canonical one $\tilde{x}(k_0) = k_0(x)$, $k_0 \in K_0$. Since as remarked above every $k_0 \in K_0$ can be identified with (at least) one point of $K \cup -K$ it follows that also $\tilde{X}$ satisfies (6.12). Thus by Lemma 6.8 $\tilde{X}$ is a sublattice of $C(K_0)$ and hence an M space with a unit. By a representation theorem of Kakutani [22] (Day [6, p. 103]), X is isometric to $C(K_1)$ for some compact Hausdorff K_1.

Corollary. The sequence space defined after the proof of Theorem 6.6 is not a G space though it satisfies (1) - (13) of Theorem 6.1.

This disproves the conjecture of Grothendieck [15] mentioned above.

We shall now apply Theorem 6.1 to prove that a Banach space which is a $\mathcal{P}_{1+\varepsilon}$ space for every $\varepsilon > 0$ is already a $\mathcal{P}_1$ space. Actually we shall prove a somewhat stronger assertion. For the statement of this assertion we need the following notion which was introduced by Grünbaum [17] (in a slightly different notation).

A Banach space X is called an E_λ space $(\lambda \geq 1)$ if for every collection of mutually intersecting cells $\{s(x_\alpha, r_\alpha)\}$ in X we have

$$\bigcap_\alpha S(x_\alpha, \lambda r_\alpha) \neq \emptyset .$$

By Lemmas 5.2 and 5.3 it follows easily that X is an E_λ space iff for every $Z \supset X$ with $\dim Z/X = 1$ there is a projection P from Z onto X with $\|P\| \leq \lambda$ (cf. [17]). It is clear therefore that every $\mathcal{P}_\lambda$ space is an E_λ space. The converse assertion holds only for $\lambda = 1$.

<u>Theorem 6.10</u>. A Banach space X which is an $E_{1+\varepsilon}$ space for every $\varepsilon > 0$ is already an E_1 space (i.e. a $\mathcal{P}_1$ space).

<u>Proof</u>. Let X be an $E_{1+\varepsilon}$ space for every $\varepsilon > 0$. As remarked by Grünbaum [17], it follows from Theorem 4 in Section 3 of Aronszajn and Panitchpakdi [2] that every collection of mutually intersecting cells in X with uniformly bounded radii has a non empty intersection. Inspecting the proof given by Aronszjan and Panitchpakdi [2] to the fact that the unit cell of a $\mathcal{P}_1$ space has an extreme point we see that they used only the fact that $\mathcal{P}_1$ spaces have the intersection property stated in the previous sentence. Hence S_X has an extreme point. Since it is clear that X has the F.2.I.P we infer from Theorem 6.1 that X is (isometric to) a subspace of some $C(K)$ which contains 1_K. We show now that in its natural order (induced by $C(K)$) X is a lattice (not necessarily a sublattice of $C(K)$). Let $f_1, f_2 \in X$, we have to show that there is a $g \in X$ satisfying

$$(6.13) \qquad g \geq f_1,\ g \geq f_2, \text{ and } h \geq f_1,\ h \geq f_2,\ h \in X \Longrightarrow h \geq g .$$

Without loss of generality we may assume that $\|f_1\|, \|f_2\| \leq 1$. We remark first that if P is a projection from a subspace Y of $C(K)$, containing X, onto X with $\|P\| \leq 1+\varepsilon$ then

$$y \in Y, \quad y \geq 0 \Longrightarrow Py \geq -\varepsilon\|y\|\cdot 1_K .$$

Indeed,

$$y \geq 0 \Longrightarrow \left\|1_K - \frac{y}{\|y\|}\right\| \leq 1 \Longrightarrow \left\|P\left(1_K - \frac{y}{\|y\|}\right)\right\| \leq 1+\varepsilon \Longrightarrow$$

$$\Longrightarrow \left\|1_K - \frac{Py}{\|y\|}\right\| \leq 1+\varepsilon \Longrightarrow Py \geq -\varepsilon\|y\|\cdot 1_K .$$

Let $F = \max(f_1, f_2) \in C(K)$. If $F \in X$ then $g = F$ satisfies (6.13). Otherwise let Y be the subspace of $C(K)$ spanned by F and X, and let $1 \geq \varepsilon > 0$. Since X is an $E_{1+\varepsilon}$ space there is a projection P_ε from Y onto X with $\|P_\varepsilon\| \leq 1+\varepsilon$. Put $g_\varepsilon = P_\varepsilon F$. We have

(i) $\|g_\varepsilon\| \leq 2$

(ii) $2\varepsilon\cdot 1_K + g_\varepsilon \geq F$

(iii) $h \in X,\ h \geq F \Longrightarrow h + \varepsilon(\|h\|+1)\cdot 1_K \geq g_\varepsilon$.

Indeed, $\|F\| \leq \max(\|f_1\|, \|f_2\|) \leq 1$ and (i) follows. $F \geq f_1$ and hence $P_\varepsilon(F - f_1) \geq -\varepsilon\|F - f_1\|\cdot 1_K$ or $g_\varepsilon \geq f_1 - 2\varepsilon\cdot 1_K$. Similarly $g_\varepsilon \geq f_2 - 2\varepsilon\cdot 1_K$ and (ii) follows. (iii) holds since $h - F \geq 0$ implies $h - g_\varepsilon \geq -\varepsilon\|h - F\|\cdot 1_K$.

Let now ε_n be the sequence $1/n(n+3)$, and put $g_n = g_{\varepsilon_n} + 2\varepsilon_n 1_K$. We have

(1) $g_n \in X$ (2) $\|g_n\| \leq 3$

(3) $g_n \geq F$ (4) $h \in X,\ h \geq F,\ \|h\| \leq n \Longrightarrow h + \frac{1}{n}\cdot 1_K \geq g_n$.

Taking in (4) $h = g_m$ we obtain $g_m + \frac{1}{n}\cdot 1_K \geq g_n$ $(n \geq 3)$. Similarly $g_n + \frac{1}{m}\cdot 1_K \geq g_m$ $(m \geq 3)$ and hence $\|g_n - g_m\| \leq \max(\frac{1}{n}, \frac{1}{m})$ for every $n, m \geq 3$. The function $g = \lim_{n\to\infty} g_n$ satisfies (6.13).

We have thus proved that X is a lattice. It follows that it is isometric to a $C(K_1)$ space for some compact Hausdorff K_1 (see e.g. Kadison [21, Theorem 4.1]). Amir [1] has proved that if a $C(K)$ space is an E_λ space for some $\lambda < 2$ then it is already a $\mathcal{P}_1$ space. This concludes the proof of the theorem.

Theorem 6.10 solves a problem raised by Grünbaum [17] and Semadeni. Grünbaum [17] gave an example of a space which is an $E_{2+\varepsilon}$ space for every $\varepsilon > 0$ but not an E_2 space. Isbell and Semadeni [19] gave an example of a $\mathcal{P}_{2+\varepsilon}$ space for every $\varepsilon > 0$ which is not a $\mathcal{P}_2$ space. In [35] we gave an example of two Banach spaces $Z \supset X$ with $\dim Z/X = 2$ such that there is no projection of norm 1 from Z onto X but for every $\varepsilon > 0$ there is a projection of norm $\leq 1 + \varepsilon$ from Z onto X and for every Y with $Z \supset Y \supset X$ such that $\dim Y/X = 1$ there is a projection of norm 1 from Y onto X. (Simpler examples of this type can be given by using the methods of Section 2 of [34]).

CHAPTER VII. NORM PRESERVING EXTENSIONS

In this chapter we shall treat the following question (as well as some variants of it): Given a compact operator T from a Banach space Y to a Banach space X which has properties (1) - (13) of Theorem 6.1; under what conditions is it possible to extend T in a norm preserving manner to an operator $\tilde{T}$ from $Z(Z \supset Y)$ to X? We shall consider especially the case where X is a G space. For such X it is convenient to use the explicit form of the general bounded (or compact) operator having X as range space.

<u>Lemma 7.1</u>. Let X be a closed subspace of $C(K)$ consisting of all the functions satisfying a set Ω of conditions

$$(7.1) \qquad f(k_\alpha^1) = \lambda_\alpha f(k_\alpha^2), \quad k_\alpha^1, k_\alpha^2 \in K; \ \lambda_\alpha \text{ real}; \ \alpha \in \Omega.$$

Let T be a bounded operator from a Banach space Y into X. Then there is a continuous function F from K to Y^* (with the w^* topology) such that

$$(7.2) \qquad Ty(k) = F(k)y \qquad k \in K,\ y \in Y,$$

$$(7.3) \qquad \|T\| = \sup_{k \in K} \|F(k)\|,$$

and

$$(7.4) \qquad F(k_\alpha^1) = \lambda_\alpha F(k_\alpha^2) \qquad k_\alpha^1, k_\alpha^2 \in K, \quad \lambda_\alpha \text{ real}, \quad \alpha \in \Omega .$$

Conversely, to every continuous function F from K into Y^* (with the w^* topology) which satisfies (7.4) there corresponds by (7.2) a bounded operator T from Y into X, and (7.3) holds. T is compact [weakly compact] iff the corresponding F is continuous with the norm [resp. w] topology in Y^*.

Proof. The Lemma is an immediate consequence of the similar and well known result for operators which map into $C(K)$ spaces (cf. Dunford-Schwartz [8] pp. 490-491).

The extension problem for operators into G spaces reduces thus to the following: Let $Z \supset Y$ be Banach spaces and let K be a compact Hausdorff space. Let F be a continuous function from K to Y^* (with one of its three standard topologies) which satisfies (7.4). Does there exist a continuous function $\hat{F}$ from K to Z^* which satisfies (7.4) and for which $\hat{F}(k)_{|Y} = F(k)$ for every $k \in K$? If we are interested in norm preserving extensions we have to add the requirement that $\sup_{k \in K} \|\hat{F}(k)\| = \sup_{k \in K} \|F(k)\|$.

The fact that a G space satisfies (3) of Theorem 6.1 (and hence (1) - (13), see also Theorem 6.9) can now be proved directly by using the following result of Bartle and Graves [3] (cf. also Michael [36]):

Let $U \supset U_0$ be Banach spaces and let V be the quotient space U/U_0 with the usual norm. Let φ be the canonical map from U onto V.

Then for every $\varepsilon > 0$ there is a continuous function ψ_ε from V into U satisfying

(i) $\varphi\psi_\varepsilon(v) = v$, $v \in V$

(ii) $\|\psi_\varepsilon(v)\| \leq (1+\varepsilon)\|v\|$, $v \in V$

(iii) $\psi_\varepsilon(\lambda v) = \lambda\psi_\varepsilon(v)$, λ scalar and $v \in V$.

In general (if U_0 is not complemented in U) ψ_ε will not be additive.

Let now $Z \supset Y$ be Banach spaces. The restriction map φ from Z^* onto Y^* is the canonical map from Z^* onto its quotient space Y^*. Let F be a norm continuous function from K to Y^* corresponding to a compact operator T from Y into the G space X. Let ψ_ε be a function from Y^* to Z^*, corresponding to a given $\varepsilon > 0$ and to the restriction map φ, whose existence is ensured by the theorem of Bartle-Graves. The function $\hat{F} = \psi_\varepsilon F$ from K to Z^* corresponds to a compact extension $\tilde{T}$ of T from Z to X for which $\|\tilde{T}\| \leq (1+\varepsilon)\|T\|$.

In view of the importance of continuous norm preserving extension of functionals for our discussion we find it convenient to use the following terminology. Let $Z \supset Y$ be Banach spaces. A map Ψ from Y^* to Z^* is called a <u>continuous norm preserving extension</u> (C.N.P.E.) <u>map</u> if it is continuous (taking in Y^* and Z^* the norm topologies) and satisfies

$$(7.5) \qquad \Psi(y^*)_{|Y} = y^* \quad \text{and} \quad \|\Psi(y^*)\| = \|y^*\|, \quad y^* \in Y^* .$$

If $Z \supset Y$ are Banach spaces and if each functional on Y has a unique norm preserving extension to Z we say that Y is a <u>U subspace</u> of Z (this is the terminology of Phelps [40]).

With these notations we state now an easy consequence of Lemma 7.1 concerning the existence of norm preserving extensions of operators.

<u>Lemma 7.2</u> (a) Let $Z \supset Y$ be Banach spaces such that there is a C.N.P.E. map from Y^* to Z^*. Then every compact operator from Y to a G space X has a compact norm preserving extension from Z to X.

(b) Let Y be a U subspace of a Banach space Z. If every compact operator from Y to the space of convergent sequences c has a compact norm preserving extension from Z to c then there exists a C.N.P.E. map from Y^* to Z^*.

Proof. (a) Use Lemma 7.1 and the fact that if Ψ is a C.N.P.E. map then the map $\tilde{\Psi}$ defined by

$$\tilde{\Psi}(y^*) = \begin{cases} \frac{\|y^*\|}{2} \left(\Psi\left(\frac{y^*}{\|y^*\|}\right) - \Psi\left(\frac{-y^*}{\|y^*\|}\right)\right) & \text{if } y^* \neq 0 \\ 0 & \text{if } y^* = 0 \end{cases}$$

is also a C.N.P.E. map and satisfies $\tilde{\Psi}(\lambda y^*) = \lambda \tilde{\Psi}(y^*)$, $y^* \in Y^*$ and λ real.

(b) We have to prove that if Ψ is the (uniquely determined) map which satisfies (7.5) and if $\|y_n^* - y^*\| \to 0$ then $\|\Psi(y_n^*) - \Psi(y^*)\| \to 0$. This is obvious if $y^* = 0$, and hence we may assume that $\|y_n^*\| = 1$ for every n. The operator T from Y to c defined by

$$Ty = (y_1^*(y), y_2^*(y), \ldots), \qquad y \in Y$$

is compact and of norm 1. Let $\tilde{T}$ be a norm preserving compact extension of T from Z to c. Then

$$\tilde{T}z = (z_1^*(z), z_2^*(z), \ldots), \qquad z \in Z$$

where $\|z_n^*\| \leq 1$, $z_n^*|_Y = y_n^*$ for every n and $\|z_n^* - z^*\| \to 0$ for some $z^* \in Z^*$. Hence $z_n^* = \Psi(y_n^*)$ and $z^* = \Psi(y^*)$ and this concludes the proof of the lemma.

Remark. It is easy to construct examples of spaces $Z \supset Y$ such that Y is a U subspace of Z and such that the uniquely determined map Ψ which satisfies (7.5) is not continuous (in the norm topologies of Z^* and Y^*). Take for example $Z = C(0,1)$ and Y the 2-dimensional subspace of Z spanned by the functions $f_1(x) = x$ and $f_2(x) = x^2$ (we omit the simple details). If Z^* is locally uniformly convex (cf. Day [6, p. 113] for this and related properties) then for every subspace Y of Z there is a C.N.P.E. map from Y^* into Z^*. Indeed, Y is a U subspace of Z and if ψ is the (uniquely determined) map satisfying (7.5) then $\|y_n^* - y^*\| \to 0 \Rightarrow \|y_n^* + y^*\| \to 2\|y^*\|$ and hence $\|\psi(y_n^*) + \psi(y^*)\| \to 2\|\psi(y^*)\| = 2\|y^*\|$ and therefore by our assumption on Z^*, $\|\psi(y_n^*) - \psi(y^*)\| \to 0$.

We next give examples of Banach spaces X such that for every $Z \supset X$ there is a C.N.P.E. map from X^* into Z^*.

Theorem 7.3. (a). Let X satisfy (1) - (13) of Theorem 6.1. Then for every $Z \supset X$ there is a C.N.P.E. map from X^* to Z^*.

(b) Let X be a finite-dimensional Banach space whose unit cell is a polyhedron. Then for every $Z \supset X$ there is a C.N.P.E. map from X^* to Z^*.

(c) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of Banach spaces such that for every n and every $Z \supset X_n$ there is a C.N.P.E. map from X_n^* to Z^*. Then for every $Z \supset X = (X_1 \oplus X_2 \oplus \cdots)_{c_0}$ there is a C.N.P.E. map from X^* to Z^*.

Proof. (a) For every Banach space X the canonical embedding of X^* in X^{***} is a C.N.P.E. map from X^* to $(X^{**})^*$. Another observation we need is that if $Z \supset Y$ are Banach spaces and if there is a projection P of norm 1 from Z onto Y then $\psi(y^*) = P^*(y^*)$ is a C.N.P.E. map from Y^* to Z^*. Hence if X^{**} is a $\mathcal{P}_1$ space and if $Z \supset X^{**} \supset X$ then by the preceding remarks there is a C.N.P.E. map from X^* to Z^*. Let now W be an arbitrary Banach space containing X. Let $Y = (W \oplus X^{**})_{\ell_1^2}$ and let V be the subspace of Y consisting of all the vectors of the form $(x,-x)$, $x \in X$. Let Z be the quotient space Y/V and denote by T the quotient map from Y to Z. The restrictions of T to the subspaces $(W,0)$ and $(0,X^{**})$ of Y are isometries and $T(x,0) = T(0,x)$,

$x \in X$. Hence (with obvious identifications) $Z \supset X^{**}$, $Z \supset W$ and $X^{**} \cap W = X$. As remarked above there is a C.N.P.E. map from X^* to Z^*. Applying the restriction map from Z^* to W^* we get a C.N.P.E. map from X^* to W^*.

(b) Let $\{x_i^*\}_{i=1}^n$ be the extreme points of S_{X^*}. There exist continuous real-valued functions $\lambda_i(x^*)$, $1 \le i \le n$, defined on S_{X^*} such that

$$x^* = \sum_{i=1}^n \lambda_i(x^*) x_i^* , \qquad x^* \in X^* ;$$

$$\sum_{i=1}^n \lambda_i(x^*) = 1; \quad \lambda_i(x^*) \ge 0, \qquad 1 \le i \le n .$$

This is an assertion on convex polyhedra which is easily proved by induction on the dimension (cf. also [24]). Let z_i^* $(1 \le i \le n)$ be norm preserving extensions of x_i^* to Z. Then

$$\psi(x^*) = \begin{cases} \|x^*\| \sum_{i=1}^n \lambda_i(x^*/\|x^*\|)\, z_i^* & \text{if } x^* \neq 0 \\ 0 & \text{if } x^* = 0 \end{cases}$$

is a C.N.P.E. map from X^* to Z^*.

(c) Observe that if $Y_n \supset X_n$, $n = 1,2,\ldots$, then by the assumptions of part (c) there is a C.N.P.E. map from X^* to Y^* where $Y = (Y_1 \oplus Y_2 \oplus \cdots)_{c_0}$. If all the Y_n^{**} are $\mathcal{P}_1$ spaces then also Y^{**} is a $\mathcal{P}_1$ space. The proof of (c) is now concluded by using the same argument as at the end of the proof of part (a).

The question of the existence of C.N.P.E. maps is dual to the question of the existence of continuous best approximations. We have

Lemma 7.4. Let $Z \supset Y$ be Banach spaces. There is a C.N.P.E. map from Y^* to Z^* iff there exists a norm continuous function φ from Z^* to $Y^{\perp}$ (the annihilator of Y in Z^*) satisfying

$$\|z^* - \varphi(z^*)\| = \min_{u^* \in Y^{\perp}} \|z^* - u^*\| .$$

Proof. Let φ be a mapping of Z^* into itself and let ψ be a mapping from Y^* into Z^* so that

$$\varphi(z^*) = z^* - \psi(z^*_{|Y}) \ .$$

It is easily checked that if this relation holds $\varphi(z^*)$ is a nearest point to z^* in $Y^{\perp}$ iff $\psi(z^*_{|Y})$ is a norm preserving extension of $z^*_{|Y}$ to Z. Hence if a C.N.P.E. map ψ exists the formula above gives a suitable φ. The converse follows similarly by using the fact that a norm continuous function $\tilde{\psi}$ from Y^* to Z^* such that $\tilde{\psi}(y^*)_{|Y} = y^*$ always exists (by the theorem of Bartle and Graves). Hence

$$\psi(y^*) = \tilde{\psi}(y^*) - \varphi(\tilde{\psi}(y^*)) \ , \qquad y^* \in Y^*$$

is a C.N.P.E. map from Y^* to Z^* if φ is a norm continuous nearest point map into $Y^{\perp}$.

We return now to the study of norm preserving extension of operators. Our next result is obviously related to the characterization of $\mathcal{P}_1$ spaces.

Theorem 7.5. Let X be a Banach space and let K be the w^* closure of the extreme points of S_{X^*}. The following statements are equivalent.

(1) K is extremally disconnected.

(2) For every $Y \supset X$ there is an operator T with $\|T\| = 1$ from Y into $C(K)$ such that $Tx(k) = k(x)$, $k \in K$, $x \in X$.

(3) The same as (2) but only for $Y \supset X$ with $\dim Y/X = 1$.

Proof. (1) ==> (2) follows from the fact that if (1) holds $C(K)$ is a $\mathcal{P}_1$ space and hence from every $Y \supset X$ there is a norm preserving extension of the canonical embedding of X in $C(K)$ $(x(k) = k(x)$, $x \in X$, $k \in K)$. (2) ==> (3) is also obvious. We show that (3) ==> (1). Let G

be an open subset of K for which $G \cap (-G) = \emptyset$. Let $\tilde{S}$ be the w^* closed convex hull of $K \sim G$ and let E be those points of G which are extreme points of S_{X^*}. Since $K \sim G$ is w^* compact it follows from a theorem of Milman (cf. Day [6, p. 80]) that every point of $\tilde{S}$ which is an extreme point of S_{X^*} belongs to $K \sim G$. Hence

$$\tilde{S} \cap E = \emptyset \ . \tag{7.6}$$

Since $G \cap (-G) = \emptyset$ it follows that

$$S_{X^*} = \mathrm{co}\,(\tilde{S} \cup (-\tilde{S})) \ . \tag{7.7}$$

In $X^* \oplus R$ let S_0 be the symmetric (with respect to the origin) convex hull of $\{(x^*,1);\ x^* \in \tilde{S}\}$. S_0 is compact, taking in $X^* \oplus R$ the product of the w^* topology in X^* with the usual topology of R. By (7.7)

$$\|x\| = \sup_{(x^*,\lambda) \in S_0} (x^*(x) + \lambda \cdot 0) \qquad x \in X \ .$$

It follows that there is a Banach space $Y \supset X$ with $\dim Y/X = 1$ such that the mapping $(x^*,\lambda) \longrightarrow x^*$ from $X^* \oplus R$ (with S_0 as unit cell) onto X^* is exactly the restriction map from Y^* onto X^*. Let $x^* \in E$, then by (7.6) the only point of the form (x^*,λ) which belongs to S_0 is the point $(x^*,-1)$. In other words $(x^*,-1)$ is the unique norm preserving extension of x^* to Y. Similarly $(x^*, 1)$ is the unique norm preserving extension of x^* to Y if $x^* \in -E$. Hence, since by (3) and Lemma 7.1 there is a mapping F from K to Y^* continuous in the w^* topology of Y^* and satisfying $\|F(k)\| \leq 1$, $F(k)_{|X} = k$, $k \in K$, it follows that $\bar{E} \cap \overline{(-E)} = \emptyset$ (closures are taken in K, that is in the w^* topology which is used as the topology of K). Since $\bar{G} = \bar{E}$ we get that $\bar{G} \cap \overline{(-G)} = \emptyset$. Taking in particular G to be a maximal open subset for which $G \cap (-G) = \emptyset$ holds we get that $K = \bar{G} \cup -\bar{G}$ where $\bar{G} \cap -\bar{G} = \emptyset$. In order to show that (1) holds we have to prove that $\bar{G}$ is extremally disconnected. Let G_1 and G_2 be disjoint open subsets of $\bar{G}$. Put $G_0 = G_1 \cup (-G_2)$. Then $G_0 \cap (-G_0) = \emptyset$ and hence by what we have proved above $\overline{G_0} \cap (\overline{-G_0}) = \emptyset$. It follows that $\overline{G_1} \cap \overline{G_2} = \emptyset$ and this concludes the proof of the theorem.

Corollary. Let X be a finite-dimensional Banach space. The following statements are equivalent

(1) S_X is a polyhedron.

(2) For every $Z \supset X$ there is a C.N.P.E. map from X^* into Z^*.

(3) For every $Z \supset X$ with $\dim Z/X = 1$ there is a C.N.P.E. map from X^* into Z^*.

Proof. (1) ==> (2) is assertion (b) of Theorem 7.3. (2) ==> (3) is obvious. We show that (3) ==> (1). Let X be a finite-dimensional space which satisfies (3). By Lemma 7.2 (a) X has also property (3) of Theorem 7.5. The w^* closure of the extreme points of S_{X^*} is compact metric (this is true for every separable X) and extremally disconnected by Theorem 7.5 (1). Hence S_{X^*} has only a finite number of extreme points and therefore S_X is a polyhedron.

We next use the same idea as in the proof of Theorem 7.5 to prove a similar result for compact operators. For finite-dimensional spaces X Theorem 7.6 will give the same information as Theorem 7.5. This is the case also with some theorems we are going to prove later on (Theorems 7.8 and 7.9). All these results reduce in the finite-dimensional case to an assertion which is essentially the implication (3) ==> (1) (or (2) ==> (1)) of the Corollary to Theorem 7.5. However in the infinite-dimensional case each of the theorems gives some information which cannot be deduced from the other theorems.

Theorem 7.6. Let X be a Banach space and assume that the extreme points of S_{X^*} are not isolated in the norm topology of X^*. Then there exist a compact operator T from X to c and a Banach space $Y \supset X$ with $\dim Y/X = 1$ such that T has no norm preserving extension from Y to c.

Proof. Let $\{x_n^*\}_{n=1}^\infty$ be a sequence of extreme points of S_{X^*} such that $\|x_n^*-u^*\| \to 0$ for some $u^* \in X^*$. We may assume that $u^* \neq \pm x_n^*$ for every n and every choice of signs. We choose now inductively a subsequence $\{n_i\}_{i=1}^\infty$ of the integers, elements $x_i \in X$ and positive numbers λ_i as follows. We take $n_1 = 1$ and choose x_1 and λ_1 so that $|u^*(x_1)| < \lambda_1$ and $x_{n_1}^*(x_1) > \lambda_1$. We next choose $n_2 > n_1$ so that $|x_{n_2}^*(x_1)| < \lambda_1$. The set

$$\mathrm{Co}\,(\{\pm u^*\} \cup \{x^*;\ \|x^*\| \le 1,\ |x^*(x_1)| \ge \lambda_1\})$$

is w^* compact and does not contain $x_{n_2}^*$ (since $x_{n_2}^*$ is an extreme point of S_{X^*}). Hence by the separation theorem there is an $x_2 \in X$ and a $\lambda_2 > 0$ so that $x_{n_2}^*(x_2) > \lambda_2$, $|u^*(x_2)| < \lambda_2$ and

$$S_{X^*} \cap \{x^*;\ |x^*(x_1)| \ge \lambda_1\} \cap \{x^*;\ |x^*(x_2)| \ge \lambda_2\} = \emptyset\ .$$

We continue in a similar manner and get that

$$x_{n_i}^*(x_i) > \lambda_i \qquad i = 1,2,\ldots$$

and

$$S_{X^*} \cap \{x^*;\ |x^*(x_i)| \ge \lambda_i\} \cap \{x^*;\ |x^*(x_j)| \ge \lambda_j\} = \emptyset\ ,\quad i \neq j\ .$$

Put

$$\tilde{S} = S_{X^*} \bigcap_{i=1}^{\infty} \{x^*;\ x^*(x_{2i}) \ge \lambda_{2i}\} \bigcap_{i=1}^{\infty} \{x^*;\ x^*(x_{2i+1}) \le -\lambda_{2i+1}\}\ .$$

With this choice of $\tilde{S}$ (7.7) holds. We continue as in the proof of Theorem 7.5. In $X^* \oplus R$ we introduce a norm whose unit cell is the symmetric convex hull of $\{(x^*,1);\ x^* \in \tilde{S}\}$. Then $X^* \oplus R = Y^*$ where $Y \supset X$, $\dim Y/X = 1$ and the map $(x^*,\lambda) \to x^*$ is exactly the restriction map from Y^* to X^*. Since

$$\{x^*_{n_{2i}}\}_{i=1}^{\infty} \cap (-\tilde{S}) = \emptyset$$

and the x^*_n are extreme points of S_{X^*} we get that $(x^*_{n_{2i}}, 1)$ is the unique norm preserving extension of $x^*_{n_{2i}}$ to Y, $i = 1,2,\ldots$. Similarly $(x^*_{n_{2i+1}}, -1)$ is the unique norm preserving extension of $x^*_{n_{2i+1}}$ to Y, $i=1,2,\ldots$. It follows (use Lemma 7.1) that the compact operator T from X to c defined by $Tx = (x^*_1(x), x^*_2(x), \ldots)$ does not have a norm preserving extension from Y to c.

<u>Corollary</u>. If a Banach space X has the property that for every $Z \supset X$ there is a C.N.P.E. map from X^* to Z^* then the extreme points of S_{X^*} are isolated (in the norm topology of X^*).

For our next result on the extension of operators we need first a characterization of finite-dimensional spaces whose unit cell is a polyhedron.

<u>Theorem 7.7</u>. Let X be a Banach space. X is finite-dimensional and its unit cell is a polyhedron iff there does not exist a sequence $\{x_i\}_{i=1}^{\infty}$ in X such that for every choice of signs

$$(7.8) \qquad \|x_i \pm x_j\| \leq \|x_i\| + \|x_j\| - 1, \qquad i \neq j.$$

<u>Proof</u>. If X is finite-dimensional and S_X has n faces then clearly any set of vectors $\{x_i\}$ for which (7.8) holds has at most n elements. This proves the "only if" part of the theorem. To prove the other part assume first that X is finite-dimensional and its unit cell is not a polyhedron. X has a 2-dimensional subspace whose unit cell is not a polyhedron (cf. Klee [27]), and hence we may assume that $\dim X = 2$. It is easily seen that there is a sequence $\{y_i\}_{i=1}^{\infty}$ in X such that $\|y_i - y\| \to 0$ for some $y \in X$, $\|y_i\| = 1$ for every i and

$\| x^* \| = |x^*(y_i)| = 1$ implies $\sup_{j \neq i} |x^*(y_j)| < 1$. Hence there exist $\delta_i > 0$ such that the sets

$$S_{X^*} \cap \{x^*;\ |x^*(y_i)| \geq 1 - \delta_i\}\ , \quad i = 1,2,\ldots$$

are mutually disjoint. Put $x_i = y_i/\delta_i$, $1 \leq i \leq \infty$. For every choice of signs, every $x^* \in S_{X^*}$ and every $i \neq j$

$$x^*(x_i \pm x_j) \leq 1/\delta_j + 1/\delta_i - 1 = \| x_i \| + \| x_j \| - 1\ .$$

Hence these $\{x_i\}_{i=1}^{\infty}$ satisfy (7.8) .

Let now X be an infinite-dimensional Banach space. If X has a finite-dimensional subspace whose unit cell is not a polyhedron the existence of a sequence satisfying (7.8) follows from what we have already proved. Hence it remains to prove that if X is infinite-dimensional and if the unit cells of all its finite-dimensional subspaces are polyhedra then there is a sequence $\{x_i\}_{i=1}^{\infty}$ in X which satisfies (7.8). We have not found, however, a simple argument which applies to this special class of spaces X. We give therefore a (rather complicated) proof which holds for every infinite-dimensional Banach space X.

From the well known theorem of Borsuk on antipodal mappings of spheres it follows that if B is an $(n+1)$-dimensional Banach space and if $\{u_i\}_{i=1}^{n}$ are n points in B then there is a $u \in B$ such that $\| u\| = 1$ and $\| u-u_i \| = \| u+u_i \|$ for every i. Hence in the infinite-dimensional space X we can choose inductively a sequence $\{y_i\}_{i=1}^{\infty}$ such that $\| y_i \| = 1$ for every i and

$$(7.9) \quad \| \theta_1 y_1 + \theta_2 y_2 + \ldots + \theta_k y_k - y_{k+1} \| = \| \theta_1 y_1 + \theta_2 y_2 + \ldots + \theta_k y_k + y_{k+1} \|$$

for every k and every choice of θ_i, allowing each θ_i to take one of the three values 0, $+1$ and -1. Having chosen such a sequence $\{y_i\}_{i=1}^{\infty}$ we have to consider separately two cases

(i) There is an increasing sequence $\{i_n\}_{n=1}^{\infty}$ of integers and a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of signs such that

$$\| \varepsilon_1 y_{i_1} + \varepsilon_2 y_{i_2} + \cdots + \varepsilon_k y_{i_k} \| \geq k - 1/2, \quad k = 1,2\ldots .$$

(ii) There are no such sequences $\{i_n\}$ and $\{\varepsilon_n\}$.

Suppose that (i) holds and let $\{i_n\}$ and $\{\varepsilon_n\}$ be suitable sequences. Put

$$x_k = \varepsilon_1 y_{i_1} + \varepsilon_2 y_{i_2} + \cdots + \varepsilon_k y_{i_k} - \varepsilon_{k+1} y_{i_{k+1}}, \qquad k = 1,2,\ldots .$$

By (7.9) and (i) $\| x_k \| \geq k + 1/2$ for every k. Hence for $h > k$

$$\| x_h + x_k \| = \| 2 \sum_{j=1}^{k} \varepsilon_j y_{i_j} + \sum_{j=k+2}^{h} \varepsilon_j y_{i_j} - \varepsilon_{h+1} y_{i_{h+1}} \|$$
$$\leq 2k + (h-k-1) + 1 \leq \| x_k \| + \| x_h \| - 1 .$$

Similarly

$$\| x_k - x_h \| \leq h-k+2 \leq \| x_k \| + \| x_h \| - 1 .$$

This concludes the proof if (i) holds.

We assume now that (ii) holds. In this case there are $i_1 < i_2 \ldots < i_m$, with $i_1 = 1$ and $m \geq 1$, and signs $\{\varepsilon_j\}_{j=1}^{m}$ such that

$$\| \varepsilon_1 y_{i_1} + \varepsilon_2 y_{i_2} + \cdots + \varepsilon_m y_{i_m} \| > m - 1/2$$

and such that for every $i > i_m$ and every sign ε

$$\| \varepsilon_1 y_{i_1} + \varepsilon_2 y_{i_2} + \cdots + \varepsilon_m y_{i_m} + \varepsilon y_i \| \leq (m+1) - 1/2 .$$

Put

$$n_1 = m, \quad z_1 = \sum_{j=1}^{n_1} \varepsilon_j y_{i_j}, \quad \sigma_1 = \| z_1 \| - n_1 + 1/2 .$$

Clearly $1/2 \geqslant \sigma_1 > 0$. By the assumption that (ii) holds it follows that there are $i_{n_1+1} < i_{n_1+2} < \ldots < i_{n_2}$, with $i_{n_1+1} = i_{n_1} + 1$ and $n_2 \geq n_1 + 1$, and signs $\varepsilon_{n_1+1}, \ldots, \varepsilon_{n_2}$ such that

$$\left\| \sum_{j=n_1+1}^{n_2} \varepsilon_j y_{i_j} \right\| > n_2 - n_1 - \sigma_1/2$$

and such that for every $i > i_{n_2}$ and every sign ε

$$\left\| \sum_{j=n_1+1}^{n_2} \varepsilon_j y_{i_j} + \varepsilon y_i \right\| \leq n_2 - n_1 + 1 - \sigma_1/2 .$$

Put

$$z_2 = \sum_{j=n_1+1}^{n_2} \varepsilon_j y_{i_j} , \quad \sigma_2 = \| z_2 \| - n_2 + n_1 + \sigma_1/2 .$$

Continuing in this manner we get sequences $\{i_j\}$, $\{\varepsilon_j\}$, $\{n_k\}$, $\{z_k\}$ and $\{\sigma_k\}$ such that for every $k > 1$

(7.10) $\quad i_j < i_{j+1}$, ε_j is either $+1$ or -1

(7.11) $\quad z_k = \sum_{j=n_{k-1}+1}^{n_k} \varepsilon_j y_{i_j}$

(7.12) $\quad \| z_k \| = n_k - n_{k-1} - \sigma_{k-1}/2 + \sigma_k$

(7.13) $\quad 1/2 \geq \sigma_{k-1} \geq 2\sigma_k > 0$

(7.14) $\quad \| z_k + \varepsilon y_i \| \leq n_k - n_{k-1} - \sigma_{k-1}/2 + 1, \quad i > i_{n_j}, \quad \varepsilon = \pm 1.$

Let now $k < h$. By (7.10), (7.11) and (7.14)

$$\| z_k \pm z_h \| \leq (n_k - n_{k-1} + 1 - \sigma_{k-1}/2) + (n_h - n_{h-1} - 1) .$$

Hence by (7.12) and (7.13)

$$\| z_k \pm z_h \| \leq \| z_k \| + \| z_h \| - \sigma_k/2 .$$

Finally, put $x_k = 2z_k/\sigma_k$. Then for $k < h$

$$\| x_k \pm x_h \| \leq 2 \| z_h \| (\sigma_h^{-1} - \sigma_k^{-1}) + 2 \| z_k \pm z_h \| \sigma_k^{-1}$$

$$\leq 2 \| z_h \| (\sigma_h^{-1} - \sigma_k^{-1}) + 2(\| z_k \| + \| z_h \| - \sigma_k/2) \sigma_k^{-1}$$

$$= \| x_h \| + \| x_k \| - 1 .$$

Hence the sequence $\{x_k\}_{k=1}^{\infty}$ satisfies (7.8) and this concludes the proof of the theorem.

<u>Remarks</u>. In general the sequence $\{x_i\}_{i=1}^{\infty}$ which satisfies (7.8) cannot be chosen to be bounded. This is clearly the case if X is finite-dimensional (even if S_X is not a polyhedron). There are also infinite-dimensional spaces X in which there is no bounded sequence which satisfies (7.8). $X = \ell_1$ is such a space (the verification of this fact is quite simple but somewhat long and therefore we omit it). It is easily seen that in an infinite-dimensional uniformly convex space there is always a bounded sequence for which (7.8) holds. If a bounded sequence satisfying (7.8) exists in a Banach space X then clearly $\| x_i \pm x_j \| \leq \lambda(\| x_i \| + \| x_j \|)$ for $i \neq j$ and some $\lambda < 1$. This observation can be used for proving a stronger version of Theorem 7.8 for such spaces X.

<u>Theorem 7.8</u>. Let $Z \supset X$ be Banach spaces and assume that S_{Z^*} is w^* sequentially compact. If for every $Y \supset X$ with $\dim Y/X = 1$ there is an operator with norm 1 from Y into Z whose restriction to X is the identity then X is finite-dimensional and its unit cell is a polyhedron.

<u>Proof</u>. The assumptions in the statement of the theorem imply that every collection of mutually intersecting cells in Z whose centers belong to X has a non empty intersection (use Lemma 5.3). If X is not a finite-dimensional space with S_X a polyhedron there exists, by Theorem 7.7, a sequence $\{x_i\}_{i=1}^{\infty}$ in X for which (7.8) holds. We may assume that $\| x_i \| \geq 1$ for every i. Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$ be a sequence of signs and consider the sequence of cells

$$S_Z(\varepsilon_i x_i, \; \| x_i \| - 1/2) \qquad i = 1,2,\dots .$$

By (7.8) they are mutually intersecting and hence there is a $z_\varepsilon \in Z$ such that

$$(7.15) \qquad \| z_\varepsilon - \varepsilon_i x_i \| \leq \| x_i \| - 1/2, \quad \varepsilon = (\varepsilon_1, \varepsilon_2, \dots), \quad i = 1,2,\dots .$$

Let $z_i^* \in Z^*$ satisfy $z_i^*(x_i) = \|x_i\|$ and $\|z_i^*\| = 1$ for every i. By (7.15) $z_i^*(z_\varepsilon) \geq 1/2$ if $\varepsilon_i = +1$ and $z_i^*(z_\varepsilon) \leq -1/2$ if $\varepsilon_i = -1$. It follows that the z_i^* cannot have a w^* convergent subsequence and this contradicts our assumption on Z^*.

Remarks. The assumption that Z^* is w^* sequentially compact is satisfied, in particular, if Z is separable or reflexive or a direct sum of such spaces. If we drop the assumption that Z^* is w^* sequentially compact Theorem 7.8 will no longer hold — take for example the case in which Z is an infinite dimensional $\mathcal{P}_1$ space and X any subspace of Z. It should be remarked perhaps that if $Z \supset X$ are such that every collection of mutually intersecting cells in Z whose centers belong to X has a non empty intersection it does not follow that there is a $\mathcal{P}_1$ space Z_0 with $X \subset Z_0 \subset Z$. Take for example $Z = (m \oplus Y)_{\ell_\infty^2}$ where Y is a separable non reflexive subspace of m (denote the embedding map from Y into m by T). Let X be the subspace of Z consisting of the points $(Ty, y/3)$, $y \in Y$. Every space Z_0 satisfying $X \subset Z_0 \subset Z$ has Y as a quotient space and hence Z_0 cannot be a $\mathcal{P}$ space. However it is easily seen that any collection of cells in Z whose centers belong to X has a non empty intersection (cf. [34, Section 2]).

Our next three theorems give characterizations of some classes of spaces which have properties (3) or (4) of Theorem 6.1 (or related properties) with $\varepsilon = 0$. Another theorem of this kind in which the $C(K)$ spaces are treated is given in [32].

Theorem 7.9. Let X be a Banach space which satisfies (1) - (13) of Theorem 6.1, and let Y be a finite-dimensional Banach space.

(a) If S_Y is a polyhedron then for every $Z \supset Y$ every operator from Y to X has a compact norm preserving extension from Z to X.

(b) If Y is a subspace of X and if S_Y is not a polyhedron then the identity operator from Y to X does not have a compact norm preserving extension from X into itself.

Proof. (a) Since S_Y is a polyhedron we may consider Y as a subspace of ℓ_∞^n for some n. Let T be an operator from Y into X. Since X has property (6) of Theorem 6.1 there is a norm preserving extension T_0 of T from ℓ_∞^n into X. ℓ_∞^n is a $\mathcal{P}_1$ space and hence the identity operator from Y into ℓ_∞^n has an extension T_1 from Z to ℓ_∞^n with $\|T_1\| = 1$. The operator $\tilde{T} = T_0T_1$ has the desired properties.

(b) Suppose there is a compact operator $\tilde{T}$ from X into itself with norm 1 whose restriction to Y is the identity. By the ergodic theorem (Dunford-Schwartz [8, p. 711]) the sequence $(I + \tilde{T} + \tilde{T}^2 + \cdots + \tilde{T}^n)/n$ converges to a projection P from X onto the subspace Y_0 of X consisting of all the points x for which $\tilde{T}x = x$. $\|P\| = 1$ and Y_0 is finite-dimensional. Hence Y_0 is a $\mathcal{P}_1$ space (Corollary 3 to Theorem 2.1). Since $Y_0 \supset Y$ S_Y is a polyhedron and this contradicts our assumptions.

Corollary 1. Let X satisfy (1) - (13) of Theorem 6.1, and let Y be a finite-dimensional subspace of X such that S_Y is not a polyhedron. Let also $Z \supset X$. Then the identity operator from Y to X does not have a compact norm preserving extension from X into Z.

Proof. Suppose there exists such an extension and denote it by $\tilde{T}$. Let V be a $\mathcal{P}_1$ space such that $V \supset Z \supset X$. By property (8) of Theorem 6.1 there is a norm preserving and compact extension $\tilde{T}_0$ of $\tilde{T}$ from V into Z ($\subset V$). This contradicts Theorem 7.9 (b).

This corollary shows that Theorem 2.3 does not hold with $\varepsilon = 0$.

Corollary 2. Let X satisfy (1) - (13) of Theorem 6.1, and let Y be a finite-dimensional subspace of X whose unit cell is a polyhedron.

Then there is a finite-dimensional $\mathcal{P}_1$ space Y_0 for which $Y \subset Y_0 \subset X$.

Proof. This follows from Theorem 7.9 (a) and the proof of Theorem 7.9 (b).

A Banach space X is called polyhedral (Klee [28]) if every finite-dimensional subspace of X has a polyhedron as its unit cell.

Corollary 3. Let X be a polyhedral Banach space. X satisfies (1) - (13) of Theorem 6.1 iff it is an $\mathcal{N}_1$ space

Proof. This follows from Corollary 2 above and Corollary 1 to Theorem 6.1

Theorem 7.10. Let X be a Banach space. The following three statements are equivalent.

(1) X is polyhedral and satisfies (1) - (13) of Theorem 6.1.

(2) Every operator T from Y into X with a range of dimension ≤ 3 has a compact and norm preserving extension from Z $(Z \supset Y)$ to X.

(3) Every operator T from Y to X with a finite dimensional range has a norm preserving extension $\tilde{T}$ from Z $(Z \supset Y)$ to X such that the range of $\tilde{T}$ is finite-dimensional.

Proof. (1) $\Rightarrow$ (3). Let T be an operator from Y into X for which $T(Y)$ is finite-dimensional. By Corollary 2 to Theorem 7.9 there is a finite-dimensional $\mathcal{P}_1$ space Y_0 with $T(Y) \subset Y_0 \subset X$. There is a norm preserving extension of T from Z into Y_0 and this proves (3). (3) $\Rightarrow$ (2) is clear. We show next that (2) $\Rightarrow$ (1). It is clear that if X satisfies (2) of the present theorem it satisfies (1) - (13) of Theorem 6.1 (see property (4) there). By Theorem 7.9 (b) every 3-dimensional subspace of X (assuming X satisfies (2)) has a polyhedron as its unit cell. Klee [27] proved that this implies that X is polyhedral.

Remark. If we replace (2) (or (3)) by the weaker property which is obtained from it by requiring that $\dim Z/Y = 1$ then the theorem will no longer hold. Every $\mathcal{P}_1$ space X satisfies this weaker version of (2) (or (3)), but an infinite-dimensional $\mathcal{P}_1$ space is not polyhedral (in fact since c is not polyhedral no infinite-dimensional $C(K)$ space is polyhedral).

Theorem 7.11. Let X be a Banach space such that S_{X^*} is w^* sequentially compact. The following three statements are equivalent.

(1) X is polyhedral and satisfies (1) - (13) of Theorem 6.1.

(2) Every operator T from Y into X with range of dimension ≤ 3 has a norm preserving extension from Z $(Z \supset Y, \dim Z/Y = 1)$ to X.

(3) Every operator T from Y into X with a finite-dimensional range has a norm preserving extension from Z $(Z \supset Y)$ into X.

Proof. By Theorem 7.10 (1) implies even a stronger version of (3) (i.e. statement (3) of Theorem 7.10). (3) ==> (2) is clear. By Theorem 7.8 if X satisfies (2) every 3-dimensional subspace of X has a polyhedron as unit cell, and hence X is polyhedral (Klee [27]).

We do not know whether a polyhedral space X for which X^* is an L_1 space satisfies (3) of Theorem 6.1 with $\varepsilon = 0$. (Theorem 7.10 shows that if X satisfies (3) of Theorem 6.1 with $\varepsilon = 0$ then X is polyhedral and X^* is an L_1 space.) We shall now construct a class of spaces which have property (3) of Theorem 6.1 with $\varepsilon = 0$. We first give a sufficient condition for a space to be polyhedral.

Lemma 7.12. Let X be a Banach space such that for every point $x \neq 0$ there is a finite number of extreme points $\{x_i^*\}_{i=1}^{n_x}$ of S_{X^*} and a number $\theta(x) < 1$ such that $|x^*(x)| \leq \theta(x) \cdot \|x\|$ for every extreme point x^* of S_{X^*} which does not belong to $\{x_i^*\}_{i=1}^{n_x}$. Then X is polyhedral.

Proof. The Lemma follows from the compactness of the unit cells of finite-dimensional spaces and from the fact that for every finite set Ω of functionals in X^* the set

$$\{ x ; x \in X, \sup_{x^* \in \Omega} |x^*(x)| < \|x\| \}$$

is an open subset of X.

By using this lemma it is possible to give some examples of infinite-dimensional spaces which satisfy (1) - (3) of Theorems 7.10 and 7.11. Let X be a G space, i.e. X is (isometric to) a subspace of C(K) consisting of all the functions which satisfy (7.1). If for every $f \in X$ there is a finite subset $\{k_i\}_{i=1}^{n_f}$ of K such that

$$\sup_{k \neq k_i} |f(k)| < \|f\| ,$$

then by Lemma 7.12 X is polyhedral and since X is a G space $X^* = L_1(\mu)$. Moreover, for such X every compact operator T from Y to X has, for every $Z \supset Y$, a compact and norm preserving extension from Z to X. Indeed, let F be the function from K to Y^* corresponding by (7.2) to T. From our assumptions on X and from the compactness of F(K) it follows that there is a finite set $\{k_i\}_{i=1}^{n} \subset K$ such that

$$\alpha = \max_{k \in K \sim \{k_i\}_{i=1}^{n}} \|F(k)\| < \max_{k \in K} \|F(k)\| = \|T\|$$

By a selection theorem of Michael [36, Example 1.3 and Proposition 7.2] there is a norm continuous function Ψ from Y^* to Z^* which satisfies

(i) $\Psi(y^*)_{|Y} = y^*, \quad y^* \in Y^*$,
(ii) $\Psi(\lambda y^*) = \lambda \Psi(y^*), \quad y^* \in Y^*, \quad \lambda$ scalar,
(iii) $\|\Psi(F(k_i))\| = \|F(k_i)\| , \quad i = 1,\dots,n$,
(iv) $\|\Psi(y^*)\| \leq \|T\| \, \|y^*\| / \alpha , \; y^* \in Y^*$.

The operator $\tilde{T}$ which corresponds to the function ψF from K to Z^* is an extension with the desired properties.

The simplest example of a space of the category considered above is c_0 (the fact that c_0 is polyhedral is due to Klee [28]).

We conclude by giving a sufficient condition for a compact operator T into a G space to have a norm preserving extension. Unlike the preceding theorems we do not consider here the extension problem for a general class of operators but rather restrict ourselves to a single given operator.

Theorem 7.13. Let T be a compact operator from Y into X (X a G space). Let F be the mapping from K to Y^* corresponding to T by (7.2). Let $Z \supset Y$ and put

$$A = F(K) \cap \{y^* ; \|y^*\| = \|T\|\} .$$

If every functional in A has a unique norm preserving extension to a functional on Z then there is a norm preserving extension of T from Z to X. If, in addition, one of the following two conditions holds

(a) $\dim Z/Y < \infty$,

(b) $1 = \|z^*\| = \|z_n^*\|$, $n = 1,2,\ldots$, and $z_n^* \xrightarrow{w^*} z^*$ $(z_n^*, z^* \in Z^*)$ $\Longrightarrow \|z_n^* - z^*\| \longrightarrow 0$,

then T has even a compact and norm preserving extension from Z to X.

Proof. Without loss of generality we may assume that $\|T\| = 1$. The sets $F(K)$ and A are compact in the norm topology of Y^*. Let B be the set

$$B = \{y^*;\ y^* = \pm F(k)/\|F(k)\| ,\quad k \in K,\quad F(k) \neq 0\} .$$

Clearly $B \supset A \cup -A$. To every $y^* \in B$ we assign a number $p(y^*)$ by

$$p(y^*) = \min(2, \min_{y^*=\pm F(k)/\|F(k)\|} \|F(k)\|^{-1}) \quad .$$

(The inner min, which is taken over all the k for which $F(k)/\|F(k)\|$

is equal either to y^* or to $-y^*$, exists since $F(K)$ is compact).
The function p satisfies

(i) $1 \leq p(y^*) \leq 2$

(ii) $p(y^*) = 1 \Longleftrightarrow y^* \in A \cup -A$

(iii) p is norm lower semi continuous (n.l.s.c) i.e.

$$\| y_n^* - y^* \| \longrightarrow 0 \text{ implies } p(y^*) \leq \underline{\lim}\, p(y_n^*) .$$

(i) and (ii) are clear. We prove (iii). We may assume that $\lim p(y_n^*)$ exists and that $p(y_n^*) < 2$ for every n (if $\lim p(y_n^*) = 2$ there is nothing to prove). Let $k_n \in K$ and let ε_n be signs such that

$$p(y_n^*) = \frac{1}{\|\Gamma F(k_n)\|}, \quad y_n^* = \varepsilon_n \frac{F(k_n)}{\|\Gamma F(k_n)\|} .$$

Since $F(K)$ is compact we may assume that

$$\varepsilon_n F(k_n) \longrightarrow \varepsilon F(k) \quad , \ |\varepsilon| = 1, \quad k \in K .$$

Hence also $\| F(k_n)\| \longrightarrow \| F(k)\|$ and therefore

$$y^* = \varepsilon F(k)/\| F(k)\| .$$

Thus $p(y^*) \leq \| F(k)\|^{-1} = \lim p(y_n^*)$ and this proves (iii). Consider now the function s defined on B by

$$s(y^*) = 1 + \tfrac{1}{2} d(y^*, A \cup -A) \cdot (p(y^*) - 1) ,$$

where $d(y^*, A \cup -A)$ denotes the distance of y^* from $A \cup -A$. s has the following properties

$(i)_o$ $1 \leq s(y^*) \leq p(y^*) \leq 2$,

$(ii)_o$ $s(y^*) = 1 \Longleftrightarrow y^* \in A \cup -A$,

$(iii)_o$ $s(y^*)$ is n.l.s.c. ,

$(iv)_o$ $s(y^*)$ is norm continuous at all the points of $A \cup -A$.

$(i)_o$, $(ii)_o$ and $(iii)_o$ follow immediately from the corresponding properties of p. $(iv)_o$ holds since $\| y_n^* - y^*\| \longrightarrow 0$, $y^* \in A \cup -A$

implies that $d(y_n^*, A \cup -A) \longrightarrow 0$ and hence $s(y_n^*) \longrightarrow 1 = s(y^*)$. Let $B_0 = B \sim (A \cup -A)$. To every $y^* \in B_0$ we correspond the closed and convex subset of Z^* consisting of all the extensions of y^* to functionals on Z. By a selection theorem of Michael [36, Lemma 7.1] there is a norm continuous function θ from B_0 into Z^* which satisfies

$$\|\theta(y^*)\| \leq s(y^*) \ , \quad \theta(y^*)_{|Y} = y^* \ , \quad y^* \in B_0 \ .$$

We extend θ to a function defined on B (not necessarily norm continuous) by taking as $\theta(y^*)$ for $y^* \in A \cup -A$ the (unique) norm preserving extension of y^* to a functional on Z. Finally we define the following function on $F(K) \cup -F(K)$:

$$\Psi(y^*) = \begin{cases} \frac{\|y^*\|}{2} \left(\theta\left(\frac{y^*}{\|y^*\|}\right) - \theta\left(\frac{-y^*}{\|y^*\|}\right)\right) \ , & \text{if } y^* \neq 0 \\ 0 \ , & \text{if } y^* = 0 \text{ and } 0 \in F(K) \end{cases}$$

We shall prove that Ψ has the following properties:

(1) $\|\Psi(y^*)\| \leq 1$,

(2) $\Psi(y^*)_{|Y} = y^*$,

(3) $\Psi(\lambda y^*) = \lambda \Psi(y^*)$, λ a scalar,

(4) $\|y_n^* - y^*\| \longrightarrow 0 \implies \|\Psi(y_n^*)\| \longrightarrow \|\Psi(y^*)\|$, and

$\Psi(y_n^*) \xrightarrow{w^*} \Psi(y^*)$.

(We shall prove that these properties hold whenever all the terms appearing in them are defined i.e. whenever all the arguments of Ψ belong to $F(K) \cup -F(K)$.)

Properties (2) and (3) are immediate. We show first that (1) holds. If $y^* = 0$, (1) is clear. For $y^* \neq 0$ let $y_0^* = y^*/\|y^*\|$. $y_0^* \in B$ and $p(y_0^*) \leq \|y^*\|^{-1}$, hence

$$\|y^*\| \cdot \|\theta(y_0^*)\| \leq s(y_0^*)\|y^*\| \leq p(y_0^*)\|y^*\| \leq 1 \ .$$

Similarly $\|y^*\| \cdot \|\theta(-y_0^*)\| \leq 1$ and hence $\|\Psi(y^*)\| \leq 1$.

We turn to the proof of (4). We consider first the case when $y^* = 0$. We may assume that $y_n^* \neq 0$ for every n and we have

$$\|\psi(y_n^*)\| \leq \tfrac{1}{2}\|y_n^*\| \left(\left\|\theta\left(\frac{y_n^*}{\|y_n^*\|}\right)\right\| + \left\|\theta\left(\frac{-y_n^*}{\|y_n^*\|}\right)\right\|\right) \leq 2\|y_n^*\| \longrightarrow 0 .$$

For $y^* \neq 0$ we consider $y_0^* = y^*/\|y^*\|$ and assume first that $y_0^* \in B_0$ (and hence also $-y_0^* \in B_0$). In this case we have even $\|\psi(y_n^*) - \psi(y^*)\| \longrightarrow 0$, since θ is norm continuous on B_0 (B_0 is relatively open in B).

There remains only the case where y_0^* (and hence also $-y_0^*$) belongs to $A \cup -A$. Let $y_{n,0}^* = y_n^*/\|y_n^*\|$, $n = 1,2,\ldots$. From $(\text{iv})_0$ we infer that

$$\|\theta(y_{n,0}^*)\| \longrightarrow 1 = \|\theta(y_0^*)\| , \quad \|\theta(-y_{n,0}^*)\| \longrightarrow 1 = \|\theta(-y_0^*)\| ,$$

and hence

$$\overline{\lim}\ \|\psi(y_n^*)\| \leq \|y^*\| = \|\psi(y^*)\| .$$

On the other hand

$$\|\psi(y_n^*)\| \geq \|\psi(y_n^*)_{|Y}\| = \|y_n^*\| \longrightarrow \|y^*\| ,$$

and thus $\|\psi(y_n^*)\| \longrightarrow \|\psi(y^*)\|$.

We have also to show that $\psi(y_n^*) \xrightarrow{w^*} \psi(y^*)$. This will follow once $\theta(y_{n,0}^*) \xrightarrow{w^*} \theta(y_0^*)$ is proved (since by symmetry the same holds for $-y_{n,0}^*$ and $-y_0^*$). Since S_{Z^*} is w^* compact we have only to prove that every limiting point of the sequence $\theta(y_{n,0}^*)$ coincides with $\theta(y_0^*)$. Let z^* be a w^* limiting point of the sequence $\theta(y_{n,0}^*)$. Then $\|z^*\| \leq \lim\|\theta(y_{n,0}^*)\| = 1$, and $z^*_{|Y} = \lim \theta(y_{n,0}^*)_{|Y} = y^*$. Since we assumed that y^* has a unique norm preserving extension $z^* = \theta(y_0^*)$ and this concludes the proof of (4).

Having established the properties of ψ the existence of a suitable extension $\tilde{T}$ follows immediately. We may take as $\tilde{T}$ the operator corresponding to the function $\hat{F} = \psi F$ from K to Z^* (continuous in the w^* topology).

The second part of the theorem follows also easily. If (a) is satisfied then any bounded extension of a compact operator is necessarily compact. If (b) is satisfied then the function Ψ constructed above will be continuous in the norm topologies of Y^* and Z^* (this is a consequence of property (4) of Ψ). Hence the operator corresponding to ΨF will be compact.

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