1. **5 points** Let $V$ be a subspace of $\mathbb{R}^n$ of dimension 3. If $\beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $V$, then is $\gamma = \{c\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3\}$ necessarily a basis of $V$ for $c \neq 0$? Justify your answer.

(Remark: Only a ‘Yes’ or ‘No’ answer without any justification earns you a ‘0’ point. You must justify your answer.)

**Ans. Yes.**

**Proof:** Since dim $V = 3$ and there are 3 vectors in $\beta$, it is enough to prove that $c\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ are linearly independent.

$$c_1 (c\vec{v}_1) + c_2 (\vec{v}_1 + \vec{v}_2) + c_3 (\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \vec{0}.$$ 

$$\Rightarrow (c_1 c + c_2 + c_3) \vec{v}_1 + (c_2 + c_3) \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}.$$ 

Since $\beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $V$, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

$$c_1 c + c_2 + c_3 = 0 \Rightarrow c_1 = 0 \Rightarrow c_1 = 0 \neq 0.$$ 

$$c_2 + c_3 = 0 \Rightarrow c_2 = 0.$$ 

$$c_3 = 0 \Rightarrow c_3 = 0.$$ 

$$c_1 = c_2 = c_3 = 0.$$ 

Hence, $c\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ are linearly independent. In particular, $\gamma$ is a basis of $V$. 

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2. **5 points** Let $A$ be a $m \times n$ matrix. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be linearly independent vectors in $\mathbb{R}^n$. If $\text{Ker}(A) = \{ \vec{0} \}$, then are the vectors $A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_k$ necessarily linearly independent? Justify your answer.

(Remark: Only a ‘Yes’ or ‘No’ answer without any justification earns you a ‘0’ point. You must justify your answer.)

\[ \text{Ans. Yes} \]

\[ \text{Proof:} \]
\[ c_1(A\vec{v}_1) + c_2(A\vec{v}_2) + \cdots + c_k(A\vec{v}_k) = \vec{0} \quad \star \]
\[ = A(c_1\vec{v}_1) + A(c_2\vec{v}_2) + \cdots + A(c_k\vec{v}_k) = \vec{0} \]
\[ = A(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) = \vec{0} \]
\[ = (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) \in \text{Ker}(A) = \{ \vec{0} \} \]
\[ = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0} \]

Since $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent, this implies that $c_1 = c_2 = \cdots = c_k = 0$

From the equation $\star$ we conclude that $A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_k$ are linearly independent.
3. **5 points** Find an example of a $2 \times 2$ matrix $A$ such that $(\text{Ker}(A))^\perp = \text{Im}(A)$.

Consider the line $L: y = x$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$T(x) = \text{Proj}_L(x)$$

Then it is clear that, $\text{Im}(T) = L$, and $\text{Ker}(T)$ is the line perpendicular to $L$ and passing through the origin. In particular,

$$\text{Ker}(T) = \{ x \in \mathbb{R}^2 \mid \text{Proj}_L(x) = 0 \}$$

$$= \{ x \in \mathbb{R}^2 \mid y = -x \}$$

A = The matrix of $\text{Proj}_L(x)$

$\mathbf{u} = \frac{1}{\sqrt{2}} (1)$ is a unit vector along $L$.

Let $\mathbf{x} = (x_1, x_2)$.

Then $T(x) = \text{Proj}_L(x) = (\mathbf{u} \cdot \mathbf{x}) \mathbf{u}$

$$= \left[ \frac{1}{\sqrt{2}} (1) \cdot \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \right] \left( \begin{array}{c} 1 \\ 1 \end{array} \right)$$

$$= \frac{1}{2} \left( x_1 + x_2 \right) \left( \begin{array}{c} 1 \\ 1 \end{array} \right)$$

$$= \left( \frac{x_1 + x_2}{2} \right) = \left( \frac{1}{2} \frac{1}{2} \right) \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)$$

$$= A \mathbf{x}$$

where $A = \left( \begin{array}{cc} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right)$.
4. **10 points** For each of the following statements, determine whether it is true or false.

1. \( W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \geq 0, y \geq 0 \right\} \) is a subspace of \( \mathbb{R}^2 \).

2. Let \( V \) and \( W \) be two subspaces of \( \mathbb{R}^n \). Then \( V \cup W \) is never a sub-space of \( \mathbb{R}^n \).

3. Let \( A \) and \( B \) be two square matrices of size \( m \times n \). Then \( \text{Ker}(A) \cap \text{Ker}(B) \) is always contained in \( \text{Ker}(A + B) \).

4. Let \( A \) be a \( n \times n \) matrix. If \( \text{rank}(A) < n \), then \( A \) is never invertible.

5. Let \( A \) and \( B \) be two \( n \times n \) matrices. If \( A = P^{-1}BP \) for every invertible matrix \( P \), then it necessarily follows that \( A = B \).

*(Remark: Only a ‘True’ or ‘False’ answer without any justification earns you a ‘0’ point. You must justify your answer.)*

1. **False**

   Since \( (1) \in W \) but \( (2) \) \( \neq \) \( (3) \) \( \neq \) \( W \)

   \( (1) \in W \), \( (2) \in W \), \( (3) \in W \), but \( (1) + (2) = (1) \in W \).

   \( W \) is not stable under addition.

2. **False**

   Let consider \( \mathbb{R}^3 \), let \( V = xy \)-plane.

   Then \( V U W = xy \)-plane, \( w \)-axis is not a subspace of \( \mathbb{R}^3 \).

3. **True**

   Let \( x \in \text{Ker}(A) \cap \text{Ker}(B) \)

   \( x \in \text{Ker}(A) \cap \text{Ker}(B) \)

   \( \Rightarrow AX = 0 \) \( \ldots \) (1)

   \( \text{and} \quad BX = 0 \) \( \ldots \) (2)

   By \((1) + (2)\) we get

   \( AX + BX = 0 \)

   \( \Rightarrow (A + B)x = 0 \)

   \( \Rightarrow x \in \text{Ker}(A + B) \)

   Thus every vector of \( \text{Ker}(A) \cap \text{Ker}(B) \) is contained in \( \text{Ker}(A + B) \)
4.) True. If rank(A) < n, \(*\) since A is a m x n matrix, by the Rank-Nullity theorem, we get 
\[ \dim(\ker(A)) > 0. \]
\[ \Rightarrow \ker(A) \neq \{0\} \]
\[ \Rightarrow A \text{ is not invertible.} \]

5.) True. Since A = P^{-1}BP for all invertible matrix P, we may choose \( P = I_n \).
Thus, \[ A = I_n^{-1}B I_n = I_n B = B. \]