Midterm 1
Linear Algebra and Applications
(Math 33A-001)

Show your work to receive partial credits. Use of calculator is NOT allowed for this exam.

Name: ___________________________ U ID: ______________

TA’s Name: ______________________ TA Meeting Day: ______________

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1. **5 points** Let $T$ be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

\[
T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} \quad \text{and} \quad T \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}.
\]

Compute the matrix of $T$.

**Solution:** Since $T$ is a linear transformation, $T \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = T(3e_2) = 3T(e_2)$ and $T \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = T(2e_3) = 2T(e_3)$. Thus we have

\[
T(e_1) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad T(e_2) = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 0 \end{pmatrix} \quad \text{and} \quad T(e_3) = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 2 \end{pmatrix}.
\]

Therefore the matrix of $T$ is given by

\[
A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{2} \\ -1 & 0 & 0 \\ 0 & -2 & 2 \end{pmatrix}.
\]
2. 5 points Find a $3 \times 3$ matrix $A$ such that $A\vec{x}$ is parallel to the vector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ for all $\vec{x} \in \mathbb{R}^3$.

Solution: Let $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. The unit vector $\vec{u}$ parallel to $\vec{v}$ is given by $\vec{u} = \frac{1}{\sqrt{3}} \vec{v}$.

The projection of the vector $\vec{x}$ onto the line $L$ passing through the origin in the direction of the vector $\vec{u}$ is given by

$$\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u}) \vec{u} = \begin{pmatrix} \frac{x_1 - x_2 + x_3}{3} \\ \frac{-x_1 + x_2 - x_3}{3} \\ \frac{x_1 - x_2 + x_3}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$ 

Then $A = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$ is a matrix satisfying the given properties.

Remark: Note that this problem doesn’t have a unique solution, i.e., there are other matrices obtained via different methods which also satisfy the required properties and all of those are correct answers as well.
3. [5 points] Let $A$ be a $4 \times 4$ matrix, $\vec{b}$ is a non-zero vector in $\mathbb{R}^4$, and $\vec{0}$ is the zero vector in $\mathbb{R}^4$. We are told that the linear system $A\vec{x} = \vec{0}$ has \textit{infinitely} many solutions. What can you say about the number of solutions of the system $A\vec{x} = \vec{b}$? You must explain your answer.

\textbf{Solution:} Since $A$ is a $4 \times 4$ matrix and $A\vec{x} = \vec{0}$ has infinitely many solutions, the reduced row-echelon form of the augmented matrix $[A \mid \vec{0}]$ contains a row which is identically equal to zero, i.e., a row of the from $(0, 0, \cdots, 0 \mid 0)$. Now for the system $A\vec{x} = \vec{b}$, the corresponding row of the reduced row-echelon form of the augmented matrix $[A, \mid \vec{b}]$ looks like $(0, 0, \cdots, 0 \mid b'_i)$, $1 \leq i \leq 4$. If $b'_i \neq 0$, then the system $A\vec{x} = \vec{b}$ is inconsistent, i.e., doesn’t have any solution. If $b'_i = 0$, then we will have 3 or less independent equations and 4 variables. Since the number of variables is more than the number of equations, there will be infinitely many solutions in this case.
4. Let $T$ and $S$ be two linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T$ is defined by the matrix $A = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$, i.e., $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$, and $S$ is given by $S \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 2x_2 \end{pmatrix}$ for all vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$.

(a) 2 points Compute the matrix of $S$.

(b) 3 points Compute the matrix of $S \circ T$, where $S \circ T$ means $(S \circ T)(\vec{x}) = S(T(\vec{x}))$ for all $\vec{x} \in \mathbb{R}^2$.

Solution: $S(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $S(e_2) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Therefore the matrix of $S$ is given by

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

The matrix of $S \circ T$ is given by

$$BA = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}.$$