# Practice Final
Algebra (Math 110B)

Name: _______________________________ U ID: _______________  

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1. [5 points] Prove that a group of order 351 must have a normal Sylow $p$-subgroup for some prime $p$ dividing its order.
2. 5 points Let $G$ be a finite group such that it has exactly 2 conjugacy classes. Then prove that $o(G) = 2$. 
3. **5 points** Let \( p \) and \( q \) be two primes such that \( p < q \) and \( p \) does not divide \( q - 1 \). Then prove that every group of order \( pq \) is cyclic.
4. 5 points Let $G$ be a **non-abelian** group of order $p^3$, where $p$ is a prime. Then prove that $o(Z(G)) = p$, where $Z(G)$ is the center of $G$. 
5. 5 points Let $G$ be a group and $N$ a normal subgroup of $G$ such that every element of $N$ and $G/N$ has finite order. Then prove that every element in $G$ has finite order.

(Warning: Do not assume that $G$ is finite.)
6. Let $G$ be a finite group and $H, K, N$ are three subgroups of $G$.

(a) [5 points] If $K \subset H$, then prove that $[G : K] = [G : H][H : K]$.

(b) [5 points] If $N$ is normal in $G$, then prove that $HN/N \cong H/(N \cap H)$.

(c) [5 points] If $o(H)$ and $[G : N]$ are relatively prime, then prove that $H \subset N$.

(Hint: For Part (2), define a homomorphism $\varphi : H \to HN/N$ by $\varphi(h) = hN$. Show that $\ker(\varphi) = N \cap H$. Then apply the First Isomorphism Theorem. For Part (3), note that $H \subset N \Leftrightarrow N \cap H = H \Leftrightarrow [H : (N \cap H)] = 1$. Then use Part (1) and (2) to get Part (3).)
7. 5 points If $H$ and $K$ are two subgroups of a finite group $G$ such that $[G : H] = p$ and $[G : K] = q$, where $p$ and $q$ are distinct primes, then prove that $pq$ divides $[G : (H \cap K)]$. 
8. 5 points Let $G$ be a finite abelian group containing two distinct elements of order 2. Then prove that $o(G)$ is a multiple of 4. Show that this statement is not true if $G$ is non-abelian.