Definition: Let $A$ be a non-empty set. A permutation of the elements of $A$ is a bijection $f: A \rightarrow A$.

Let $S$ be a set of $n$ distinct symbols. We can choose these symbols to number, alphabetize, catz, dogs, kinds, etc. etc. as you like.

For simplicity assume that these symbols are the integers $1, 2, \ldots, n$. 

i.e. $S = \{1, 2, \ldots, n\}$.

Let $S^n$ be the set of all permutations of the elements of $S$, i.e. $S^n$ is the set of all bijections from $S$ to $S$.

Let $f: S \rightarrow S$ be a bijection.

- $f(1) = \text{higher}$
- $f(2, 3, \ldots, n) = \text{lower}$
- $f(2) = n - 1$
- $f(3) = n - 2$
- $f(n) = 1$

So, there are $n!$ bijections.

The number of bijections from $S$ to $S$ is $n!$.

Therefore, $|S^n| = n!$.
Let $n = 3$, $S = \{1, 2, 3\}$.

$|S_3| = \frac{3!}{2!} = 3 \cdot 2 \cdot 1 = 6$

Let $f: S \rightarrow S$ be given by:

$f(1) = 2, \quad f(2) = 1, \quad f(3) = 3$

We can represent $f$ by $(1 \ 2 \ 3)$

First row is the domain of $f$ and the second row is the range.

$(1 \ 2 \ 3)$ is a permutation of $S$.

$\sigma_1 = (1 \ 2 \ 3), \quad \sigma_2 = (2 \ 3 \ 1), \quad \sigma_3 = (1 \ 3 \ 2), \quad \sigma_4 = (2 \ 1 \ 3), \quad \sigma_5 = (1 \ 2 \ 3), \quad \sigma_6 = (1 \ 3 \ 2)$

$S_3 = \{ \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6 \}$

Product of two permutations:

Since permutations are bijections, the product of two permutations can be defined as the composition of functions, which is again a bijection.
\[ P_2 P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \]
\[ = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = P_6 \]

\[ P_1 P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \]
\[ = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = P_6 \]

\[ P_6 P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix} \]
\[ = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = P_6 \]

In general, \( P_i P_j = P_k \) if \( i \leq j \) and \( i \leq k \leq j \).

\( P_6 \) is the identity permutation.

\( \therefore (S_3, \cdot) \) is a monoid.
A Inverse Formulation

\[ P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \]

\[ P_1^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = P_1 \]

\[ P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \]

\[ P_2^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = P_2 \]

Since inversion of a bijection is again a bijection, inversion of permutation is again a permutation.

\( S_4 \) is the first non-abelian group of order 24.

Def: A permutation \( P \in S_n \) is called a cycle of length \( k \) if

\[ P = (i_1, i_2, \ldots, i_k) \]

\[ P(i_1) = i_2, \quad P(i_2) = i_3, \quad \ldots, \quad P(i_k) = i_1 \]

\[ P(i_j) = i_j \quad \text{for all } j \neq i_1, \ldots, i_k \]

In this case we denote \( P \) by \((i_1, i_2, \ldots, i_k)\).
Example: 1. \( S_3 \) \( \rho = (2 \; 3) \) is a cycle.

\[ \rho = (2 \; 3) = (1 \; 2 \; 3) \]

\[ \rho = (1 \; 2), \quad \rho = (1 \; 3) \]

\[ e = \rho_1 = (1 \; 2 \; 3) \]

- (2) In \( S_4 \), \( 15! = 41 \times 24 \)

\[ \{1, 2, 3, 4\} \]

\[ \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = (1 \; 2 \; 4) \]

\[ \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (1 \; 3) \]

**Def:** Two cycles in \( S_n \) are called **disjoint** if they don't share any common elements.

**Example:** (1.3) or (2.4) are *not* disjoint cycles in \( S_4 \). But (1.2.4.3) and (1.3) are disjoint in \( S_4 \).