Thus: Let \( G \) be a group and \( H \) a subgroup.

Then there is a bijection between the set of all distinct left cosets of \( H \) and the set of all distinct right cosets of \( H \).

Proof: Exercise.

Definition: Let \( G \) be a group and \( H \) a subgroup. The number of distinct left cosets of \( H \) in \( G \) is called the \text{index of } H \text{ in } G \text{ and}

It is denoted by \([G: H]\) or \( G : H \).

Remark 1: In general \([G: H]\) is countable.

Remark 2: If \( G \) is a finite group, then from the proof of Lagrange's theorem it follows that

\[
[G : H] = \frac{o(G)}{o(H)}
\]
Def: Let $G$ be a group. A subgroup $H$ is called a normal subgroup of $G$ if

$$aH = Ha \quad \forall a \in G.$$ 

Warning: $aH = Ha$ does not mean $aHa^{-1} = H$.

For example, $G = S_3 = \{ (1), (12), (13), (123), (132) \}$

$A_3 \leq S_3$, $A_3 = \{ (1), (123), (132) \}$

$A_3 = A_3(1) = A_3
A_3 = A_3 = A_3(123)
A_3 = A_3 = A_3(132)
A_3(12) = \{ (12), (23), (13) \}
A_3(13) = \{ (13), (12), (23) \}
A_3(12) = \{ (12), (13), (23) \}
A_3(12) = A_3(12)
A_3 = A_3(13)
A_3 = A_3(13)
A_3 = A_3(13)
A_3 = A_3(13)$
\[(23)A_3 = \{ (23)(1), (23)(2), (23)(3), (23)(123) \}\]
\[= \{ (23), (123), (12) \}\]

\[A_3 \langle 23 \rangle = \{ (1)(23), (123)(23), (132)(23) \}\]
\[= \{ (23), (12), (13) \}\]

\[\therefore (23)A_3 = A_3 \langle 23 \rangle\]
\[\therefore 6A_3 = A_3 0 \forall \sigma \in S_3\]
\[A_3 \text{ is a normal subgroup.}\]

\[S_3, \quad H = \langle (12) \rangle \langle 13 \rangle \]
\[\langle 13 \rangle H = \{ (13)(1), (13)(2) \} = \{ (13), (123) \}\]
\[H \langle 13 \rangle = \{ (1)(13), (12)(13) \} = \{ (13), (132) \}\]

\[\therefore \langle 13 \rangle H \neq H \langle 13 \rangle\]
\[\therefore H \text{ is NOT normal in } S_3\]

Then: Every normal subgroup of an abelian group is normal.

Proof: Let \( G \) be an abelian group and \( H \) is an arbitrary subgroup of \( G \).

Since \( G \) is abelian,
\[ah = ha \quad \forall \ a, h \in G\]
\[\implies aH = Ha \quad \forall \ a, h \in G\]
\[\therefore H \text{ is normal in } G\]
Theorem: Let $G$ be a group and $H$ a subgroup. If the index of $H$ in $G$ is 2, i.e., $[G:H] = 2$, then $H$ is normal.

Proof: Since $[G:H] = 2$, there are exactly 2 distinct left cosets (and right cosets) $H$ in $G$.

We know that distinct left cosets of $H$ form a partition of $G$. Since $H$ is always a left coset of itself, the other left coset is $gH$.

We need to show that $gH = Ha \neq aH$.

If $gH$ is $H$, then $gH = H$ and $Ha = H$.

\[ gH = Ha \neq aH \quad (i) \]

If $g \notin H$, then $gH \neq H$, i.e., $gH$ is the other left coset, which is $gH = gH$.

\[ gH = gH \]

Again, $a \in gH$, $Ha \neq H$, i.e., $Ha$ is the other right coset, i.e., $Ha = gH$.

\[ aH = Ha \neq gH \quad (ii) \]

Repeating (ii) giving $aH = Ha \neq gH$.

$H$ is normal in $G$. 
Example 1

- \( o(S_3) = 3! = 6 \)
- \( o(A_3) = \frac{3!}{2} \)
- \( \left[ S_3 : A_3 \right] = \frac{6}{3} = 2 \)
- \( A_3 \) is normal in \( S_3 \).

In general, \( o(S_n) = n! \) and \( o(A_n) = \frac{n!}{2} \)

\[ \left[ S_n : A_n \right] = \frac{o(S_n)}{o(A_n)} = \frac{n!}{\frac{n!}{2}} = 2 \]

\( A_n \) is a normal subgroup of \( S_n \) if \( n \geq 4 \).

**Theorem (Test of Normality):**

Let \( G \) be a group and \( a H \) a subgroup.

Then \( H \) is normal in \( G \) \( \iff gH = Hg \) for all \( g \in G \) and for all \( h \in H \).

**Proof:** Assume that \( H \) is normal.

Then \( gH = Hg \) for all \( g \in G \).

Now for all \( g \in G \) and \( h \in H \), we have

\( gh \in Hg \)

\[ \implies gh \in Hg \quad (\because gh = Hg) \]

\[ \implies gh \in Hg \quad \text{for some } h' \in H \]

\[ \implies ghg^{-1} = h' \in H \]

\[ \implies ghg^{-1} \in H \quad \text{for all } g \text{ in } G \]
Assume that \( G H \subseteq H \) for a group \( H \).

Let \( a \in H \).
Let \( x = ax \). Then \( x = ah \) for some \( h \in H \).

\[
(xh)a = (ax)h = ah \cdot h = a(hh) \quad \text{(since } hh = h \cdot h \text{)} \\
(xh)a = a(hh) = a \cdot a = a^2 \\
\text{Let } j \in H. \text{ Then } y = ha \text{ for some } a \in H.
\]

\[
y = yh = ha \text{ for some } a \in H \\
(yh)a = (ya)h = h(aa) \\
(yh)a = ahh = ahh = ah \\
\text{by yeah.} \\
ha = ah \\
\text{so yeah!} \\
ha = aH \\
\]

(1) \& (2) giving \( aH = Ha \).