Lecture 10

Cosets:

Def: Let \( G \) be a group and \( H \) a subgroup of \( G \). Let \( a \in G \), then set \( \text{left coset of } \ aH \) is called a left coset of \( H \) in \( G \).

- Similarly, a right coset \( Ha \) is defined as 
\[
Ha := \{ ha : h \in H \}
\]

Example: (1) \( G = \mathbb{Z}^+ \) and \( H = 6 \mathbb{Z}^+ \).

\[
0 + H = \{ 0 + 3n : h \in \mathbb{Z} \} = 3\mathbb{Z} = H \\
1 + H = \{ 1 + 3n : h \in \mathbb{Z} \} = 3n + 1 : n \in \mathbb{Z} \\
2 + H = \{ 2n + 2 : n \in \mathbb{Z} \} \\
3 + H = 0 + H, \quad 4 + H = \{ 3n + 4 : n \in \mathbb{Z} \} \\
\quad = \{ 3(n+1) + 1 : n \in \mathbb{Z} \} = H + H
\]

\( H, \quad 6H \) and \( 2 + H \) are all distinct left cosets of \( H \).

(2) \( G = S_3 \), \( H = A_3 = \{ (1), (123), (132) \} \)

(1) \( A_3 = \{ (1), (123), (132) \} \) \( = A_3 \) since \( (1) \) is identity element in \( S_3 \).

(12) \( A_3 = \{ (12), (1)(23), (12)(132) \} \) \( = \{ (12), (23), (13) \} \)

(13) \( A_3 = \{ (13), (1)(123), (13)(123) \} \) \( = \{ (13), (12), (23) \} \)
\[(2 \, 3) A_3 = \{ (2 \, 3 \, 1), \ (2 \, 3 \, 1 \, 2 \, 3), \ (2 \, 3) \, (1 \, 3 \, 2) \} \]
\[= \{ (2, 3), \ (13), \ (12) \} \]

\[(1 \, 2 \, 3) A_3 = \{ (1 \, 2 \, 3 \, 1), \ (1 \, 2 \, 3 \, 1 \, 2 \, 3), \ (1 \, 2 \, 3) \, (1 \, 3 \, 2) \}\]
\[= \{ (1 \, 2 \, 3), \ (1 \, 3 \, 2), \ (1) \} = A_3 \]

\[(1 \, 3 \, 2) A_3 = \{ (1 \, 3 \, 2 \, 1), \ (1 \, 3 \, 2 \, 1 \, 2 \, 3), \ (1 \, 3 \, 2) \, (1 \, 2 \, 3) \}\]
\[= \{ (1 \, 2 \, 3), \ (1), \ (1 \, 2 \, 3) \} \]
\[= A_3 \]

\[(1) A_3 = (1 \, 2 \, 3) A_3 = (1 \, 3 \, 2) A_3 = A_3 \]

\[\text{As has 2 distinct left cosets: } 1, \ A_3 \]

Thus, let \( G \) be a group and \( H \) a subgroup of \( G \).
If \( x \in H \), then \( xH = H \), i.e., \( H \) is a left coset of itself.
- Similarly, \( \text{HH} = H \).

**Proof:** Since \( H \leq H \) and \( H \) is a subgroup, clearly
\[HH \leq H \leq \text{H.} \text{ Clearly} \]

but \( xH \), then \( x = h (h^{-1}x) \in hH \).

\[hH \leq H \leq hH \]

Since \( x \in H \) and \( H \) is a subgroup.

(1) and (ii) give \( hH = H \).
Theorem: Let $G$ be a group and $H$ a subgroup. Then any two left cosets of $H$ have either equal or disjoint. i.e. for $a, b \in G$ either $aH = bH$ or $aH \cap bH = \emptyset$.

Proof: If $aH \cap bH = \emptyset$, then we are done.

Let $aH \cap bH \neq \emptyset$. Then $\exists x_0 \in aH \cap bH$.

$\Rightarrow x_0 = ah_1 = bh_2$ for some $h_1, h_2 \in H$.

$\Rightarrow a = x_0h_1^{-1}$, and $b = x_0h_2^{-1}$.

Let $x \in aH$. Then $x = ah$ for some $h \in H$.

$\Rightarrow x = (x_0h_1h)$

$\Rightarrow x = x_0(h_1h)$

$\Rightarrow x = bh_2(h_1h)$

$\Rightarrow x = b(h_2^{-1}h)$

$\Rightarrow x = bh_3$, where $h_3 = h_2^{-1}h \in H$.

$\therefore x \in bH$.

Hence $aH \subseteq bH$.  \(\square\)
Similarly, we can show that $bH = aH$. 
(i) and (ii) imply $aH = bH$.

Thus, let $G$ be a group and $H$ a subgroup of $G$. Let $a, b \in G$. Then $aH = bH \iff a^{-1}b \in H$.

**Proof:** Assume $aH = bH$.

Note that $a = ae \in aH$; $e \in H$.

\[ a \in bH, \quad (\because aH = bH) \]

\[ \Rightarrow a = bh \quad \text{for some } h \in H. \]

\[ \Rightarrow b^{-1}a = h \]

\[ \Rightarrow (b^{-1}a)^{-1} = h^{-1} \]

\[ \Rightarrow a^{-1}b = h^{-1} \in H. \quad \text{(QED)} \]

Assume that $a^{-1}b \in H$.

Let $a^{-1}b = h$.

\[ \Rightarrow b = ah \in aH \]

But $b = be \in bH$.

\[ \text{i.e. } b \in aH, bH \quad \text{i.e. } aH = bH \text{ i.e. } aH \neq bH \]

\[ \Rightarrow aH = bH \quad \text{by previous discussion} \]
Theorem: Let $G$ be a group and $H$ a subgroup of $G$. Then any two left cosets of $H$ have the same cardinality.

Proof: Let $aH$ and $bH$ are two distinct left cosets of $H$. We need to prove that there is a bijection between them.

Define a function $f : aH \rightarrow bH$ as follows:

$$f(ah) = bh, \quad \forall h \in H.$$ 

Let $f(ah_i) = f(ah_j)$

$\Rightarrow bh_i = bh_j$

$\Rightarrow h_i = h_j$

$\Rightarrow a_{h_i} = a_{h_j}$

$\Rightarrow f$ is injective.

From the definition of $f$ and $bH$ it is clear that $f$ is bijective.

$f$ is bijective (Porn)