HOMEWORK 6
MATH 110B
(GROUP THEORY)

(A problem with a ‘*’ or ‘**’ mark means we will make references to these problems in the future and thus you should memorize their statements.)

Due Date: Friday, May 25.

*(1) Using Lagrange’s theorem prove the following results:
   (a) (Fermat’s Little Theorem) Let $p$ be a prime number and $a$ an integer such that $p$ does not divide $a$. Then $a^{p-1} \equiv 1 \pmod{p}$.

   (b) (Euler’s Theorem) Let $a$ and $n$ be two positive integers such that $\gcd(a, n) = 1$. Then prove that $a^{\phi(n)} \equiv 1 \pmod{n}$, where $\phi(n)$ is the Euler’s phi function.

   (Hint: For Part (a), consider the multiplicative group $(\mathbb{Z}_p \setminus \{0\}, \cdot)$ which has order $p - 1$. For the Part (b), consider the group $U_n$ which has order $\phi(n)$. Recall that $U_n$ is the group of all units in the ring $(\mathbb{Z}_n, +, \cdot)$.)

*(2) Let $G$ be a group and $H$ a normal subgroup of $G$.
   (a) If $K$ is a subgroup of $G$ such that $H \subseteq K \subseteq G$, then prove that $K/H$ is a subgroup $G/H$.

   (b) Prove that every subgroup of $G/H$ is equal to $M/H$ for some subgroup $M$ of $G$ such that $H \subseteq M \subseteq G$.

   (Hint: For Part (b), let $L$ be an arbitrary subgroup of $G/H$. Consider the map $\varphi : G \to G/H$ defined by $\varphi(a) = aH$ for all $a \in G$. First show that $\varphi$ is a group homomorphism and also that $\varphi$ is surjective. Then $\varphi^{-1}(L)$ is a subgroup of $G$, say $M = \varphi^{-1}(L)$. Then since $\varphi$ is surjective, $\varphi(\varphi^{-1}(L)) = L$, i.e., $\varphi(M) = L \Rightarrow M/H = L$.)

**(3)** (a) Let $G$ be a group and $H$ a subgroup of index 2. Then show that $g^2 \in H$ for all $g \in G$.

(b) Show that $A_4$ does not have any subgroup of order 6. This example shows that the converse of the Lagrange’s theorem is false.

*(Hint:)* For Part (b), to the contrary assume that $A_4$ has a subgroup $H$ of order 6. Then $[A_4 : H] = 12/6 = 2$. Then by Part (a) we know that $\sigma^2 \in H$ for all $\sigma \in A_4$. Now consider the set of all 3-cycles in $A_4$, there are total $\binom{4}{3} \cdot 2! = 8$ 3-cycles in $A_4$. Then get a contraction from this on the order of $H$.

(4) Let $G$ be a group of order 8 and $x \in G$ such that $o(x) = 4$. Prove that $x^2 \in Z(G)$.

*(Hint:)* Use a similar idea as in the proof of the previous problem Part (a).

(5) Let $H$ be a normal subgroup of a group $G$ and $[G : H] = m$. Prove that $a^m \in H$ for all $a \in G$.

(6) Let $H$ be a normal subgroup of a group $G$ such that $o(H) = 3$ and $[G : H] = 10$. If $a \in G$ and $o(a) = 3$, then prove that $a \in H$.

*(7)* Let $H$ be a normal subgroup of a group $G$. Then prove that the quotient group $G/H$ is abelian if and only if $xyx^{-1}y^{-1} \in H$ for all $x, y \in G$.

(8) Let $H$ be a subgroup of a group $G$ such that $x^2 \in H$ for all $x \in G$. Then prove that $H$ is normal in $G$ and $G/H$ is abelian.

*(9)* Let $G$ be a group and $H$ a normal subgroup of $G$. If $K$ is a normal subgroup of $G$ containing $H$, i.e., $H \subseteq K \subseteq G$, then prove that the quotient group $K/H$ is normal in $G/H$.

(10) Prove that a non-abelian group of order 10 must have a trivial center.