(A problem with a ‘*’ or ‘**’ mark means we will make references to these problems in the future and thus you should memorize their statements.)

**Due Date:** Friday, May 11.

1. (a) Describe the elements of the group $D_3$, which is the group of all symmetries of an equilateral triangle on a plane.
   (b) Show that $D_3$ is the same group as $S_3$.
   (c) Describe $D_3$ as an abstract group using the elements $r$ and $s$ as it was done in the lecture. Describe what $r$ and $s$ each represents in $D_3$, what are their orders in $D_3$ and what is the relation between them.

2. Express the elements of $S_3$ as cycle decomposition and find all cyclic subgroups of $S_3$.

3. (a) Describe the elements of $S_4$ in terms of disjoint cycle decomposition.
   (b) Describe the elements of $A_4$ in terms of disjoint cycle decomposition.
   (c) How many distinct left cosets of $A_4$ are there in $S_4$?

4. Find all subgroups of $A_4$.

5. Let $B_n$ be the set of all odd permutations of $S_n$. Define a function $f : A_n \rightarrow B_n$ by $f(\sigma) = (12)\sigma$ for all $\sigma \in A_n$, where $A_n$ is the set of all even permutations of $S_n$ known as the alternating subgroup of $S_n$.
   (a) Prove that $f$ is injective.
   (b) Prove that $f$ is surjective. Then $f$ is bijective and thus $A_n$ and $B_n$ both have same number of elements.
(c) Prove that \( o(A_n) = \frac{n!}{2} \).

**Hint:** For Part (a), recall that every transposition has order 2, in particular, \( (12)(12) = \text{Identity} \). For Part (b) recall the definition of even and odd permutation and the fact that \( (12)(12) = \text{Identity} \). For Part (c), notice that every elements of \( S_n \) is either in \( A_n \) or in \( B_n \) and not in both of them at the same time.

(6) Describe the elements of \( D_2 \) and \( S_2 \). Show that \( D_2 \) and \( S_2 \) are the same group and they are both cyclic.

(7) Let \( G \) be a group such that the intersection of its subgroups which are different from \( \{e\} \) is again a subgroup different from \( \{e\} \). Then prove that every element in \( G \) has finite order.

(8) Let \( G \) be a group such that \( o(G) > 1 \) and it does not have any non-trivial proper subgroup. Then prove that \( G \) must be a cyclic group of prime order.

**Warning:** Note that \( o(G) > 1 \) does not mean that \( o(G) \) is finite, it is a possibility that \( o(G) = \infty \). So do not assume in this problem that \( G \) is a finite group.

(9) Let \( G \) be a group and \( H \) a subgroup of \( G \).

(a) Prove that \( gHg^{-1} \) is a subgroup of \( G \) for any \( g \in G \).

(b) Prove that \( N = \cap_{g \in G} gHg^{-1} \) is subgroup of \( G \). Furthermore, show that \( aNa^{-1} = N \) for all \( a \in G \).

(10) Prove that a group of order 9 must have a subgroup of order 3.

(11) Let \( G \) be a group of order \( 2p \), where \( p \) is an odd prime. Prove that \( G \) must have an element of order \( p \).

(12) Let \( G \) be a group and \( H \) and \( K \) are two subgroups such that \( o(H) \) is relatively prime to \( o(K) \). Then prove that \( H \cap K = \{e\} \).

(13) Let \( G \) be a finite group of odd order. Then prove that for every element \( x \in G \) there is an element \( y \in G \) such that \( x = y^2 \).

(14) Prove the every abelian group of order 15 is cyclic.