HOMEWORK 3
MATH 110B
(GROUP THEORY)

(A problem with a ‘*’ or ‘**’ mark means we will make references to these problems in the future and thus you should memorize their statements.)

Due Date: Monday, April 30.

*(1) Let $G$ be a group and $a \in G$.
   (a) Let $o(a) = n$. Then for any positive integer $m \in \mathbb{N}$ prove that $o(a^m) = \frac{n}{\gcd(m,n)}$.
   (b) Let $o(a) = n$. Prove that for a positive integer $k \in \mathbb{N}$, $o(a^k) = n$ if and only if $k$ is relatively prime to $n$, i.e., $\gcd(k,n) = 1$.

(2) Let $G$ be a group. Then prove that a non-empty subset $H \subseteq G$ is a subgroup of $G$ if and only if $a^{-1}b \in H$ for all $a, b \in H$.

(3) Let $G$ be a group.
   (a) If $H$ and $K$ are two subgroups of $G$, then prove that $H \cap K$ is also a subgroup of $G$.
   (b) Let $\{H_\alpha : \alpha \in \Lambda\}$ be an arbitrary collection of subgroups (e.g., infinitely many subgroups) of $G$. Then $\cap_{\alpha \in \Lambda} H_\alpha$ is a subgroup of $G$.
   (c) If $H$ and $K$ are two subgroups of $G$, then is $H \cup K$ necessarily a subgroup of $G$? Under what condition(s) $H \cup K$ is a subgroup of $G$? Prove you claim.

(Remark: Problem (3) is very similar to the corresponding problems for Subspaces from 115A course. So looking at those proofs will help here.)

**(4) Let $H$ and $K$ be two subgroup of a group $G$. We define the product $HK$ of two subgroups $H$ and $K$ as $HK := \{hk : h \in H, k \in K\}$.
$H, k \in K$; similarly we define the product $KH$ as $KH := \{kh : h \in H, k \in K\}$.
Prove that $HK$ is a subgroup of $G$ if and only if $HK = KH$.

**Warning:** Here $HK = KH$ does not mean that $hk = kh$ for all $h \in H$ and $k \in K$. It means that for every $h \in H$ and $k \in K$ there exist some $h' \in H$ and $k' \in K$ such that $hk = k'h'$ holds.)

**(5)** Let $H$ and $K$ be two finite subgroups $G$. Then prove that $o(HK) = \frac{o(H)o(K)}{o(H \cap K)}$.

**Warning:** Note that you can not assume here that $HK$ is a subgroup of $G$, since we don’t know whether $HK = KH$ holds or not.)

**Remark:** Problem (4) and (5) will be extremely useful in the future when we do the applications of Lagrange’s theorem in the coming weeks.

*(6) Let $G$ be a group.
(a) Prove that $G$ is abelian if and only if $Z(G) = G$.
(b) Let $a \in G$. Then prove that the centralizer $C(a)$ of $a$ is a subgroup of $G$.
(c) Prove that $Z(G)$ be a subgroup of $C(a)$ for any $a \in G$.
(d) Prove that $Z(G) = \cap_{a \in G} C(a)$.

(7) Prove that a cyclic group of prime order does not have any non-trivial proper subgroup.

**Remark:** Recall that a subgroup $H$ is called trivial if $H = \{e\}$ and not proper subgroup if $H = G$. In other words, $H$ is called a non-trivial proper subgroup if $H \neq \{e\}$ and $H \neq G$.)

(8) Let $G$ be a cyclic group generated by $a \in G$. Prove that $o(G)$ is infinite if and only if $o(a)$ is infinite.

(9) Prove that every non-trivial subgroup of an infinite cyclic group is infinite.
*(10) (a) Let $G$ be a finite cyclic group of order $n > 1$ generated by $a \in G$. Then for any positive integer $k$, $a^k$ is a generator of $G$ if and only if $k$ is relatively prime to $n$.
(b) Conclude that for the group $G$ is Part (a) the number of generators is $\varphi(n)$, where $\varphi$ is the Euler phi-function.
(c) Find the total number of distinct generators of $(\mathbb{Z}_{1000}, +)$.

(Hint: In Part (a), for one of the direction of use Problem (1)(b).)

*(11) Let $G$ be a cyclic group of order $n$. Then prove that for every positive integer $d$ such $d | n$ there exists one and exactly one subgroup $H$ of order $d$.

(12) Prove that $(\mathbb{Q}, +)$ is not cyclic. Then conclude that $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$ are not cyclic either.

(Hint: On the contrary assume that $(\mathbb{Q}, +)$ is cyclic and generated by $a$. Then consider the element $\frac{1}{2}a$.)

*(13) Fix a positive integer $n \in \mathbb{N}$. Consider the set $G = \{z \in \mathbb{C} : z^n = 1\}$. Then show that $G$ is a cyclic group under the operation of multiplication of complex numbers. Find all generators of $G$.

(Hint: Recall the famous Euler’s theorem $e^{i\theta} = \cos \theta + i \sin \theta$ for all $\theta \in \mathbb{R}$.)

*(14) Find all subgroups of $(\mathbb{Z}, +)$. 