(A problem with a ‘*’ or ‘**’ mark means we will make references to these problems in the future and thus you should memorize their statements.)

Due Date: Friday, April 20.

(1) Find the order of the group $U_{20}$ and also the order of the each elements of $U_{20}$.

(2) Find the order of every element of $\mathbb{Z}_4 \times \mathbb{Z}_2$.

(3) Let $G$ be a group.
   (a) Let $a \in G$ such that $a^{12} = e$. What are possible values of $o(a)$?
   (b) If $x \in G$ such that $x \neq e$ and $x^p = e$ for some prime number $p$, then what is $o(x)$? Prove or disprove your answer.

(4) Let $G$ be a group.
   (a) If $a \in G$ such that $o(a) = 12$. Find the orders of each of the elements $a, a^2, a^3, \ldots, a^{11}$.
   (b) Based on the evidence in part (a), first make a conjecture about the order of $a^k$ when $o(a) = n$. Then prove your conjecture.

(5) Let $G$ be a finite group of order $n$. Then prove that $o(a) \leq n$ for all $a \in G$.
   (Hint: Consider $e, a, a^2, \ldots, a^n \in G$ and notice that these give $n + 1$ elements in $G$.)
   (Remark: This exercise shows that order of an element of a finite group is finite.)

(6) Let $S$ be an infinite set and $\mathcal{P}(S)$ be the power set of $S$, i.e., set of all subsets of $S$. We define an operation $\Delta$ on $\mathcal{P}(S)$ as
follows: \( A \Delta B := (A \setminus B) \cup (B \setminus A) \) for all \( A, B \in \mathcal{P}(S) \).

(a) Show that \((\mathcal{P}(S), \Delta)\) is a group.
(b) Compute the order of every element of \( \mathcal{P}(S) \) with rest to the operation \( \Delta \).

(\textbf{Remark:} This exercise shows that there exist groups \( G \) such that \( o(G) \) is infinite but every element of \( G \) has finite order.)

*(7) Let \( G \) be a group such that every non-identity element of \( G \) has order 2. Then prove that \( G \) is an abelian group.
(\textbf{Hint:} First show that \( x = x^{-1} \) for all \( x \in G \). Now for \( a, b \in G \) set \( x = ab \).)

(8) Let \( G \) be a group and \( a, b \in G \). Prove that \( o(ab) = o(ba) \).
(\textbf{Warning:} You must consider two cases based on whether \( o(ab) \) if finite or infinite.)

*(9) Let \( G \) be a group and \( a, g \in G \). Prove that \( o(gag^{-1}) = o(a) \).
(\textbf{Warning and Hint:} Again you need to consider two cases like the previous problem. Look at the first problem in the example set I provided.)

(10) Let \( G \) be a group such that the following property holds: For any \( a, b, c \in G \), \( ab = ca \Rightarrow b = c \). Prove that \( G \) is an abelian group.

(11) Let \( G \) be a group and \( a, b \in G \) such that \( b^6 = e \) and \( ab = b^4a \). Then prove that \( b^3 = e \) and \( ab = ba \).

(12) Let \( G \) be a group and \( a, b \in G \) such that \( o(a) = 5 \), \( b \neq e \) and \( aba^{-1} = b^2 \). Find \( o(b) \).

(13) Let \( G \) be a group such \( (ab)^n = a^n b^n \) holds for three consecutive values of \( n \) for all \( a, b \in G \). Then prove that \( G \) is an abelian group.
(\textbf{Hint:} Compare this problem with the second problem in the example set I provided.)

**(14) Let \( G \) be a group and \( a, b \in G \) such that \( ab = ba \). Let \( o(a) = m \) and \( o(b) = n \). If \( m \) and \( n \) are relatively prime integers, then prove that \( o(ab) = mn \).
(\textbf{Hint:} Let \( o(ab) = k \). First notice that \( (ab)^{mn} = a^{mn} b^{mn} = e \).
This implies that \( k \mid mn \). Again \( (ab)^k = e \Rightarrow a^k = b^{-k} \). Then \( a^{kn} = b^{-nk} = e \). From this conclude that \( m \mid k \). Similarly show that \( n \mid k \).

(15) Consider the group \( GL(2, \mathbb{R}) \) and following two elements of it:

\[
A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

(a) Prove that \( o(A) = 3 \) and \( o(B) = 4 \) but \( o(AB) \) is infinite.
(b) Notice that this example shows that the conclusion of Problem (13) fails if \( ab \neq ba \). More generally it also shows that in a group \( G \) there can be elements \( a \) and \( b \) each of finite order but \( o(ab) \) is infinite if \( a \) and \( b \) do not commute with each other.