(1) Fix a positive integer \( n \in \mathbb{N} \). Give an example of a groupoid \((G, \circ)\) such that it has exactly \( n \) number of left identities.

(Warning and Hint: Do not give the example of \( a \circ b = b \) for all \( a, b \in G \) with \(|G| = n\); it will not be acceptable, since it’s a stupid answer :-). Instead, consider the following: \( G = \{ (a \ b) \mid a, b \in \mathbb{C}, a^n = 1 \text{ and } b^n = 1 \} \) with a binary composition \( \circ \) defined as the usual matrix multiplication. Now do a similar computation as we did during the lecture.)

(2) Let \( X \) and \( Y \) be two finite sets with \(|X| = m\) and \(|Y| = n\). Let \( f : X \rightarrow Y \) be an injective (i.e., one-to-one) function. If \( 1 < m < n \), then prove that \( f \) has at least two left inverses. (Remark: A function \( g : Y \rightarrow X \) is called a left inverse of \( f \) if \( g(f(x)) = x \) for all \( x \in X \).)

(3) Let \( G \) be the set of all functions from \( \mathbb{N} \) to \( \mathbb{N} \). Define a binary composition \( \circ \) on \( G \) by \( f \circ g \) as the usual composition of functions, i.e., for all \( f, g \in G \), \((f \circ g)(x) = f(g(x))\) for all \( x \in \mathbb{N} \).

(a) Show that \((G, \circ)\) is a monoid.
(b) Give an example of an injective function \( f : \mathbb{N} \rightarrow \mathbb{N} \) which is not surjective (i.e., not onto).
(c) Prove that this \( f \) has infinitely many left inverses. (Hint: Use the same idea as in the proof of the Problem (2) to construct the left inverses of \( f \).)

(4) Let \( X \) and \( Y \) be two finite sets with \(|X| = m\) and \(|Y| = n\). Let \( f : X \rightarrow Y \) be a surjective (i.e., onto) function. If \( m > n \), then prove that \( f \) has at least two right inverses. (Remark: A function \( g : Y \rightarrow X \) is called a right inverse of \( f \).)
if \( f(g(y)) = y \) for all \( y \in Y \).

(5) Give an example of a surjective function \( f : \mathbb{N} \to \mathbb{N} \) which is **not injective at infinitely many points**.

(Remark: A function \( f : A \to B \) is called not injective at infinitely many points if the set

\[ S = \{ b \in B \mid \text{the inverse image set } f^{-1}(b) \text{ contains at least 2 points of } A \} \]

is an **infinite subset** of \( B \)).

(6) Prove that this \( f \) has **infinitely many right inverses** in the monoid \((G, \circ)\) defined in Problem (3).

(Hint: Use the same idea as in the proof of the Problem (4) to construct the right inverses of \( f \)).

(7) Let \((S, \circ)\) be a **finite semigroup** and \( a \in S \). Then prove that there exist two positive integers \( m > 0 \) and \( n > 0 \) such that \( a^{m+n} = a^m \). Further deduce that \( a^{mn} \) is an **idempotent element** in \((S, \circ)\).

(Remark: An element \( x \in S \) is called an idempotent element if \( x^2 = x \)).

(8) Let \((M, \circ)\) be a **finite monoid**. If the identity element \( e \in M \) is the **only idempotent element** of \( M \), then prove that every element of \( M \) is invertible.

(Hint: Use the result from Problem (7)).

(9) Let \((M, \circ)\) be a monoid. If each element of \( M \) is left invertible then prove that every element of \( M \) is also right invertible.

(Hint: Let \( a \in M \). Let \( a' \in M \) be a left inverse of \( a \), i.e., \( a' \circ a = e \). Since every element of \( M \) has a left inverse, \( a' \in M \) must have a left inverse too. Let \( a'' \in M \) be a left inverse of \( a' \), i.e., \( a'' \circ a' = e \). Since \( a'' \) is an inverse of an inverse \( a' \) of \( a \), our intuition tells us that \( a'' \) should be equal to \( a \). Prove that it is indeed the case, i.e., \( a'' = a \)).

(10) Prove that \((\mathbb{Z} \times \mathbb{Z}, *)\) is a commutative monoid, where \(*\) is defined as \((a, b) \ast (c, d) = (ac, bd)\) for all \((a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z} \).

(a) Find all units (invertible elements) in \((\mathbb{Z} \times \mathbb{Z}, *)\).

(b) Find all idempotent elements in \((\mathbb{Z} \times \mathbb{Z}, *)\).