**Examples**

1. Let $G$ be a group. Let $a, b$ be any elements of $G$ and $n$ is any positive integer. Then prove that $(aba^{-1})^n = aba^{-1}^n$.

   **Proof:**
   \[
   \text{LHS} = (aba^{-1})^n \\
   = aba^{-1}aba^{-1}aba^{-1} \ldots \text{n times} \\\n   = ab (a^{-1}a)b (a^{-1}a)b (a^{-1}a) \ldots \text{(Since G is associative)} \\
   = abababab \ldots \text{ebeba}^{-1} \\
   = ababa^{-1} \\
   = ab^n \cdot aba^{-1} \\
   = \text{RHS (Proved)}.
   \]

2. Let $G$ be a group such that $(ab)^2 = aba^2 + ab + c$. Then prove $G$ is abelian.

   **Proof:**
   \[
   (ab)^2 = a^2b^2 \\
   \Rightarrow aba = aba^{-1}b^2 \]
   \[
   \Rightarrow ba = ab + ab + c \text{ (since G is abelian)}
   \]
   i.e. $G$ is abelian (proved).

[In the margin] It's better to check at least 2 cases of $a$ and $b$. Furthermore, prove $G$ is a commutative group.
3) Prove that a group $G$ is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

Proof: $G$ is abelian $\iff ab = ba \forall a, b \in G$

$\iff (ab)^{-1} = (ba)^{-1} \forall a, b \in G$

$\iff (ab)^{-1} = b^{-1}a^{-1} \forall a, b \in G$

(Proved).

4) If $a \in G$, prove that $o(a) = o(a^{-1})$.

Proof: Let $o(a) = n$.

Then $a^n = e$ and $a^k \neq e$ for $0 < k < n$.

$\Rightarrow (a^n)^{-1} = e^{-1}$

$\Rightarrow a^{-n} = e$ (since $e^{-1} = e$).

$\Rightarrow (a^{-1})^n = e$. ........... (i)

Since $a^k \neq e$ for $0 < k < n$

$\Rightarrow a^{-k} \neq e^{-1}$

$\Rightarrow (a^{-1})^k \neq e$ for $0 < k < n$ .... (ii)

Combining (i) and (ii) we get

$o(a^{-1}) = n = o(a)$. 
Let $G$ be a group such that $o(G)$ is an **even** integer.

Then prove that $G$ contains an **odd** number of elements of order 2. In particular, $G$ contains at least one element of order 2.

**Proof:** $G$ has a unique element of order 1, namely the identity element $e$.

If $a$ is an element of $G$ of order 2, i.e., $o(a) = 2$,
then $a^2 = e \Rightarrow a = a^{-1}$.

If $o(a) > 3$,
then $a \neq a^{-1}$, otherwise $a = a^{-1}$ would imply that $a^2 = e$, which contradicts that $o(a) > 3$. We also have $o(a^{-1}) > 3$ in this case.

For all $x \in G$ such that $o(x) > 3$, we consider the pairs $\{x, x^{-1}\}$.

These show that there are **even number** of elements in $G$ of order $\geq 3$.

Now every element of $G$ falls into one of the following three categories:

- It is an element of order 1.
- It is an element of order $\geq 3$.
- It is an element of order $\geq 2$.

In this case, $a = a^{-1}$.

And there are **even number** of such elements in $G$ as explained above.
Let \( m \) be the number of elements of order 2 in \( G \).

Since \( o(G) \) is even, we have

\[ +m + \text{an even number} = o(G) = \text{an even integer} \]

(for the identity e)

\[ \Rightarrow m + (1 + \text{an even number}) = \text{even number} \]

\[ \Rightarrow m + \text{an odd number} = \text{even number} \]

\[ \therefore m = \text{an odd number} \]

\[ \Rightarrow \text{even} - \text{odd} = \text{odd} \]

\[ \text{e.g., } 4 - 1 = 3, \]
\[ 100 - 69 = 31 \]

etc.

\[ \therefore G \text{ contains odd number of elements of order 2.} \]

Since 1 is smallest positive odd integer, there is at least one element of \( G \) of order 2.