Cantor's *Grundlagen*

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A manifold (*Mannigfaltigkeit) is a set (*Menge).

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W.W. Tait:

"Given such a rich assortment of original material and given the prominence anyway of the problem of the infinite in the history of philosophy, one would *a priori* have expected the *Grundlagen* to be regarded as one of the great philosophical classics of all time."

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The absolute cannot be *determined*. This implies, in particular, that absolutely infinite totalities cannot be numbered.

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Numbers are also the *Anzahlen* of well-ordered sets, thus they play the role of ordinal numbers.

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The *first principle of generation* requires that whenever a number has just been created an immediate successor of that number should be created.

The Second Principle of Generation

Cantor states this principle as follows: "If any definite succession of defined numbers is put forward of which no greatest exists, a new number is created by means of this second principle of generation, which is thought of as the *limit* of those numbers; that is, it is defined as the next number greater than all of them."

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By a "definite succession" he means a set. Equivalently, he means a finite or transfinite totality.

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Cantor regards it a law of logic that every set can be well-ordered, so this theorem impies that every set has the same power as some number class.

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Cantor regards it as always justified to apply the second principle of generation to start a new number class when an old one becomes complete.

In other words, he implicitly assumes that each number class is a *set.* This is, in effect, the ordinal analogue of the Power Set Axiom.

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This argument generalizes to prove to show that Cantor's principles and assumptions imply the theorem that the γth number class exists for every γ. The generalization also yields what can be roughly stated by: "The length of the sequence of all numbers is weakly inaccessible."

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In the last paragraph of Section 12, he says that the third principle “consisted in the demand that a new integer could be made with the help of one of the two other principles of creation *only* if the totality of all previous numbers had the power of a defined number class which was already *in existence* over its entire extent.”

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He says that, with the three principles, "one can attain with the greatest certainty and obviousness, ever newer number classes, . . . and the new numbers obtained in this way are then always of the same determinacy and objective reality as the earlier ones."

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Major problem: The third principle is too restrictive. The creation of ω_ω is not allowed.

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The terminology in these letters is rather different from the terminology of the letter to Hilbert. The latter is much closer to the terminology of *Grundlagen*.

In 1883 and 1899, sets are characterized as multiplicities that can be thought of as unities. But what he says in 1899 is that sets can *without contradiction* be thought of as unities, and "set" is taken to be synonymous with "consistent multiplicity." Totalities that too big to be sets are in 1899 called *inconsistent multiplicities*.

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I regard the complaint about sequential choices as unjustified. Sequential choice seems more natural than simultaneous choice.

Cantor's reductio argument did involve a proper class of choices, but this could have easily been avoided. He could simply have remarked that the number of choices was less than or equal to the number of elements of the set W and hence—by Separation and Cantor's version of Replacement—only a set of ordinals would be assigned.

Both the version of Replacement he stated and the fact that he didn't avoid making absolutely infinitely many choices illustrate that Cantor in 1899 treated absolute infinity more like sethood than he had in 1883.