The Two Versions of the Proof

The posted "A Purely inductive proof of Borel determinacy" is, except for a difference in page numbers, identical with the paper appearing in the out-of-print AMS *Proceedings of Symposia in Pure Mathematics*, Volume 12, 1985. The other posted file contains the proof of Borel determinacy in my unpublished book manuscript on determinacy theorems. I'll call this "the book proof" below.

The only comments I'll make about the 1985 proof concern the "curiosity" at the end. The supposed appeal to Uniformization is bogus. One gets only a multi-valued ϕ component of the covering. See Exercise 2.1.5 in the book proof file. The problem about unraveling $\boldsymbol{\Pi^1_1}$ sets has been solved positively by Itay Neeman. See Itay Neeman, Unraveling Π^1_1 sets. Ann. of Pure and Applied Logic, vol. 106 (2000), pp. 151205, and Itay Neeman, Unraveling Π^1_1 sets, revisited. Israel J. of Math., vol. 152 (2006), pp. 181203.

The most noticeable difference between the two proofs is that the 1985 proof uses trees without terminal nodes and the book proof allows and uses terminal nodes. But the most important difference between the two is that the book proof incorporates an idea of Tonny Hurkens from his 1993 dissertation, *Borel determinacy without the axiom of choice*. There Hurkens uses multi-valued strategies to avoid the axiom of choice, but to my mind the dissertations main contribution is something else. Hurkens has a way of presenting the Borel determinacy proof's central construction (unraveling closed sets) in a way that makes the motivation for the construction completely clear. Hurkens' idea works best for trees with terminal nodes, and for me that tips the balance in favor of terminal nodes.

In the book proof, every terminal node is *taboo* one player or the other. If a terminal node is reached, then the player for whom it is taboo loses.

Hurkens' idea is used in the proof of the central Lemma 2.1.6 in the book proof. The remark at the middle of the page numbered 65 explains the idea.

Here is another, slightly less formal, explanation of the idea for the case $k = 0$ of the lemma. Suppose that players I and II are playing a game in a tree T with some set B of plays as I's winning set. Let A be a closed set. Let Z be the set of non-terminal positions that minimally avoid A: the set of positions p that are such that no play extending p belongs to A but this is false of every proper initial segment of p. Play in $\tilde{T}$ begins by I's choosing a subset X of Z. I is claiming that every $p \in X$ is a winning position for I in the B game and conceding that every position in $Z \setminus X$ is a win for II. Player

II can challenge I's claim by choosing a $p \in Z$. When this happens, play in $\tilde{T}$ continues with play in T starting at p. If II does not challenge I's claim, then play in $\tilde{T}$ continues with play in T starting at the initial position—except that positions in X are taboo for II and the other positions in Z taboo for I.

In $\tilde{T}$, the players thus first negotiate about who will be declared the winner if and when a position incompatible with A is first reached. If the negotiation leads to agreement, then there is no need to continue play after such a position is reached. If the negotiation is unsuccessful, then play starts at the disputed position chosen by II.

Note: I plan to post—gradually over the coming months—a large portion of the book manuscript.