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A Purely Inductive Proof of Borel Determinacy

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In [2] we proved that all infinite Borel games of perfect information are determined. The proof had two parts: (1) a basic construction showing that Σ_k determinacy implies Σ_{k+1} determinacy; (2) for each Borel set of Borel rank α, an α-fold iteration of the basic construction, reducing the corresponding Borel game to an open game. In the basic construction there occurred an auxiliary game involving nested sequences of trees. This was fairly complex and necessitated a priority argument. Step (2) produced more serious difficulties for the reader, since the argument was not purely inductive but rather involved directly considering the α-fold iteration of the basic construction.

In this paper we present a new proof in which we have made two important changes:

(a) The basic construction handles only a single closed set (instead of infinitely many closed sets). Thus our auxiliary game has only two auxiliary moves.

(b) We state a property of sets which implies determinacy and prove, by transfinite induction on Borel rank, that all Borel sets have the property.

Our "new" proof is really only the old proof reorganized, but we expect that the reader will find it much simpler.

By a *tree* we mean a set T of finite sequences such that

(i) $(\sigma \in T \,\&\, \tau \subseteq \sigma\ [\sigma \text{ extends } \tau]) \Rightarrow \tau \in T$;

(ii) $\sigma \in T \Rightarrow \exists\tau(\sigma \subsetneqq \tau \,\&\, \tau \in T)$.

(ii) says that T has no terminal nodes. If T is a tree, $[T]$ is the set of all infinite sequences x such that $\forall n(x \restriction n \in T)$, where $x \restriction n = \langle x(0), \ldots, x(n-1)\rangle$. A *game on* T is played as follows:

$$\begin{array}{lccccc} \text{I} & a_0 & & a_2 & & \cdots \\ \text{II} & & a_1 & & a_3 & \cdots \end{array}$$

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It is required that $\langle a_0,\dots,a_n\rangle \in T$ for each n. If $A \subseteq [T]$, then $G(A,T)$ is the game on T with the following winning condition: I wins a play $\langle a_0, a_1,\dots\rangle$ if and only if $\langle a_0, a_1,\dots\rangle \in A$. Otherwise II wins. The notions of *strategies* and *winning strategies* for I and II are defined in the obvious way. $G(A,T)$ is *determined* if either I or II has a winning strategy. Let $S(T)$ be the set of all strategies for either player for games on T.

We give $[T]$ a topology by letting the basic open sets be those of the form $\{x: \sigma \subseteq x\}$ for $\sigma \in T$. For α a countable ordinal > 0, we define the classes Σ_α and Π_α as follows: $\Sigma_1 =$ the class of all open sets. $\Pi_\alpha = \{A: A$ is the complement of a set in $\Sigma_\alpha\}$. For $\alpha > 1$, $\Sigma_\alpha = \{A: A$ is a countable union of sets in $\bigcup_{\beta<\alpha}\Pi_\beta\}$. A is *Borel* if $A \in \Sigma_\alpha$ for some $\alpha < \omega_1$.

A *covering* of a tree T is a triple $(\tilde{T}, \pi, \varphi)$ where

(1) $\tilde{T}$ is a tree;

(2) $\pi: [\tilde{T}] \to [T]$;

(3) $\varphi: S(\tilde{T}) \to S(T)$ and each $\varphi(\tilde{s})$ is a strategy for the same player as $\tilde{s}$;

(4) if x is a play consistent with $\varphi(\tilde{s})$, there is a play $\tilde{x}$ consistent with $\tilde{s}$ such that $\pi(\tilde{x}) = x$.

A covering $(\tilde{T}, \pi, \varphi)$ of T *unravels* a set $A \subseteq [T]$ if $\pi^{-1}(A)$ is clopen (closed and open).

We recall that Gale and Stewart [1] proved that $G(A,T)$ is determined if A is an open or closed subset of $[T]$.

LEMMA 1. *Let $(\tilde{T}, \pi, \varphi)$ be a covering which unravels $A \subseteq [T]$. $G(A,T)$ is determined.*

PROOF. $\pi^{-1}(A)$ is clopen, so $G(\pi^{-1}(A), \tilde{T})$ is determined. Let $\tilde{s}$ be a winning strategy, say for I. We show that $\varphi(\tilde{s})$ is a winning strategy for I for $G(A,T)$. Let x be a play consistent with $\varphi(\tilde{s})$. Let $\tilde{x}$ be as given by (4). Since $\tilde{x}$ is consistent with $\tilde{s}$, $\tilde{x} \in \pi^{-1}(A)$. Thus $x = \pi(\tilde{x}) \in A$.

LEMMA 2. *Let (T_1, π_1, φ_1) be a covering of T_0 and let (T_2, π_2, φ_2) be a covering of T_1. $(T_2, \pi_1 \circ \pi_2, \varphi_1 \circ \varphi_2)$ is a covering of T_0.*

Note that if $(\tilde{T}, \pi, \varphi)$ is a covering of T, $A \subseteq [T]$, $A \in \Sigma_\alpha$, and π is continuous, then $\pi^{-1}(A) \in \Sigma_\alpha$. In particular, if we compose coverings as in Lemma 2 and π_2 is continuous, then if (T_1, π_1, φ_1) unravels A it follows that $(T_2, \pi_1 \circ \pi_2, \varphi_1 \circ \varphi_2)$ unravels A.

We want to prove by induction on α (simultaneously for all T) that every set in Σ_α can be unraveled by a covering. Continuity helps, but to carry out an induction we need a stronger condition:

A covering $(\tilde{T}, \pi, \varphi)$ of T is a *k-covering* if

(a) $(\pi(\tilde{x})) \upharpoonright n$ depends only on $x \upharpoonright n$;

(b) $\varphi(\tilde{s})$ restricted to positions of length $\leqslant n$ depends only on $\tilde{s}$ restricted to positions of length $\leqslant n$.

(c) By (a) we may think of π as π: $\tilde{T} \to T$. We demand that $\pi \restriction \tilde{T}^k$ be one-one and onto T^k, where $T^k = \{\sigma \in T\text{: length}(\sigma) = k\}$.

LEMMA 3. *Let $A \subseteq [T]$ be closed and let $k \in \omega$. There is a k-covering of T which unravels A.*

PROOF. We describe $\tilde{T}$ implicitly by describing how games on $\tilde{T}$ are played. By increasing k if necessary, we may assume k is even.

$$\begin{array}{lccccccccccc} \mathrm{I} & a_0 & & a_2 & \cdots & a_{k-2} & & (a_k, T_\mathrm{I}) & & a_{k+2} & & \cdots \\ \mathrm{II} & & a_1 & & \cdots & & a_{k-1} & & (T_\mathrm{II}, a_{k+1}) & & a_{k+3} & \cdots \end{array}$$

All $\langle a_0,\ldots,a_j\rangle$ must belong to T. T_I must be a I-*imposed subtree* of T: i.e., if $\langle b_0,\ldots,b_j\rangle \in T_\mathrm{I}$ and $\langle b_0,\ldots,b_j, b_{j+1}\rangle \in T$ and j is even, then $\langle b_0,\ldots,b_j, b_{j+1}\rangle \in T_\mathrm{I}$. (I-imposed subtrees are now often called *quasi-strategies* for I.) Furthermore we require that $\sigma \subseteq \langle a_0,\ldots,a_k\rangle$ or $\langle a_0,\ldots,a_k\rangle \subseteq \sigma$ for each $\sigma \in T_\mathrm{I}$. There are two options for II:

First Option. T_II can be II-imposed subtree of T_I such that $[T_\mathrm{II}] \subseteq A$. (II-*imposed* is defined as is I-*imposed*, with "odd" replacing "even".)

Second Option. T_II can be $\{\sigma \in T\text{: } \sigma \subseteq \tau \text{ or } \tau \subseteq \sigma\}$ for some $\tau \in T_\mathrm{I}$ such that $\langle a_0,\ldots,a_k\rangle \subseteq \tau$ and $\{x\text{: } x \supseteq \tau\} \cap A = \varnothing$.

For $j > k$, $\langle a_0,\ldots,a_j\rangle$ must belong to T_II.

Note that each player has a legal move at every position, so we have indeed described a tree $\tilde{T}$ in our sense of "tree". The function π is the obvious one. Note that (a) and (c) in the definition of a k-covering are satisfied.

First let $\tilde{s} \in S(\tilde{T})$ be a strategy for I. Let $\varphi(\tilde{s})$ agree with $\tilde{s}$ on positions of length $< k$. Let $\langle a_0,\ldots,a_{k-1}\rangle$ be a position consistent with $\varphi(\tilde{s})$, and so with $\tilde{s}$. Let (a_k, T_I) be the move given by $\tilde{s}$. Let $\varphi(\tilde{s})$ play a_k at $\langle a_0,\ldots,a_{k-1}\rangle$.

Consider the game $G([T_\mathrm{I}] - A, T_\mathrm{I})$. $[T_\mathrm{I}] - A$ is open, so this game is determined. If II has a winning strategy, let T_II be the II-imposed subtree of T_I consisting of positions in T_I which are not lost for II in $G([T_\mathrm{I}] - \mathrm{A}, \mathrm{T}_\mathrm{I})$. If II plays a_{k+1} at $\langle a_0,\ldots,a_k\rangle$, $\varphi(\tilde{s})$ assumes that the First Option is taken at $\langle a_0,\ldots,a_{k-1},(a_k, T_\mathrm{I})\rangle$ and (T_II, a_{k+1}) is played and follows $\tilde{s}$ until (if ever) a position $\sigma \notin T_\mathrm{II}$ is reached. When this happens, or immediately if T_II does not exist or $\langle a_0,\ldots,a_{k+1}\rangle \notin T_\mathrm{II}$, $\varphi(\tilde{s})$ proceeds as follows: First a winning strategy for $G([T_\mathrm{I}] - A, T_\mathrm{I})$ is played, reaching—since A is closed—a position $\tau \in T_\mathrm{I}$ such that $A \cap \{x\text{: } x \supseteq \tau\} = \varnothing$. Now $\varphi(\tilde{s})$ assumes that II took the Second Option at $\langle a_0,\ldots,a_{k-1},(a_k, T_\mathrm{I})\rangle$ and played $T_\mathrm{II} = \{\sigma \in \tau\text{: } \sigma \subseteq \tau \text{ or } \tau \subseteq \sigma\}$. $\varphi(\tilde{s})$ proceeds according to $\tilde{s}$.

Now let $\tilde{s} \in S(\tilde{T})$ be a strategy for II. At positions of length $\leqslant k$, $\varphi(\tilde{s})$ follows $\tilde{s}$. Let $\langle a_0,\ldots,a_k\rangle$ be consistent with $\varphi(\tilde{s})$, so that $\langle a_0,\ldots,a_{k-1}\rangle$ is consistent with $\tilde{s}$.

Consider the game $G(B,T)$ where

$$B = \{x: \neg\exists\tau[\tau \subseteq x \;\&\; \exists T'_{\mathrm{I}}\,(\text{if I plays } (a_k, T'_{\mathrm{I}}) \text{ at } \langle a_0,\ldots,a_{k-1}\rangle, \text{ then } \tilde{s} \text{ calls for II to take the Second Option and play } T_{\mathrm{II}} = \{\sigma \in T: \sigma \subseteq \tau \text{ or } \tau \subseteq \sigma\})]\}.$$

II then wins $G(B,T)$ if a position $\tau \supseteq \langle a_0,\ldots,a_k\rangle$ is reached such that $\{x: x \supseteq \tau\} \cap A = \varnothing$ and

$$\langle a_0,\ldots,a_{k-1},(a_k,T'_{\mathrm{I}}),(\{\sigma \in T: \sigma \subseteq \tau \text{ or } \tau \subseteq \sigma\}, a_{k+1})\rangle$$

is consistent with $\tilde{s}$ for some T'_{I} and some a_{k+1}. B is closed. Suppose that $\langle a_0,\ldots,a_k\rangle$ is a winning position for I in $G(B,T)$. Let $T_{\mathrm{I}} = \{\sigma \in T: \langle a_0,\ldots,a_k\rangle \supseteq \sigma$ or $(\langle a_0,\ldots,a_k\rangle \subseteq \sigma$ and σ is not lost for I in $G(B,T))\}$. Suppose I plays (a_k, T_{I}) at $\langle a_0,\ldots,a_{k-1}\rangle$. Clearly $\tilde{s}$ cannot call for II to take the Second Option, since the associated τ would be a loss for I in $G(B,T)$ but would belong to T_{I}. $\varphi(\tilde{s})$ thus proceeds by assuming that (a_k, T_{I}) is played and following $\tilde{s}$ (omitting of course actually to play T_{II}). If I ever departs from T_{I}, or immediately if $\langle a_0,\ldots,a_k\rangle$ is a winning position for II in $G(B,T)$, $\varphi(\tilde{s})$ proceeds to play a winning strategy for $G(B,T)$. Thus a position τ is reached such that, for some T'_{I} and a_{k+1}, if I plays (a_k, T'_I) at $\langle a_0,\ldots,a_{k-1}\rangle$ then $\tilde{s}$ calls for II to take the Second Option and play $(\{\sigma \in T: \sigma \subseteq \tau \text{ or } \tau \subseteq \sigma\}, a_{k+1})$. $\varphi(\tilde{s})$ then follows $\tilde{s}$, assuming that (a_k, T'_{I}) was played by I.

It is easy to check that φ satisfies (b) in the definition of a k-covering. Our construction of φ has in effect shown that π and φ satisfy (4) in the definition of a covering.

We must finally show that $\pi^{-1}(A)$ is clopen. $\tilde{x} \in \pi^{-1}(A) \Rightarrow$ II takes the First Option $\Leftrightarrow [T_{\mathrm{II}}] \subseteq A$.

LEMMA 4. *Let $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$ be $(k+i)$-coverings of T_i, for each $i \in \omega$. There is a tree $\tilde{T}$ and there are $\tilde{\pi}_i$, $\tilde{\varphi}_i$ for $i \in \omega$ such that for each $i \in \omega$: $(\tilde{T}, \tilde{\pi}_i, \tilde{\varphi}_i)$ is a $(k+i)$-covering of T_i and $\tilde{\pi}_i = \pi_{i+1} \circ \tilde{\pi}_{i+1}$ and $\tilde{\varphi}_i = \varphi_{i+1} \circ \tilde{\varphi}_{i+1}$.*

PROOF. We show in effect that the inverse limit of the system of coverings exists. Using (c) in the definition of a $(k+i)$-covering, we may assume—replacing the T_i's by isomorphic trees if necessary—that $(T_i)^{k+1} = (T_{i+1})^{k+i}$ and $\pi_{i+1} \restriction (T_{i+1})^{k+i} =$ identity for each i. Let $\sigma \in \tilde{T} \Leftrightarrow \sigma \in T_i$ for all large i ($\Leftrightarrow \sigma \in T_i$ for i minimal such that $k + i \geqslant \text{length}(\sigma)$). Let $j \in \omega$. Let $\tilde{\pi}_j(\sigma) = \pi_{j+1} \circ \cdots \circ \pi_{i-1} \circ \pi_i(\sigma)$, where $k + i \geqslant \text{length}(\sigma)$. (If $j = i$, we intend $\tilde{\pi}_j(\sigma) = \sigma$.) Condition (4) in the definition of a covering and condition (b) in the definition of a $(k+i)$-covering imply (since $\pi_{i+1} \restriction (T_{i+1})^{k+i} =$ identity), that $\varphi_{i+1}(s_{i+1})$ agrees with s_{i+1} on positions of length $\geqslant k + i$. Thus we can let $\tilde{\varphi}_j(\tilde{s}) = \varphi_{j+1} \circ \cdots \circ \varphi_i(\tilde{s})$ on positions of length $< k + i$. (Once again we intend $\tilde{\varphi}_j(\tilde{s}) = \tilde{s}$ on positions of length $< k + i$, when $j = i$. Note also that "$\varphi_i(\tilde{s})$" makes sense on positions of length $< k + i$, since $\tilde{T}^{k+i} = (T_i)^{k+i}$.)

We need only check (4). Let x_j be consistent with $\tilde{\varphi}_j(\tilde{s})$. Let $x_{j+1}, x_{j+2}, \ldots$ be successively given by (4) for the coverings $(T_{j+1}, \pi_{j+1}, \varphi_{j+1}), (T_{j+2}, \pi_{j+2}, \varphi_{j+2}), \ldots$. Since $\pi_{i+1} \upharpoonright (T_{i+1})^{k+i}$ is the identity, we may let $\sigma \subseteq \tilde{x} \Leftrightarrow (\sigma \subseteq x_i$ for all large $i)$. Since $\tilde{s}$ agrees with $\tilde{\varphi}_i(\tilde{s})$ on positions of fixed length, for all large i, $\tilde{x}$ is consistent with $\tilde{s}$. Also $\tilde{\pi}_i(\tilde{x} \upharpoonright n) = \tilde{x}_i \upharpoonright n$ for all large i, so $\tilde{\pi}_j(\tilde{x} \upharpoonright n) = \pi_{j+1} \circ \cdots \circ \pi_i(x_i \upharpoonright n) = x_j \upharpoonright n$.

THEOREM. *If A is a Borel subset of $[T]$ and $k \in \omega$, there is a k-covering of T which unravels A.*

PROOF. By Lemma 3, the theorem holds for all $A \in \Pi_1$, for all T. Obviously any covering which unravels A unravels the complement of A. Assume that $\alpha < \omega_1$ and that, for all T, the theorem holds for each set in Σ_{β_i} for $\beta < \alpha$. Let $A \in \Sigma_\alpha$. Then $A = \bigcup_{i \in \omega} A_i$ with each $A_i \in \Pi_\beta$, $\beta_i < \alpha$. Let (T_1, π_1, φ_1) be a k-covering of $T_0 = T$ which unravels A_0. Let (T_2, π_2, φ_2) be a $(k+1)$-covering of T_1 which unravels $\pi_1^{-1}(A_1)$. In general, let $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$ be a $(k+i)$-covering of T_i which unravels $\pi_i^{-1} \circ \pi_{i-1}^{-1} \circ \cdots \circ \pi_1^{-1}(A_i)$. Let $\tilde{T}$ and the $\tilde{\pi}_i$, $\tilde{\varphi}_i$ be given by Lemma 4. $(\tilde{T}, \tilde{\pi}_0, \tilde{\varphi}_0)$ unravels each of $A_0, A_1, \ldots$. Since $\tilde{\pi}_0^{-1}(A) = \bigcup_{i \in \omega} \tilde{\pi}_0^{-1}(A_i)$, $\tilde{\pi}_0^{-1}(A)$ is open. Let (T^*, π^*, φ^*) be a k-covering of $\tilde{T}$ which unravels $\tilde{\pi}_0^{-1}(A)$. $(T^*, \tilde{\pi}_0 \circ \pi^*, \tilde{\varphi}_0 \circ \varphi^*)$ is a k-covering of T which unravels A.

COROLLARY. *If $A \subseteq [T]$ is Borel, $G(A, T)$ is determined.*

REMARKS. (1) The priority construction of [2] has disappeared. It seems possible that considering the infinitely many closed sets at once, as in [2], might be necessary for a sharp calculation of complexity of strategies. Superficially, it might also appear that this could be necessary for getting a sharp bound on the size of the covering trees, but a little thought shows this is not the case.

(2) In [2] we said that our proof did not need the axiom of choice in the case T is countable. Several people have pointed out that countable choice is necessary to get a Borel code for each Borel subset of $[T]$. That is also true here, though our definition of "Borel" is more restrictive here.

(3) Moschovakis (who incidentally kept suggesting that a purely inductive proof of Borel determinacy should be possible) simplified our original proof (see [3]) by using trees with terminal nodes, removing the necessity for $G([T_1] - A, T_1)$ and $G(B, T)$ used above. We do not know whether this idea can be mixed with out new proof.

(4) Does the unraveling property hold for any class beyond the Borel sets? We first note the following curiosity.

Curiosity. Assume the Axiom of Determinacy plus Uniformization for sets of pairs of elements of ω^{ω}. There is a single 0-covering $(\tilde{T}, \pi, \varphi)$ which unravels every $A \subseteq \omega^\omega$.

PROOF. We describe games on $\tilde{T}$. I begins by playing a strategy s for I a game on T. II next plays an element x of ω^ω consistent with s. The players then amuse themselves playing, say, natural numbers to satisfy our definition of a tree. Let $\pi(s, x, \ldots) = x$. (a) is fulfilled.

If $\tilde{s}$ is a strategy for I, let $\varphi(\tilde{s})$ be the first move given by $\tilde{s}$.

Suppose $\tilde{s}$ is a strategy for II. Consider the following game $G^{\tilde{s}}$ on T. II wins a play x of $G^{\tilde{s}}$ if and only if there is an s such that, if I plays s then $\tilde{s}$ calls for II to play x. If s is a strategy for I for $G^{\tilde{s}}$, then II can defeat s by playing the x given by $\tilde{s}$. Thus, by AD, II has a winning strategy for $G^{\tilde{s}}$. By uniformization, we can pick for each $\tilde{s}$ a winning strategy $\varphi(\tilde{s})$ for II for $G^{\tilde{s}}$.

Uniformization is needed only because we required that φ is single-valued.

We know of no proof, from any large cardinal assumption consistent with choice, that every Π_1^1 set can be unraveled by a covering. If we could show that, for any countable family $\mathscr{A}$ of Π_1^1 sets, there is a covering which unravels every number of $\mathscr{A}$, then we could prove determinacy for the σ-algebra generated by the Π_1^1 sets. Results of J. Steel show that one needs at least (approximately) a measurable cardinal κ of order κ^{++}.

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