Chapter 2

General Borel Games

In this chapter we introduce the technical concept of a *covering* of a game tree, and we use this concept to prove the determinacy of all Borel games and—in uncountable trees—the determinacy of all games in a larger class.

Borel determinacy is proved in §2.1. In countable trees, the Borel sets are the same as the the $\boldsymbol{\Delta}^1_1$ sets (to be defined in §2.2). In general, however, $\boldsymbol{\Delta}^1_1$ is a larger class, the class of what we shall call *quasi-Borel* sets. In §2.2 we prove this and also prove that all quasi-Borel games are determined. §2.1 and §2.2 depend only on §1.1 and §1.2 (and not on the rest of Chapter 1).

Readers interested only in main results may confine themselves to §2.1 (though §2.2 should present no extra difficulties).

In §2.3 we work again in the weak set theory of §§1.3–1.4. We use the proofs of §2.1 and the results of §1.4 to get $\boldsymbol{\Sigma}^0_\alpha$ determinacy with the minimal possible amount of Power Set and Replacement (allowed by refinements—given in the exercises—of results of Harvey Friedman).

In §2.4 we consider a class of infinite games of imperfect information called *Blackwell games* after David Blackwell, who initiated their study. We introduce the basic theory of imperfect information games, and then we prove the determinacy of Borel Blackwell games by showing that it follows from ordinary Borel deteminacy. This is done by proving a general theorem reducing the the determinacy of Blackwell games of any reasonably closed class to the determinacy of ordinary games of that class. Thus all our determinacy results in later chapters will imply corresponding determinacy results for Blackwell games.

2.1 Borel Determinacy

Almost all the determinacy results in the remainder of this book will be proved by the technique of *auxiliary games*: To prove $G(A;T)$ determined, we will associate with $G(A;T)$ another game $G(A^*;T^*)$. This auxiliary game we will know to be determined. Moreover the two games will be so related that the determinacy of $G(A;T)$ will follow from that of $G(A^*;T^*)$. In a sense we have already seen this technique. To prove Theorem 1.3.1, for example, we made use of the closed games $G(C;T)$ occurring in the proof of Lemma 1.3.2. Such games were used also in proving Theorems 1.3.3, 1.4.9, and 1.4.17. The auxiliary game technique as we will use it later differs from these examples in two important ways: (1) The determinacy of the given game $G(A;T)$ will be reduced to the determinacy of a *single* game $G(A^*;T^*)$. (2) T^* will be *larger* than T, whereas the auxiliary game trees in the earlier examples were all subtrees of the given T. Indeed the results of Friedman [1971] show that some use of existence principles for sets larger than T is necessary to prove the determinacy of Borel games in T. (See Exercises 1.4.1–1.4.5 and Exercises 2.3.2–2.3.10.)

In using the auxiliary game technique, one can think of moves in the auxiliary tree as being moves in T together with extra components. In later chapters the extra components will be elements of measure spaces. Winning strategies for the main game will be derived from winning strategies for the auxiliary game by *integration*. In this chapter the extra components of moves in the auxiliary tree will be, in the basic case, (a) subtrees of T and (b) decisions about whether the element of $\lceil T \rceil$ being produced will belong to certain subsets of $\lceil T \rceil$. Exercise 2.1.2 illustrates this technique, reproving Theorem 1.3.1 with the help of an auxiliary game. However, components of the form (b) do not appear in this example. In more general cases, auxiliary trees will come from iterations of the process that gives the basic case.

Remark. The first proof of $\mathbf{\Sigma}^0_4$ determinacy, that in [Paris, 1972], used an auxiliary game technique modeled on the one we shall present in Chapter 4. James Baumgartner had earlier found, adapting the method of Chapter 4, a new proof of $\mathbf{\Sigma}^0_3$ determinacy,

Our proof of Borel determinacy will be like that in [Martin, 1985] in that we shall prove inductively that all Borel sets have a certain property, the property of being reducible in a certain way to a clopen set (of plays in a different tree). The determinacy of a set with this property will follow easily from the determinacy of a set related to the clopen set. In the details there we will be several differences between the proof in [Martin, 1985] and the proof as we shall present it below. Our presentation will be similar to that in [Hurkens, 1993]. This similarity is partly coincidental and partly by choice. When the first draft of this section was written around 1990, it was influenced by an idea of Moschovakis (found in the proof of Theorem 6F.1 of [Moschovakis, 1980]). Moschovakis' idea eliminates from the original proof of Borel determinacy (the proof in [Martin, 1975]) part of its use of quasistrategies and subsidiary games. In writing the present chapter, the author wished to go further: (a) to combine Moschovakis' idea with the purely inductive proof in [Martin, 1985] and (b) to eliminate from the proof every vestige of the use of quasistrategies. To accomplish these aims, the author introduced *game trees with taboos*, game trees in which each terminal position is automatically lost for one player or the other—is *taboo* for one player or the other—independently of the payoff set. (In the first draft of the section, non-taboo terminal positions were also permitted.) Hurkens, who explicitly had aim (a), produced a proof that has essentially all the ingredients in the author's draft (which Hurkens had not seen). Hurkens' proof introduces one additional idea, an idea that both simplifies and helps motivate the main construction of the proof. Although the author had in his possession a copy of [Hurkens, 1993], he learned about this idea only indirectly, in conversation with Marco Vervoort. Afterwards he actually consulted [Hurkens, 1993] and discovered the similiarities between Hurkens' proof and his own. Hurkens' additional idea seemed too valuable to omit, so the author has revised his draft to incorporate that idea (and to make some other modifications). In the course of giving the proof, we shall explain Hurkens' idea and we shall comment on relations between the two proofs.

A *game tree with taboos* is a triple $\mathbf{T} = \langle T, \mathcal{T}_{\mathrm{I}}, \mathcal{T}_{\mathrm{II}} \rangle$, where

(1) T is a game tree;

(2) $\mathcal{T}_{\mathrm{I}}$ and $\mathcal{T}_{\mathrm{II}}$ are disjoint sets of terminal positions in T;

(3) every terminal position in T belongs to $\mathcal{T}_{\mathrm{I}}$ or to $\mathcal{T}_{\mathrm{II}}$.

Recall that terminal positions in T are members of T that are also finite plays in T. Infinite plays are not positions, and so are not terminal positions.

Convention. We always use boldface letters, perhaps with other markings, for game trees with taboos. For the underlying game trees, we use the corresponding italic lightface letters, with the same markings; for the other two components, we use the corresponding calligraphic letters, with the same markings and with subscripts "I" and "II." For example, $\tilde{\mathbf{T}}^i$ will be $\langle \tilde{T}^i, \tilde{\mathcal{T}}^i_{\mathrm{I}}, \tilde{\mathcal{T}}^i_{\mathrm{II}} \rangle$.

If $\mathbf{T}$ is a game tree with taboos, then *positions*, *moves*, *plays*, *strategies*, etc. in $\mathbf{T}$ are just positions, moves, plays, strategies, etc. in T. If $p \in T$, then $\mathbf{T}_p$ is the game tree with taboos $\langle T_p, \mathcal{T}_{\mathrm{I}} \cap T_p, \mathcal{T}_{\mathrm{II}} \cap T_p \rangle$.

For any game tree T, we let $[T]$ be the set of all infinite plays in T. Note that $[T]$ is a closed subset of $\lceil T \rceil$.

Let $\mathbf{T}$ be a game tree with taboos. Plays belonging to $\mathcal{T}_{\mathrm{I}}$ are *taboo* for I in $\mathbf{T}$, and plays belonging to $\mathcal{T}_{\mathrm{II}}$ are *taboo* for II in $\mathbf{T}$. Hence $[T]$ is the set of all plays that are not taboo for either player in $\mathbf{T}$, i.e., that are not *taboo* in $\mathbf{T}$. For $A \subseteq [T]$, we define the game $G(A; \mathbf{T})$ as follows: A finite play of $G(A; \mathbf{T})$ is lost by the player for whom it is taboo. A play $x \in [T]$ is won by I if and only if $x \in A$. Thus $G(A; \mathbf{T})$ is the same game as $G((A \cup \mathcal{T}_{\mathrm{II}}) \setminus \mathcal{T}_{\mathrm{I}}; T)$. The notions, for $G(A; \mathbf{T})$, of *winning strategy* and being *determined* are the same as those for $G((A \cup \mathcal{T}_{\mathrm{II}}) \setminus \mathcal{T}_{\mathrm{I}}; T)$.

Remark. [Hurkens, 1993] does not have game trees with taboos, but it has a device that does the same work. It has a move function of the sort we discussed on page 2. The move function is defined even in terminal positions, and whichever player has the impossible task of moving in a terminal position loses the that play of the game.

We could have omitted clause (3) from the definition of game trees with taboos, i.e., we could have permitted the existence of finite non-taboo plays. Indeed, this would have been the more natural definition, since we permitted finite plays throughout Chapter 1. The reason why we include clause (3) is that without it many of our definitions and proofs would have been more complicated, since we would have had to worry about whether any given finite play was taboo or not.

Remark. While Hurkens' use of a move function does all the work done by game trees with taboos, it would not in a straightforward way do the work of game trees with taboos in the more liberal sense just discussed.

It is important to make sure that proving determinacy results only for game trees with taboos in our restricted sense involves no loss of generality. First note that, for ordinary game trees (without taboos), nothing is lost by considering only trees without finite plays. To see this, let T be a game tree.

Consider the tree

$$T' = T \cup \{p^\frown\langle \underbrace{0,\ldots,0}_{n}\rangle \mid n \in \omega \wedge p \text{ is terminal in } T\}.$$

The tree T' has no terminal postions. The obvious bijection $f : \lceil T' \rceil \to \lceil T \rceil$ is a homeomorphism such that, for each $A \subseteq \lceil T \rceil$, $G(A;T)$ is determined if and only if $G(f^{-1}(A);T')$ is determined. Similarly, let $\mathbf{T}$ be an game tree with taboos in the unrestricted sense (possibly not satifying clause (3)). Set

$$T' = T \cup \{p^\frown\langle \underbrace{0,\ldots,0}_{n}\rangle \mid n \in \omega \wedge p \text{ is terminal in } T \text{ and not taboo in } \mathbf{T}\}.$$

Let $\mathbf{T}' = \langle T', \mathcal{T}_{\mathrm{I}}, \mathcal{T}_{\mathrm{II}}\rangle$. Then $\mathbf{T}'$ is a game tree with taboos. Furthermore, the obvious homeomorphism $f : \lceil T' \rceil \to \lceil T \rceil$ restricts to a homeomorphism (in the sense of the definition below, adapted to allow for game trees with taboos in the unrestricted sense) from $[T']$ to the set of all non-taboo plays in $\mathbf{T}$. Moreover, for any set A of non-taboo plays in $\mathbf{T}$, $G(A;\mathbf{T})$ is determined if and only if $G(f^{-1}(A);\mathbf{T}')$ is determined (under the obvious definition).

We give $[T]$ the relative topology: A subset A of $[T]$ is open just in case there is an open $B \subseteq \lceil T \rceil$ such that $A = B \cap [T]$. We shall construe our topological definitions as making sense even in the case $[T]$ is empty, so that the unique subset $\emptyset$ of $[T]$ is open, Borel, etc. The following easy lemma will allow us usually not to worry about the distinction between the Borel hierarchy on $[T]$ and that on $\lceil T \rceil$.

Lemma 2.1.1. *Let $\mathbf{T}$ be a game tree with taboos. For all ordinals $\alpha \geq 1$ and all subsets A of $[T]$, A belongs to $\mathbf{\Pi}^0_\alpha$ as a subset of $[T]$ if and only if A belongs to $\mathbf{\Pi}^0_\alpha$ as a subset of $\lceil T \rceil$. For all ordinals $\alpha > 1$ and all subsets A of $[T]$, A belongs to $\mathbf{\Sigma}^0_\alpha$ as a subset of $[T]$ if and only if A belongs to $\mathbf{\Sigma}^0_\alpha$ as a subset of $\lceil T \rceil$.*

Proof. We prove the lemma by induction on $\alpha \geq 1$.

By definition of the relative topology, every subset of $[T]$ closed as a subset of $\lceil T \rceil$ is closed as a subset of $[T]$. Since $[T]$ is closed as a subset of $\lceil T \rceil$, every subset of $[T]$ closed as a subset of $[T]$ is closed as a subset of $\lceil T \rceil$.

Let $\alpha > 1$ and assume that the lemma holds for all $\beta < \alpha$. The fact that any subset of $[T]$ belongs to $\mathbf{\Sigma}^0_\alpha$ as subset of $[T]$ if and only if it belongs to $\mathbf{\Sigma}^0_\alpha$ as a subset of $\lceil T \rceil$ follows directly from the definition of $\mathbf{\Sigma}^0_\alpha$ and our induction hypothesis. Suppose that $A \in \mathbf{\Pi}^0_\alpha$ as a subset of $[T]$. Thus $[T] \setminus A \in \mathbf{\Sigma}^0_\alpha$ as a subset of $[T]$ and so also as a subset of $\lceil T \rceil$. $\lceil T \rceil \setminus [T] \in \mathbf{\Sigma}^0_1 \subseteq$ (by Lemma 1.1.1) $\mathbf{\Sigma}^0_\alpha$. By Lemma 1.1.1 again, $\lceil T \rceil \setminus A = ([T] \setminus A) \cup (\lceil T \rceil \setminus [T])$ belongs to $\mathbf{\Sigma}^0_\alpha$. By the definition of $\mathbf{\Pi}^0_\alpha$, $A \in \mathbf{\Pi}^0_\alpha$ as a subset of $\lceil T \rceil$. Suppose now that $A \subseteq [T]$ and that $A \in \mathbf{\Pi}^0_\alpha$ as a subset of $\lceil T \rceil$. Thus $\lceil T \rceil \setminus A \in \mathbf{\Sigma}^0_\alpha$. By Lemma 1.1.1, the closed set $[T]$ belongs to $\mathbf{\Sigma}^0_\alpha$ as a subset of $\lceil T \rceil$. By Lemma 1.1.1 again, $[T] \setminus A = (\lceil T \rceil \setminus A) \cap [T]$ belongs to $\mathbf{\Sigma}^0_\alpha$ as a subset of $\lceil T \rceil$. Thus $[T] \setminus A \in \mathbf{\Sigma}^0_\alpha$ as a subset of $[T]$, and so $A \in \mathbf{\Pi}^0_\alpha$ as a subset of $[T]$. □

There is another way to characterize the topology on $[T]$. Note that $[T] = \lceil \bar{T} \rceil$, where $\bar{T} = \{p \in T \mid (\exists x \supseteq p)\, x \in [T]\}$. If $[T]$ is nonempty, then $\bar{T}$ is a game tree, and our topology for $[T]$ is the same as the topology it has as $\lceil \bar{T} \rceil$. Thus Lemma 1.1.1 holds for the Borel hierarchy on $[T]$. (One can also see this using Lemma 2.1.1.)

Let us now show that determinacy for games in game trees with taboos is level by level equivalent to determinacy for games in ordinary game trees. By the remark above, determinacy in ordinary game trees is equivalent level by level to determinacy in ordinary game trees that have no terminal positions, so we need only consider the latter. In one direction, note that any game tree without terminal nodes can be considered a game tree with taboos by setting $\mathcal{T}_{\mathrm{I}} = \mathcal{T}_{\mathrm{II}} = \emptyset$. In the other direction, let $\mathbf{T}$ be a game tree with taboos. If $G(\lceil T \rceil \setminus \mathcal{T}_{\mathrm{I}}; T)$ is a win for II, then all games in $\mathbf{T}$ are wins for II. Assume otherwise and let R be I's non-losing quasistrategy for $G(\lceil T \rceil \setminus \mathcal{T}_{\mathrm{I}}; T)$. If $G(\mathcal{T}_{\mathrm{II}}; R)$ is a win for I, then all games in $\mathbf{T}$ are wins for I. Assume otherwise and let S be II's non-losing quasistrategy for $G(\mathcal{T}_{\mathrm{II}}; R)$. The game subtree S of T satisfies $\lceil S \rceil \subseteq [T]$. Moreoever, for any $A \subseteq [T]$, the games $G(A \cap \lceil S \rceil; S)$ and $G(A; \mathbf{T})$ are completely equivalent; in particular, the latter is determined if the former is. Finally, we have that $A \cap \lceil S \rceil$ is as simple topologically as A. One consequence of this is that our previous determinacy results hold also for game trees with taboos:

Lemma 2.1.2. *Theorems 1.2.4, 1.3.1, 1.3.3, 1.4.9, and 1.4.17 and Corollaries 1.2.3, 1.4.14, and 1.4.18, hold for games in game trees with taboos.*

Proof. The argument given in the paragraph preceding the statement of the lemma goes through in ZC$^-$ + Σ_1 Replacement. Thus the Theorems listed in the statement of the lemma holds for games in trees with taboos. Corollary 1.2.3 follows from Theorem 1.2.4. To see that Corollaries 1.4.14

and 1.4.18 follow, it is suffices to show that Theorems 1.4.2 and 1.4.16 hold in each $[T]$. This in turn follows from the original Theorems 1.4.2 and 1.4.16 for $\lceil \bar{T} \rceil$, where $\bar{T}$ is as above. $\square$

Remark. Since games in $\mathbf{T}$ are equivalent to games in the S defined above, we could avoid dealing with game trees with taboos by replacing each $\mathbf{T}$ with the corresponding S. In a sense, that is what is done in [Martin, 1975] and [Martin, 1985]. Here, however, we are interested in avoiding the nuisance of quasistrategies, and so we put up with the nuisance of taboos.

If $\tilde{\mathbf{T}}$ and $\mathbf{T}$ are game trees with taboos, we write $\pi : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}$ to mean that

(i) $\pi : \tilde{T} \to T$;

(ii) $\tilde{p} \subseteq \tilde{q} \;\to\; \pi(\tilde{p}) \subseteq \pi(\tilde{q})$ for all $\tilde{p}$ and $\tilde{q}$ belonging to $\tilde{T}$;

(iii) $\ell\mathrm{h}(\pi(\tilde{p})) = \ell\mathrm{h}(\tilde{p})$ for all $\tilde{p} \in \tilde{T}$.

(iv) $\pi(\tilde{p}) \in \mathcal{T}_{\mathrm{I}} \;\to\; \tilde{p} \in \tilde{\mathcal{T}}_{\mathrm{I}}$ for all $\tilde{p} \in \tilde{T}$;

(v) $\pi(\tilde{p}) \in \mathcal{T}_{\mathrm{II}} \;\to\; \tilde{p} \in \tilde{\mathcal{T}}_{\mathrm{II}}$ for all $\tilde{p} \in \tilde{T}$;

Note that it is allowed that $\tilde{p}$ be terminal in $\tilde{T}$ (and so taboo in $\tilde{\mathbf{T}}$) even though $\pi(\tilde{p})$ is not terminal in T.

Let $\pi : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}$. If $\tilde{x}$ is a play in $\tilde{T}$, then clause (ii) implies that $\bigcup_{\tilde{p} \subseteq \tilde{x}} \pi(\tilde{p})$ is either a position or a play in T. If $\tilde{x}$ is finite, then $\bigcup_{\tilde{p} \subseteq \tilde{x}} \pi(\tilde{p}) = \pi(\tilde{x})$. Thus we can extend π to a function, which we also denote by "π," from $\tilde{T} \cup \lceil \tilde{T} \rceil$ to $T \cup \lceil T \rceil$. By clause (iii), $\ell\mathrm{h}(\pi(\tilde{x})) = \ell\mathrm{h}(\tilde{x})$ for all plays $\tilde{x}$, where we recall that $\ell\mathrm{h}(\tilde{x}) = \omega$ if x is infinite. If $\tilde{x}$ is an infinite play in $\tilde{T}$, then $\pi(\tilde{x})$ is an infinite play in T. Thus π induces a function

$$\boldsymbol{\pi} : [\tilde{T}] \to [T].$$

The function $\boldsymbol{\pi}$ is continuous and satisfies a "Lipschitz condition," i.e. $\boldsymbol{\pi}(\tilde{x}) \restriction n$ depends only on $\tilde{x} \restriction n$.

If $\tilde{\mathbf{T}}$ and $\mathbf{T}$ are game trees with taboos, we write $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}$ to mean that

(i) $\phi : \mathcal{S}(\tilde{T}) \to \mathcal{S}(T)$;

(ii) each $\phi(\tilde{\sigma})$ is a strategy for the same player as is $\tilde{\sigma}$;

(iii) for each $n \in \omega$, the restriction of $\phi(\tilde{\sigma})$ to positions of length $< n$ depends only on the restriction of $\tilde{\sigma}$ to positions of length $< n$.

If T is a game tree and $k \in \omega$, let

$${}_kT \;=\; \{p \in T \mid \ell\mathrm{h}(p) \leq k\}.$$

By clause (iii) of the definition, we can think of a ϕ such that $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}$ as acting on $\bigcup_{k\in\omega} \mathcal{S}({}_k\tilde{T})$ so that, for each k, $\phi \restriction \mathcal{S}({}_k\tilde{T}) : \mathcal{S}({}_k\tilde{T}) \to \mathcal{S}({}_kT)$.

We are now ready to give the main technical definition of this chapter. If $\mathbf{T}$ is a game tree with taboos, then a *covering* of $\mathbf{T}$ is a quadruple $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ such that

(a) $\tilde{\mathbf{T}}$ is a game tree with taboos;

(b) $\pi : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}$;

(c) $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}$;

(d) $\Psi : \{\langle \tilde{\sigma}, x\rangle \mid \tilde{\sigma} \in \mathcal{S}(\tilde{T}) \wedge x \in \lceil T\rceil \wedge x \text{ is consistent with } \phi(\tilde{\sigma})\} \to \lceil \tilde{T}\rceil$, and, for all $\langle \tilde{\sigma}, x\rangle \in \text{domain}\,(\Psi)$,

 (i) $\Psi(\tilde{\sigma}, x)$ is consistent with $\tilde{\sigma}$;

 (ii) $\pi(\Psi(\tilde{\sigma}, x)) \subseteq x$;

 (iii) either $\pi(\Psi(\tilde{\sigma}, x)) = x$ or $\Psi(\tilde{\sigma}, x)$ is taboo for the player for whom $\tilde{\sigma}$ is a strategy.

With regard to clause (d)(iii), note that $\pi(\Psi(\tilde{\sigma}, x)) = x$ implies $\ell\mathrm{h}(\Psi(\tilde{\sigma}, x)) = \ell\mathrm{h}(x)$; and this in turn implies that $\Psi(\tilde{\sigma}, x)$ and x are both finite or both infinite. Note also that if both are finite then, by clauses (iv) and (v) of the definition of $\pi : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}$, both are taboo for the same player.

Remarks:

(a) A variant definition, and one that has some advantages which we shall point out later, would replace the quadruple $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ by the triple $\langle \tilde{\mathbf{T}}, \pi, \phi, \rangle$ and replace clause (d) by

(d′) if $\tilde{\sigma} \in \mathcal{S}(\tilde{T})$ and x is consistent with $\tilde{\sigma}$, then there is an $\tilde{x} \in \lceil \tilde{T}\rceil$ such that

 (i) $\tilde{x}$ is consistent with $\tilde{\sigma}$;

 (ii) $\pi(\tilde{x}) \subseteq x$;

 (iii) either $\pi(\tilde{x}) = x$ or $\tilde{x}$ is taboo for the player for whom $\tilde{\sigma}$ is a strategy.

(b) Although the fact will not be directly used by us, the π of a covering is a surjection. Indeed, every play in T is in the range of the extended π. (Exercise 2.1.4). For an example and an almost-example of coverings, see Exercises 2.1.3 and 2.1.5.

We say that a covering $\langle\tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ *unravels* a subset A of $[T]$ if the preimage $\boldsymbol{\pi}^{-1}(A)$ is a clopen subset of $[\tilde{T}]$.

We prove at once the basic lemma connecting coverings and unraveling with determinacy:

Lemma 2.1.3. *Let* $\mathbf{T}$ *be a game tree with taboos. If there is a covering of* $\mathbf{T}$ *that unravels* $A \subseteq [T]$*, then* $G(A; \mathbf{T})$ *is determined.*

Proof. Let $\langle\tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ be a covering of $\mathbf{T}$ that unravels $A \subseteq [T]$. By Lemma 2.1.2 (as applied to Corollary 1.2.3), $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$ is determined. Let us call the player for whom $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$ is a win the *good player* and let us call the other player the *bad player.* Let $\tilde{\sigma}$ be a winning strategy for the good player for $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$. We show that $\phi(\tilde{\sigma})$ is a winning strategy for the good player for $G(A; \mathbf{T})$. Let x be a play in T consistent with $\phi(\tilde{\sigma})$. We must prove that x is a win for the good player in $G(A; \mathbf{T})$. We may assume that x is not taboo for the bad player.

It is enough to show that $\pi(\Psi(\tilde{\sigma}, x)) = x$ and that $\Psi(\tilde{\sigma}, x)$ is infinite. If this is true then, since $\boldsymbol{\pi} = \pi \restriction [\tilde{T}]$ and $\boldsymbol{\pi} : [\tilde{T}] \to [T]$,

$$\Psi(\tilde{\sigma}, x) \in \boldsymbol{\pi}^{-1}(A) \leftrightarrow \pi(\Psi(\tilde{\sigma}, x)) \in A \leftrightarrow x \in A.$$

Because $\Psi(\tilde{\sigma}, x)$ is a win for the good player in $G(\boldsymbol{\pi}^{-1}(A); \tilde{\mathbf{T}})$, it follows that x is a win for the good player in $G(A; \mathbf{T})$.

By clause (d)(i) in the definition of a covering, $\Psi(\tilde{\sigma}, x)$ is a play in $\tilde{T}$ that is consistent with $\tilde{\sigma}$. Since $\tilde{\sigma}$ is a winning strategy, $\Psi(\tilde{\sigma}, x)$ cannot be taboo for the good player. Thus clause (d)(iii) gives that $\pi(\Psi(\tilde{\sigma}, x)) = x$. By the observations after the definition of a covering, x and $\Psi(\tilde{\sigma}, x)$ are both finite or both taboo for the same player. They cannot both be taboo for the same player, for $\Psi(\tilde{\sigma}, x)$ is not taboo for the good player, and we are assuming that x is not taboo for the bad player. $\square$

Borel determinacy will be proved if we can show that every Borel set is unraveled by a covering. To do this, we need to do two things: (i) We must show that every open set can be unraveled. (ii) We must find some operations on coverings corresponding to the operations that generate the

Borel sets from the open sets. (i) is the heart of the proof. We begin with the more routine (ii).

Let $\mathbf{T}$ be a game tree with taboos and let $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ be a covering of $\mathbf{T}$. For $k \in \omega$, $\mathcal{C}$ is a *k-covering* of $\mathbf{T}$ if

(i) ${}_k\tilde{T} = {}_kT$, ${}_k\tilde{T} \cap \tilde{\mathcal{T}}_{\mathrm{I}} = {}_kT \cap \mathcal{T}_{\mathrm{I}}$, and ${}_k\tilde{T} \cap \tilde{\mathcal{T}}_{\mathrm{II}} = {}_kT \cap \mathcal{T}_{\mathrm{II}}$;

(ii) $\pi \restriction {}_k\tilde{T}$ is the identity;

(iii) $\phi \restriction \mathcal{S}({}_k\tilde{T})$ is the identity.

Suppose that $\mathcal{C}_1 = \langle \mathbf{T}_1, \pi_1, \phi_1, \Psi_1 \rangle$ is a covering of $\mathbf{T}_0$ and that $\mathcal{C}_2 = \langle \mathbf{T}_2, \pi_2, \phi_2, \Psi_2 \rangle$ is a covering of $\mathbf{T}_1$. We define the *composition* $\mathcal{C}_1 \circ \mathcal{C}_2$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ to be

$$\langle \mathbf{T}_2, \pi_1 \circ \pi_2, \phi_1 \circ \phi_2, \Psi \rangle,$$

where $\Psi(\sigma, x) = \Psi_2(\sigma, \Psi_1(\phi_2(\sigma), x))$. We omit the routine proof of the following lemma.

Lemma 2.1.4. *The composition of coverings is a covering. For natural numbers k_1 and k_2, the composition of a k_1-covering and a k_2-covering is a $\min\{k_1, k_2\}$-covering.*

The next lemma gives us a sufficient condition that the limit of a sequence of k-coverings exist and be a k-covering. It is for constructing such limits that the concept of k-covering was introduced.

Lemma 2.1.5. *Let $k \in \omega$, let $\mathbf{T}_i$, $i \in \omega$, be game trees with taboos, and let $\langle k_{j,i}, \pi_{j,i}, \phi_{j,i}, \Psi^{i,j} \mid i \leq j \in \omega \rangle$ be such that*

(1) if $i \leq j \in \omega$ then $\mathcal{C}_{j,i} = \langle \mathbf{T}_j, \pi_{j,i}, \phi_{j,i}, \Psi^{i,j} \rangle$ is a $k_{j,i}$-covering of $\mathbf{T}_i$;

(2) if $i_1 \leq i_2 \leq i_3 \in \omega$ then $\mathcal{C}_{i_3,i_1} = \mathcal{C}_{i_2,i_1} \circ \mathcal{C}_{i_3,i_2}$;

(3) $\inf_{i \leq j \in \omega} k_{j,i} \geq k$;

(4) $\underline{\lim}_{j \in \omega} \inf_{j' \geq j} k_{j',j} = \infty$; i.e., for all $n \in \omega$ there is an $i \in \omega$ such that $k_{j',j} \geq n$ for all $j' \geq j \geq i$.

Then there is a $\mathbf{T}_\infty$ with $|T_\infty| \leq \sum_{i \in \omega} |T_i|$ and there is a system

$$\langle \pi_{\infty,i}, \phi_{\infty,i}, \Psi^{i,\infty} \mid i \in \omega \rangle$$

such that each $\mathcal{C}_{\infty,i} = \langle \mathbf{T}_\infty, \pi_{\infty,i}, \phi_{\infty,i}, \Psi^{i,\infty} \rangle$ is a k-covering of $\mathbf{T}_i$ and such that, for $i \leq j \in \omega$, $\mathcal{C}_{\infty,i} = \mathcal{C}_{j,i} \circ \mathcal{C}_{\infty,j}$.

Proof. The idea is that, because of (4), what is in essence the inverse limit exists. For $n \in \omega$, let i_n be the least number i such that, for all $j' \geq j \geq i$, $k_{j',j} \geq n$. Thus ${}_nT_j$, ${}_nT_j \cap (\mathcal{T}_j)_{\mathrm{I}}$, and ${}_nT_j \cap (\mathcal{T}_j)_{\mathrm{II}}$, are independent of j for $j \geq i_n$. For any finite sequence p, let

$$\begin{aligned} p \in T_\infty &\leftrightarrow p \in T_{i_{\ell\mathrm{h}(p)}}; \\ p \in (\mathcal{T}_\infty)_{\mathrm{I}} &\leftrightarrow p \in (\mathcal{T}_{i_{\ell\mathrm{h}(p)}})_{\mathrm{I}}; \\ p \in (\mathcal{T}_\infty)_{\mathrm{II}} &\leftrightarrow p \in (\mathcal{T}_{i_{\ell\mathrm{h}(p)}})_{\mathrm{II}}. \end{aligned}$$

Clearly $\mathbf{T}_\infty$ is a game tree with taboos and $|T_\infty| \leq \sum_{i\in\omega} |T_i|$. Since (3) gives that $i_n = 0$ for $n \leq k$, we have that ${}_kT_\infty = {}_kT_j$, ${}_kT_\infty \cap (\mathcal{T}_\infty)_{\mathrm{I}} = {}_kT_j \cap (\mathcal{T}_j)_{\mathrm{I}}$, and ${}_kT_\infty \cap (\mathcal{T}_\infty)_{\mathrm{II}} = {}_kT_j \cap (\mathcal{T}_j)_{\mathrm{II}}$ for each j, as required by clause (i) of the definition of a k-covering.

For $p \in {}_nT_\infty$, we let

$$\pi_{\infty,j}(p) = \begin{cases} p & \text{if } j \geq i_n; \\ \pi_{i_n,j}(p) & \text{if } j < i_n. \end{cases}$$

It is routine to check that each $\pi_{\infty,j}$ is well-defined, that $\pi_{\infty,j} : \mathbf{T}_\infty \Rightarrow \mathbf{T}_j$, and that $\pi_{\infty,j} = \pi_{j',j} \circ \pi_{\infty,j'}$ whenever $j \leq j' \in \omega$. Clearly $\pi_{\infty,j} \restriction {}_nT_\infty$ is the identity for each $j \geq i_n$, and so the fact that $i_n = 0$ for $n \leq k$ guarantees that every $\pi_{\infty,j} \restriction {}_kT_\infty$ is the identity, as required by clause (ii) of the definition of a k-covering.

Similarly, for $\sigma \in \mathcal{S}({}_nT_\infty)$, we let

$$\phi_{\infty,j}(\sigma) = \begin{cases} \sigma & \text{if } j \geq i_n; \\ \phi_{i_n,j}(\sigma) & \text{if } j < i_n. \end{cases}$$

We omit the verifications that each $\phi_{\infty,j}$ is well-defined, that each $\phi_{\infty,j}$: $\mathbf{T}_\infty \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}_j$, and that $\phi_{\infty,j} = \phi_{j',j} \circ \phi_{\infty,j'}$ for all $j \leq j' \in \omega$. Since $\phi_{\infty,j} \restriction \mathcal{S}({}_nT_\infty)$ is the identity whenever $j \geq i_n$, the fact that $i_n = 0$ for $n \leq k$ guarantees that clause (iii) of the definition of a k-covering holds.

It remains to define the $\Psi^{j,\infty}$ and to verify clause (d) in the definition of a covering.

First note that we always have $(\Psi^{j,j'}(\sigma, x)) \restriction k_{j',j} = x \restriction k_{j',j}$; for $(\Psi^{j,j'}(\sigma, x)) \restriction k_{j',j} = \pi_{j',j}((\Psi^{j,j'}(\sigma, x)) \restriction k_{j',j}) \subseteq x \restriction k_{j',j}$, and $(\Psi^{j,j'}(\sigma, x)) \restriction k_{j',j} \subsetneq x \restriction k_{j',j}$ is impossible. Let $j \in \omega$ and let $\sigma \in \mathcal{S}(T_\infty)$. For $x \in \lceil T_j \rceil$ and x consistent with $\phi_{\infty,j}(\sigma)$, we can set

$$\Psi^{j,\infty}(\sigma, x) = \lim_{j'\to\infty} \Psi^{j,j'}(\phi_{\infty,j'}(\sigma), x),$$

since, for each $n \in \omega$,

$$\lim_{j' \to \infty}((\Psi^{j,j'}(\phi_{\infty,j'}(\sigma), x)) \restriction n) = \begin{cases} x \restriction n & \text{if } j \geq i_n; \\ (\Psi^{j,i_n}(\phi_{\infty,i_n}(\sigma), x)) \restriction n & \text{if } j < i_n. \end{cases}$$

If some $(\Psi^{j,\infty}(\sigma, x)) \restriction n$ is not consistent with σ, then, for any j' such that $j \leq j'$ and $i_n \leq j'$, the same position $(\Psi^{j,j'}(\phi_{\infty,j'}(\sigma), x)) \restriction n$ is not consistent with $\phi_{\infty,j'}(\sigma)$, which agrees with σ on positions of length $< n$. This contradicts property (d)(i) of the covering $\mathcal{C}_{j',j}$, so and property (d)(i) is verified for $\mathcal{C}_{\infty,j}$. For (d)(ii) and (d)(iii), note that we have, for each $n \in \omega$, for each j' such that $j \leq j'$ and $j' \geq i_n$, for each $\sigma \in \mathcal{S}(T_\infty)$, and for each $x \in \lceil T_j \rceil$ consistent with $\phi_{\infty,j}(\sigma)$, that

$$(\pi_{\infty,j}(\Psi^{j,\infty}(\sigma, x))) \restriction n = (\pi_{j',j}(\Psi^{j,j'}(\phi_{\infty,j'}(\sigma), x))) \restriction n.$$

Property (d)(ii) for $\mathcal{C}_{\infty,j}$ thus follows from property (d)(ii) for $\mathcal{C}_{j',j}$. Moreover, since $j' \geq i_n$ implies that $(\Psi^{j,\infty}(\sigma, x)) \restriction n = (\Psi^{j,j'}(\phi_{\infty,j'}(\sigma), x)) \restriction n$, property (d)(iii) for $\mathcal{C}_{\infty,j}$ also follows from property (d)(iii) for $\mathcal{C}_{j',j}$. We omit the verification that $\Psi^{j,\infty}(\sigma, x) = \Psi^{j',\infty}(\sigma, \Psi^{j,j'}(\phi_{\infty,j'}(\sigma), x))$ for all $j \leq j'$ and all $\langle \sigma, x \rangle$ in domain $(\Psi^{j,\infty})$. □

Remark. One advantage of adopting the alternative definition of covering considered in remark (a) on page 60 would be that the construction of the proof of Lemma 2.1.5 would literally be the construction of the inverse limit of the given system of coverings.

Lemma 2.1.6. *Let* $\mathbf{T}$ *be a game tree with taboos. If* $A \subseteq [T]$ *is open or closed and* $k \in \omega$*, then there is a* k*-covering of* $\mathbf{T}$ *that unravels* A*.*

Proof. Since any covering that unravels a set also unravels its complement, it is enough to prove that every closed subset of $[T]$ is, for each $k \in \omega$, unraveled by some k-covering of $\mathbf{T}$. Let then $A \subseteq [T]$ be closed. Recall that A is also closed as a subset of $\lceil T \rceil$. Let $k \in \omega$ and, since every $(k+1)$-covering is also a k-covering, assume without loss of generality that k is even.

We shall define $\mathcal{C} = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ and show that $\mathcal{C}$ is a k-covering and that $\mathcal{C}$ unravels A.

We begin with $\tilde{\mathbf{T}}$. Because we have to make $\mathcal{C}$ a k-covering, we let ${}_k\tilde{T} = {}_kT$, ${}_k\tilde{T} \cap \tilde{\mathcal{T}}_{\mathrm{I}} = {}_kT \cap \mathcal{T}_{\mathrm{I}}$, and ${}_k\tilde{T} \cap \tilde{\mathcal{T}}_{\mathrm{II}} = {}_kT \cap \mathcal{T}_{\mathrm{II}}$. All moves in $\tilde{T}$ will be moves in T, except for move k and move $k+1$. Each of these two moves will consist of a move in T together with one or two extra components.

To describe move k, let $p \in \tilde{T}$ with $\ell\mathrm{h}(p) = k$. Thus $p \in T$ also. If p is terminal in T—and so taboo in $\mathbf{T}$—then p is taboo in $\tilde{\mathbf{T}}$—and so terminal in $\tilde{T}$; and hence there is no move k. Assume therefore that p is not terminal in T. Since k is even, it is I's turn to move at p. We stipulate that I's move at p in $\tilde{T}$ must be of the form

$$\langle a, X\rangle,$$

where a is a move legal in T at p and X is a subset of the set Z of all $q \in T$ satisfying the following conditions:

(i) $p^\frown\langle a\rangle \subsetneq q$.

(ii) q is not terminal in T.

(iii) $[T_q] \cap A = \emptyset$.

(iv) $(\forall r)(p^\frown\langle a\rangle \subsetneq r \subsetneq q \;\rightarrow\; [T_r] \cap A \neq \emptyset)$.

Remark. Here is how to think of the move X. Suppose that the players are considering playing some game $G(B; \mathbf{T})$. Player I is asserting that he can win $G(B; \mathbf{T}_q)$ for every $q \in X$ and is conceding that II can win $G(B; \mathbf{T}_q)$ for every $q \in Z \setminus X$. If $x \in [T]_{p^\frown\langle a\rangle}$ then $x \notin A$ if and only if x extends some $q \in Z$. I is proposing that if and when a position $q \in Z$ is reached the play be terminated immediately, with I declared the winner if $q \in X$ and II declared the winner otherwise. In proposing this, I is proposing that the players should play out an infinite play only when that play belongs to A. The idea of having I play subsets of Z, rather than quasistrategies for I in $T_{p^\frown\langle a\rangle}$ or subtrees of $T_{p^\frown\langle a\rangle}$ is the idea of Hurkens mentioned on page 55.

If $p^\frown\langle a\rangle$ is taboo in $\mathbf{T}$, then we must let $p^\frown\langle\langle a, X\rangle\rangle$ be taboo for the same player in $\tilde{\mathbf{T}}$. Suppose that $p^\frown\langle a\rangle$ is not taboo in $\mathbf{T}$, and so is not terminal, in T. Then we make $p^\frown\langle\langle a, X\rangle\rangle$ not terminal in $\tilde{T}$. We allow II, in principle, two options for move $k+1$ in $\tilde{T}$, though the second option is available only if $X \neq \emptyset$.

Option (1). II may *accept* X. If II accepts X, then II's move in $\tilde{T}$ at $p^\frown\langle\langle a, X\rangle\rangle$ must be of the form

$$\langle 1, b\rangle,$$

where b is a legal move for II in T at $p^\frown\langle a\rangle$. We stipulate that the positions in $\tilde{T}$ that extend the resulting position $p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 1, b\rangle\rangle$ are the finite sequences of the form

$$p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 1, b\rangle\rangle^\frown s$$

with $p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s \in T$. A position of this form is to be terminal in $\tilde{T}$ if and only if one of the following holds.

(i) $p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s$ is terminal in T.

(ii) $p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s \in Z$.

If (i) holds, then we make $p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 1, b\rangle\rangle^\frown s$ is taboo in $\tilde{\mathbf{T}}$ for the player for whom $p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s$ is taboo in $\mathbf{T}$. If (ii) holds, then we let $p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 1, b\rangle\rangle$ be taboo in $\tilde{\mathbf{T}}$ for II if $p^\frown\langle a\rangle^\frown\langle b\rangle \in X$ and for I otherwise. Note that (i) and (ii) cannot both hold, and note that either might hold for $s = \emptyset$.

Option (2). II may *challenge* X. If II challenges X, then II's move in $\tilde{T}$ at $p^\frown\langle\langle a, X\rangle\rangle$ must be of the form

$$\langle 2, r, b\rangle,$$

where $r \in X$ and $b = r(k+1)$ (so that $p^\frown\langle a\rangle^\frown\langle b\rangle \in T_r$). The positions in $\tilde{T}$ that extend $p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 2, r, b\rangle\rangle$ are to be precisely those of the form

$$p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 2, r, b\rangle\rangle^\frown s$$

with $p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s \in T_r$. Such a position in $\tilde{T}$ is taboo for a player in $\tilde{\mathbf{T}}$ if and only if $p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s$ is taboo for that player in $\tilde{\mathbf{T}}$.)

Remark. Here is the way to think about II's two options. If II accepts X, then II accepts the proposal of I that was described in the remark on page 65. If II challenges X and makes the move $\langle 2, r, b\rangle$, then II is denying I's contention that I can win the game $G(B; \mathbf{T}_r)$. The players then play that game to decide who is right. (Remember, of course, that the set B is entirely imaginary. We imagine it only to motivate the definition of $\tilde{\mathbf{T}}$.)

The definition of π is the obvious one:

$$(\pi(\tilde{p}))(i) = \begin{cases} \tilde{p}(i) & \text{if } i \neq k \text{ and } i \neq k+1; \\ a & \text{if } i = k \text{ and } \tilde{p}(k) = \langle a, X\rangle; \\ b & \text{if } i = k+1 \text{ and } \tilde{p}(k+1) = \langle 1, b\rangle; \\ b & \text{if } i = k+1 \text{ and } \tilde{p}(k+1) = \langle 2, r, b\rangle. \end{cases}$$

In other words, $\pi(\tilde{p})$ is obtained from $\tilde{p}$ by deleting the components X, 1, 2, and r that occur in $\tilde{p}$.

Before defining the rest of our covering, let us pause to verify that $\boldsymbol{\pi}^{-1}(A)$ is clopen, so that our covering will unravel A. If a $\tilde{x}$ is an infinite play in $\tilde{T}$

in which II accepts I's X, then no position in $\pi(\tilde{x})$ belongs to the associated Z. Hence $\lceil T \rceil_q \cap A \neq \emptyset$ for all $q \subseteq \pi(\tilde{x})$. Since A is closed, $\pi(\tilde{x}) \in A$. If $\tilde{x}$ is any play in $\tilde{T}$ of length $> k+1$ in which II challenges I's X, then $\pi(\tilde{x}) \notin A$, for $\pi(\tilde{x})$ must extend the r played by II at move $k+1$, and this r belongs to X and so to Z. Define $\tilde{A} \subseteq \lceil \tilde{T} \rceil$ by stipulating, for $\tilde{x} \in \lceil \tilde{T} \rceil$, that

$$\tilde{x} \in \tilde{A} \ \leftrightarrow (\ell\mathrm{h}(\tilde{x}) > k+1 \ \wedge \text{II accepts I's } X).$$

Clearly $\tilde{A}$ is clopen. Moreover $\tilde{A} \cap [\tilde{T}] = \boldsymbol{\pi}^{-1}(A)$, as required for unraveling.

Next we define ϕ and Ψ simultaneously. It will be clear from the definitions that clauses (c) and (d) in the definition of a covering and clause (iii) in the definition of a k-covering are satisfied.

First let $\tilde{\sigma} \in \mathcal{S}_{\mathrm{I}}(\tilde{T})$. Here is the idea: The strategy $\tilde{\sigma}$ supplies us with values of $(\phi(\tilde{\sigma}))(p)$ for $\ell\mathrm{h}(p) \leq k$. Furthermore $\tilde{\sigma}$ supplies us with an X, and thus we have a clear choice for $\Psi(\tilde{\sigma}, x)\lceil k{+}1$. As long as no position is reached that belongs to X, we get subsequent values of $\phi(\tilde{\sigma})$ from values of $\tilde{\sigma}$ gotten by assuming that II accepts X. If no position belonging to X is ever reached, then this assumption gives us $\Psi(\tilde{\sigma}, x)$ also. Suppose we reach a position $r \in X$. If we were to define $\Psi(\tilde{\sigma}, x)$ using the assumption that II accepts X, then we would make $\Psi(\tilde{\sigma}, x)$ taboo for II, in violation of clause (d)(iii) in the definition of a covering. But we can avoid such a violation, for $\langle 2, r, r(k+1)\rangle$ is a legal move $k+1$ in $\tilde{T}$ in the position $\Psi(\tilde{\sigma}, x) \restriction k+1$. We get subsequent values of $\phi(\tilde{\sigma})$, and we get $\Psi(\tilde{\sigma}, x)$, by assuming that this move is made.

Here are the formal details. We describe $\phi(\tilde{\sigma}) = \sigma$ by describing an arbitrary play x consistent with σ. We thus omit the definition of $\sigma(p)$ for p inconsistent with σ. Such values can be assigned arbitrarily, except for the easily met constraints from clause (iii) in the definition of $\phi : \tilde{\mathbf{T}} \overset{\mathcal{S}}{\Rightarrow} \mathbf{T}$ and clause (iii) in the definition of a k-covering.

At each position $p \subseteq x$, either we shall have a guess $\psi(p)$ for $\Psi(\tilde{\sigma}, x)\restriction\ell\mathrm{h}(p)$ or else there will be a $q \subsetneq p$ such that $\psi(q)$ is taboo for I in $\tilde{\mathbf{T}}$ and we shall have already set $\Psi(\tilde{\sigma}, x) = \psi(q)$. Each $\psi(p)$ will be such that $\psi(p) \in \tilde{T}$, $\psi(p)$ is consistent with $\tilde{\sigma}$, and $\pi(\psi(p)) = p$. At most once during the construction we shall contradict our previous guesses: for at most one $p \subseteq x$, $\psi(p)$ will be defined but will not be an extension of the $\psi(p \restriction i)$ for $i < \ell\mathrm{h}(p)$.

We shall arrange that $\psi(p)$ is taboo for II in $\tilde{\mathbf{T}}$ only if p is taboo for II in $\mathbf{T}$. If we reach a p such that $\psi(p)$ is terminal in $\tilde{T}$, then we set $\Psi(\tilde{\sigma}, x) = \psi(p)$. In such a case, if p is not terminal then we define σ on extensions of p to agree with some fixed (independent of x) strategy σ_p in T_p.

To begin, we let σ agree with $\tilde{\sigma}$ and $\psi(p) = p$ until (if ever) we have reached a position p of length k. At this point we still let $\psi(p) = p$. If p is not terminal and $\tilde{\sigma}(p) = \langle a, X\rangle$, then set $\sigma(p) = a$ and $\psi(p^\frown\langle a\rangle) = p^\frown\langle\langle a, X\rangle\rangle$. If the position $p^\frown\langle a\rangle$ is not terminal, let b be II's next move.

As long as no position is reached that belongs to X, we proceed as follows. For positions $q = p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s$, let $\tilde{q} = p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 1, b\rangle\rangle^\frown s$. If $\tilde{q} \in \tilde{T}$, then let $\psi(q) = \tilde{q}$ and, if $\tilde{q}$ is non-terminal and of even length, let $\sigma(q) = \tilde{\sigma}(\tilde{q})$. If there is a last $q \subseteq x$ such that the associated $\tilde{q}$ belongs to $\tilde{T}$, then there are two possibilities for this last q.

(a) q is terminal. Then $q = x$ and we let $\Psi(\tilde{\sigma}, x) = \psi(q)$.

(b) $q \in Z \setminus X$. Then $\psi(q)$ is taboo for I and we let $\Psi(\tilde{\sigma}, x) = \psi(q)$.

If there is no last q such that the associated $\tilde{q} \in \tilde{T}$, then the play x is infinite. In this case we set $\Psi(\tilde{\sigma}, x) = \bigcup_{q \subseteq x} \psi(q)$.

Suppose that there is a position $r \subseteq x$ that belongs to X. For some s, $r = p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s$. We let $\tilde{r} = p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 2, r, b\rangle\rangle^\frown s$. Note that $\tilde{r}$ is a legal postion in $\tilde{T}$. Note also that $\tilde{r}^\frown t \in \tilde{T}$ for any t such that $r^\frown t \in T$. For positions $r^\frown t$, we set $\psi(r^\frown t) = \tilde{r}^\frown t$ and, for $r^\frown t$ of even length and not terminal, we let $\sigma(r^\frown t) = \tilde{\sigma}(\tilde{r}^\frown t)$. If the play x is infinite, we let $\Psi(\tilde{\sigma}, x) = \bigcup_{n \geq \ell\mathrm{h}(r)} \psi(x \restriction n)$.

Next let $\tilde{\tau} \in \mathcal{S}_{\mathrm{II}}(\tilde{T})$. Here is the idea: When we reach a position $p^\frown\langle a\rangle$ in T of length $k+1$, there is a subset Y of the Z associated with $p^\frown\langle a\rangle$ such that

(i) τ calls for II to accept Y;

(ii) for any $r \in Z \setminus Y$, there is an $X \subseteq Z$ such that $\tau(p^\frown\langle\langle a, X\rangle\rangle) = \langle 2, r, r(k+1)\rangle$.

As long as no position is reached that belongs to $Z \setminus Y$, we get subsequent values of $\phi(\tilde{\tau})$ from valuses of $\tilde{\tau}$ gotten by assuming that I plays $\langle a, Y\rangle$. If no position belonging to $Z \setminus Y$ is ever reached, then this assumption gives us $\Psi(\tilde{\tau}, x)$ also. Suppose we reach a position $r \in Z \setminus Y$. If we were to define $\Psi(\tilde{\tau}, x)$ using the assumption that I plays $\langle a, Y\rangle$, then we would make $\Psi(\tilde{\tau}, x)$ taboo for I, in violation of clause (d)(iii) in the definition of a covering. We can avoid such a violation by using property (ii) of Y. If X is as given by (ii), then we get subsequent values of $\phi(\tilde{\tau})$, and we get $\Psi(\tilde{\tau}, x)$, by assuming that the moves $\langle a, X\rangle$ and $\langle 2, r, r(k+1)\rangle$ are made.

Now we give the formal details. As in the preceding case, we describe $\phi(\tilde{\tau}) = \tau$ by describing an arbitrary play x consistent with τ. At each position $p \subseteq x$, either we shall have a guess $\psi(p)$ for $\Psi(\tilde{\tau}, x) \restriction \ell\text{h}(p)$ or else there will be $q \subsetneq p$ such that $\psi(q)$ is taboo for II in $\tilde{\mathbf{T}}$ and we shall have already set $\Psi(\tilde{\tau}, x) = \psi(q)$. Each $\psi(p)$ will be such that $\psi(p) \in \tilde{T}$, $\psi(p)$ is consistent with $\tilde{\tau}$, and $\pi(\psi(p)) = p$. As before, there will be at most one $p \subseteq x$ such that $\psi(p)$ is defined but is not an extension of the $\psi(p \restriction i)$ for $i < \ell\text{h}(p)$.

We shall arrange that $\psi(p)$ is not taboo for I in $\tilde{\mathbf{T}}$ unless p is taboo for I in $\mathbf{T}$. If we reach a p such that $\psi(p)$ is terminal, then we set $\Psi(\tilde{\tau}, x) = \psi(p)$. We use the same method as we used before for σ to define τ on extensions of p when $\psi(p)$ is terminal in $\tilde{T}$ but p is not terminal in T.

To begin, we let τ agree with $\tilde{\tau}$ and $\psi(p) = p$ until (if ever) we have reached a position p of length k. For this p also, we let $\psi(p) = p$. If p is not terminal, let a be I's move at p. Let Z be the set associated with $p^\frown\langle a\rangle$, the set of which the second component of move k must be a subset. Let

$$Y \;=\; \{r \in Z \mid (\forall X \subseteq Z)\, \tilde{\tau}(p^\frown\langle\langle a, X\rangle\rangle) \neq \langle 2, r, r(k+1)\rangle\}.$$

The move $\langle a, Y\rangle$ is legal for I in $\tilde{T}$ at p, and so we can let $\psi(p^\frown\langle a\rangle) = p^\frown\langle\langle a, Y\rangle\rangle$. Assume that $p^\frown\langle a\rangle$ is not terminal in T. Then $p^\frown\langle\langle a, Y\rangle\rangle$ is not terminal in $\tilde{T}$.

It is obvious from the definition of Y that Y has property (ii) above. Let us show that Y has property (i), i.e., that $\tilde{\tau}$ cannot call for II to challenge Y at $p^\frown\langle\langle a, Y\rangle\rangle$. Assume the contrary and let $\tilde{\tau}(p^\frown\langle\langle a, Y\rangle\rangle) = \langle 2, r, r(k+1)\rangle$. By the definition of Y, we have that $r \notin Y$. But challenging Y requires that $r \in Y$, so we have a contradiction.

Thus $\tilde{\tau}(p^\frown\langle\langle a, Y\rangle\rangle) = \langle 1, b\rangle$ for some b with $p^\frown\langle a\rangle^\frown\langle b\rangle \in T$. We let $\tau(p^\frown\langle a\rangle) = b$.

As long as no position is reached that belongs to $Z \setminus Y$, we proceed as follows. For positions $q = p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s$, let $\tilde{q} = p^\frown\langle\langle a, Y\rangle\rangle^\frown\langle\langle 1, b\rangle\rangle^\frown s$. If $\tilde{q} \in \tilde{T}$, then let $\psi(q) = \tilde{q}$ and, if $\tilde{q}$ is non-terminal and of odd length, let $\tau(q) = \tilde{\tau}(\tilde{q})$. If there is a last $q \subseteq x$ such that the associated $\tilde{q}$ belongs to $\tilde{T}$, then there are two possibilities for this last q.

(a) q is terminal. Then $q = x$ and we let $\Psi(\tilde{\tau}, x) = \psi(q)$.

(b) $q \in Y$. Then $\psi(q)$ is taboo for II and we let $\Psi(\tilde{\tau}, x) = \psi(q)$.

If there is no last q such that the associated $\tilde{q} \in \tilde{T}$, then the play x is infinite. In this case we set $\Psi(\tilde{\tau}, x) = \bigcup_{q \subseteq x} \psi(q)$.

Suppose that there is a position $r \subseteq x$ that belongs to $Z \setminus Y$. By property (ii) of Y, let $X \subseteq Z$ be such that $\tilde{\tau}(p^\frown\langle\langle a, X\rangle\rangle) = \langle 2, r, r(k+1)\rangle$. For some s, $r = p^\frown\langle a\rangle^\frown\langle b\rangle^\frown s$. We let $\tilde{r} = p^\frown\langle\langle a, X\rangle\rangle^\frown\langle\langle 2, r, b\rangle\rangle^\frown s$. Note that $\tilde{r}$ is a legal postion in $\tilde{T}$. Note also that $\tilde{r}^\frown t \in \tilde{T}$ for any t such that $r^\frown t \in T$. For positions $r^\frown t$, we set $\psi(r^\frown t) = \tilde{r}^\frown t$ and, for $r^\frown t$ of odd length and not terminal, we let $\tau(r^\frown t) = \tilde{\tau}(\tilde{r}^\frown t)$. If the play x is infinite, we let $\Psi(\tilde{\tau}, x) = \bigcup_{n \geq \ell\mathrm{h}(r)} \psi(x \restriction n)$. □

Theorem 2.1.7. ([Martin, 1985]) *Let* $\mathbf{T}$ *be a game tree with taboos. If* $A \subseteq [T]$ *is Borel and* $k \in \omega$*, then there is a* k*-covering of* $\mathbf{T}$ *that unravels* A*.*

Proof. By induction on countable ordinals $\alpha \geq 1$, we prove

$(\dagger)_\alpha$ *For all* $\mathbf{T}$*, for all* $A \subseteq [T]$ *such that* $A \in \mathbf{\Sigma}^0_\alpha$*, and for all* $k \in \omega$*, there is a* k*-covering of* $\mathbf{T}$ *that unravels* A*.*

$(\dagger)_1$ is equivalent with Lemma 2.1.6. Assume then that $\alpha > 1$ and that $(\dagger)_\beta$ holds for all β with $1 \leq \beta < \alpha$. Let $k \in \omega$ and let $A \subseteq [T]$ with $A \in \mathbf{\Sigma}^0_\alpha$. By the definition of $\mathbf{\Sigma}^0_\alpha$, there are B_n, $n \in \omega$, such that each B_n belongs to $\mathbf{\Pi}^0_{\beta_n}$ for some $\beta_n < \alpha$ and such that $A = \bigcup_{n\in\omega} B_n$.

Let $\mathbf{T}_0 = \mathbf{T}$. By induction on $j' \in \omega$, we define $\mathbf{T}_{j'}$ and

$$\mathcal{C}_{j',j} = \langle \mathbf{T}_{j'}, \pi_{j',j}, \phi_{j',j}, \Psi^{j,j'}\rangle$$

for $j \leq j'$ such that $\mathcal{C}_{j',i} = \mathcal{C}_{j,i} \circ \mathcal{C}_{j',j}$ for all $i \leq j \leq j'$. We do this in such a way that each $\mathcal{C}_{j',j}$ is a $(k+j)$-covering of $\mathbf{T}_j$ and $\mathcal{C}_{j',0}$ unravels B_j for each $j \leq j'$. Note that $\mathcal{C}_{j',j'}$ must be the trivial covering, with $\pi_{j',j'}$ and $\phi_{j',j'}$ the identities and $\Psi^{j',j'}(\sigma, x) = x$ for all σ and x.

Suppose that we have defined $\mathbf{T}_{j'}$ and the $\mathcal{C}_{j',j}$ for all $j' \leq n$. By the continuity of $\boldsymbol{\pi}_{n,0}$, we have that $\boldsymbol{\pi}_{n,0}{}^{-1}(B_n) \in \mathbf{\Pi}^0_{\beta_n}$. By $(\dagger)_{\beta_n}$, let $\mathcal{C}_n = \langle \tilde{\mathbf{T}}, \pi, \phi, \Psi\rangle$ be a $(k+n)$-covering of $\mathbf{T}_n$ that unravels $\boldsymbol{\pi}_{n,0}{}^{-1}([T] \setminus B_n)$ and so unravels $\boldsymbol{\pi}_{n,0}{}^{-1}(B_n)$. Let $\mathbf{T}_{n+1} = \tilde{\mathbf{T}}$. For $j \leq n$, let $\mathcal{C}_{n+1,j} = \mathcal{C}_{n,j} \circ \mathcal{C}_n$; let $\mathcal{C}_{n+1,n+1}$ be the trivial covering. The required properties of the $\mathcal{C}_{n+1,j}$ follow directly from Lemma 2.1.4 and the continuity of the $\boldsymbol{\pi}_{n,j}$.

If we let $k_{j,i} = k + i$, then the hypotheses of Lemma 2.1.5 hold. Let $\mathbf{T}_\infty$ and, for $i \in \omega$, $\mathcal{C}_{\infty,i} = \langle \mathbf{T}_\infty, \pi_{\infty,i}, \phi_{\infty,i}, \Psi^{i,\infty}\rangle$ be given by that lemma. For each n, $\boldsymbol{\pi}_{\infty,0}{}^{-1}(B_n)$ is clopen, by the continuity of $\boldsymbol{\pi}_{\infty,n+1}$. Thus $\boldsymbol{\pi}_{\infty,0}{}^{-1}(A)$ is open. By Lemma 2.1.6, let $\tilde{\mathcal{C}}$ be a k-covering of $\mathbf{T}_\infty$ that unravels $\boldsymbol{\pi}_{\infty,0}{}^{-1}(A)$. $\mathcal{C}_{\infty,0} \circ \tilde{\mathcal{C}}$ is a k-covering of T that unravels A. □

Theorem 2.1.8. ([Martin, 1975]) *All Borel games are determined.*

Proof. The theorem follows immediately from Lemma 2.1.3 and Theorem 2.1.7. □

Exercise 2.1.1. Consider the following two strengthenings of AD.

(1) $\mathrm{AD}_{\mathbb{R}}$, the assertion that all games in ${}^{<\omega}({}^{\omega}\omega)$ are determined;

(2) $\mathrm{AD}(\omega^2)$, the assertion that all games of length ω^2 with moves in ω are determined.

Prove that $\mathrm{AD}_{\mathcal{R}}$ and $\mathrm{AD}(\omega_2)$ are equivalent.

Hint. In the non-trivial direction, consider a game of length ω in which I's individual moves are strategies for games in ${}^{<\omega}\omega$ and II's moves are plays consistent with these strategies.

Remarks:

(a) This result was proved independently by Andreas Blass and Jan Mycielski. (See [Blass, 1975].) Until the author learned of it in 1974, his and others' attempts to prove Borel determinacy involved auxiliary games with individual moves that were ordinal numbers. (See [Paris, 1972] for a partial success.) The Blass–Mycielski proof suggested trying games with individual moves that were strategies (or something similar). In [Martin, 1975] and [Martin, 1985], there are individual moves that are quasistrategies. In the version of the proof we have just presented, however, the quasistrategies have disappeared.

(b) Oddly enough, the determinacy of all games of countable length, with real or natural number moves, follows from $\mathrm{AD}_{\mathbb{R}}$. This fact is a consequence of a theorem independently proved by Hugh Woodin and the author, together with another theorem of Woodin. (The work is unpublished.)

Exercise 2.1.2. Let $A \subseteq \lceil T \rceil$ and suppose that $A = \bigcup_{i\in\omega} A_i$, with each A_i closed. Consider the following game $G^* = G(A^*; T^*)$. I begins by picking a strategy σ_0 for I in T. II then chooses a position $p_0 \in T$ consistent with σ_0. If the position in T^* is not terminal (as defined below), I next picks a strategy σ_1 for I in T_{p_0}; II picks $p_1 \in T_{p_0}$ consistent with σ_1 such that $p_1 \supseteq p_0$; etc. If some $\lceil T_{p_i} \rceil$ is not disjoint from A_i, then the position just after II has picked p_i is terminal. This is the only way terminal positions in G^* arise. A play of G^* is a win for I if and only if the play is finite. Prove using G^* and Theorem 1.2.4 that $G(A; T)$ is determined.

Exercise 2.1.3. Modify the T^* of Exercise 2.1.2 to get a covering of $\mathbf{T} = \langle T, \emptyset, \emptyset \rangle$ that unravels the A of Exercise 2.1.2.

Exercise 2.1.4. Let $\langle \tilde{\mathbf{T}}, \pi, \phi, \Psi \rangle$ be a covering of $\mathbf{T}$. Show that the extended $\pi : \tilde{T} \cup \lceil \tilde{T} \rceil \to T \cup \lceil T \rceil$ is a surjection.

Hint. Let $x \in \lceil T \rceil$. Consider the game in $\tilde{T}$ that I wins unless someone makes a Move $\tilde{p}$ such that $\pi(\tilde{p}) \nsubseteq x$ and I is the first player to do so. Prove that this game is a win for I. Prove that the analogous game with the roles of the players reversed is a win for II.

Exercise 2.1.5. Work in ZF and assume AD. Let $T = {}^{<\omega}\omega$. Let games in $\tilde{T}$ be played as follows:

$$\begin{array}{lllllllll} \mathrm{I} & \langle \sigma, n_0 \rangle & & n_2 & & n_4 & & \ldots & \\ \mathrm{II} & & \langle x, n_1 \rangle & & n_3 & & n_5 & & \ldots \end{array}$$

Here σ must be a strategy for I in T with $\sigma(\emptyset) = n_0$, and x must be a play in T consistent with σ. Each n_i must be $x(i)$. (Thus only σ and x matter.) Use $\tilde{T}$ to get a $(\tilde{\mathbf{T}}, \pi, \phi, \Psi)$ that fails to be a covering of $\mathbf{T} = \langle T, \emptyset, \emptyset \rangle$ unraveling every subset of ${}^{\omega}\omega$ only in that ϕ is not single-valued.

In [Martin, 1985] it is asserted that a certain uniformization hypothesis permits one to get a single-valued ϕ. The hypothesis is mistated in [Martin, 1985], but the intended one does not work. In [Neeman, 2000] it is shown that every $\mathbf{\Pi}^1_1$ subset of ${}^{\omega}\omega$ can be unraveled by a covering of $\langle {}^{<\omega}\omega, \emptyset, \emptyset \rangle$. See Exercise **??**.

Exercise 2.1.6. Under the hypotheses of Lemma 2.1.5, let $\mathbf{T}_\infty$ and $\langle \mathcal{C}_{\infty,i} \mid i \in \omega \rangle$ be the tree and sequence of coverings constructed in the proof of that lemma. Suppose that $\mathbf{T}'$ and $\langle \mathcal{C}'_{\infty,i} \mid i \in \omega \rangle$ are such that each $\mathcal{C}'_{\infty,i}$ is a k-covering of $\mathbf{T}_i$ with first component $\mathbf{T}'$ and such that, for $i \leq j \in \omega$, $\mathcal{C}'_{\infty,i} = \mathcal{C}'_{j,i} \circ \mathcal{C}'_{\infty,j}$. Show that there are π',ϕ', and Ψ' such that $\mathcal{C}' = \langle \mathbf{T}', \pi', \phi', \Psi' \rangle$ is a k-covering of $\mathbf{T}_\infty$ and, for each $i \in \omega$, $\mathcal{C}'_{\infty,i} = \mathcal{C}_{\infty,i} \circ \mathcal{C}'$.