The main subject of this book is games in which two players are given a set $A$ of infinite sequences of natural numbers and take turns choosing natural numbers, producing an infinite sequence. The player who moves first wins if this sequence belongs to $A$; otherwise the opponent wins. Such a game is determined if one of the players has a winning strategy.

If $A$ belongs to a set $\Gamma$ of sets of infinite sequences of natural numbers, then we call the game a $\Gamma$ game. We will present proofs of theorems of the following form: Under hypothesis $H$, all $\Gamma$ games are determined. In Chapter 1, the sets $\Gamma$ are the first few levels of the Borel hierarchy and the hypotheses $H$ are the axioms of second-order arithmetic or slightly more. For most of Chapter 2, $\Gamma$ is the set of all Borel sets and $H$ is ZFC. In the remaining chapters, the sets $\Gamma$ get larger and larger, and the hypotheses $H$ are large cardinal hypotheses.

Many of these theorems have converses or quasi-converses. These are presented as exercises with hints that are essentially sketches of proofs.

The reader should have basic familiarity with set theory, but the book assumes no familiarity with games, descriptive set theory, or large cardinals.

Eight of the nine chapters of the book are included in the current posting. Chapter 5 will be added after editing and the addition of the proofs of some of the converses. Though Chapter 9 is included, the reader should be aware that it has not been seriously proofread, and it—especially the last part of it—might have significant errors. Corrections and suggestions for Chapter 9 (and for the other chapters) would be welcome.
Chapter 1

Elementary Methods

In this chapter we introduce the basic concepts of our subject and prove as much determinacy as, roughly speaking, can be proved without appealing to the existence of infinite sets larger than the sets of legal positions in our games.

Readers interested primarily in the main results may wish to read just the introductory Section 1.1 and the basic Section 1.2, where the determinacy of open games is proved.

The proofs in §1.1 and §1.2 do not really need the Power Set or Replacement Axioms of set theory, though this fact is not mentioned in those sections. In §1.3 and in much of §1.4, we explicitly work in a set theory without the Power Set Axiom and with only a fragmentary Replacement Axiom (adopted mostly to avoid complexities). We try to do this in a sufficiently unobtrusive way that readers unfamiliar with axiomatic set theory should be able to follow the proofs as ordinary proofs. In §1.4 we discuss the optimal determinacy result for this theory, due to Antonio Montalban and Richard Shore. In a slightly stronger theory, we prove the determinacy of all $\Delta^0_i$ games (games that are both $G_{\delta\sigma\delta}$ and $F_{\sigma\delta\sigma}$). In the exercises we discuss Harvey Friedman’s methods, which show that the determinacy of all $\Sigma^0_i$ games is not provable in the usual ZFC set theory if the Power Set Axiom is dropped, and we mention an improvement by Montalban and Shore showing the optimality of their positive results. Later (in §2.3) we will use the results from §1.4 in analyzing level by level how much of the Power Set and Replacement Axioms is needed for our proof of the determinacy of Borel games.
1.1 Basic Definitions

We begin by discussing rules of play of our games and afterward take up such matters as winning, winning strategies, and determinacy.

Plays of our games will be finite or infinite sequences of moves. Rules of play are given by specifying a game tree. A game tree is a nonempty tree of finite sequences, i.e. is a set $T$ of finite sequences such that if $p \in T$ and $p$ extends $q$ then $q \in T$. Members of $T$ are called legal positions in $T$ or simply positions in $T$. When there is no danger of confusion, we will call them legal positions or positions. A position in $T$ is terminal in $T$ if it has no proper extension in $T$. If $p$ is a non-terminal position, then a legal move at $p$ in $T$ or simply a move at $p$ in $T$ is an $a$ such that $p \langle a \rangle \in T$, where $\langle \cdot \rangle$ is the concatenation operation on sequences. A play in $T$ is a finite or infinite sequence every initial part of which belongs to $T$ and which is a terminal position in $T$ if finite. All our games will have two players, I and II. ("I" and "II" are not very imaginative names, but they have become traditional.)

Play of a game in $T$ begins at the initial position (the empty sequence $\emptyset$, which must belong to every game tree). I moves first, and moves alternate between the two players. Thus a play of a game is produced as follows:

<table>
<thead>
<tr>
<th>I</th>
<th>$a_0$</th>
<th>$a_2$</th>
<th>$a_4$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>$a_1$</td>
<td>$a_3$</td>
<td></td>
<td>...</td>
</tr>
</tbody>
</table>

Each $\langle a_0, a_1, \ldots, a_n \rangle$ must be a position in $T$. If a terminal position is reached, then we have a play of the game and no further moves are made. If no terminal position is reached, then the play is infinite.

There are various ways in which we could have chosen a more general notion of game tree, even in our context of two-person games of perfect information:

(1) We did not have to require that our players alternate moves. Instead we could have introduced a move function $M$, defined on all non-terminal positions, with $M(p)$ giving the player who moves at $p$. There are two reasons we did not do this. First, it is not really more general, since we can get the same effect in our more restricted set-up. Suppose, for instance that we want to simulate a game in which player II makes the first two moves. To do so we introduce a new tree in which (a) the first move must be the empty sequence and (b) the second move must be a sequence of length 2 that is a legal position in the original tree. (See Exercise 1.1.5.) The second and more important reason why we do not introduce a move function is that it would
1.1. BASIC DEFINITIONS

make the notion of a game tree more complicated. A game tree would be a pair \( (T, M) \) and we would continually have to pay attention to the extra object \( M \) in situations where it played no significant role.

(2) There is a more general notion of game tree which we could have chosen. By a tree we mean a partial ordering with wellordered initial segments. That is to say, a tree is a pair \( (T, <) \) such that \( T \) is a set and \( < \) partially orders \( T \) and

\[(\forall p \in T)(< \text{ wellorders } \{ q \in T \mid q < p \}).\]

A tree of finite sequences becomes a special case of a tree if we define \( p < q \) to mean that \( p \) is properly extended by \( q \). Why did we not define game trees to be arbitrary trees rather than trees of finite sequences? One reason has to do with finite sequences. (See (3) below.) The other reason has to do with sequences: We want our positions to be sequences of moves. With general trees, we have no extra objects to be our moves. This isn’t really a serious problem, however. We will always be assuming that our players have complete knowledge about the position whenever they make a move. Thus making a move is essentially the same as choosing the new position that will result when the move has been made. With general trees as game trees, \textit{move} could be defined by changing “essentially the same as” into “identical with.” In other words, a legal move at \( p \) could be defined to be a position \( q \) that is an immediate successor of \( p \), i.e. a \( q \in T \) with \( p < q \) such that there is no \( r \) with \( p < r < q \). In this chapter such a solution would be quite satisfactory for us. Indeed it would work somewhat more smoothly than our actual definitions (and so we do after all keep it as an “actual” definition, as the reader will see two paragraphs hence). In later chapters, however, we will often be concerned with properties of individual moves in our official sense. For example, it may be important that certain moves are chosen from a countable set or that other moves come from a space that carries a measure. Our choice was thus made in view of these later chapters. We confess that we were also influenced by a desire to conform to real games: in chess a move involves changing the placement of one or two individual pieces; it does not involve the complete history of the game.

(3) As we indicated above, our game trees are special not only in that they are trees of sequences but also in that we demand that the sequences be finite. (In the context of general trees the corresponding restriction would be a requirement that each member of \( T \) have only finitely many predecessors.) If we removed this restriction we would be dealing not merely with games
of infinite length—games that take forever to play—but also with games of 
transfinite length—games that aren’t finished even after the players have 
played forever. Such games are indeed of interest and a good deal of theory 
about them has been developed. We will occasionally discuss these longer 
games, both in the text and in exercises. Nevertheless, such games are in 
some ways essentially more complex than merely infinite games, and we chose 
this case simplicity in our subject matter.

(4) Other possible generalizations of our notion of game tree allow for 
such things as simultaneous moves by the two players. Though there are 
ways to get the effect of such generalizations, we actually use simultaneous 
moves when we study games of imperfect information in §2.4.

We said in (2) above that there is a possible definition of “move” according 
to which a move is a position. Though we did not choose this definition, 
there will be a number of occasions at which it would have been notationally 
simpler if we had chosen it. Let us then compromise and define a Move at 
p in T to be a position q such that, for some move a at p in T, \( q = p^\prec(a) \).

This usage would produce ambiguity if we were ever to write “Move” at the 
beginning of a sentence, but we will have no reason to do so.

It is time to be more precise about some of our terminology and to intro-
duce some basic notation. By a finite sequence we mean a function whose 
domain is the set of all predecessors of some natural number. We adopt 
the convention from set theory that a natural number is the set of all its 
predecessors, so that a finite sequence is a function whose domain is a natu-
ral number. We also adopt the set-theoretic notion of function, identifying a 
function with its graph. With this convention, a finite sequence \( p \) is extended 
by a finite sequence \( q \) if and only if \( p \subseteq q \), and so “\( \subseteq \)” will be our standard 
notation for “is extended by.” The length of a finite sequence \( p \) is the domain 
of \( p \). We denote the length of \( p \) by \( \ell h(p) \). Infinite sequences will be treated 
similarly. An infinite sequence is a function with domain \( \omega \), the set of all 
natural numbers. The length of an infinite sequence is \( \omega \). Infinite ordinal 
numbers are also considered to be the set of all their predecessors. If \( x \) is a 
finito or infinite play in \( T \), then \( p \subseteq x \) just means that \( p \) is extended by \( x \). 
We will denote the set of all plays in \( T \) by \( \lceil T \rceil \).

The most important example of a game tree is \( <\omega \omega \), the set of all finite 
sequences of natural numbers. The set of all plays in this tree is \( \omega \omega \), the set 
of all infinite sequences of natural numbers. Note that \( \omega \) is the set of all 
functions \( f : x \rightarrow y \). (It is sometimes important to distinguish, e.g. \( \omega \) from 
the ordinal number \( \omega^\omega \).) The notation \( \subset x y \), for ordinal numbers \( x \), stands
for $\bigcup_{x' < x} x'y$. The tree $<\omega\omega$ is an example of a tree all of whose plays are infinite. If we wished, we could deal only with such trees, extending what are now terminal positions by adjoining infinitely many irrelevant moves.

A strategy for I in $T$ is a function $\sigma$ whose domain is

$$\{p \in T \mid \ell(h(p)) \text{ is even and } p \text{ is not terminal}\}$$

such that $\sigma(p)$ is always a legal move in $T$ at $p$. A strategy for II in $T$ is similarly a function $\tau$ with domain $\{p \in T \mid \ell(h(p)) \text{ is odd and } p \text{ is not terminal}\}$ such that $\tau(p)$ is always a legal move at $p$. By $S_I(T)$ we mean the set of all strategies for I in $T$; by $S_{II}(T)$ we mean the set of all strategies for II in $T$. We let $S(T) = S_I(T) \cup S_{II}(T)$. Just as we defined Moves as well as ordinary moves, we could define Strategies which are like strategies except that their values are Moves instead of moves. We refrain from doing so: it turns out not to be as useful as the Move move. A position $p$ in $T$ is consistent with a strategy $\sigma$ for I if $p(n) = \sigma(p \upharpoonright n)$ for every even $n < \ell(h(p))$. (Here $p \upharpoonright n$ is the restriction of the function $p$ to the set $n = \{0, 1, \ldots, n - 1\}$, i.e. $p \upharpoonright n$ is the initial part of $p$ of length $n$.) A play $x$ in $T$ is consistent with $\sigma$ if every position $p \subseteq x$ is consistent with $\sigma$. Being consistent with a strategy for II is similarly defined.

For each game tree $T$ and each $A \subseteq [T]$, i.e., for each set of plays in $T$, we have a game $G(A; T)$. I wins a play $x$ of $G(A; T)$ just in case $x \in A$. Otherwise II wins $x$. It will be convenient to have $G(A; T)$ defined even for sets $A$ that are not subsets of $[T]$. In this case $G(A; T)$ will be the same game as $G(A \cap [T]; T)$. A strategy $\sigma$ for I is a winning strategy for $G(A; T)$ if I wins each play consistent with $\sigma$. Winning strategies for II are similarly defined. $G(A; T)$ is determined if either I or II has a winning strategy for $G(A; T)$. Note that it is impossible for both players to have winning strategies, since there would be a play consistent with both strategies.

In Chapter 2 we will find it useful to introduce a variant notion of a game tree in which there are some built-in winning conditions: Certain terminal positions are designated as losing for one or the other of the players independently of $A$. We defer making the definition until we have some use for it.

Not all games in our sense are determined. (See [Gale and Stewart, 1953], [Mycielski, 1964], page 114 of [Mauldin, 1981], and also Exercises 1.1.2 and 1.1.4.) To get determinacy results it is necessary to impose conditions of some kind on the games.
CHAPTER 1. ELEMENTARY METHODS

One way to do this is to impose conditions of size on the game tree. As we will see in the next section, all games $G(A; T)$ with $T$ finite are determined. There are, however, undetermined games $G(A; T)$ with $T$ countable (See Exercise 1.1.2.) Most of the concern in this book will be with determinacy results in the case $T$ is countable.

Remark: All known proofs of the existence of undetermined games in countable trees make use of the Axiom of Choice. [Mycielski and Steinhaus, 1962] proposes as an alternative to the Axiom of Choice an assertion, there called the Axiom of Determinateness and now called the Axiom of Determinacy or simply AD.

AD: All games in countable trees are determined.

Large cardinal axioms imply that the Axiom of Determinacy is consistent with the axioms of set theory other than Choice. This will be proved in Chapter 9. (In this book we make free use of the Axiom of Choice, though we will make occasional remarks about whether or not particular theorems require it.)

Are there other conditions on $T$ implying determinacy? In §1.2 we will see that the absence of infinite plays is such a condition. But the absence of infinite plays in $T$ is for practical purposes equivalent with a simple topological condition on I’s winning set $A$. Such topological conditions will be the hypotheses of almost all our determinacy theorems, and so it is to topology that we now turn.

For $p \in T$ let

$$T_p = \{q \in T \mid q \subseteq p \lor p \subseteq q\}.$$  

$T_p$ is a game subtree of $T$, i.e. $T_p$ is a subtree of $T$ (a subset of $T$ that is a game tree), and every position terminal in $T_p$ is terminal in $T$. Games in $T_p$ are played just as are those in $T$, except that the first $\ell h(p)$ moves are fixed in advance so as to produce the position $p$. We give $[T]$ a topology by taking as basic open sets the $[T_p]$ for $p \in T$. For $A \subseteq [T]$, let us say that the game $G(A; T)$ is open, closed, etc. just in case $A$ is open, closed, etc. respectively.

Remark. If $p$ is a position in $T$, we will never create ambiguity by using the notation “$T_p$” with any meaning other than that given it in the preceding paragraph. The reader should be warned, however, that we will take such liberties as denoting elements of an infinite sequence of game trees by “$T_i$.”
In this chapter we will prove determinacy results for games in low levels of the Borel hierarchy. We now define that hierarchy and prove some basic facts about it.

We use the logical notation for the Borel hierarchy in a topological space. \( \Sigma^0_1 \) is the class of open sets; \( \Pi^0_1 \) is the class of closed sets; \( \Delta^0_1 \) the class of clopen (closed and open) sets. For ordinals \( \alpha > 1 \), \( \Sigma^0_\alpha \) is the class of all countable unions of sets belonging to \( \bigcup_{\beta<\alpha} \Pi^0_\beta \), \( \Pi^0_\alpha \) is the class of complements of sets belonging to \( \Sigma^0_\alpha \), and \( \Delta^0_\alpha = \Sigma^0_\alpha \cap \Pi^0_\alpha \). A set is Borel if it belongs to the smallest class containing the open sets and closed under countable unions and complements. The following lemma gives some basic facts about the Borel hierarchy in \( \lceil T \rceil \).

**Lemma 1.1.1.** (1) The following hold in spaces \( \lceil T \rceil \) for every ordinal number \( \alpha \geq 1 \):

(a) \((\forall \beta)(\alpha < \beta \to \Sigma^0_\alpha \cup \Pi^0_\alpha \subseteq \Delta^0_\beta)\).

(b) \( \Sigma^0_\alpha \) is closed under countable unions and finite intersections.

(c) \( \Pi^0_\alpha \) is closed under countable intersections and finite unions.

(2) A set is Borel if and only if it belongs to \( \bigcup_{1 \leq \alpha < \omega_1} \Sigma^0_\alpha \), where \( \omega_1 \) is the least uncountable ordinal number.

**Proof.** (1)(a). If \( A \in \Pi^0_\alpha \), then \( A = \bigcup\{A\} \); thus \( A \in \Sigma^0_\beta \) for all \( \beta > \alpha \). This shows that \( \Pi^0_\alpha \subseteq \Sigma^0_\beta \) for all \( 1 \leq \alpha < \beta \). It follows directly that, for all such \( \alpha \) and \( \beta \), \( \Sigma^0_\alpha \subseteq \Pi^0_\beta \). If \( 1 < \alpha < \beta \), then it is immediate from the definition that \( \Sigma^0_\alpha \subseteq \Sigma^0_\beta \). Let \( A \in \Sigma^0_1 \). Since \( A \) is open,

\[
A = \bigcup\{[T_p] \mid p \in T \land [T_p] \subseteq A\}.
\]

For \( n \in \omega \), let

\[
A_n = \bigcup\{[T_p] \mid p \in T \land h(p) = n \land [T_p] \subseteq A\}.
\]

Each \( A_n \) is closed as well as open, since

\[
\neg A_n = \bigcup\{[T_p] \mid p \in T \land h(p) = n \land [T_p] \not\subseteq A\}.
\]

(\( \neg A \) is \( \lceil T \rceil \setminus A \).) Since \( A = \bigcup_{n \in \omega} A_n \), we have that \( A \) is a countable union of \( \Pi^0_1 \) sets and so that \( A \in \Sigma^0_\beta \) for every \( \beta > 1 \). We have now shown that
\( \Sigma^0_\alpha \subseteq \Sigma^0_\beta \) whenever \( 1 \leq \alpha < \beta \). Combining this with our first observation, we have that \( \Sigma^0_\alpha \subseteq \Delta^0_\beta \) for all such \( \alpha \) and \( \beta \). Since complements of \( \Delta^0_\beta \) sets are also \( \Delta^0_\beta \), we get the other half of (1)(a).

(1)(b) and (1)(c). The open sets of any space are closed under arbitrary unions. For \( \alpha > 1 \), the closure of \( \Sigma^0_\alpha \) under countable unions is immediate from the definition. The closure of \( \Pi^0_\alpha \) under countable intersections follows from the closure of \( \Sigma^0_\alpha \) under countable unions. The open sets of any space are closed under finite intersections. Let \( \alpha > 1 \) and \( j \in \omega \) and let \( A_i \in \Sigma^0_\alpha \) for \( i < j \). For each \( i < j \), let \( A_{i,n} \subseteq \Pi^0_\gamma \) for all such \( \alpha \) and \( \beta \). Since complements of \( \Delta^0_\beta \) sets are also \( \Delta^0_\beta \), we get the other half of (1)(a).

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(2). By (1)(a),

\[
\bigcup_{1 \leq \alpha < \omega_1} \Sigma^0_\alpha \subseteq \bigcup_{1 \leq \alpha < \omega_1} \Pi^0_\alpha.
\]

Hence \( \bigcup_{1 \leq \alpha < \omega_1} \Sigma^0_\alpha \) is closed under complements. If \( A \) is a countable subset of \( \bigcup_{1 \leq \alpha < \omega_1} \Sigma^0_\alpha \), then there is a countable ordinal \( \delta \) such that \( A \subseteq \bigcup_{1 \leq \alpha < \delta} \Sigma^0_\alpha \subseteq \bigcup_{1 \leq \alpha < \delta+1} \Pi^0_\alpha \). Hence \( \bigcup A \in \Sigma^0_{\delta+1} \). We have then that \( \bigcup_{1 \leq \alpha < \omega_1} \Sigma^0_\alpha \) is a class containing the open sets and closed under countable unions and complements. By definition, this means that every Borel set belongs to \( \bigcup_{1 \leq \alpha < \omega_1} \Sigma^0_\alpha \).

The fact that every \( \Sigma^0_\alpha \), \( 1 \leq \alpha < \omega_1 \), is Borel is proved by an easy induction on \( \alpha \).

It follows from part (2) of the lemma that, for all \( \alpha \geq \omega_1 \), \( \Sigma^0_\alpha = \Pi^0_\alpha = \Delta^0_\alpha \) is the class of all Borel sets. If, e.g., \( T = \mathcal{X} \) and the cardinal number \( |X| \) of \( X \) is at least 2, then the Borel hierarchy does not collapse before \( \omega_1 \), i.e. \( \Delta^0_\alpha \subseteq \Pi^0_\alpha \subseteq \Delta^0_\beta \) whenever \( 1 \leq \alpha < \beta < \omega_1 \). (See Exercise 1.F.6 of [Moschovakis, 2009].)

The “\( 0 \)” in “\( \Sigma^0_\alpha \)” means that the sets in the class are definable by quantification over objects of type 0, i.e. natural numbers: Countable unions
correspond to existential quantification over natural numbers; countable intersections correspond to universal quantification over natural numbers. The “\(\alpha\)” in “\(\Sigma^0_\alpha\)” means that there are \(\alpha\) alternations of universal and existential quantifiers, and the “\(\Sigma\)” means that the first quantifier is existential.

For example the \(\Sigma^0_2\) sets are just those sets \(A\) such that there is a clopen \(B \in [T] \times \omega^2\) such that

\[
(\forall x \in [T])(x \in A \leftrightarrow (\exists m_1)(\forall m_2)(x, m_1, m_2) \in B).
\]

In later chapters we will introduce classes \(\Sigma^1_n\).

In the exercises we will sometimes deal with the effective Borel hierarchy of subsets of \(\omega^\omega\). (The reader not familiar with recursion theory can skip the definition that follows and skip also the relevant exercises.) For simplicity we stick to finite levels of that hierarchy. \(A \subseteq \omega^\omega\) belongs to \(\Sigma^0_n\), for \(n \geq 1\), if there is a recursive \(B \subseteq \omega^\omega \times n^\omega\) such that

\[
(\forall x)(x \in A \leftrightarrow (\exists m_1)(\forall m_2)(\exists m_3) \cdots (Q m_n)(x, m_1, m_2, m_3, \ldots m_n) \in B).
\]

\(A \in \Pi^0_n\) if \(\neg A \in \Sigma^0_n\). \(\Delta^0_n = \Sigma^0_n \cap \Pi^0_n\). It is fairly easy to see that if we replace “recursive” by “clopen” in this definition, we get the ordinary finite Borel hierarchy. If \(x \in \omega^\omega\), then we define \(\Sigma^0_n(x)\), \(\Pi^0_n(x)\), and \(\Delta^0_n(x)\) by replacing “recursive” by “recursive in \(x\)”. It is fairly easy to see that, e.g., \(\Sigma^0_n = \bigcup_{x \in \omega^\omega} \Sigma^0_n(x)\). (See page 160 of [Moschovakis, 1980].)

We end this section by listing the formal ZFC (Zermelo–Fraenkel, with Choice) axioms for set theory. These axioms will play no explicit role until §1.3, and even there and in §1.4, all the proofs in the text should be readable by someone unfamiliar with formal set theory and ZFC.

First order logic has the symbols

\[
(, , \neg, \land, \exists, =,
\]

together with variables

\[
v_0, v_1, v_2, \ldots.
\]

We assume the reader has enough familiarity with symbolic logic to know that, e.g., “\(\land\)” is interpreted to mean “and.” We will often be careless about what are our official variables, connectives, and quantifiers. One make think of use of symbols other than the official ones as abbreviation. We will also be careless about parentheses.

The language of set theory has, in addition to the symbols of first order logic, the two-place predicate symbol \(\in\). The formulas of the language of set theory are defined inductively as the smallest class satisfying the following.
(a) If \( x \) and \( y \) are variables, then \( x = y \) and \( x \in y \) are (atomic) formulas.

(b) If \( \varphi \) is a formula, then so is \( \neg \varphi \).

(c) If \( \varphi \) and \( \psi \) are formulas, then \( (\varphi \land \psi) \) is a formula;

(d) If \( \varphi \) is a formula and \( x \) is a variable, then \( (\exists x) \varphi \) is a formula.

An occurrence of a variable in a formula is free if it is not in the scope of a quantifier, i.e., if it is not in a subformula of the form \( (\exists x) \varphi \). When we write, e.g., \( \varphi(v_1, \ldots, v_n) \), we imply that only variables among \( v_1, \ldots, v_n \) occur free in \( \varphi \).

Following are the formal ZFC axioms. In stating them we make use of some standard abbreviations, whose definitions the reader should be able to give. For example, we write \( \emptyset \) for the empty set (whose existence and uniqueness follows from the Axioms of Empty Set, Comprehension, and Extensionality), so that \( x = \emptyset \) abbreviates \( \neg(\exists y) y \in x \). A perhaps less familiar abbreviation is \( (\exists! x) \varphi(x, z_1, \ldots, z_n) \), which abbreviates

\[
(\exists x)(\varphi(x, z_1, \ldots z_n) \land (\forall y)(\varphi(y, z_1, \ldots, z_n) \rightarrow y = x)).
\]

We precede the statement of each formal axiom by a parenthetical informal version of the axiom.

**Empty Set:** (There is a set with no members.)

\[
(\exists x)(\forall y) y \notin x.
\]

**Extensionality:** (Two sets with the same members are identical.)

\[
(\forall x)(\forall y)((\forall z)(z \in x \leftrightarrow z \in y) \rightarrow x = y).
\]

**Comprehension** (Axiom Schema): (Every definable subcollection of a set is a set.) For formulas \( \varphi(x, u, w_1, \ldots, w_n) \),

\[
(\forall w_1) \cdots (\forall w_n)(\forall u)(\exists v)(\forall x)(x \in v \leftrightarrow x \in u \land \varphi).
\]

**Foundation:** (Every nonempty set has a \( \in \)-minimal member.)

\[
(\forall x)(x \neq \emptyset \rightarrow (\exists y \in x) y \cap x = \emptyset).
\]

**Pairing:** (For any sets \( x \) and \( y \), there is a set whose members are precisely \( x \) and \( y \).)

\[
(\forall x)(\forall y)(\exists z)(\forall w)(w \in z \leftrightarrow (w = x \lor w = y)),
\]
1.1. BASIC DEFINITIONS

Union: (For any set \( x \), there is a set of all the members of members of \( x \).)

\[
(\forall x)(\exists u)(\forall z)(z \in u \leftrightarrow (\exists y)(z \in y \land y \in x)).
\]

Infinity: There is a set \( x \) such that \( \emptyset \in x \) and such that \( x \) is closed under the operation \( y \mapsto y \cup \{ y \} \).

\[
(\exists x)(\emptyset \in x \land (\forall y \in x) y \cup \{ y \} \in x).
\]

Replacement (Axiom Schema): (If \( F \) is a definable operation and the domain of \( F \) is a set, then the range of \( F \) is a set.) For formulas \( \varphi(x, y, u, w_1, \ldots, w_n) \),

\[
(\forall w_1) \cdots (\forall w_n)(\forall u)((\forall x \in u)(\exists! y) \varphi
\rightarrow (\exists v)(\forall y)(y \in v \leftrightarrow (\exists x \in u) \varphi)).
\]

Power Set: For any set \( x \), there is a set of all the subsets of \( x \).

\[
(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow z \subseteq x).
\]

Choice: If \( x \) is any set of disjoint nonempty sets, then there is a set \( u \) that has exactly one member in common with each member of \( x \).

\[
(\forall x)((\forall y \in x)(y \neq \emptyset \land (\forall z \in x)(y \neq z \rightarrow y \cap z = \emptyset)))
\rightarrow (\exists u)(\forall y \in x)((\exists! w) w \in y \cap u)).
\]

In formal logic the Empty Set Axiom is superfluous; for the existence of some object is provable, and so the existence of \( \emptyset \) follows by Comprehension.

Exercise 1.1.1. Let \( A \subseteq \omega^\omega \) with \( |A| < 2^{\aleph_0} \). Prove that II has a winning strategy for \( G(A; \prec_\omega \omega) \).

Exercise 1.1.2. Prove that not every game \( G(A; \prec_\omega \omega) \) is determined.

**Hint.** Use the Axiom of Choice to wellorder the set of all strategies in \( \prec_\omega \omega \) in a sequence of order type \( 2^{\aleph_0} \). Now diagonalize to get an \( A \) for which no strategy is winning. This is the proof in [Gale and Stewart, 1953], and it is the most direct one. There are many other proofs. Unpublished work of Banach and Mazur gives a proof which proceeds by showing that AD implies that all sets of reals have the property of Baire. See pages 298-300 of [Moschovakis, 1980], page 114 of [Mauldin, 1981], and [Oxtoby, 1957]. For another proof, see Exercise 1.1.3.
CHAPTER 1. ELEMENTARY METHODS

Exercise 1.1.3. Let $T^*$ be the game tree plays in which are as follows:

<table>
<thead>
<tr>
<th></th>
<th>$s_0$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>$a_0$</td>
<td>$a_1$</td>
<td></td>
<td>...</td>
</tr>
</tbody>
</table>

where each $s_i \in \omega \omega$ and each $a_i \in \{0, 1\}$. For any $A \subseteq \omega^2$, let

$$A^* = \{\langle s_0, a_0, s_1, a_1, \ldots \rangle \mid s_0 \upharpoonright \langle a_0 \rangle \upharpoonright s_1 \upharpoonright \langle a_1 \rangle \upharpoonright \ldots \in A\}$$

and let $G^*(A) = G(A^*; T^*)$.

(a) Prove that I has a winning strategy for $G^*(A)$ if and only if $A$ has a perfect subset (a non-empty closed subset without isolated points).

(b) Prove that II has a winning strategy for $G^*(A)$ if and only if $A$ is countable.

(c) Use the Axiom of Choice to construct an uncountable subset of $\omega^2$ with no perfect subset.

Remark. This is a result of [Davis, 1964].

Exercise 1.1.4. Prove, in ZF (i.e., in ZFC without the Axiom of Choice) that not every game $G(A; \omega \omega_1)$ is determined. (Recall that $\omega_1$ is the least uncountable ordinal number, i.e. the set of all countable ordinal numbers.) This result appears in [Mycielski, 1964].

Hint. Use Exercise 1.1.3 to show that it follows from AD that there is no one-one $f : \omega_1 \rightarrow \omega^2$. (Assume such an $f$ exists and get a one-one $g : \omega_1 \rightarrow \mathbb{R}$. Then use the existence of a perfect subset of the range of $g$ to get a one-one $h : \mathbb{R} \rightarrow \omega_1$, and show that this contradicts AD.) Now consider the game $G(A; \omega \omega_1)$, where $A$ is the set of all $x : \omega \rightarrow \omega_1$ such that $x(0) \geq \omega$ and \{\langle m, n \rangle \mid x(2m + 1) < x(2n + 1)\} is not a wellordering of $\omega$ of order type $x(0)$.

Exercise 1.1.5. Assume that all $\Sigma^0_7$ games in countable trees are determined, and prove that this still holds when we broaden our notion of games to allow a move function as on page 4 above. (Obviously $\Sigma^0_7$ is just an example.)

1.2 Open Games

The main result of this section is Theorem 1.2.4, the important basic theorem of [Gale and Stewart, 1953] that all open games are determined. We will also
introduce and study the technical concept of a quasistrategy, a concept that will be the main tool in the rest of this chapter.

The fact that all games in finite trees are determined is usually attributed to [Zermelo, 1913]. (See page 371 of [Kanamori, 1994] for a discussion.) The proof of this fact works with very little change to give a proof of determinacy for the case of trees without infinite plays.

**Theorem 1.2.1.** If there are no infinite plays in $T$, then $G(A; T)$ is determined for every $A \subseteq [T]$.

**Proof.** The Theorem follows easily from the following lemma.

**Lemma 1.2.2.** If $G(A; T_p)$ is not determined, then there is a Move $q$ at $p$ such that $G(A; T_q)$ is not determined. (Recall the definition of “Move” on page 4 and recall that $G(A; T_q)$ is $G(A \cap [T_q]; T_q)$.)

**Proof of Lemma.** Assume that $G(A; T_p)$ is not determined. Assume for definiteness that $p$ has even length. (The other case is similar.)

If $q$ is a legal Move at $p$, then I does not have a winning strategy for $G(A; T_q)$. If he had such a strategy $\sigma'$, then that strategy together with the move $q$ would give him a winning strategy $\sigma$ for $G(A; T_p)$:

$$\sigma(r) = \begin{cases} q(fh(r)) & \text{if } r \subseteq p; \\ \sigma'(r) & \text{if } q \subseteq r. \end{cases}$$

(technically we should also define $\sigma(r)$ in the third case: $p \subseteq r \land q \not\subseteq r$. The reason we omitted this case is that such positions $r$ are not consistent with $\sigma$. We could have defined strategy so that strategies take as arguments only positions consistent with them. see Exercise 1.2.3 for a minor reason for doing so.)

It suffices then to show that there is a Move $q$ at $p$ such that II does not have a winning strategy for $G(A; T_q)$. If there is no such $q$, then for each Move $q$ at $p$ there is a winning strategy $\tau_q$ for II for $G(A; T_q)$. We then get a winning strategy $\tau$ for II for $G(A; T_p)$ by setting

$$\tau(r) = \begin{cases} p(fh(r)) & \text{if } r \subseteq p; \\ \tau_q(r) & \text{if } q \subseteq r \land q \text{ is a Move at } p. \end{cases}$$

(We can describe $\tau$ more briefly as $\bigcup q \tau_q$.) This contradiction shows that $q$ must exist and completes the proof of the lemma. □
Now let us prove the theorem by proving its contrapositive. Suppose that $G(A; T)$ is not determined. Repeated applications of the lemma give us a sequence $p_0 \subset p_1 \subset p_2 \subset \ldots$ of elements of $T$. There is an infinite play $x$ such that $x \supseteq p_i$ for all $i$. □

Remark. Both the proof of the lemma and the proof of the theorem from the lemma use the Axiom of Choice. If we strengthen the hypothesis of the theorem to make $T$ wellfounded (equivalent in the presence of Choice to our hypothesis that $T$ has no infinite plays), then the latter use of the Axiom of Choice is avoided. (See Exercise 1.2.1.) The former use is necessary even for trees which contain only positions of length $\leq 2$. (See Exercise 1.2.2.) Of course, Choice is not needed to prove the theorem for a $T$ that has a canonical wellordering, as does our main example $<\omega$.$\omega$.

Corollary 1.2.3. All clopen games are determined.

Proof Let $A \subseteq [T]$ with $A$ clopen. For each $x \in [T]$ there is a $p \subseteq x$ such that $[T_p] \subseteq A$ or $[T_p] \subseteq \neg A$. This is because both $A$ and $\neg A$ are open and so are the unions of their basic open subsets. Let

$$T^* = \{ q \in T \mid (\forall p \subset q) ([T_p] \cap A \neq \emptyset \land [T_p] \cap \neg A \neq \emptyset) \}. $$

The game tree $T^*$ has no infinite plays: If $x$ is an infinite play in $T^*$, then $x$ is a play in $T$ and so there is a $p \subseteq x$ such that $[T_p] \subseteq A$ or $[T_p] \subseteq \neg A$. But the definition of $T^*$ gives the contradiction that $p$ is terminal in $T^*$. Let

$$A^* = \{ x \in [T^*] \mid (\exists y \in [T])(x \subseteq y \land y \in A) \}. $$

By Theorem 1.2.1, $G(A^*; T^*)$ is determined. Assume for definiteness that $\sigma^*$ is a winning strategy for I for $G(A^*; T^*)$. Let $\sigma$ be any strategy for I in $T$ such that $\sigma$ agrees with $\sigma^*$ on non-terminal positions in $T^*$. We show that $\sigma$ is a winning strategy for $G(A; T)$. Let $x \in [T]$ be consistent with $\sigma$. There is a $p \subseteq x$ that is terminal in $T^*$. Either $[T_p] \subseteq A$ or else $[T_p] \subseteq \neg A$. But $p$ is consistent with $\sigma^*$, so $[T_p] \cap A \neq \emptyset$ and this means that $x \in A$. □

The following terminology will be convenient in many of the proofs that follow. By $G$ is a win for I we mean that there is a winning strategy for I for $G$. Similarly define $G$ is a win for II.
Theorem 1.2.4. ([Gale and Stewart, 1953]) All open games are determined. All closed games are determined.

Proof The first assertion implies the second: If $A \subseteq [T]$ is closed, let

$$T' = \{\emptyset\} \cup \{(0)^{-p} \mid p \in T\}; \quad A' = \{(0)^{-x} \mid x \notin A\}.$$ 

The open game $G(A'; T')$ is just $G(A; T)$ with the roles of the players reversed via the dummy initial move 0. If the former is determined then so is the latter.

Lemma 1.2.5. Let $T, A,$ and $p \in T$ be arbitrary and assume that $G(A; T_p)$ is not a win for $I$.

(i) If $\ell_h(p)$ is even then there is no Move $q$ at $p$ such that $G(A; T_q)$ is a win for $I$.

(ii) If $\ell_h(p)$ is odd then there is a Move $q$ at $p$ such that $G(A; T_q)$ is not a win for $I$.

Proof of Lemma. The proof of Lemma 1.2.2 essentially contains the proof of the present lemma, so we will be brief. (i) If there is a Move $q$ at $p$ such that $G(A; T_q)$ is a win for $I$, then $I$ can win $G(A; T_p)$ by first playing $q$ and then playing (the moves given by) a winning strategy for $G(A; T_q)$. (ii) If $\sigma_q$ is a winning strategy for $I$ for $G(A; T_q)$ for each Move $q$ at $p$, then $\bigcup_q \sigma_q$ is a winning strategy for $I$ for $G(A; T_p)$. □

Returning to the proof of the theorem, let us assume that $A \subseteq [T]$ is open and that $G(A; T)$ is not a win for $I$. We will prove that there is a winning strategy $\tau$ for $\Pi$ for $G(A; T)$. For each position $p$ of odd length such that $G(A; T_p)$ is not a win for $I$, choose a move $\tau(p)$ at $p$ such that $G(A; T_p^\tau(p))$ is not a win for $I$. Part (ii) of the lemma gives the existence of such a move. For other positions of odd length, let $\tau(p)$ be arbitrary. Let $x$ be a play consistent with $\tau$. By induction, using part (i) of the lemma, we get that each $p \subseteq x$ is such that $G(A; T_p)$ is not a win for $I$. But $A$ is open. If $x \in A$ then $x \in [T_p]$ for some $p$ such that $[T_p] \subseteq A$. For any such $p$, $p \subseteq x$ and $G(A; T_p)$ is obviously a win for $I$. This contradiction gives that $x \notin A$. This in turn shows that $\tau$ is a winning strategy for $\Pi$. □

Lemma 1.2.5 has other applications besides Theorem 1.2.4. For making such applications, it will be useful to reformulate the lemma, which we now do.

A quasistrategy for $\Pi$ in $T$ is a game subtree $T'$ of $T$ such that
(a) if \( p \in T' \) and \( \ell h(p) \) is even, then every Move in \( T \) at \( p \) belongs to \( T' \);

(b) if \( p \in T' \), \( \ell h(p) \) is odd, and \( p \) is not terminal in \( T \), then some Move at \( p \) in \( T \) belongs to \( T' \).

(Note that a subtree \( T' \) of \( T \) satisfying (a) and (b) is automatically a game subtree of \( T \) and so a quasistrategy for II in \( T \).) Quasistrategies for I are similarly defined.

Every strategy \( \tau \) for II in \( T \) gives rise to a quasistrategy for II in \( T \): Let \( T' = \{ p \in T \mid p \) is consistent with \( \tau \} \). Except for irrelevancies, \( T' \) determines \( \tau \): \( T' \) determines \( \tau(p) \) for all positions \( p \) consistent with \( \tau \). The special property distinguishing the quasistrategy determined by a strategy from a general quasistrategy is that in (b) “some” can be replaced by “one and only one.” Thus we may think of a quasistrategy as a many-valued strategy. Quasistrategies are often useful in situations where one is not assuming the Axiom of Choice. But they are also useful in proofs of determinacy, as the rest of this chapter will show. Quasistrategies for II in \( T \) are sometimes called II-imposed subtrees (or subgames) of \( T \).

The following Lemma is really just a reformulation of Lemma 1.2.5 (and its dual).

**Lemma 1.2.6.** (1) If \( G(A;T) \) is not a win for I, then

\[ \{ q \in T \mid (\forall p \subseteq q) \ G(A;T_p) \text{ is not a win for I} \} \]

is a quasistrategy for II.

(2) If \( G(A;T) \) is not a win for II, then

\[ \{ q \in T \mid (\forall p \subseteq q) \ G(A;T_p) \text{ is not a win for II} \} \]

is a quasistrategy for I.

**Proof.** For (1), assume that \( G(A;T) \) is not a win for I and let \( T' = \{ q \in T \mid (\forall p \subseteq q) \ G(A;T_p) \text{ is not a win for I} \} \). Clearly \( T' \) is a subtree of \( T \). Property (a) for \( T' \) follows from (i) of Lemma 1.2.5. Property (b) follows from (ii). (2) similarly follows from the obvious variant of Lemma 1.2.5. \( \square \)

Whenever \( G(A;T) \) is not a win for I, let us call

\[ \{ q \in T \mid (\forall p \subseteq q) \ G(A;T_p) \text{ is not a win for I} \} \]
II’s non-losing quasistrategy for $G(A; T)$, . Similarly define I’s non-losing quasistrategy for $G(A; T)$ when $G(A; T)$ is not a win for II. The proof of Theorem 1.2.4 from Lemma 1.2.5 amounted to showing that, for $A$ open, II’s non-losing quasistrategy for $G(A; T)$ is a winning quasistrategy, in the obvious sense.

**Lemma 1.2.7.** (1) If $G(A; T)$ is not a win for I and $T'$ is II’s non-losing quasistrategy, then $G(A; T')$ is not a win for I.

(2) If $G(A; T)$ is not a win for II and $T'$ is I’s non-losing quasistrategy, then $G(A; T')$ is not a win for II.

**Proof.** We prove (1). Suppose that $\sigma$ is a winning strategy for I for $G(A; T')$. Then I can win $G(A; T)$ by playing $\sigma$ until (if ever) II first departs from $T'$ at some position $p$ and then playing a winning strategy for $G(A; T_p)$. □

**Exercise 1.2.1.** A game tree $T$ is wellfounded if for every nonempty $Y \subseteq T$ there is a terminal element $p$ of $T \cap Y$, i.e. a $p \in T \cap Y$ such that no $q$ properly extending $p$ belongs to $T \cap Y$.

(a) Prove that $T$ is wellfounded if and only if there are no infinite plays in $T$. (The “if” direction will require the Axiom of Choice.)

(b) Assume that Lemma 1.2.2 holds for $A$ and $T$ and prove in ZF (i.e., don’t use the Axiom of Choice) that if $T$ is wellfounded then $G(A; T)$ is determined.

**Exercise 1.2.2.** Show that the Axiom of Choice is equivalent in ZF with the determinacy of all games in trees $T$ such that every $p \in T$ has length $\leq 2$.

**Exercise 1.2.3.** Working in ZF, assume that the Axiom of Choice is false. Prove that there are $A$ and $T$ such that (i) there is a play of length 1 belonging to $A$ but (ii) $G(A; T)$ is not determined. (This shows that, in the absence of Choice, it would be more natural to define strategy as suggested during the proof of Lemma 1.2.2.)

**Exercise 1.2.4.** Let $A \subseteq [T]$ be open. For each ordinal number $\alpha$, we define $P_\alpha$, a set of positions of even length in $T$. The definition proceeds by transfinite induction on $\alpha$. Let $p \in P_\alpha$ if and only if $[T_p] \subseteq A$. For $\alpha > 0$, $p \in P_\alpha$ if and only if $p \in P_0$ or there is a Move $q$ at $p$ such that either $q \in A$ or $q$ is not terminal and, for every Move $r$ at $q$, $r \in \bigcup_{\beta < \alpha} P_\beta$. First show that there is an $\alpha$ such that $(\forall \gamma \geq \alpha) P_\gamma = P_\alpha$. Now let $P_\infty$ be this limiting value.
of $P_\alpha$. Show that $G(A; T)$ is a win for I if the initial position $\emptyset \in P_\infty$ and that $G(A; T)$ is a win for II if $\emptyset \notin P_\infty$. (This is a more constructive proof of Theorem 1.2.4. It was independently noticed by several people. See Blass [1972] for a related result.)

**Exercise 1.2.5.** Use the construction of Exercise 1.2.4 to prove that, if $A \subseteq \omega_\omega$, $A \in \Sigma^0_1$, and $G(A; \leq \omega_\omega)$ is a win for I, then there is a winning strategy for I belonging to $L(\beta)$ for $\beta$ the least admissible ordinal greater than $\omega$. Prove also for such $A$ and for the same $\beta$, that if $G(A; \leq \omega_\omega)$ is a win for II then there is a winning strategy for II belonging to $L(\beta + 1)$. (The literal construction of Exercise 1.2.4 doesn’t quite work; modify the definition of $P_0$ to get $P_0 \in L(\beta)$.)

### 1.3 The Theorems of Wolfe and Davis

In §2.1 we will prove that all Borel games are determined. Nevertheless, the remaining two sections of this chapter will be devoted to proofs of partial results that will not be used in the proof in §2.1. What is of interest about these proofs is that in essence they do not use the Power Set and Replacement Axioms of ZFC (though one of them does use something that goes beyond the other standard ZFC axioms). A striking result of [Friedman, 1971], proved before Borel determinacy, implies that both Power Set and Replacement are needed to prove that all Borel games (even in countable trees) are determined. This is surprising because almost all theorems of mathematics can be proved in *Zermelo Set Theory* (ZC): ZFC without the Axiom of Replacement but with Comprehension. Moreover the assertion that all Borel games in countable trees are determined concerns only countable objects, whereas Friedman’s result might be described as implying that Borel determinacy cannot be proved without invoking principles about uncountable objects.

In the next two sections and in §2.3, we want to avoid using Power Set and Replacement whenever we can. However, in order not to get embroiled in technicalities, it is convenient to have available always a small part of the Axiom of Replacement. To describe the appropriate theory, we need to introduce the *Lévy hierarchy* of formulas of the language of set theory.

First we define the *bounded* formulas as constituting the smallest class satisfying the following:

(a) Every atomic formula is bounded.
(b) If $\varphi$ is bounded, then so are $(\exists x)(x \in y \land \varphi)$ and $(\forall x)(x \in y \rightarrow \varphi)$.

A formula is called $\Sigma_0$ and also $\Pi_0$ if it is bounded. For $n \in \omega$, a formula is $\Sigma_{n+1}$ if it is $(\exists x)\varphi$ for some variable $x$ and some $\Pi_n$ formula $\varphi$; it is $\Pi_{n+1}$ if it is $(\forall x)\varphi$ for some variable $x$ and some $\Sigma_n$ formula $\varphi$.

The theory in which we will work in most of the next two sections is $\text{ZC}^- + \Sigma_1$ Replacement: ZFC without the Axiom of Power Set and with the Axiom of Replacement only for $\Sigma_1$ formulas. Another way to describe this theory is that it is Kripke-Platek set theory with Choice (KPC) plus Comprehension. The point of $\Sigma_1$ Replacement is that it gives us cartesian products, enough ordinal numbers, and some simple definitions by transfinite recursion. We could get by without $\Sigma_1$ Replacement, but then we would have to be careful how we formulate some of our theorems as well as how we prove them. With respect to the absence of the Power Set axiom, the reader not familiar with formal axiomatic set theory should notice that the sets we deal with in proofs about games in a tree $T$ are subsets of $T$ or are other sets of no greater size than $T$. Sometimes we mention larger sets, e.g. $[T]$ and subsets of $[T]$. Talk of $[T]$ is eliminable in simple ways: for example, instead of “$(\forall x)(x \in [T] \rightarrow \ldots)$,” we can say “$(\forall x)((\forall p \subseteq x) p \in T) \rightarrow \ldots$.” Our talk of subsets of $[T]$ will be almost always be eliminable because the subsets in question will be Borel sets, and therefore they can be specified by countable systems of subsets of $T$: To specify a Borel set, it is enough to describe how it is built up a countable family of open sets; the open sets $A$ themselves are given by the set of $p \in T$ such that $[T_p] \subseteq A$. Lemma 1.4.1 gives another way to specify a Borel set: via a clopen subset of $[T] \times [S]$, with $S$ a countable tree.

*The proofs of all results in §1.1 and §1.2 go through in $\text{ZC}^- + \Sigma_1$ Replacement.*

In this section we will prove, in $\text{ZC}^- + \Sigma_1$ Replacement, determinacy for Borel levels through $\Sigma_0^3$.

**Theorem 1.3.1.** ([Wolfe, 1955]; $\text{ZC}^- + \Sigma_1$ Replacement) All $\Sigma_0^3 (F_\sigma)$ games are determined.

**Proof.** We first prove the following lemma.

**Lemma 1.3.2.** Let $B \subseteq A \subseteq [T]$ with $B$ closed. If $G(A;T)$ is not a win for $I$, then there is a strategy $\tau$ for $\Pi$ such that every play consistent with $\tau$ contains a position $p$ with these properties:


(i) $[T_p] \cap B$ is empty.  
(ii) $G(A; T_p)$ is not a win for $I$.

**Proof of Lemma.** Assume that $G(A; T)$ is not a win for $I$. Let $C$ be the set of all $x \in [T]$ such that no $p \subseteq x$ satisfies both (i) and (ii). The lemma asserts precisely that $G(C; T)$ is a win for $II$. Assume for a contradiction that this is false. $C$ is closed, so Theorem 1.2.4 implies that $G(C; T)$ is a win for $I$. Let $T'$ be $II$'s non-losing quasistrategy for $G(A; T)$. By Lemma 1.2.7, $G(A; T')$ is not a win for $I$. But $T'$ does not restrict $I$’s moves in $T$, so $G(C; T')$ is a win for $I$. Let $x \in [T']$ be consistent with $\sigma$. For every $p \in T'$, and so for every $p \subseteq x$, $G(A; T_p)$ is not a win for $I$; i.e., (ii) holds for every $p \subseteq x$. Thus (i) fails for every $p \subseteq x$. In other words $[T_p] \cap B$ is nonempty for every $p \subseteq x$. But $B$ is closed, so this implies that $x \in B$. $B \subseteq A$ and hence $x \in A$ also. Since $x$ was an arbitrary play consistent with $\sigma$, we have derived the contradiction that $\sigma$ is a winning strategy for $I$ for $G(A; T')$. □

For the proof of the theorem, let $A \subseteq [T]$ with $A \in \sum_0^3$. Then $A$ can be written as $A = \bigcup_{i \in \omega} A_i$ with each $A_i$ closed. Assume that $G(A; T)$ is not a win for $I$. We get a winning strategy $\tau$ for $II$ as follows. Here and on other occasions, we describe (the essential part of) a strategy by describing an arbitrary play consistent with the strategy. Let $\tau_0$ be as given by the lemma with $B = A_0$. Let $\tau$ agree with $\tau_0$ until a position $p_0$ is first reached satisfying (i) and (ii). Now apply the lemma with $B = A_1$ and $T_{p_0}$ for $T$, getting $\tau_1$. Let $\tau$ agree with $\tau_1$ from $p_0$ until a $p_1$ is first reached satisfying (i) and (ii). Continue in this way. If $\bigcup_{i \in \omega} p_i$ is finite and non-terminal, let $\tau$ be arbitrary on positions extending $\bigcup_{i \in \omega} p_i$. Let $x$ be consistent with $\tau$. For each $i$, there is a $p_i \subseteq x$ with $[T_p] \cap A_i = \emptyset$. Hence $x \notin \bigcup_{i \in \omega} A_i$; i.e., $x \notin A$. □

**Theorem 1.3.3.** ([Davis, 1964]; ZC$^-$ + $\Sigma_1$ Replacement) All $\sum_0^3 (G_{\delta \sigma})$ games are determined.

**Proof** We first prove a lemma analogous to Lemma 1.3.2.

**Lemma 1.3.4.** Let $B \subseteq A \subseteq [T]$ with $B \in \Pi_2^0$. If $G(A; T)$ is not a win for $I$, then there is a quasistrategy $T^*$ for $II$ with the following properties:

(i) $[T^*] \cap B$ is empty.
1.3. THE THEOREMS OF WOLFE AND DAVIS

(ii) \( G(A; T^*) \) is not a win for I.

Proof of Lemma. Assume that \( G(A; T) \) is not a win for I. Let \( T' \) be II's non-losing quasistrategy for \( G(A; T) \). Note that, for each \( p \in T' \), \( T'_p \) is II's non-losing quasistrategy for \( G(A; T_p) \); thus by Lemma 1.2.7 every \( p \in T' \) is such that \( G(A; T'_p) \) is not a win for I.

Let us call a position \( p \) in \( T' \) good if there is a quasistrategy \( T^* \) for II in \( T'_p \) such that (i) \( [T^*] \cap B \) is empty and (ii) \( G(A; T^*) \) is not a win for I. The lemma will be proved if we can show that the initial position \( \emptyset \) is good. Define a quasistrategy \( T \) such that (i) \( \emptyset \) is good. Let \( T \) be a winning strategy for I for \( G \). Thus there is an \( B \in \Pi_2^0 \), so let \( B = \cap_{n \in \omega} D_n \) with each \( D_n \) open. For each \( n \) let

\[
E_n = A \cup \{ x \in [T'] \mid (\exists q \subseteq x)([T'_q] \subseteq D_n \land p \text{ is not good}) \}.
\]

Fix \( n \) and assume that \( G(E_n; T') \) is not a win for I. We show that \( \emptyset \) is good. Define a quasistrategy \( T^* \) for II in \( T' \) as follows: \( T^* \) agrees with II's non-losing quasistrategy \( T'' \) for \( G(E_n; T') \) until first (if ever) a position \( p \) is reached with \( [T'_p] \subseteq D_n \). Consider a first such \( p \) reached. Since \( p \) belongs to \( T'' \), \( p \) must be good. Choose a quasistrategy \( \hat{T}(p) \) for II witnessing that \( p \) is good. Let \( T^* \) agree with \( \hat{T}(p) \) for \( q \supseteq p \). We will show that \( T^* \) witnesses that \( \emptyset \) is good. If \( x \in [T^*] \) then either \( x \notin D_n \) or else \( x \) belongs to some \([\hat{T}(p)]\) and so \( x \notin B \) by (i). Thus \([T^*] \subseteq \neg D_n \cup \neg B = \neg B \), and we need only show that \( G(A; T^*) \) is not a win for I. Suppose to the contrary that \( \sigma \) is a winning strategy for I for \( G(A; T^*) \). If there is a position \( p \) consistent with \( \sigma \) such that \([T'_p] \subseteq D_n \), then there is such a \( p \) such that \( T^* = \hat{T}(p) \). \( \hat{T}(p) \) has property (ii) and so \( G(A; T'_p) \) is not a win for I. But then \( \sigma \) cannot be a winning strategy for \( G(A; T^*) \). Hence no such \( p \) can exist, and so every play consistent with \( \sigma \) belongs to \([T'']\). By Lemma 1.2.7, \( G(E_n; T'') \) is not a win for I. Thus there is an \( x \in [T''] \) such that \( x \) is consistent with \( \sigma \) and \( x \notin E_n \). \( A \subseteq E_n \), and so \( x \notin A \). Therefore \( \sigma \) is not a winning strategy. This contradiction completes the proof that \( T^* \) witnesses that \( \emptyset \) is good.

The argument just given has shown that \( \emptyset \) is good unless, for each \( n \in \omega \), \( G(E_n; T') \) is a win for I. For \( p \in T' \) and \( n \in \omega \), let

\[
E^p_n = A \cup \{ x \in [T'_p] \mid (\exists q \subseteq x)(p \subseteq q \land [T'_q] \subseteq D_n \land q \text{ is not good}) \}.
\]

The same argument shows that, for all \( p \in T' \) and all \( n \in \omega \), \( p \) is good unless \( G(E^p_n; T'_p) \) is a win for I.
CHAPTER 1. ELEMENTARY METHODS

Assume that the lemma is false. We get a strategy $\sigma$ for I as follows. Let $\sigma_0$ be a winning strategy for I for $G(E_0; T')$. $\sigma$ agrees with $\sigma_0$ until first (if ever) a $p_0$ is reached with $[T'_p] \subseteq D_0$ and $p_0$ not good. If such a $p_0$ is reached, choose a winning strategy $\sigma_1$ for I for $G(E_1; T'_p)$. Let $\sigma$ agree with $\sigma_1$ from $p_0$ until a $p_1$ is first reached with $[T'_p] \subseteq D_1$ and $p_1$ not good. Continue in this way, letting $\sigma$ be arbitrary on positions extending $\bigcup_{n \in \omega} p_n$ if the latter is a non-terminal position. If some $p_n$ does not exist, then the play $x$ belongs to $E_n$ if $n = 0$ but there is no $p \subseteq x$ with $p_{n-1} \subseteq p$ if $n > 0$ and $[T'_p] \subseteq D_n$ and $p$ not good. By the definition of $E_n$, $x \in A$. If all $p_n$ exist, then the play $x$ belongs to $\bigcap_{n \in \omega} D_n = B \subseteq A$. Thus every play consistent with $\sigma$ belongs to $A$, contrary to the hypothesis that $G(A; T')$ is not a win for I.

Now let us prove the theorem. Let $A \subseteq [T]$ with $A \in \Sigma^0_3$. Let $A = \bigcup_{n \in \omega} A_n$ with each $A_n \in \Pi^0_2$. We get a strategy $\tau$ for II as follows. Apply the Lemma with $B = A_0$ to get $T^*(\emptyset)$. For positions $p_1 \in T$ of length 1, let $\tau(p_1)$ be an arbitrary move legal in II’s non-losing quasistrategy for $G(A; T^*(\emptyset))$. For any position $p_2$ consistent with $\tau$ and with $\ell h(p_2) = 2$, apply the lemma with $B = A_1$ and with $(T^*(\emptyset))_{p_2}$ for $T$, getting $T^*(p_2)$. For any position $p_3 \in T^*(p_2)$ with $\ell h(p_3) = 3$, let $\tau(p_3)$ be an arbitrary move legal in II’s non-losing quasistrategy for $G(A; T^*(p_3))$. Continue in this way. Let $x$ be a play consistent with $\tau$. If $x$ is finite, then $x$ belongs to II’s non-losing quasistrategy for $G(A; T^*(\emptyset))$, hence $(T^*(\emptyset))_x \not\subseteq A$, and so $x \notin A$. If $x$ is infinite, then $x \in \bigcap_{n \in \omega} [T^*(x \upharpoonright n)] \subseteq \bigcap_{n \in \omega} \neg A_n$, so $x \notin A$. Thus $\tau$ is a winning strategy for II for $G(A; T)$.

**Exercise 1.3.1.** Let $A \subseteq [T]$ with $A \in \Sigma^0_3$. Let $A = \bigcup_{n \in \omega} A_n$ with each $A_n$ closed. For each ordinal number $\alpha$, we define $P_{\alpha}$, a set of positions of even length in $T$. For $p \in T$ and $[T_p] \cap A \neq \emptyset$, let $n(p)$ be the least $n$ such that $[T_p] \cap A_n \neq \emptyset$. For each ordinal $\alpha$, let

$$B^\alpha_p = \{ x \in [T_p] \mid x \in A_{n(p)} \land (\exists q)(p \subseteq q \subseteq x \land q \in \bigcup_{\beta < \alpha} P_\beta) \}. $$

Let $p \in P_\alpha$ if and only if $n(p)$ is defined and $G(B^\alpha_p; T_p)$ is a win for I. As with Exercise 1.2.4, first show that there is an $\alpha$ such that $(\forall \gamma \geq \alpha) P_\gamma = P_\alpha$, and let $P_\infty$ be this limiting value of $P_\alpha$. Now show that $G(A; T)$ is a win for I if $\emptyset \in P_\infty$ and that $G(A; T)$ is a win for II if $\emptyset \notin P_\infty$. 

Exercise 1.3.2. Use the construction of Exercise 1.3.1 to prove Solovay’s result (see pages 414–415 of [Moschovakis, 1980]) that, if $A \subseteq \omega^\omega$, $A \in \Sigma^0_2$, and $G(A; T)$ is a win for I, then there is a winning strategy for I belonging to $L_\beta$ for $\beta$ the closure ordinal for $\Sigma^1_1$ monotone inductive definitions. Prove also that, for such $A$ and for the same $\beta$, that if $G(A; T)$ is a win for II then there is a winning strategy for II belonging to $L(\beta')$, where $\beta'$ the least admissible ordinal $> \beta$.

1.4 $\Delta^0_4$ Games

In this section we prove the determinacy of all $\Delta^0_4$ games. For countable trees, $\Delta^0_4$ coincides with the difference hierarchy on $\Pi^0_3$. (Theorem 1.4.2, a result in [Kuratowski, 1958]). For uncountable trees, $\Delta^0_4$ coincides with what we call the generalized difference hierarchy on $\Pi^0_3$. Because the proofs in this section are somewhat complicated, we first deal fully with the case of countable trees. We prove, in the countable case, the equality of the difference hierarchy with $\Delta^0_4$, and we prove (in the general case) determinacy for the difference hierarchy. Then we take up general trees, showing how to modify the definitions and proofs from the countable case to make them work in the general case.

In earlier versions of this chapter, we mistakenly claimed that our proof of determinacy for the difference hierarchy on $\Pi^0_3$ went through in $\text{ZC}^- + \Sigma_1$ Replacement. In [Montalban and Shore, 2012], the authors point out that only for fixed finite levels of that difference hierarchy does our proof go through in $\text{ZC}^- + \Sigma_1$ Replacement. They go on to demonstrate that the assertion that determinacy holds in countable trees for all finite levels cannot be proved in $\text{ZFC}^-$ (ZFC minus Power Set). This improves the known theorem, proved using the methods of [Friedman, 1971], that the determinacy of all $\Sigma^0_4$ games in countable trees cannot be proved in $\text{ZFC}^-$. (See Exercise 1.4.1.) Before giving our proof of determinacy for the full difference hierarchy, we present the simplification of that proof that results from adapting it to the case of a fixed finite level of the hierarchy. For this proof, we will not need to treat the case of countable trees separately.

For the both the fixed-finite-level case and the full difference hierarchy case, we first give determinacy proofs without paying attention to what hypotheses are being used. Afterward we discuss hypotheses. The determinacy of the full difference hierarchy needs a theory stronger than $\text{ZC}^- + \Sigma_1$ Re-
placement, but this theory does not imply the existence of uncountable sets. Moreover it, like ZC$^-$ + $\Sigma_1$ Replacement, does not imply that all $\Sigma^0_1$ games in countable trees are determined. (See Exercise 1.4.6.)

If $\Gamma$ is a class of sets (e.g., $\Pi^0_3$) and $\alpha > 0$ is a countable ordinal, then a set $A \subseteq \lceil T \rceil$ belongs to $\alpha$-$\Gamma$, the $\alpha$th level of the difference hierarchy on $\Gamma$, just in case there is a sequence $\langle A_\beta \mid \beta < \alpha \rangle$ with each $A_\beta \in \Gamma$ and such that

$$\forall x \in \lceil T \rceil)(x \in A \leftrightarrow \mu \beta(x \notin A_\beta \lor \beta = \alpha) \text{ is odd},$$

where “$\mu$” means “the least” and “odd” is defined in the natural way. (Limit ordinals are even.) The difference hierarchy is ordinarily defined only for classes $\Gamma$ closed under countable intersections (so that $\bigcap_{\gamma < \beta} A_\gamma \in \Gamma$) and we will consider it only for such classes. 1-$\Gamma = \Gamma$, 2-$\Gamma$ is the class of differences of sets belonging to $\Gamma$, etc. Let $\text{Diff}(\Gamma) = \bigcup_{\alpha < \omega_1} \alpha$-$\Gamma$. We are interested in the difference hierarchy because of Theorem 1.4.2, which states that $\text{Diff}(\Pi^0_\alpha) = \Delta^0_{\alpha+1}$. Before proving this result of Kuratowski we prove a characterization of $\Sigma^0_\alpha$ that will be useful for the proof.

A game tree $T$ is wellfounded if, for every nonempty $Y \subseteq T$, there is a $p$ in $T \cap Y$ such that no $q \supseteq p$ belongs to $T \cap Y$. Using the Axiom of Choice, one can show that $T$ is wellfounded if and only if there are no infinite plays in $T$. (Exercise 1.2.1.) For wellfounded $T$, the plays in $T$ are thus exactly the same as the terminal positions in $T$. If $T$ is wellfounded, then we can define functions with domain $T$ by transfinite recursion. (See Theorem 5.6 of [Kunen, 1980] or pages 82–83 of [Moschovakis, 1980].) To define an $f$ with domain $(f) = T$, it is enough to define $f(p)$ in terms of the restriction of $f$ to the proper extensions of $p$, i.e. to define an operation $G$ (which may be a proper class) and let

$$f(p) = G(f \upharpoonright \{q \mid p \subset q\})$$

for each $p \in T$. (Since the existence of $f$ is proved by using the Axiom of Replacement, we will in this section use definition by transfinite recursion only for $G$’s definable by a simple enough formula that $\Sigma_1$ Replacement suffices to get $f$.) When we make such a definition, we will say that we are defining $f$ by induction on $T$. One can talk in more general terms of wellfounded relations: A relation $\prec$ is wellfounded if, for every nonempty set $Y$, there is an element of $Y$ minimal with respect to $\prec$. Thus a game tree $T$ is wellfounded if and only if $\not\exists T$ is a wellfounded relation. In general, the wellfoundedness of a relation is equivalent with the non-existence of infinite
descending chains with respect to $\prec$. Definition by transfinite recursion is applicable to general wellfounded relations. When we use it for $\prec$ we will say we are making a definition by induction on $\prec$.

Let $\text{Ord}$ be the (proper) class of all ordinal numbers, and let $T$ be a wellfounded game tree. By induction on $T$, we define $\parallel T \parallel : T \to \text{Ord}$ by

$$\parallel p \parallel_T = \sup\{\parallel q \parallel_T + 1 \mid p \subset q\}.$$ 

Note that the supremum in this definition might as well be restricted to $q$ with $h(q) = h(p) + 1$. Define also $\parallel T \parallel = \parallel \emptyset \parallel_T$.

Note that $\parallel T \parallel$ and each $\parallel p \parallel_T$ are ordinals smaller than $|T|^+$, the least cardinal number greater than $|T|$, the cardinal number of $T$. (Of course, since we now are working in the weak theory $\text{ZC}^{-} + \Sigma_1$ Replacement, we don’t know that $|T|^+$ exists. Nevertheless, we will use the notation $|T|^+$, construing it as proper class if it is not a set.)

**Lemma 1.4.1.** ($\text{ZC}^{-} + \Sigma_1$ Replacement) $A \subseteq [T]$ is Borel if and only if there is a tree $S \subseteq \omega \omega$ and there is a clopen $B \subseteq [T] \times [S]$ such that $S$ is wellfounded and

$$(\forall x \in [T])(x \in A \leftrightarrow G(B(x); S) \text{ is a win for } I),$$

where $B(x) = \{q \in [S] \mid \langle x, q \rangle \in B\}$. Moreover, for $\alpha > 0$ and $A \subseteq [T]$, $A \in \Sigma^\alpha_\|S\|$ if and only if such an $S$ and $B$ exist with $\|S\| \leq \alpha$.

**Proof.** If such $S$ and $B$ exist with $\|S\| \leq \alpha$, define $A_q \subseteq [T]$ for $q \in S$ by

$$x \in A_q \leftrightarrow G(B(x); S_q) \text{ is a win for } I.$$ 

We prove by induction on $\|q\| = \|q\|^S > 0$ that

$$A_q \in \begin{cases} \Sigma_{\|q\|}^0 & \text{if } \ell h(q) \text{ is even;} \\ \Pi_{\|q\|}^0 & \text{if } \ell h(q) \text{ is odd.} \end{cases}$$ 

If $\|q\| = 0$ then $q$ is terminal, and so $x \in A_q \leftrightarrow \langle x, q \rangle \in B$, so $A_q$ is clopen. Let $\|q\| > 0$. If $\ell h(q)$ is even, then

$$A_q = \bigcup\{A_{q'} \mid q \subseteq q' \land \ell h(q') = \ell h(q) + 1\};$$
If $\ell h(q)$ is odd, then

$$A_q = \bigcap \{ A_{q'} \mid q \subseteq q' \land \ell h(q') = \ell h(q) + 1\}.$$ 

In both cases, the desired conclusion follows directly by induction.

We prove the converse by induction on $\alpha$. If $\alpha = 1$ then $A$ is open. Let $S = 1^\omega$ and let $\langle x, \langle k \rangle \rangle \in B \leftrightarrow [T_{x|k}] \subseteq A$. Let $\alpha > 1$ and assume $A \in \Sigma^0_\alpha$. Then there are $A_i, i \in \omega$, such that $A = \bigcup_{i \in \omega} A_i$ and such that each $A_i \in \Pi^0_{\beta_i}$ for some $\beta_i < \alpha$. Let $S_i \subseteq \omega^\omega$ and $B_i$ be given by induction for $\neg A_i$ and $\beta_i$, for each $i$. Let

$$S = \{ \langle i \rangle \neg q \mid q \in S_i \};$$

$$B = \{ \langle x, \langle i \rangle \neg q \rangle \mid q \in [S_i] \land \langle x, q \rangle \notin B_i \}.$$ 

It is easy to see that $S$ and $B$ are as required. \qedsymbol

**Theorem 1.4.2.** ([Kuratowski, 1958], §33 III; ZC$^-$ + $\Sigma_1$ Replacement) For countable $T$, $\text{Diff}(\Pi^0_\xi) = \Delta^0_{\xi+1}$, for each countable ordinal $\xi \geq 1$.

**Proof** We first show $\text{Diff}(\Pi^0_\xi) \subseteq \Delta^0_{\xi+1}$. (For this part, we do not need the assumption that $T$ is countable.) Let $\langle A_\beta \mid \beta < \alpha \rangle$ witness that $A \in \text{Diff}(\Pi^0_\xi)$. Assume for definiteness that $\alpha$ is even.

$$A = \bigcup_{\beta < \alpha} (\neg A_\beta \cap \bigcap_{\gamma < \beta} A_\gamma)$$

$$= \bigcup_{\beta < \alpha} \neg A_\beta \cap \bigcap_{\beta < \alpha} \bigcup_{\gamma < \beta} (A_\beta \cup \bigcup_{\gamma < \beta} \neg A_\gamma)$$

Since each $A_\gamma \in \Pi^0_\xi$ and $\Pi^0_\xi$ is closed under countable intersections, $\bigcap_{\gamma < \beta} A_\gamma \in \Pi^0_{\xi+1}$. Furthermore, $\neg A_\beta \in \Sigma^0_\xi \subseteq \Sigma^0_{\xi+1}$. Since $\Sigma^0_{\xi+1}$ is closed under finite intersections, $\neg A_\beta \cap \bigcap_{\gamma < \beta} A_\gamma \in \Sigma^0_{\xi+1}$. The first equation then gives that $A \in \Sigma^0_{\xi+1}$. An analogous calculation using the second equation gives that $A$ is the intersection of a member of $\Sigma^0_\xi$ and a member of $\Pi^0_{\xi+1}$; hence $A \in \Pi^0_{\xi+1}$ also. By the definition of $\Delta^0_{\xi+1}$, $A \in \Delta^0_{\xi+1}$. 


We now turn to the more difficult other half of the theorem. Let $A \in \Delta^0_{\xi+1}$. There are sets $C_i, i \in \omega$, such that each $C_i \in \Pi^0_{\xi}$ and such that

$$\neg A = \bigcup_{i \in \omega} C_{2i};$$
$$A = \bigcup_{i \in \omega} C_{2i+1}.$$  

For each $i \in \omega$ let $S_i$ and $B_i$ be given by Lemma 1.4.1 with $\neg C_i$ for the $A$ of that lemma and $\xi$ for the $\alpha$. Let

$$S = \{\langle i, \neg q \rangle \mid q \in S_i\};$$
$$B = \{\langle x, \langle i, \neg q \rangle \rangle \mid q \in [S_i] \land \langle x, q \rangle \notin B_i\}.$$  

Note that $G(B(x); S)$ is a win for I if and only if $x \in \bigcup_{n \in \omega} C_n = A \cup \neg A = \lceil T \rceil$, and so each $G(B(x); S)$ is a win for I. For $n \in \omega$ let

$$S^n = \{q \in S \mid \text{lh}(q) \leq 2n \land (\forall n' < n)(n' \text{ even } \rightarrow q(n') < n)\}.$$  

and let

$$U_n = \{\langle p, t \rangle \mid p \in T \land \text{lh}(p) = n \land t \in S_{II}(S^n) \land (\forall q \in [S] \cap S^n)(q \text{ consistent with } t \rightarrow [T_p] \times \{q\} \not\subseteq B)\}.$$ 

Recall (page 5) that $S_{II}(S^n)$ is the set of all strategies for II in $S^n$. Let $U = \bigcup_{n \in \omega} U_n$. Partially order $U$ by

$$\langle p, t \rangle < \langle p', t' \rangle \iff \langle p' \subseteq p \land t' \subseteq t \rangle.$$  

The relation $<$ is wellfounded, since if

$$\cdots < \langle p_2, t_2 \rangle < \langle p_1, t_1 \rangle < \langle p_0, t_0 \rangle$$

is an infinite descending chain with respect to $<$, then $\bigcup_{n \in \omega} t_n$ is a winning strategy for II for $G(B(\bigcup_{n \in \omega} p_n); S)$, and this game is a win for I. By induction on $<$ define $\text{ord}(p, t) = \sup\{\text{ord}(p', t') + 1 \mid \langle p', t' \rangle < \langle p, t \rangle\}$. Note that the supremum in this definition might as well be restricted to $\langle p', t' \rangle \in U_{n+1}$ if $\langle p, t \rangle \in U_n$. The unique member of $U_0$ is $\langle \emptyset, \emptyset \rangle$. Set $\gamma = \text{ord}(\langle \emptyset, \emptyset \rangle)$. By the hypothesis that $T$ is countable, it follows that $U$ is countable also; hence $\gamma$ is a countable ordinal.
If \( x \in [T] \), \( t \in S_{II}(S^n) \), and \( i < n \), then \( t \) gives a fragment \( s_i^t \) of a strategy for \( I \) for \( G(B_i(x); S_i) \): Let \( s_i^t(q) = t(i \sim q) \) for each \( q \) such that \( i \sim q \in S^n \). Let us say that \( t \in S_{II}(S^n) \) is \( n \)-wrong for \( x \in [T] \) if there is an \( i < n \) such that, for all \( \sigma \in S_1(S_i) \), if \( s_i^t \subseteq \sigma \) then \( \sigma \) is not a winning strategy for \( I \) for \( G(B_i(x); S_i) \). Note that if \( t \in S_{II}(S^n) \) is not \( n \)-wrong for \( x \) then \( \langle x \upharpoonright n, t \rangle \in U_n \).

Remark. The reason we had to define “\( n \)-wrong” and not simply “\( n \)” is that it is possible to have \( S^n = S^{n'} \) with \( n \neq n' \).

Now define, for \( \alpha \leq \gamma \),
\[
A_{2\alpha} = \{ x \in [T] \mid (\forall n)(\forall t)((n \text{ even } \land \langle x \upharpoonright n, t \rangle \in U_n \land \text{ord}(x \upharpoonright n, t) = \alpha) \rightarrow t \text{ is } n\text{-wrong for } x \}\}.
\]
Similarly define \( A_{2\alpha+1} \) for \( \alpha < \gamma \), with “odd” replacing “even.”

We show that \( \langle A_{\beta} \mid \beta \leq 2\gamma \rangle \) witnesses that \( A \in \text{Diff}(\Pi_\xi^0) \). Let \( x \in [T] \) and let \( \alpha \) be the least ordinal such that there exist \( n \) and \( t \in S_{II}(S^n) \) with \( t \) not \( n \)-wrong for \( x \) and with \( \text{ord}(x \upharpoonright n, t) = \alpha \). Such \( \alpha \), \( n \), and \( t \) exist with \( \alpha \leq \gamma \), for \( \emptyset \in S_1(S^n) \), and the fact that there is no \( i < 0 \) guarantees that \( \emptyset \) is not \( 0 \)-wrong for \( x \). Let \( n \) and \( t \in S_{II}(S^n) \) be such that \( t \) is not \( n \)-wrong for \( x \) and \( \text{ord}(x \upharpoonright n, t) = \alpha \). Choose \( n \) to be even, if possible. Note that \( \alpha < \gamma \) if \( n \) is odd, since \( n > 0 \). We show that
\[
n \text{even} \rightarrow x \in \bigcap_{\beta < 2\alpha} (A_{\beta} \setminus A_{2\alpha}) \land x \notin A;
\]
\[
n \text{odd} \rightarrow x \in \bigcap_{\beta \leq 2\alpha} (A_{\beta} \setminus A_{2\alpha+1}) \land x \notin A.
\]
We do the case that \( n \) is even; the other case is similar. Since \( n \) is even, \( t \) witnesses that \( x \notin A_{2\alpha} \). But the minimality of \( \alpha \) implies that, for all \( \alpha' < \alpha \), \( x \) belongs both to \( A_{2\alpha'} \) and to \( A_{2\alpha'+1} \). Thus \( x \in \bigcap_{\beta < 2\alpha} A_{\beta} \setminus A_{2\alpha} \). Since \( t \) is not \( n \)-wrong for \( x \), there is for each \( i < n \) a winning strategy \( \sigma_i \) for \( I \) for \( G(B_i(x); S_i) \) such that \( s_i^t \subseteq \sigma_i \). This means, first of all, that \( G(B_i(x); S_i) \) is a win for \( I \) for each \( i < n \) and so that \( (\forall i < n) x \notin C_i \). Suppose for a contradiction that also \( x \notin C_n \). Let \( \sigma_n \) be a winning strategy for \( I \) for \( G(B_n(x); S_n) \). Let \( t' \in S_{II}(S^{n+1}) \) be such that \( s_i^{t'} \subseteq \sigma_i \) for each \( i \leq n \). Clearly \( t' \) is not \( n \)-wrong for \( x \). This implies, in particular, that \( x \upharpoonright n + 1, t' \in U_{n+1} \). But \( x \upharpoonright n + 1, t' \prec x \upharpoonright n, t \), and so \( \text{ord}(x \upharpoonright n + 1, t') < \text{ord}(x \upharpoonright n, t) = \alpha \). This contradiction gives us that \( x \in C_n \). Since \( n \) is even, it follows that \( x \notin A \).

To complete the proof of the theorem, we show that each \( A_{\beta} \in \Pi_\xi^0 \). Assume for definiteness that \( \beta = 2\alpha \). Since \( \Pi_\xi^0 \) is closed under countable
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intersections, it is enough to show that, for any fixed $t$ and even $n$, the set $A_{2\alpha,n,t}$ defined by

$$A_{2\alpha,n,t} = \{ x \mid (\langle x \upharpoonright n, t \rangle \in U_n \land \text{ord}(x \upharpoonright n, t) = \alpha) \rightarrow t \text{ is } n\text{-wrong for } x \}$$

belongs to $\Pi^0_\xi$. We apply Lemma 1.4.1. Fix such $n$ and $t$. For $i < n$ let

$$S'_i = \{ q \in S_i \mid q \text{ is consistent with } s'_i \}.$$ 

Let

$$B'_i = \{ \langle x, q \rangle \in B_i \mid \langle x \upharpoonright n, t \rangle \in U_n \land \text{ord}(x \upharpoonright n, t) = \alpha \}.$$ 

Now $x \in A_{2\alpha,n,t}$ if and only if there is an $i < n$ such that $G(B'_i(x); S'_i)$ is not a win for $I$. Since $\|S'_i\| \leq \|S_i\| \leq \xi$ for each $i$ and since each $B'_i$ is clopen, the fact that $A_{2\alpha,n,t} \in \Pi^0_\xi$ follows by Lemma 1.4.1.

We will prove the determinacy of $\Delta^0_4$ games by proving that all $\text{Diff}(\Pi^0_3)$ games are determined and then applying Theorem 1.4.2. This proof is rather complicated, so we first deal with the simpler case of $\alpha$-$\Pi^0_3$ games for $\alpha$ finite. The following well known fact gives an equivalent characterization of the games of this form.

**Lemma 1.4.3.** Let $\Gamma$ be a collection of subsets of $\lceil T \rceil$ closed under finite unions and intersections. A subset $A$ of $\lceil T \rceil$ is $k\cdot \Gamma$ with $k \in \omega$ if and only if $A$ is a Boolean combination of finitely many $\Gamma$ sets.

**Proof.** The “only if” direction is clear.

For the “if” direction, let $A$ be a finite Boolean combination of finitely many sets in $\Gamma$. For some positive integer $k$, $A = \bigcup_{i< k} E_i$ with each $E_i$ the intersection of finitely many sets each a member of $\Gamma$ or the complement of a member of $\Gamma$. Since $\Gamma$ is closed under finite unions and intersections, we may assume that

$$A = \bigcup_{i< k} (B_i \setminus C_i),$$

where $1 \leq k \in \omega$ and where the $B_i$ and $C_i$ belong to $\Gamma$. We may also assume that $B_i \supseteq C_i$ for each $i < k$.

For $x \in \lceil \Gamma \rceil$ and $j < k$ let

- $x \in A_{2j}$ \iff $x$ belongs to at least $j + 1$ of the sets $B_i$;
- $x \in A_{2j+1}$ \iff $x$ belongs to at least $j + 1$ of the sets $C_i$. 


We will show that $\langle A_j \mid i < 2k \rangle$ witnesses that $A \in 2k\cdot \Gamma$. Let $x \in [T]$. Let $m$ be least such that $m = 2k$ or $x \notin A_m$.

First assume that $m$ is an odd number $2j + 1$. Then $x$ belongs to exactly $j$ of the $C_i$ and to at least $j + 1$ of the $B_i$. Thus there is an $i$ such that $x \in B_i \setminus C_i$, and so $x \in A$.

Next assume that $m$ is an even number $2j$. Then $x$ belongs to exactly $j$ of the $B_i$ and to at least $j$ of the $C_i$. Since $B_i \supseteq C_i$ for each $i$, it follows that $\{i \mid x \in B_i\} = \{i \mid x \in C_i\}$. Thus there is no $i$ such that $x \in B_i \setminus C_i$, and so $x \notin A$. \hfill \Box

**Theorem 1.4.4.** For each finite $k$, $\mathbf{ZC}^+ + \Sigma_1$ Replacement $\vdash \text{“All } k\cdot \Pi^0_3 \text{ games are determined, and therefore } G(A;T) \text{ is determined for every } A \text{ in the Boolean algebra generated by } \Pi^0_3 \text{ subsets of } [T].”$

**Remark.** The proof that follows was gotten by specializing—and so somewhat simplifying—the proof of Theorem 1.4.10 to the case $\gamma = k$. The proof of Theorem 1.1 of [Montalban and Shore, 2012] was gotten in the same way, and at heart the two proofs are the same. Montalban’s and Shore’s context differs from ours in that they are dealing with second order arithmetic instead of set theory, and so only countable trees are involved. Also they pay attention to what fragment of the axiomatic theory of second order arithmetic is needed for the proof. We won’t pay attention to this in giving our proof, but we will discuss it afterward.

**Proof.** We give the proof for the case that $k$ is odd. From this, the proof for the case of even $k$ can be gotten by exchanging “I” and “II.”

Let $A \subseteq [T]$ with $A \in k\cdot \Pi^0_3$. Let $\langle A_n \mid n < k \rangle$ witness that $A \in k\cdot \Pi^0_3$. Also without loss of generality, we assume that

$$m \leq n < k \rightarrow A_m \supseteq A_n.$$ 

For each $n < k$, let $A_{n,i}, i \in \omega$, be $\Sigma^0_2$ sets such that $A_n = \bigcap_{i \in \omega} A_{n,i}$. For each $n < k$ and each $i \in \omega$, let $A_{n,i,j}, j \in \omega$, be closed sets such that $A_{n,i} = \bigcup_{j \in \omega} A_{n,i,j}$.

For $s \in \leq k\omega$ and for game subtrees $S$ of $T$, we define, by induction on $\ell h(s)$, the assertion $P^s(S)$.

1. $P^0(S)$ holds if and only if $G(A;S)$ is a win for $\Pi$;
(2) If $lh(s) = n + 1$ and $n$ is even, then $P^s(S)$ holds if and only if there is a quasistrategy $U$ for $I$ in $S$ such that

- $[U] \subseteq A \cup A_{k - n - 1, s(n)}$;
- $P^{s|n}(U)$ fails;

(3) If $lh(s) = n + 1$ and $n$ is odd, then $P^s(S)$ holds if and only if there is a quasistrategy $U$ for $II$ in $S$ such that

- $[U] \subseteq \neg A \cup A_{k - n - 1, s(n)}$;
- $P^{s|n}(U)$ fails;

Note that $k - n - 1$ is even if and only if $n$ is even.

For $lh(s) > 0$, we say that $U$ witnesses $P^s(S)$ if the obvious conditions hold. For $lh(s) = 0$, $U$ witnesses $P^s(S)$ if $U$ is (the quasistrategy corresponding to) a winning strategy for $II$ for $G(A; S)$.

For $lh(s) = n + 1$, we say that a quasistrategy $U$ (for $I$ if $n$ is even and for $II$ if $n$ is odd) locally witnesses $P^s(S)$ if there is a subset $D$ of $S$ and there are for each $d \in D$ quasistrategies $R^d$ in $S_d$, for $II$ if $n$ is even and for $I$ if $n$ is odd, such that

1. for each $d \in D \cap U$, $U_d \cap R^d$ witnesses $P^s(R^d)$;
2. $[U] \setminus \bigcup_{d \in D} [R^d] \subseteq \begin{cases} A & \text{if } n \text{ is even;} \\ \neg A & \text{if } n \text{ is odd;} \end{cases}$
3. for each $p \in S$, there is at most one $d \in D$ such that $d \subseteq p$ and $p \in R^d$.

If $U$ witnesses $P^s(S)$, then $U$ locally witnesses $P^s$: Let $D = \{\emptyset\}$ and let $R^d = S$. The next lemma, which will be an important technical tool, is the converse.

**Lemma 1.4.5.** Let $s \in \mathbb{N}$ with $0 < lh(s) = n + 1$. Assume that $U$ locally witnesses $P^s(S)$. Then $U$ witnesses $P^s(S)$.

**Proof of Lemma.** We prove the lemma by induction on $n$ (actually on odd and even $n$ separately).

Note first that $U$ cannot fail to have property (a) (i.e. (2)(a) or (3)(a), whichever is appropriate). To see this, assume for definiteness that $n$ is even and let $x \in [U]$. By (ii), we may assume that $x \in [R^d]$ for some $d \in D$. But then (i) implies that $x \in A \cup A_{k - n - 1, s(n)}$. 


We now turn to property (b), for which we really need induction.

Suppose first that \( n = 0 \). Assume for a contradiction that (b) fails. Since \( n = 0 \), let \( \tau \) be a winning strategy for II for \( G(A; U) \).

We show that there is a \( d \in D \) consistent with \( \tau \) such that if \( x \supseteq d \) is a play consistent with \( \tau \) then \( x \) belongs to \([R^d]\). Assume this is false. Then for each \( d \in D \) such that \( d \) is consistent with \( \tau \), let \( f(d) \supseteq d \), \( f(d) \) consistent with \( \tau \), \( f(d) \notin R^d \), and \((\forall q)(d \subseteq q \subseteq f(d) \rightarrow q \in R^d)\). By (iii) there are no members \( d \) and \( d' \) of \( D \) that are consistent with \( \tau \) and such that \( d \subseteq d' \subseteq f(d) \). It follows that there is a play \( x \) consistent with \( \tau \) such that \( f(d) \subseteq x \) whenever \( d \subseteq x \). Clearly \( x \) cannot belong to \( \bigcup_{d \in D} [R^d] \). But (ii) gives the contradiction that \( x \in A \).

Let then \( d \) be consistent with \( \tau \) such that \( x \) belongs to \([R^d]\) for every play \( x \supseteq d \) such that \( x \) is consistent with \( \tau \). Then the obvious restriction of \( \tau \) is a winning strategy for II for \( G(A; U_d \cap R^d) \). Hence \( P_0(U_d \cap R^d) \), contradicting (i).

Next suppose that \( n > 0 \) is even. Assume for a contradiction that (b) fails. Let \( S' \) witness \( P^{s|n}(U) \). We define \( D' \subseteq S' \) as follows:

\[
d \in D' \iff \exists \left\{ \begin{array}{ll}
d \in S' \\
d \in D \\
G(\neg[R^d]; S'_d) \text{ is a win for II.}
\end{array} \right.\]

For \( d \in D' \), let \( R^d \) be II’s non-losing quasistrategy for \( G(\neg[R^d]; S'_d) \). Note that \( R^d \subseteq R^d \).

Let \( d \in D' \). Since \( R^d \) is a quasistrategy for II in \( S'_d \) and \( S'_d \) is a quasistrategy for II in \( U_d \), it follows that \( R^d \) is a quasistrategy for II in \( U_d \). Since \([R^d] \subseteq [S'] \subseteq \neg A \cup A_{k-n+s(n-1)} \), condition (3)(a) holds for \( R^d \). By (i), \( R^d \) cannot witness \( P^{s|n}(U_d) \), so (3)(b) must fail for \( R^d \). Let then \( U^d \) witness \( P^{s|n-1}(R^d) \).

We define a quasistrategy \( U' \) in \( S' \) as follows:

(1) If \( p \in U' \) and there is no \( d \in D \) such that \( d \subseteq p \) and \( p \in R^d \), then let any move legal in \( S' \) at \( p \) be legal in \( U' \) at \( p \).

(2) For each \( q \in S' \) and \( d \in D \) such \( d \subseteq q \), such that \( q \in R^d \setminus R^d \) (taking \( R^d = \emptyset \) for \( d \notin D' \)), and such that every \( q' \subseteq q \) belongs to \( R^d \), let \( \sigma_q \) be a winning strategy for I for \( G(\neg[R^d]; S'_q) \). Whenever such a \( q \) belongs to \( U' \), we let \( U'_q \) agree with \( \sigma_q \) until a position \( p \notin R^d \) is reached.

(3) For \( d \in D' \cap U' \), let \( U'_d \cap R^d = U^d \).
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Using $\mathcal{D}'$ and $\langle R^d \mid d \in \mathcal{D}' \rangle$, we now show that $U'$ locally witnesses $P^{s[n-1]}(S')$. Induction will then give that $U'$ witnesses $P^{s[n-1]}(S')$, contradicting property (3)(b) of $S'$. Property (i) follows from clause (3) in the definition of $U'$ and the fact that $U'^d$ witnesses $P^{s[n-1]}(R^d)$. For (ii), note first that clause (2) in the definition of $U'$ guarantees that, for each $d \in \mathcal{D}$,

$$[U'] \cap [R^d] \subseteq [R^d].$$

Thus $[U'] \setminus \bigcup_{d \in \mathcal{D}'} [R^d] = [U'] \setminus \bigcup_{d \in \mathcal{D}} [R^d] \subseteq A$. (iii) follows from the facts that $\mathcal{D}' \subseteq \mathcal{D}$ and that $(\forall d \in \mathcal{D}') R^d \subseteq R^d$.

For the remaining case, that of an odd $n > 0$, we make the same definitions as for the case of even $n > 0$, except that we exchange I and II, $A$ and $\neg A$, and $G(\neg [R^d]; S'_d)$ and $G([R^d]; S'_d)$. The argument is exactly the same, except for a minor change in the case $n = 1$: In that case, the $U'^d$ are the quasistrategies corresponding to winning strategies for II for $G(A; R^d)$, and we must prove that $[U'] \subseteq \neg A$. As before, $[U'] \setminus \bigcup_{d \in \mathcal{D}'} [R^d] \subseteq \neg A$. Moreover for $d \in \mathcal{D}'$ we have that $[U'] \cap [R^d] = [U'^d] \subseteq \neg A$.

We say that $P^s(S)$ fails everywhere if $P^s(S_p)$ fails for every $p \in S$.

**Lemma 1.4.6.** Let $s \in \omega^k$ and let $m = lh(s)$. If $P^s(S)$ fails, then there is a quasistrategy $W$ in $S$ for I if $m$ is even and for II if $m$ is odd such that $P^s(W)$ fails everywhere.

**Proof of Lemma.** The case $m = 0$ is Lemma 1.2.7, so assume $m = n + 1$. Suppose for definiteness that $n$ is even; the other case is similar. Let $\mathcal{D}$ be the set of all $d \in S$ such that $P^s(S_d)$ but such that, for every $p \subseteq d$, $P^s(S_p)$ fails. For each $d \in \mathcal{D}$, let $U^d$ witness $P^s(S_d)$. Let

$$B = \{ x \in [S] \mid (\exists d \in \mathcal{D}) d \subseteq x \}.$$

First assume for a contradiction that the open game $G(B; S)$ is a win for I. Let $\sigma$ be a winning strategy for I for $G(B; S)$. We define a quasistrategy $U$ for I in $S$ as follows: $U$ agrees with $\sigma$ until a position $d \in \mathcal{D}$ is reached. Then $U_d = U^d$. For $d \in \mathcal{D}$, let $R^d = S_d$. It is easy to see, using $\mathcal{D}$ and $\langle R^d \mid d \in \mathcal{D} \rangle$, that $U$ locally witnesses $P^s(S)$, so Lemma 1.4.5 gives the contradiction that $U$ witnesses $P^s(S)$.

We know then that $G(B; S)$ is a win for II. Let $W$ be II’s non-losing quasistrategy. Assume for a contradiction that $q \in W$ and that $U^*$ witnesses
$P^s(W_q)$. Let $U$ be a quasistrategy for I in $S_q$ defined as follows: Let $U \cap W_q = U^*$. When first (if ever) a position $p \notin W$ is reached, let $U$ agree with a winning strategy $\sigma_p$ for I for $G(B; S)$ until a position $d \in D$ is reached. Then let $U_d = U^d$. Let $D' = D \cup \{q\}$. Let $R^d = W_q$ and let $R^d = S_d$ for $d \in D$. It is easy to see, using $D'$ and $\langle R^d \mid d \in D' \rangle$, that $U$ locally witnesses $P^s(S_q)$. Lemma 1.4.5 gives us the contradiction that some $d \subseteq q$ belongs to $D$. 

For $n + 1 = \ell(x(s))$, we say that $W$ strongly witnesses $P^s(S)$ if, for all $p \in W$, $W_p$ witnesses $P^s(S_p)$, i.e. if $W$ witnesses $P^s(S)$ and $P^{s|n}(W)$ fails everywhere.

Lemma 1.4.7. Let $s \in \leq^k \omega$ with $0 < \ell(x(s)) = n + 1$. If $P^s(S)$, then there is a $W$ that strongly witnesses $P^s(S)$.

Proof of Lemma. Assume for definiteness that $n$ is even. Let $U$ witness that $P^s(S)$. By property (2)(b) of $U$, $P^{s|n}(U)$ fails. By Lemma 1.4.6, let $W$ be a quasistrategy for I in $U$ such that $P^{s|n}(W)$ fails everywhere. Since $W$ is a quasistrategy for I in $S$ and $W$ inherits property (2)(a) from $U$, it follows that $W$ strongly witnesses $P^s(S)$.

Lemma 1.4.8. Let $s \in \leq^k \omega$ with $0 < \ell(x(s)) = n + 1$. At least one of $P^s(S)$ and $P^{s|n}(S)$ holds.

Proof of Lemma. We prove the lemma by induction on $k - n$, simultaneously for all $s$ and $S$.

Suppose for definiteness that $n$ is even. (The case that $n$ is odd is slightly simpler, since $n + 1 = k$ is impossible.) Assume that $P^s(S)$ fails. We will define a quasistrategy $U$ for II, and also $D \subseteq S$ and $\langle R^d \mid d \in D \rangle$. Simultaneously we will define the notion of a position $q \in U$ marking stage $j$, for $j \in \omega$. For any play $x \in [U]$, the set of $j$ such that some $q \subseteq x$ marks stage $j$ will be a (not necessarily proper) initial segment of $\omega$, and, whenever $q \subseteq x$ marks stage $j$ and $q' \subseteq x$ marks stage $j'$, we will have $q \subseteq q' \leftrightarrow j < j'$.

The initial position $\emptyset$ marks stage 0. By induction, $P^{s^{-\{0\}}}(S)$ holds if $n + 1 < k$; let $W^0$ be a quasistrategy for II strongly witnessing this. If $n + 1 = k$, let $W^0$ be a quasistrategy for II in $S$ such that $P^s(W^0)$ fails everywhere.

Assume inductively that $q \in U$ marks stage $j$ and that $q$ belongs to a quasistrategy $W^q$ for II in $S_q$ such that $P^s(W^q)$ fails everywhere and such that $W^q$ strongly witnesses $P^{s^{-\{j\}}}(S_q)$ if $n + 1 < k$. 

For $n + 1 = \ell(x(s))$, we say that $W$ strongly witnesses $P^s(S)$ if, for all $p \in W$, $W_p$ witnesses $P^s(S_p)$, i.e. if $W$ witnesses $P^s(S)$ and $P^{s|n}(W)$ fails everywhere.
Assume first that $G(A_{k-n+1,s(n),j}; W^q)$ is a win for $I$. Then $q \in \mathcal{D}$. Let $\hat{R}^q$ be $\Gamma$’s non-losing quasistrategy for for $G(A_{k-n+1,s(n),j}; W^q)$. Let $R^q \cap W^q = \hat{R}^q$ and, for $p \in S_q \setminus W^q$, let $R^q_p = S_p$. Let $U^q$ witness $P^{s|n}(\hat{R}^q)$. ($U^q$ exists since $[\hat{R}^q] \subseteq A_{k-n+1,s(n),j} \subseteq A_{k-n+1,s(n)}$, and so the non-existence of $U^q$ would imply $P^s(W^q)$, whereas $P^s(W^q)$ fails everywhere.) We let $U$ agree with $U^q$ on $\hat{R}^q$. No $p \in \hat{R}^q$ with $q \subsetneq p$ belongs to $\mathcal{D}$ or marks any stage.

Suppose that either $\hat{R}^q$ exists and $p \supseteq q$ is a first position in $U^q$ not belonging to $\hat{R}^q$ or else $p = q$ and $\hat{R}^q$ does not exist (i.e., $G(A_{k-n+1,s(n),j}; W^q)$ is a win for $II$). Let $U$ agree with a winning strategy $\tau_p$ for $II$ for $G(A_{k-n+1,s(n),j}; W^q)$ until a position $q' \supseteq p$ is first reached with $[W^q] \cap A_{k-n+1,s(n),j} = \emptyset$ and $q \supseteq p$ if $p$ is not terminal. No $q^*$ with $p \subseteq q^* \subseteq q$ belongs to $\mathcal{D}$ or marks any stage. The position $q'$ marks stage $j + 1$. $P^s(W^q)$ fails everywhere, because $P^s(W^q)$ fails everywhere. If $n = k$, let $W^q = W_k^q$. If $n < k$ then, by induction, $P^{s-\langle j+1\rangle}(W_k^q)$ holds; let $W^q$ strongly witness this. The position $q'$ marks stage $j + 1$. Note that $W^q$ strongly witnesses $P^{s-\langle j+1\rangle}(S_q)$ if $n < k$, as required.

This completes the definition of $U$.

Suppose first that $n > 0$. We will show, using $\mathcal{D}$ and $\langle R^d \mid d \in \mathcal{D} \rangle$, that $U$ locally witnesses $P^{s|n}(S)$. By Lemma 1.4.5, this will show that $U$ witnesses $P^{s|n}(S)$.

Since $U_d \cap R^d = U_d \cap \hat{R}^d = U^d$ for $d \in \mathcal{D}$, condition (i) holds. For (ii), suppose that $x$ is a play in $U$ such that $x \notin \bigcup_{d \in \mathcal{D}}[R^d]$. From the definition it follows that we have either

$$\emptyset = q_0 \subsetneq q_1 \subsetneq q_2 \subsetneq \cdots \subseteq x$$

or

$$\emptyset = q_0 \subsetneq \cdots \subsetneq q_k = q_{k+1} = \cdots = x$$

such that each $q_j$ marks stage $j$. From the definition we also get that

$$j < j' \rightarrow W^{q_{j'}} \supseteq W^{q_j}.$$

Hence $x \in \bigcap_{j \in \omega}[W^{q_j}]$. Since $W^{q_j}$ witnesses $P^{s-\langle j\rangle}(S_{q_j})$ if $n + 1 < k$, it follows in that case that

$$x \in \bigcap_{j \in \omega}(-A \cup A_{k-n-2,j}) = -A \cup \bigcap_{j \in \omega}A_{k-n-2,j} = -A \cup A_{k-n-2}.$$
From the definition we also get that, for each \( j \),
\[
[W^{q_j+1}] \cap A_{k-n-1,s(n),j} = \emptyset.
\]
Hence \( x \notin \bigcup_{j \in \omega} A_{k-n-1,s(n),j} = A_{k-n-1,s(n)} \). Since \( A_{k-n-1,s(n)} \supseteq A_{k-n-1} \), we have that \( x \notin A_{k-n-1} \). If \( n + 1 = k \), this means that \( x \notin A_0 \), and so \( x \notin A \).

If \( n + 1 < k \), then \( x \in (\neg A \cup A_{k-n-2}) \setminus A_{k-n-1} \).

By our assumption that the \( A_n \) are monotonely decreasing with \( n \),
\[
x \in (\neg A \cup \bigcap_{m<k-n-1} A_m) \setminus A_{k-n-1}.
\]
Since \( k-n-1 \) is even, \( x \in \neg A \), as required by (ii). It is easy to see that (iii) holds.

Now suppose that \( n = 0 \). The argument for (ii) in the case \( n > 0 \) still works, so \( [U] \setminus \bigcup_{d \in D} [R^d] \supseteq \neg A \). Since \( U_d \cap R^d = U^d \) and \( [U^d] \subseteq \neg A \), we have that \( \bigcup_{d \in D} [R^d] \subseteq \neg A \). Thus \( U \) is a winning quasistrategy for \( G(S;A) \), and so \( P^0(S) \) holds.

We can now prove the theorem. Assume that \( G(A;T) \) is not a win for II. This means that \( P^0(T) \) fails. By Lemma 1.4.6, let \( W^0 \) be a quasistrategy for I in \( T \) such that \( P^0(W^0) \) fails everywhere.

We define a quasistrategy \( U \) for I in \( W^0 \). Assume inductively that we have defined \( \{ p \mid p \in U \land \ell h(p) \leq j \} \). Let \( p \in U \) with \( \ell h(p) = j \). Assume inductively also that \( p \in W^p \), where \( W^p \) is a quasistrategy for I in \( W^0 \) such that \( P^0(W^p) \) fails everywhere. For each \( q \supseteq p \) with \( h(q) = j + 1 \), let \( q \in U \leftrightarrow q \in W^p \). By Lemma 1.4.8, \( P^{(j)}(W^p_q) \) holds for all such \( q \). Let \( W^q \) be a quasistrategy for I in \( W^p_q \) strongly witnessing \( P^{(j)}(W^p_q) \).

We show that every play \( x \in [U] \) belongs to \( A \), and so that \( U \) is a winning quasistrategy for I for \( G(A;T) \). Let \( x \in [U] \). If \( x \) is finite, the fact that \( x \in [W^0] \) implies that \( x \in A \). Assume then that \( x \) is infinite. Since \( x \in [W^{x[j+1]}] \) for each \( j \in \omega \), it follows that
\[
x \in A \cup \bigcap_{j \in \omega} A_{k-1,j} = A \cup A_{k-1} = A.
\]
The last equality holds because \( k \) is odd and the \( A_n \) are monotonely decreasing with \( n \). \( \square \)
To get sharper form of Theorem 1.4.4, we need to pay attention to what fragment of the Comprehension schema is used for a given $k$. As we remarked on page 19, $\text{ZC}^- + \Sigma_1$ Replacement is the same as $\text{KPC} + \text{Comprehension}$. Similarly $\text{Z}^- + \Sigma_1$ Replacement is the same as $\text{KP} + \text{Comprehension}$. If, e.g., we restrict Comprehension to $\Pi_n$ formulas for $n \leq k$, then we get the theory $\text{KP} + \Pi_k\text{Comprehension}$.

**Theorem 1.4.9.** ([Montalban and Shore, 2012]) For each finite $k$, $\text{KP} + \Pi_{k+2}\text{Comprehension} \vdash \text{“All } k\text{-}\Pi^0_3 \text{ games are in countable trees are determined.”}$$

**Proof.** See [Montalban and Shore, 2012]. The proof is basically like that of Theorem 1.4.4, with careful attention to how much Comprehension the steps of that proof use. In addition to Comprehension, the proof of Theorem 1.4.4, also uses $\Sigma_{k+2}$ Dependent Choice. To justify this, Montalban and Shore use $L$ and absoluteness to prove that $\Sigma_{k+2}$ DC is conservative over $\text{KP} + \Pi_{k+2}\text{Comprehension}$ for formulas that are $\Pi^0_4$ over the reals. □

In the case of uncountable trees, the most useful version of their theorem is probably one that adds “$+ V = L$” to “$\text{KP} + \Pi_{k+2}\text{Comprehension}$.” The author has not checked whether adding this hypothesis to the hypotheses of Theorem 1.4.9 and dropping the word “countable” yields a theorem (without increasing “$k + 2$”).

**Theorem 1.4.10.** All $\text{Diff}(\Pi^0_3)$ games are determined.

**Proof.** Let $A \subseteq [T]$ with $A \in \text{Diff}(\Pi^0_3)$. Let $\langle A_\alpha \mid \alpha < \gamma \rangle$ witness that $A \in \text{Diff}(\Pi^0_3)$. Without loss of generality, we assume that $\gamma$ is odd. Also without loss of generality, we assume that $

\alpha \leq \beta < \gamma \rightarrow A_\alpha \supseteq A_\beta.

For each $\alpha < \gamma$, let $A_{\alpha,i}, i \in \omega$, be $\Sigma^0_2$ sets such that $A_\alpha = \bigcap_{i \in \omega} A_{\alpha,i}$. For each $\alpha < \gamma$ and each $i \in \omega$, let $A_{\alpha,i,j}, j \in \omega$, be closed sets such that $A_{\alpha,i} = \bigcup_{j \in \omega} A_{\alpha,i,j}$.

Let $Q$ be the set of all pairs $\langle r,s \rangle$ such that

(i) $r \in ^{<\omega} \gamma$;

(ii) $s \in ^{<\omega} \omega$. 


(iii) \( \ell h(r) = \ell h(s) \);
(iv) \((\forall i < \ell h(r))(i \text{ even } \iff r(i) \text{ even})\).
(v) \(i < j < \ell h(r) \rightarrow r(i) > r(j)\).

For \( \langle r, s \rangle \in Q \) and for game subtrees \( S \) of \( T \), we define, by induction on \( \ell h(r) \) the assertion \( P r,s(S) \):

1. \( P^0,0(S) \) holds if and only if \( G(A; S) \) is a win for II;
2. If \( \ell h(r) = n + 1 \) and \( n \) is even, then \( P r,s(S) \) holds if and only if there is a quasistrategy \( U \) for I in \( S \) such that
   (a) \( \lceil U \rceil \subseteq A \cup A_{r(n),s(n)} \);
   (b) \( P^{r,n,s|n}(U) \) fails;
3. If \( \ell h(r) = n + 1 \) and \( n \) is odd, then \( P r,s(S) \) holds if and only if there is a quasistrategy \( U \) for II in \( S \) such that
   (a) \( \lceil U \rceil \subseteq \neg A \cup A_{r(n),s(n)} \);
   (b) \( P^{r,n,s|n}(U) \) fails.

For \( \langle r, s \rangle \in Q \) and \( \ell h(r) > 0 \), \( U \) witnesses \( P r,s(S) \) if the obvious conditions hold. For \( \ell h(r) = 0 \), \( U \) witnesses \( P r,s(S) \) if \( U \) is (the quasistrategy corresponding to) a winning strategy for II for \( G(A; S) \).

For \( \langle r, s \rangle \in Q \) and \( \ell h(r) = n + 1 \), we say that a quasistrategy \( U \) (for I if \( n \) is even and for II if \( n \) is odd) locally witnesses \( P r,s(S) \) if there is a subset \( D \) of \( S \) and there are for each \( d \in D \) quasistrategies \( R_d \) in \( S_d \), for II if \( n \) is even and for I if \( n \) is odd, such that

(i) for each \( d \in D \cap U, U_d \cap R_d \) witnesses \( P r,s(R_d) \);
(ii) \( [U] \setminus \bigcup_{d \in D} [R_d] \subseteq \begin{cases} A & \text{if } n \text{ is even;} \\
\neg A & \text{if } n \text{ is odd;} \end{cases} \)
(iii) for each \( p \in S \), there is at most one \( d \in D \) such that \( d \subseteq p \) and \( p \in R_d \).

As in the special case occurring in the proof of Theorem 1.4.4, if \( U \) witnesses \( P r,s(S) \), then \( U \) locally witnesses \( P r,s \). Let \( D = \{ \emptyset \} \) and let \( R_d = S \). The next lemma, the analogue of Lemma 1.4.5, is the converse.

**Lemma 1.4.11.** Let \( \langle r, s \rangle \in Q \) with \( 0 < \ell h(r) = n + 1 \). Assume that \( U \) locally witnesses \( P r,s(S) \). Then \( U \) witnesses \( P r,s(S) \).
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Proof of Lemma. We prove the lemma by induction on $n$ (actually on odd and even $n$ separately).

Note first that $U$ cannot fail to have property (a) (i.e. (2)(a) or (3)(a), whichever is appropriate). To see this, assume for definiteness that $n$ is even and let $x \in [U]$. By (ii), we may assume that $x \in [R^d]$ for some $d \in D$. But then (i) implies that $x \in A \cup A_{r(n),s(n)}$.

We now turn to property (b), for which we really need induction.

Suppose first that $n = 0$. Assume for a contradiction that (b) fails. Since $n = 0$, let $\tau$ be a winning strategy for II for $G(A;U)$.

We show that there is a $d \in D$ consistent with $\tau$ such that if $x \supseteq d$ is a play consistent with $\tau$ then $x$ belongs to $[R^d]$. Assume this is false. Then for each $d \in D$ such that $d$ is consistent with $\tau$, let $f(d) \supseteq d$, $f(d)$ consistent with $\tau$, $f(d) \notin R^d$, and $(\forall q)(d \subseteq q \subsetneq f(d) \rightarrow q \in R^d)$. By (iii) there are no members $d$ and $d'$ of $D$ that are consistent with $\tau$ and such that $d \subseteq d' \subseteq f(d)$. It follows that there is a play $x$ consistent with $\tau$ such that $f(d) \subseteq x$ whenever $d \subseteq x$. Clearly $x$ cannot belong to $\bigcup_{d \in D} [R^d]$. But (ii) gives the contradiction that $x \in A$.

Let then $d$ be consistent with $\tau$ such that $x$ belongs to $[R^d]$ for every play $x \supseteq d$ such that $x$ is consistent with $\tau$. Then the obvious restriction of $\tau$ is a winning strategy for II for $G(A;U_d \cap R^d)$. Hence $P^{\emptyset,\emptyset}(U_d \cap R^d)$, contradicting (i).

Next suppose that $n > 0$ is even. Assume for a contradiction that (b) fails. Let $S'$ witness $P_r^{[n,s|n]}(U)$. We define $D' \subseteq S'$ as follows:

$$d \in D' \iff \begin{cases} d \in S' \land d \in D \land G(-[R^d]; S'_d) \text{ is a win for II.} 
\end{cases}$$

For $d \in D'$, let $R^d$ be II’s non-losing quasistrategy for $G(-[R^d]; S'_d)$. Note that $R^d \subseteq R^d$.

Let $d \in D'$. Since $R^d$ is a quasistrategy for II in $S'_d$ and $S'_d$ is a quasistrategy for II in $U_d$, it follows that $R^d$ is a quasistrategy for II in $U_d$. Since $[R^d] \subseteq [S'] \subseteq \neg A \cup A_{r(n-1),s(n-1)}$, condition (3)(a) holds for $R^d$. By (i), $R^d$ cannot witness $P^{[n,s|n]}(U_d)$, so (3)(b) must fail for $R^d$. Let then $U^d$ witness $P^{[n-1,s|n-1]}(R^d)$.

We define a quasistrategy $U'$ for I in $S'$ as follows:

1. If $p \in U'$ and there is no $d \in D$ such that $d \subseteq p$ and $p \in R^d$, then let any move legal in $S'$ at $p$ be legal in $U'$ at $p$. 


(2) For each \( q \in S' \) and \( d \in \mathcal{D} \) such \( d \subseteq q \), such that \( q \in R^d \setminus R^{d'} \) (taking \( R^{d'} = \emptyset \) for \( d \notin \mathcal{D}' \)), and such that every \( q' \subsetneq q \) belongs to \( R^{d'} \), let \( \sigma_q \) be a winning strategy for I for \( G(\neg[R^{d'}]; S_d') \). Whenever such a \( q \) belongs to \( U' \), we let \( U'_q \) agree with \( \sigma_q \) until a position \( p \notin R^d \) is reached.

(3) For \( d \in \mathcal{D}' \cap U' \), let \( U'_d \cap R^{d'} = U^{d'} \).

Using \( \mathcal{D}' \) and \( \langle R^{d'} \mid d \in \mathcal{D}' \rangle \), we now show that \( U' \) locally witnesses \( P_{n-1, s}^n \langle S' \rangle \). Induction will then give that \( U' \) witnesses \( P_{n-1, s}^n \langle S' \rangle \), contradicting property (3)(b) of \( S' \). Property (i) follows from clause (3) in the definition of \( U' \) and the fact that \( U^{d'} \) witnesses \( P_{n-1, s}^n \langle R^{d'} \rangle \). For (ii), note first that clause (2) in the definition of \( U' \) guarantees that, for each \( d \in \mathcal{D} \),

\[
[U'] \cap [R^d] \subseteq [R^{d'}].
\]

Thus \( [U'] \setminus \bigcup_{d \in \mathcal{D}'} [R^{d'}] = [U'] \setminus \bigcup_{d \in \mathcal{D}} [R^d] \subseteq A \). (iii) follows from the facts that \( \mathcal{D}' \subseteq \mathcal{D} \) and that \( \forall d \in \mathcal{D}' \) \( R^{d'} \subseteq R^d \).

For the remaining case, that of an odd \( n > 0 \), we make the same definitions as for the case of even \( n > 0 \), except that we exchange I and II, A and \( \neg A \), and \( G(\neg[R^d]; S_d') \) and \( G([R^d]; S_d') \). The argument is exactly the same, except for a minor change in the case \( n = 1 \): In that case, the \( U^{d'} \) are the quasistrategies corresponding to winning strategies for II for \( G(A; R^{d'}) \), and we must prove that \( [U'] \subseteq \neg A \). As before, \( [U'] \setminus \bigcup_{d \in \mathcal{D}'} [R^{d'}] \subseteq \neg A \). Moreover for \( d \in \mathcal{D}' \) we have that \( [U'] \cap [R^{d'}] = [U^{d'}] \subseteq \neg A \). \( \square \)

We say that \( P_{r,s}^n(S) \) fails everywhere if \( P_{r,s}^n(S_p) \) fails for every \( p \in S \).

**Lemma 1.4.12.** Let \( \langle r, s \rangle \in Q \) and let \( m = \text{fh}(r) \). If \( P_{r,s}^n(S) \) fails, then there is a quasistrategy \( W \) in \( S \) for I if \( m \) is even and for II if \( m \) is odd such that \( P_{r,s}^n(W) \) fails everywhere.

**Proof of Lemma.** The case \( m = 0 \) is Lemma 1.2.7, so assume \( m = n + 1 \). Suppose for definiteness that \( n \) is even; the other case is similar. Let \( \mathcal{D} \) be the set of all \( d \in S \) such that \( P_{r,s}^n(S_d) \) but such that, for every \( p \subseteq d \), \( P_{r,s}^n(S_p) \) fails. For each \( d \in \mathcal{D} \), let \( U^d \) witness \( P_{r,s}^n(S_d) \). Let

\[
B = \{ x \in [S] \mid (\exists d \in \mathcal{D}) d \subseteq x \}.
\]

First assume for a contradiction that the open game \( G(B; S) \) is a win for I. Let \( \sigma \) be a winning strategy for I for \( G(B; S) \). We define a quasistrategy
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$U$ for $I$ in $S$ as follows: $U$ agrees with $\sigma$ until a position $d \in D$ is reached. Then $U_d = U^d$. For $d \in D$, let $R^d = S_d$. It is easy to see, using $D$ and $\langle R^d \mid d \in D \rangle$, that $U$ locally witnesses $P^{r,s}(S)$, so Lemma 1.4.11 gives the contradiction that $U$ witnesses $P^{r,s}(S)$.

We know then that $G(B; S)$ is a win for $II$. Let $W$ be $II$'s non-losing quasistrategy. Assume for a contradiction that $q \in W$ and that $U^*$ witnesses $P^{r,s}(W_q)$. Let $U$ be a quasistrategy for $I$ in $S_q$ defined as follows: Let $U \cap W_q = U^*$. When first (if ever) a position $p \notin W$ is reached, let $U$ agree with a winning strategy $\sigma_p$ for $I$ for $G(B; S)$ until a position $d \in D$ is reached. Then let $U_d = U^d$. Let $D' = D \cup \{q\}$. Let $R^d = W_q$ and let $R'$ fail for $d \in D$. It is easy to see, using $D'$ and $\langle R^d \mid d \in D' \rangle$, that $U$ locally witnesses $P^{r,s}(S_q)$. Lemma 1.4.11 gives us the contradiction that some $d \subseteq q$ belongs to $D$. 

For $n + 1 = \ell h(r)$, we say that $W$ strongly witnesses $P^{r,s}(S)$ if, for all $p \in W$, $W_p$ witnesses $P^{r,s}(S_p)$, i.e. if $W$ witnesses $P^{r,s}(S)$ and $P^{r,s}[n](W)$ fails everywhere.

**Lemma 1.4.13.** Let $\langle r, s \rangle \in Q$ with $0 < \ell h(r) = n + 1$. If $P^{r,s}(S)$, then there is a $W$ that strongly witnesses $P^{r,s}(S)$.

**Proof of Lemma.** Assume for definiteness that $n$ is even. Let $U$ witness that $P^{r,s}(S)$. By property (2)(b) of $U$, $P^{r,s}[n](U)$ fails. By Lemma 1.4.12, let $W$ be a quasistrategy for $I$ in $U$ such that $P^{r,s}[n](W)$ fails everywhere. Since $W$ is a quasistrategy for $I$ in $S$ and $W$ inherits property (2)(a) from $U$, it follows that $W$ strongly witnesses $P^{r,s}(S)$. 

**Lemma 1.4.14.** Let $\langle r, s \rangle \in Q$ with $0 < \ell h(r) = n + 1$. At least one of $P^{r,s}(S)$ and $P^{r,s}[n](S)$ holds.

**Proof of Lemma.** We prove the lemma by induction on $r(n)$, simultaneously for all $n, r, s,$ and $S$. (Recall that $r$ is a strictly decreasing sequence of ordinals.)

Suppose for definiteness that $n$ is even. Assume that $P^{r,s}(S)$ fails. We will define a quasistrategy $U$ for $II$, and also $D \subseteq S$ and $\langle R^d \mid d \in D \rangle$. Simultaneously we will define the notion of a position $q \in U$ marking stage $j$, for $j \in \omega$. For any play $x \in [U]$, the set of $j$ such that some $q \in x$ marks stage $j$ will be a (not necessarily proper) initial segment of $\omega$, and, whenever $q \in x$ marks stage $j$ and $q' \in x$ marks stage $j'$, we will have $q \not\subseteq q' \iff j < j'$. 


If \( r(n) > 0 \), let \( \beta_j, m_j \), \( j \in \omega \), be an enumeration of all pairs \( \langle \beta, m \rangle \) with \( \beta \) odd, \( \beta < r(n) \), and \( m \in \omega \).

The initial position \( \emptyset \) marks stage 0. By induction, \( P^{r^{-}}(\beta_0), s^{-}(m_0)(S) \) holds if \( r(n) > 0 \); let \( W^\emptyset \) be a quasistrategy for II strongly witnessing this. If \( r(n) = 0 \), let \( W^\emptyset \) be a quasistrategy for II in \( S \) such that \( P^{r,s}(W^\emptyset) \) fails everywhere.

Assume inductively that \( q \in U \) marks stage \( j \) and that \( q \) belongs to a quasistrategy \( W^q \) for II in \( S_q \) such that \( P^{r,s}(W^q) \) fails everywhere and such that \( W^q \) strongly witnesses \( P^{r^{-}}(\beta_j), s^{-}(m_j)(S_q) \) if \( r(n) > 0 \).

Assume first that \( G(A_{r(n),s(n)}, j; W^q) \) is a win for I. Then \( q \in D \). Let \( \hat{R}^q \) be I’s non-losing quasistrategy for \( G(A_{r(n),s(n)}, j; W^q) \). Let \( \hat{R}^q \cap W^q = \hat{R}^q \) and, for \( p \in S_q \setminus W^q \), let \( R^q_p = S_p \). Let \( U^q \) witness \( P^{r[n],s[n]}(\hat{R}^q) \). \( U^q \) exists since \( [\hat{R}^q] \subset A_{r(n),s(n), j} \subset A_{r(n),s(n)} \), and so the non-existence of \( U^q \) would imply \( P^{r,s}(W^q) \), whereas \( P^{r,s}(W^q) \) fails everywhere.) We let \( U \) agree with \( U^q \) on \( \hat{R}^q \). No \( p \in \hat{R}^q \) with \( q \subset p \) belongs to \( D \) or marks any stage.

Suppose that either \( \hat{R}^q \) exists and \( p \supseteq q \) is some first position in \( U_q \) not belonging to \( \hat{R}^q \) or else \( p = q \) and \( \hat{R}^q \) does not exist (i.e., \( G(A_{r(n),s(n), j}; W^q) \) is a win for II). Let \( U \) agree with a winning strategy \( \tau_p \) for II for \( G(A_{r(n),s(n), j}; W^q) \) until a position \( q' \supseteq p \) is first reached with \( [W^q_q] \cap A_{r(n),s(n), j} = \emptyset \) and \( q \supseteq p \) if \( p \) is not terminal. No \( q^* \) with \( p \subset q^* \subset q' \) belongs to \( D \) or marks any stage. The position \( q' \) marks stage \( j + 1 \). \( P^{r,s}(W^q_q) \) fails everywhere, because \( P^{r,s}(W^q_q) \) fails everywhere. If \( r(n) = 0 \), let \( W^q_q = W^q_q \). If \( r(n) > 0 \) then, by induction, \( P^{r^{-}}(\beta_{j+1}), s^{-}(m_{j+1})(W^q_q) \) holds; let \( W^q_q \) strongly witness this. The position \( q' \) marks stage \( j + 1 \). Note that \( W^q_q \) strongly witnesses \( P^{r^{-}}(\beta_{j+1}), s^{-}(m_{j+1})(S_q) \) if \( r(n) > 0 \), as required.

This completes the definition of \( U \).

Suppose first that \( n > 0 \). We will show, using \( D \) and \( \langle R^d \mid d \in D \rangle \), that \( U \) locally witnesses \( P^{r[n],s[n]}(S) \). By Lemma 1.4.11, this will show that \( U \) witnesses \( P^{r[n],s[n]}(S) \).

Since \( U_d \cap R^d = U_d \cap \hat{R}^d = U^d \) for \( d \in D \), condition (i) holds. For (ii), suppose that \( x \) is a play in \( U \) such that \( x \notin \bigcup_{d \in D}[R^d] \). From the definition it follows that we have either

\[
\emptyset = q_0 \subsetneq q_1 \subsetneq q_2 \subsetneq \cdots \subsetneq x
\]

or

\[
\emptyset = q_0 \subsetneq \cdots \subsetneq q_k = q_{k+1} = \cdots = x
\]
such that each $q_j$ marks stage $j$. From the definition we also get that
\[ j < j' \implies W^{q_j} \supseteq W^{q_{j'}}. \]
Hence $x \in \bigcap_{j \in \omega} [W^{q_j}]$. Since $W^{q_j}$ witnesses $P_{s_j}^\gamma(S_{q_j})$ if $r(n) > 0$, it follows that
\[
x \in \bigcap_{j \in \omega} (\neg A \cup A_{\beta_j, m_j}) = \neg A \cup \bigcap_{\beta < r(n)} \bigcap_{m \in \omega} A_{\beta, m} \beta \text{ odd} = \neg A \cup \bigcap_{\beta < r(n)} A_{\beta} \beta \text{ odd}
\]
From the definition we also get that, for each $j$,
\[
[W^{q_{j+1}}] \cap A_{r(n), s(n), j} = \emptyset.
\]
Hence $x \notin \bigcup_{j \in \omega} A_{r(n), s(n), j} = A_{r(n), s(n)}$. Since $A_{r(n), s(n)} \supseteq A_{r(n)}$, we have that $x \notin A_{r(n)}$. Thus we have that
\[
x \in (\neg A \cup \bigcap_{\beta < r(n)} A_{\beta}) \setminus A_{r(n)}.
\]
Since $r(n)$ is even, $x \in \neg A$, as required by (ii). It is easy to see that (iii) holds.

Now suppose that $n = 0$. A simplification of the argument for (ii) in the case $n > 0$ still works, so $[U] \setminus \bigcup_{d \in D} [R^d] \supseteq \neg A$. Since $U_d \cap R^d = U_d$ and $[U^d] \subseteq \neg A$, we have that $\bigcup_{d \in D} [R^d] \subseteq \neg A$. Thus $U$ is a winning quasistrategy for $G(S; A)$, and so $P_{s_0}^\emptyset(S)$ holds. □

We can now prove the theorem. Assume that $G(A; T)$ is not a win for II. This means that $P_{s_0}^\emptyset(T)$ fails. By Lemma 1.4.12, let $W^\emptyset$ be a quasistrategy for I in $T$ such that $P_{s_0}^\emptyset(W^\emptyset)$ fails everywhere.

Let $\langle \beta_j, m_j \rangle$, $m \in \omega$, be an enumeration of all pairs $\langle \beta, m \rangle$ with $\beta$ even, $\beta < \gamma$, and $m \in \omega$.

We define a quasistrategy $U$ for I in $W^\emptyset$. Assume inductively that we have defined $\{ p \mid p \in U \land \ell(h(p)) \leq j \}$. Let $p \in U$ with $\ell(h(p)) = j$. Assume inductively also that $p \in W^p$, where $W^p$ is a quasistrategy for I in $W^p$ such
that $P^{\emptyset,0}(W^p) \text{ fails everywhere.}$ For each $q \supseteq p$ with $\ell h(q) = j + 1$, let $q \in U \iff q \in W^p.$ By Lemma 1.4.14, $P^{(\beta_j),(m_j)}(W^p_q)$ holds for all such $q.$ Let $W^q$ be a quasistrategy for I in $W^p_q$ strongly witnessing $P^{(\beta_j),(m_j)}(W^p_q)$.

We show that every play $x \in [U]$ belongs to $A$, and so that $U$ is a winning quasistrategy for I for $G(A;T)$. Let $x \in [U].$ If $x$ is finite, the fact that $x \in [W^q]$ implies that $x \in A$. Assume then that $x$ is infinite. Since $x \in [W^x[j+1]]$ for each $j \in \omega$, it follows that

$$x \in A \cup \bigcap_{\beta < \gamma} \bigcap_{i \in \omega} A_{\beta,i} = A \cup \bigcap_{\beta < \gamma} A_{\beta} = A.$$ 

The last equality holds because $\gamma$ is odd.

\begin{corollary}
All $\Delta^0_4$ games in countable trees are determined.
\end{corollary}

\begin{proof}
The corollary is a direct consequence of Theorems 1.4.2 and 1.4.10.
\end{proof}

Montalban and Shore demonstrate in [Montalban and Shore, 2012] the unprovability in $\text{ZC}^++\Sigma^1_1 \text{ Replacement}^-$—indeed, in $\text{ZFC}^-$—of the assertion that all games in $^{<\omega} \omega$ that are $k \cdot \Pi^0_3$ for some $k \in \omega$ are determined. (Theorem 1.4.4 says only that $k \cdot \Pi^0_3$ determinacy is provable in $\text{ZC}^++\Sigma^1_1 \text{ Replacement}$ for each fixed $k \in \omega$.) Hence neither Theorem 1.4.10 nor Corollary 1.4.15 is provable in $\text{ZC}^++\Sigma^1_1 \text{ Replacement}$.

As we will demonstrate, our proof of Theorem 1.4.10 does show that every wellfounded model of $\text{ZC}^++\Sigma^1_1 \text{ Replacement}$ satisfies “All $\text{Diff}(\Pi^0_3)$ games are determined.” A model $(M;E)$ for the language of set theory is wellfounded if $E$ is a wellfounded relation. (See page 24 for the definition of wellfoundedness for relations.) By a theorem of Mostowski, a model for the language of set theory that satisfies Extensionality is wellfounded just in case it is isomorphic to a transitive model, a model $(M;\in)$ with $M$ a transitive set.

Our proof of Theorem 1.4.10 also shows that $\gamma \cdot \Pi^0_3$ determinacy holds for a certain infinite $\gamma$ in all $\omega$-models of $\text{ZC}^++\Sigma^1_1 \text{ Replacement}$. Let us say that a model $(M;E)$ for the language of set theory is an $\omega$-model if $\omega \cup \{\omega\} \subseteq M$ and

$$(\forall x \in M)(\forall y \in \omega \cup \{\omega\})(xeY \iff x \in y).$$
If \((M; E)\) satisfies a sufficient small fragment of ZF, then \((M; E)\) is an \(\omega\)-model just in case \(\omega\) is the \(\omega\) of \((M; E)\) and the natural numbers are the natural numbers of \((M; E)\).

To prove the facts just mentioned, we need two more definitions. Let \((M; E)\) be a model for the language of set theory. If \(x \in M\), then the transitive closure of \(x\) under \(E\) is the smallest set \(w\) such that \(x \in w\) and such that \(y E z \in w \rightarrow y \in w\). The wellfounded part of \((M; E)\) is the set of all \(x \in M\) such that \(E\) restricted to the transitive closure of \(x\) under \(E\) is a wellfounded relation. It is not hard to see that the restriction of \(E\) to the wellfounded part of \((M; E)\) is a well-founded relation.

If \((M; E)\) satisfies Extensionality and a small fragment of the Comprehension Schema, then the ordinals of \((M; E)\)—i.e., those \(x \in M\) such that \((M; E) \models x\) is an ordinal—are linearly ordered by \(E\). Thus an ordinal \(x\) of \((M; E)\) belongs to the wellfounded part of \((M; E)\) if and only if \(E\) wellorders \(\{y \mid y E x\}\).

**Theorem 1.4.16.** Let \((M; E)\) be an \(\omega\)-model of ZC\(^{-}\) + \(\Sigma_1\) Replacement. Assume that \(T\) is, in \((M; E)\), a game tree. Assume that \(T\) belongs to the wellfounded part of \((M; E)\). Assume that \(\gamma\) is an ordinal of \((M; E)\) and that \(\gamma\) belongs to the wellfounded part of \((M; E)\). Then

\[(M; E) \models \text{“All } \gamma\text{-}\Pi^0_3\text{ games are determined.”}\]

**Proof.** By Mostowski’s theorem, we may assume that the wellfounded part of \((M; E)\) is a transitive set on which \(E\) agrees with membership. This implies, in particular, that \(\gamma\) is an ordinal number and that \(T\) is a game tree.

Let \(A\) be, in \((M; E)\), a \(\gamma\)-\(\Pi^0_3\) subset of \([T]\). Working in \((M; E)\), introduce \(\langle A_\alpha \mid \alpha < \gamma\rangle\), \(\langle A_\alpha, n \mid \alpha < \gamma \land n \in \omega\rangle\), and \(Q\) as in the proof of Theorem 1.4.10.

For each \(n \in \omega\), define \(P^r_s(S)\) for \(S \subseteq T\), \(\langle r, s \rangle \in Q\) and \(\ell h(r) = n\) as \(P^r_s(S)\) was defined for such objects in the proof of Theorem 1.4.10. The theory ZC\(^{-}\) + \(\Sigma_1\) Replacement does not allow us to define, by a formula of the language of set theory, the class relation \(P^r_s(S)\). What we do instead is use the inductive definition of the proof of Theorem 1.4.10 to define, for each \(n \in \omega\), a three-place relation \(P^r_n(S)\) for each \(n\). \(P_n\) is thus the definable restriction of the undefinable \(P\) to triples \(\langle S, r, s \rangle\) with \(\ell h(r) = n\).

Working in \((M; E)\) with ZC\(^{-}\) + \(\Sigma_1\) Replacement we cannot state Lemma 1.4.11. But we can replace it with individual statements for each \(n\), and our induc-
The statement of Lemma 1.4.12 also needs to be replaced by individual statements for each \( n \). The proof of Lemma 1.4.12 directly gives, for each \( n \), a proof of the statement for \( n \) from the Lemma 1.4.11 replacement statement for \( n \). Similarly the Lemma 1.4.13 replacement statement for \( n \) is proved from the Lemma 1.4.12 replacement for \( n \).

Finally, we get replacement statements for Lemma 1.4.14. The \( n \)th of these asserts that at least one of \( P_{n+1}^{r,s}(S) \) and \( P_{n,n}^{r,s}(S) \) holds for any \( S \) and any \( r \) and \( s \) of length \( n \). Assume that one of these assertions is false in \((M;E)\). Let \( S, r, \) and \( s \) have the least value of \( r(\ell h(r)) \) attained by a triple yielding a statement false in \((M;E)\). There is such a least value, since all values \( r(i) \) are ordinals of \( \mathfrak{M} \) that are \( < \gamma \). The proof of Lemma 1.4.14 shows that if \( \ell h(r) = n + 1 \) and \( P_{n,s}^{r,s}(S) \) nor \( P_{n,n}^{r,s}(S) \) holds, then there some \( P_{n+1}^{r,s}(S') \) that does not hold and has \( \beta < r(n) \). Thus that proof, applied to \((M;E)\), contradicts the assumed minimality of \( r(\ell h(r)) \).

In the proof of Theorem 1.4.10, the determinacy of \( G(A;T) \) is proved using only the \( n = 0 \) case of Lemma 1.4.14.

**Corollary 1.4.17.** Every wellfounded model of \( \text{ZC}^- + \Sigma_1 \text{ Replacement} \) satisfies “All \( \Delta^0_4 \) games in countable trees are determined.”

**Corollary 1.4.18.** If let \((M;E)\) be an \( \omega \)-model of \( \text{ZC}^- + \Sigma_1 \text{ Replacement} \). Assume that the wellfounded part of \((M;E)\) is a transitive set and that \( E \) agrees with \( \in \) on the wellfounded part of \((M;E)\). Let \( \gamma \) be an ordinal such that \( \in \upharpoonright \gamma \) is isomorphic to a recursive wellordering of \( \omega \). Then \( \gamma \) belongs to the wellfounded part of \((M;E)\) and

\[(M;E) \models \text{“All } \gamma\text{-}\Pi^0_3 \text{ games are determined.”}\]

**Proof.** All recursive relations on \( \omega \) belong to \( M \), and recursive wellorderings of \( \omega \) belonging to \( M \) are wellorderings in \((M;E)\).
1.4. $\Delta^0_4$ GAMES

licensed by ZC$^{-} + \Sigma_1$ Replacement. Our plan is to formulate a theory that does license these definitions.

Let $\mathcal{L}_R$ be the language gotten from the language of set theory by adding $R$ as a two-place predicate symbol.

For $k \geq 1$, let $\psi(v_0, v_1, \ldots, v_k)$ be a formula of the language of set theory. Let $\chi(v_0, v_1, \ldots, v_k)$ be a formula of $\mathcal{L}_R$ in which every subformula of the form $R(x, y)$ has the form $R(v_0, y)$. Let $\mathcal{L}_{R_{\psi, \chi}}$ be the language gotten from $\mathcal{L}_R$ by replacing $R$ by a symbol $R_{\psi, \chi}$. Let $\chi_{\psi, \chi}$ be the result of replacing $R$ in $\chi$ by $R_{\psi, \chi}$. Let $\mathcal{L}_{\text{rec}}$ be the union, in the obvious sense, of the $\mathcal{L}_{R_{\psi, \chi}}$.

We define a theory Rec(ZC$^{-} + \Sigma_1$ Replacement) in the language $\mathcal{L}_{\text{rec}}$.

The axioms of Rec(ZC$^{-} + \Sigma_1$ Replacement) are the same as the axioms of ZC$^{-} + \Sigma_1$ Replacement, with the following additions.

1. The Comprehension and $\Sigma_1$ Replacement Schemas apply to all formulas $\varphi(x, u, w_1, \ldots, w_n)$ of the language $\mathcal{L}_{\text{rec}}$.

2. There are axioms, described below, for each $R_{\psi, \chi}$.

(a) $R_{\psi, \chi}(x, y) \rightarrow (x \in \omega \land (\exists n \in \omega) y$ is an $n$-tuple).

(b) $R_{\psi, \chi}(0, \langle v_1, \ldots, v_k \rangle) \leftrightarrow \psi(v_1, \ldots, v_k)$.

(c) $(\forall n \in \omega)(R_{\psi, \chi}(n + 1, \langle v_1, \ldots, v_k \rangle) \leftrightarrow \chi_{\psi, \chi}(n, v_1, \ldots, v_k)$.

The difference between ZC$^{-} + \Sigma_1$ Replacement and Rec(ZC$^{-} + \Sigma_1$ Replacement) is, roughly speaking, that the latter allows formulas to be defined by recursion from formulas of the language of set theory.

Theorem 1.4.19. (Rec(ZC$^{-} + \Sigma_1$ Replacement)) All $\text{Diff}(\Pi^0_3)$ games are determined, and so all $\Delta^0_4$ games in countable trees are determined.

Proof. We specify formulas $\psi(v_1, v_2, v_3, v_4, v_5)$ and $\chi(v_0, v_1, v_2, v_3, v_4, v_5)$. Each of these formulas will say the following:

(a) $v_2$ is a countable game tree (which we will call) $T$.

(b) $v_3$ is a $\text{Diff}(\Pi^0_3)$ subset $A$ of $[T]$.

(c) $v_4$ is a function $\langle A_\alpha \mid \alpha < \gamma \rangle$ witnessing that $A \in \text{Diff}(\Pi^0_3)$.

(d) $v_5$ is a function $\langle A_{\alpha, i} \mid \alpha < \gamma \land i \in \omega \rangle$ witnessing that each $A_\alpha$ is a countable intersection of $\Sigma^0_2$ sets.

The formula $\psi(v_1, v_2, v_3, v_4, v_5)$ will also say the following.
(a) \(v_1\) is a triple \(\langle \emptyset, \emptyset, S \rangle\).
(b) \(S\) is a game subtree of \(T\).
(c) \(G(A; S)\) is a win for II.

Our axioms thus imply that \(R_{\psi, \chi}(0, \langle \emptyset, \emptyset, S \rangle) \leftrightarrow G(A; S)\) is a win for II.

The formula \(\chi_{\psi, \chi}(n, v_1, v_2, v_3, v_4, v_5)\) will say the following.

(a) \(v_1\) is a triple \(\langle r, s, S \rangle\);
(b) \(\langle r, s \rangle\) belongs to the set \(Q\) defined as on page 37;
(c) \(\ell h(r) = n + 1\).
(d) If \(n\) is even, then there is a quasistrategy \(U\) for I in \(S\) such that
   (i) \([U]\subseteq A \cup A_{r(n),s(n)}\);
   (ii) \(R_{\psi, \chi}(\langle r \mid n, s \mid n, U \rangle)\) fails.
(e) If \(n\) is odd, then there is a quasistrategy \(U\) for II in \(S\) such that
   (i) \([U]\subseteq \neg A \cup A_{r(n),s(n)}\);
   (ii) \(R_{\psi, \chi}(\langle r \mid n, s \mid n, U \rangle)\) fails.

Set
\[P_{r,s}(S) \leftrightarrow R_{\psi, \chi}(\ell h(r), \langle r, s, S \rangle).\]

It follows by induction from our axioms that \(P_{r,s}(S)\) satisfies the definition on page 38. Thus we may repeat the proof of Theorem 1.4.10. \(\square\)

Richard Shore suggested—or, more accurately, pointed out—to the author that adding a satisfaction predicate to \(\text{ZC}^- + \Sigma_1\) Replacement does the same thing as \(\text{Rec(\text{ZC}^- + \Sigma_1\text{ Replacement})}\) does. This is indeed the case, as we will explain briefly.

Fix some reasonable way of construing formulas as sets. We will define a satisfaction predicate by recursion. The defining axioms will be in the language \(L_{\text{Sat}}\) gotten from the language of set theory by adding a three-place predicate symbol Sat. The base clause \(\rho_0(x, y)\) says that Sat\((0, x, y)\) if and only if

(a) \(x\) is a a formula of the form \(v_i = v_j\) or \(v_i \in v_j\) (i.e., is a formula of the language of set theory of length 3);
(b) \(y\) is \(\langle z_0, \ldots, z_k \rangle\) for some sets \(z_0, \ldots, z_k\) with \(k \geq \max(i, j)\).
(iii) either \( x = v_i \) and \( z_i = z_j \) or else \( x \in v_j \) and \( z_i \in z_j \).

The recursion clause \( \rho(x,y) \) says that, for all \( n \in \omega \), \( \text{Sat}(n+1,x,y) \) if and only if

(a) \( x \) is a formula of the language of set theory of length \( \leq n+4 \);
(b) \( y \) is \( \langle z_0, \ldots, z_k \rangle \) for some \( z_0, \ldots, z_k \) with \( k \geq \max\{i \mid v_i \text{ is free in } x\} \);
(c) \( \sigma_n \).

Here \( \sigma_n \), which we leave to the reader, gives the definition of \( \text{Sat}(n+1,x,y) \) in terms of the relation \( \text{Sat}(n,x,y) \).

Let \( \text{Sat}(ZC^- + \Sigma_1 \text{ Replacement}) \) be the extension of \( ZC^- + \Sigma_1 \text{ Replacement} \) (1) with added axioms \( \rho_0, \rho, \) and \( (\forall z)(\forall x)(\forall y)\text{Sat}(z,x,y) \rightarrow z \in \omega \) and (2) with Comprehension and \( \Sigma_1 \text{ Replacement} \) for all formulas of \( L_{\text{Sat}} \).

We can define a satisfaction predicate \( \text{Sat} \) in \( \text{Rec}(ZC^- + \Sigma_1 \text{ Replacement}) \) and prove the axioms of \( \text{Sat}(ZC^- + \Sigma_1 \text{ Replacement}) \). To do this, first let \( \psi(v_1) \) be

\[(\exists x)(\exists y)(v_1 = \langle x, y \rangle \land (i) \land (ii) \land (iii)),\]

where (i), (ii), and (iii) are the three clauses above. Next let \( \chi(v_0,v_1) \) be

\[(\exists x)(\exists y)(v_1 = \langle x, y \rangle \land (a') \land (b) \land (c')),\]

where (a') is (a) with \( n \) replaced by \( v_0 \) and (c') is (c) with \( \text{Sat}(n,x,y) \) replaced by \( R(v_0,\langle x, y \rangle) \). Finally define \( \text{Sat}(n,x,y) \) as \( R_{\psi,\chi}(n,\langle x, y \rangle) \).

We can also define predicates \( R_{\psi,\chi} \) in \( \text{Sat}(ZC^- + \Sigma_1 \text{ Replacement}) \) and prove the axioms of \( \text{Rec}(ZC^- + \Sigma_1 \text{ Replacement}) \). Given \( \psi \) and \( \chi \), we define by recursion a sequence \( \langle \tau_n \mid n \in \omega \rangle \) of formulas of the language of set theory as follows.

(1) \( \tau_0(v_1) \) is the formula \( (\exists k \in \omega)(v_1 \text{ is a } k\text{-tuple and } \land \psi((v_1)_1, \ldots, (v_1)_k)) \).
(Here \( v_1 = ((v_1)_1, \ldots, (v_1)_k) \). We assume that we have a representation of finite sequences that lets us define the \( (v_1)_i \) from \( v_1 \).

(2) \( \tau_{n+1}(v_1) \) is \( (\exists k \in \omega)(v_1 \text{ is a } k\text{-tuple and } \land \chi'((v_1)_1, \ldots, (v_1)_k)) \), where \( \chi' \) is the result of replacing each \( R(v_0, (v_1)_i) \) in \( \chi(v_0, (v_1)_1, \ldots, (v_1)_k) \) by \( \tau_n(x) \).

Now we can define \( R_{\psi,\chi}(v_0,v_1) \) as \( v_0 \in \omega \) and \( \text{Sat}(v_0, \tau_n, v_1) \).

We finish this section by extending Corollary 1.4.15 to uncountable trees. This is work of the author, done in 1990.
For classes $\Gamma$ and ordinals $\alpha$, we say that a set $A \subseteq [T]$ belongs to the class $(\alpha, \Gamma)^*$, the $\alpha$th level of the generalized difference hierarchy on $\Gamma$, just in case there is a sequence $\langle A_\beta \ | \ \beta < \alpha \rangle$ with each $A_\beta \in \Gamma$ and there is a function $f : T \to \alpha$ such that both

(a) $(\forall x \in [T])(x \in A \iff \mu \beta(x \notin A_\beta \lor \beta = \alpha)$ is odd);
(b) $(\forall x \in [T])(\forall \beta < \alpha)(x \notin A_\beta \rightarrow (\exists n \in \omega) \beta = f(x \upharpoonright n))$.

Let $\text{Diff}^*(\Gamma) = \bigcup_{\alpha \leq |T|^+} (\alpha, \Gamma)^*$. Let $T$ be a game tree. If $A$ and $B_j$, $j \in J$, are subsets of $[T]$, then $A$ is the fully open-separated union of $\{B_j \ | \ j \in J\}$ if

(a) $A = \bigcup_{j \in J} B_j$;
(b) there are disjoint open sets $D_j$, $j \in J$, such that $\bigcup_{j \in J} D_j = [T]$ and such that $B_j \subseteq D_j$ for each $j \in J$.

If $\{D_j \ | \ j \in J\}$ witnesses that $A$ is the fully open-separated union of $\{B_j \ | \ j \in J\}$, then each $\neg D_j = \bigcup_{j' \in J \setminus \{j\}} D_{j'}$; hence each $D_j$ is in fact a clopen set.

We will use the following lemma in proving that $\text{Diff}^*(\Pi^0_\xi) \subseteq \Delta^0_{\xi+1}$.

Lemma 1.4.20. (ZC$^-$ + $\Sigma^1_1$ Replacement) Let $1 \leq \xi < \omega_1$ and let $A \subseteq [T]$ be the fully open-separated union of $\{B_j \ | \ j \in J\}$. (1) If each $B_j \in \Sigma^0_\xi$, then $A \in \Sigma^0_\xi$. (2) If each $B_j \in \Pi^0_\xi$, then $A \in \Pi^0_\xi$.

Proof. We prove the lemma by induction on $\xi$. Assume that the lemma holds for every $\xi' < \xi$. Let $\{D_j \ | \ j \in J\}$ witness that $A$ is the fully open-separated union of $\{B_j \ | \ j \in J\}$.

To prove (1) for $\xi$, assume that each $B_j \in \Sigma^0_\xi$. If $\xi = 1$ then the fact that every union of open sets is open gives that $\bigcup_{\bigcup_{J} B_j \in \Pi^0_\xi}$. Assume then that $\xi > 1$. If $\xi$ is a limit ordinal, let $\eta_0 < \eta_1 < \ldots$ be such that $\sup_{k \in \omega} \eta_k = \xi$. If $\xi = \eta + 1$, then let $\eta_k = \eta$ for each $k \in \omega$. For $j \in J$ let $D_j = \bigcup_{k \in \omega} B_{j,k}$ with each $B_{j,k} \in \bigcup_{\eta_k < \xi} \Pi^0_\eta$. Using Lemma 1.1.1 and modifying each $\langle B_{j,k} \ | \ k \in \omega \rangle$ by inserting $\emptyset$ where necessary, we may assume that each $B_{j,k} \in \Pi^0_{\eta_k}$. For $k \in \omega$, let $\bigcup_{j \in J} B_{j,k}$. Now $\{D_j \ | \ j \in J\}$ witnesses that each $A_k$ is the fully open-separated union of $\{B_{j,k} \ | \ j \in J\}$, so our induction hypothesis gives that each $A_k \in \Pi^0_{\eta_k}$. Thus $A = \bigcup_{k \in \omega} A_k$ belongs to $\Sigma^0_\xi$.

Now assume that each $B_j \in \Pi^0_\xi$. For each $j \in J$, $D_j \setminus B_j \in \Sigma^0_\xi$. (1) therefore gives that $\neg A = \bigcup_{j \in J}(D_j \setminus B_j) \in \Sigma^0_\xi$. Hence $A \in \Pi^0_\xi$, and we have proved (2) for $\xi$. \hfill \Box
1.4. $\Delta^0_4$ GAMES

Theorem 1.4.21. (ZC + $\Sigma^1_1$ Replacement) For all $T$, $\text{Diff}^*(\Pi^0_\xi) = \Delta^0_{\xi+1}$.

Proof. We first show that $\text{Diff}^*(\Pi^0_\xi) \subseteq \Delta^0_{\xi+1}$. Let $\langle A_\beta \mid \beta < \alpha \rangle$ and $f : T \to \alpha$ witness that $A \in \text{Diff}^*(\Pi^0_\xi)$. Assume for definiteness that $\alpha$ is even. It will also be convenient to assume that $(\forall p \in T)(lh(p) \text{ even} \iff f(p) \text{ even})$.

By Theorem 1.4.2, we may certainly assume that $\alpha \geq 2$, we can modify $f$ if necessary to make this assumption hold.

For $n \in \omega$ let

$$C_n = \{ x \in [T] \mid x \in A_{f(x|n)} \}.$$ 

Each $C_n$ is the fully open-separated union of

$$\{ A_{f(p)} \cap [T_p] \mid p \in T \land lh(p) = n \}.$$ 

By Lemma 1.4.20, each $C_n \in \Pi^0_\xi$. Since $\alpha$ is even,

$$A = \bigcup_{n \in \omega} (C_n \cup (C_{m \in \omega} \cap \{ x \mid f(x \upharpoonright m) \geq f(x \upharpoonright n) \}))$$

$$= \bigcup_{n \in \omega} (-C_n \cap (C_n \cup \{ x \mid f(x \upharpoonright m) < f(x \upharpoonright n) \}))$$

For each $n$ and $m$, $\{ x \mid f(x \upharpoonright m) \geq f(x \upharpoonright n) \}$ is clopen, so we can show as in the first part of the proof of Theorem 1.4.2 that $A \in \Delta^0_{\xi+1}$.

Now let $A \in \Delta^0_{\xi+1}$. Repeating the second part of the proof of Theorem 1.4.2, we get $\langle A_\alpha \mid \alpha \leq 2\gamma \rangle$, with $\gamma < |T|^+$, such that each $A_\alpha \in \Pi^0_\xi$ and such that

$$(\forall x \in [T])(x \in A \iff \mu \beta(x \notin A_\beta \lor \beta = \alpha) \text{ is odd}).$$

Moreover, in the notation of the proof of Theorem 1.4.2,

$$(\forall x \in [T])(\forall \alpha < \gamma)(x \notin A_{2\alpha} \cap A_{2\alpha+1} \to (\exists n \in \omega)(\exists t \in \mathcal{S}_{II}(S^n))(x \upharpoonright n, t) \in U_n \land \text{ord}(x \upharpoonright n, t) = \alpha)).$$

Let $\langle t_i \mid i \in \omega \rangle$ be an enumeration of $\bigcup_{n \in \omega} \mathcal{S}_{II}(S^n)$ with the property that each $t_i$ belongs to $\mathcal{S}_{II}(S^n)$ for some $n \leq i$. Define $f : T \to 2\gamma$ by

$$f(p) = \begin{cases} 2\text{ord}(p \upharpoonright n, t_i) & \text{if } lh(p) = 2i \land \langle p \upharpoonright n, t_i \rangle \in U_n; \\ 2\text{ord}(p \upharpoonright n, t_i) + 1 & \text{if } lh(p) = 2i + 1 \land \langle p \upharpoonright n, t_i \rangle \in U_n; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f$ satisfies (b) in the definition of $((2\gamma + 1) \cdot \Pi^0_\xi)^*$. \[\Box\]
Theorem 1.4.22. (Rec(ZC$^+$ + $\Sigma_1$ Replacement)) All $\text{Diff}^*(\Pi_3^0)$ games are determined.

Proof. Let $\langle A_\alpha \mid \alpha < \gamma \rangle$ and $f : T \to \gamma$ witness that $A \in \text{Diff}^*(\Pi_3^0)$. In the proof of Theorem 1.4.10—and so in the proof of Theorem 1.4.19—the countability of $T$ was used only in the proof of Lemma 1.4.14 and in the final argument after the proof of that lemma. We can adapt the proof of Lemma 1.4.14 to the present situation as follows: For $p \in T$ and $\ell h(p) = 2^k 3^m$, let

$$
\beta_p = \begin{cases} 
    f(p \upharpoonright k) & \text{if } f(p \upharpoonright k) < r(n) \land f(p \upharpoonright k) \text{ is odd;} \\
    0 & \text{otherwise.}
\end{cases}
$$

$$m_p = m.
$$

Repeat the proof of Lemma 1.4.14, except replace, in the definition of $U$, the pairs $\langle \beta_j, m_j \rangle$ by $\langle \beta_{qj}, m_{qj} \rangle$. A similar change will adapt the final argument in the proof of Theorem 1.4.10 to the present situation. \qed

Corollary 1.4.23. (Rec(ZC$^+$ + $\Sigma_1$ Replacement)) All $\Delta_4^0$ games are determined.

Theorem 1.4.16 holds for the generalized difference hierarchy, with essentially the same proof as for the original theorem.

Theorem 1.4.24. Let $(M; E)$ be an $\omega$-model of ZC$^+$ + $\Sigma_1$ Replacement. Assume that $T$ is, in $(M; E)$, a game tree. Assume that $T$ belongs to the wellfounded part of $(M; E)$. Assume that $\gamma$ is an ordinal of $(M; E)$ and that $\gamma$ belongs to the wellfounded part of $(M; E)$. Then

$$(M; E) \models \text{"All } (\gamma \cdot \Pi_3^0)^* \text{ games are determined."}$$

Corollary 1.4.25. Every wellfounded model of ZC$^+$ + $\Sigma_1$ Replacement satisfies “All $\Delta_4^0$ games are determined.”

Exercise 1.4.1. ZFC$^-$ is ZFC without the Axiom of Power Set. In other words, ZFC$^-$ is ZC$^+$ + Replacement. Show that the determinacy of all $\Sigma_4^0$ games in countable trees is not provable in ZFC$^-$. This is a refinement, due to the author, of a theorem of [Friedman, 1971]. Friedman’s theorem has $\Sigma_5^0$ instead of $\Sigma_4^0$. 
Hint. Let $\beta_0$ be the least ordinal number $\beta$ such that $L_\beta \models \text{ZC}^- + \Sigma_1$ Replacement. Show that $\beta_0$ is the least $\beta$ such that $L_\beta \models Z$ and that it is the least $\beta$ such that $L_\beta \models \text{ZFC}^-$. Show that there is no $a \subseteq \omega$ such that $a \in L_{\beta_0+1} \setminus L_{\beta_0}$, and show that $\beta_0$ is the least ordinal with this property. Show that every set in $L_{\beta_0}$ is definable in $L_{\beta_0}$.

The plan is to prove that $L_{\beta_0}$ does not satisfy the determinacy of all $\Sigma^0_4$ games in $<\omega \omega$. This is to be done by defining a $\Sigma^0_4$ game $G$ in $<\omega \omega$ such that $G$ is a win for I but the set of G"odel numbers of sentences true in $L_{\beta_0}$ is recursive uniformly in any winning strategy for I for $G$.

For a model $(M; E)$, $\text{WFP}(M; E)$, the wellfounded part of $(M; E)$, is the union of all subsets $N$ of $M$ such that

(a) $(\forall x)(\forall y)(xEy \in N \rightarrow x \in N)$;
(b) the restriction of $E$ to $N$ is wellfounded.

It is easy to show the restriction of $E$ to $\text{WFP}(M; E)$ is wellfounded, so that $\text{WFP}(M; E)$ is the largest subset $N$ of $M$ satisfying (a) and (b).

It will be convenient, in the exercises that follow and those for §2.3, to use “$\omega$-model” to mean a model $(M; E)$ such that $\omega \in \text{WFP}(M; E)$ and the restriction of $E$ to $\text{WFP}(M; E)$ is the membership relation.

If $S$ is a complete theory in the language of set theory and $S$ extends some weak fragment of ZFC and $S \vdash V = L$, then there is a canonical model of $S$, the term model. The model consists of equivalence classes of formulas $\varphi(v)$. Formulas $\varphi(v)$ and $\psi(v)$ are equivalent if $S \vdash " \text{The } <_{L}\text{-least } v \text{ such that either } \varphi(v) \text{ or else } v = 0 \text{ and } (\forall v')\neg\varphi(v') \text{ is identical with the } <_{L}\text{-least } v \text{ such that either } \psi(v) \text{ or else } v = 0 \text{ and } (\forall v')\neg\psi(v')."$ The interpretation of $\in$ is defined in the obvious way. Note that every element of the term model of $S$ is definable in the model.

Consider the following game $G$ in $<\omega \omega$: G"odel numbering all the sentences of the language of set theory, we define, for each play $x$,

$$S_1(x) = \{ \varphi \mid x(2 \#(\varphi)) = 1 \}$$
$$S_\Pi(x) = \{ \varphi \mid x(2 \#(\varphi) + 1) = 1 \}$$

If $S_1(x)$ is not the set of sentences true in an $\omega$-model of ZFC$^- + " V = L_{\beta_0},"$ then I loses. Otherwise II loses unless $S_\Pi(x)$ is also the set of sentences true in an $\omega$-model of ZFC$^- + " V = L_{\beta_0},"$ If neither player has lost for this reason,
then the term models of $S_I(x)$ and $S_{II}(x)$ are isomorphic to $\omega$-models. Let $M_I$ and $M_{II}$ be such $\omega$-models. Player I wins just in case one of the following holds:

1. The model $M_I$ is isomorphic to a initial segment of $M_{II}$.
2. There is an ordinal $a$ of $M_I$ such that $L_{M_I}^a$ is isomorphic to an initial segment of $M_{II}$ but $L_{M_I}^{a+1}$ is not.

By an ordinal of $M_I$ we mean an $a$ such that $M_I|a$ is an ordinal number. By an initial segment of $M_{II}$ we mean $\bigcup_{b \in X} L_{M_{II}}^b$, where $X$ is a (not necessarily proper) initial segment of the ordinals of $M_{II}$. It is not required that $X$ be the initial segment of any ordinal of $M_{II}$.

$G$ is a win for I, who can simply play as $S_I(x)$ the set of sentences true in $L_0^\beta$. But show that, as long as II simply copies I’s moves, I can maintain a winning position only by following this strategy. Thus the set of Gödel numbers of sentences true in $L_0^\beta$ is recursive uniformly in any winning strategy for I for $G$. It is fairly easy to see that this set of Gödel numbers does not belong to $L_0^\beta$. Thus no winning strategy for $G$ belongs to $L_0^\beta$. By absoluteness, $L_0^\beta \models \text{"G is not determined."}$

Show that $G = G(A; <_\omega \omega)$ with $A \in \Sigma^0_4$. There are two main points. First, there is a fixed $\Pi^1_1$ formula $\varphi(X,Y)$ of second order arithmetic such that, for any $\omega$-model $M$ of (a weak fragment of ZFC +) $V = L$, for any ordinal $a$ of $M$, and for any subset $b$ of $\omega$ belonging to $L^a_{\omega+1} \setminus L^a_\omega$,

(i) $L^a \cap P(\omega) \models (\exists Y)\varphi(b,Y);
(ii) for all $c \in L^a_{\omega+1} \cap P(\omega)$, $L^a \cap P(\omega) \models \varphi(b,c)$ if and only if $c$ codes a model $(\omega; E)$ isomorphic to $L^a_\omega$.

It follows that, for $\omega$-models $M$ of $V = L$ + “$\omega_1$ does not exist,” $M \cap P(\omega)$ determines the isomorphism type of $M$. This implies, for example, that (1) holds just in case the subsets of $\omega$ of $M_I$ are the same as those of some initial segment of $M_{II}$ in the sense of the $L$-hierarchy of $M_{II}$. The second main point is that, for formulas $\varphi(v)$ and $\psi(v)$ whose equivalence classes correspond to subsets of $\omega$ in $M_I$ and $M_{II}$ respectively, the condition that these subsets of $\omega$ are the same is a $\Pi^0_1$ condition. This enables one to show that (1) is a $\Pi^0_3$ condition. Similarly, the condition (2) is $\Sigma^0_4$.

Remark. In [Montalban and Shore, 2012], the authors prove a sharper result than that of Exercise 1.4.1: $ZFC^- \not\vdash (\forall n \in \omega)\text{All n-}\Pi^0_3$ games are determined.
See the remark after the hint for Exercise 1.4.2 for discussion of a still stronger result of Montalban and Shore.

**Exercise 1.4.2.** Work in $\mathcal{ZC}'\Sigma_1$ Replacement. Assume that all $\Sigma^0_4$ games are determined. Prove that there is an ordinal $\beta$ such that $L_\beta \models \text{ZFC}'$; i.e., prove that $\beta_0$ exists. Like the result of of Exercise 1.4.1, this is a refinement by the author of a result of [Friedman, 1971], replacing $\Sigma^0_5$ by $\Sigma^0_4$. One of its consequences is that the consistency of $\text{ZFC}'$ can be proved in $\mathcal{ZC}'\Sigma_1$ Replacement + “all $\Sigma^0_4$ games are determined.”

**Hint.** Show (in $\mathcal{ZC}'\Sigma_1$ Replacement) that there are arbitrarily large admissible ordinals, i.e., ordinals $\alpha$ such that $L_\alpha \models \text{Kripke–Platek set theory, KP}$. Let $T$ be the theory $\text{KP} + V = L + \lnot \beta_0$ does not exist.”

Let $Y$ be the set of ordinals $\alpha$ such that $L_\alpha \models T$ and every member of $L_\alpha$ is definable in $L_\alpha$. Prove that $Y$ is unbounded in the ordinals. Do do so, assume to the contrary that $\gamma = \sup(Y)$. Let $\alpha$ be the least admissible greater than $\gamma$. Let $X$ be the elementary submodel of $L_\alpha$ consisting of the elements of $L_\alpha$ definable in $L_\alpha$. The ordinal $\gamma$ belongs to $X$. The non-existence of $\beta_0$ implies that there is some subset $a$ of $\omega$ such that $a \in L_{\gamma+1} \setminus L_{\gamma}$. The $<_L$-least such $a$ belongs to $X$. Deduce that $X = L_\alpha$. This gives the contradiction that $\alpha \in Y$.

Consider the following game $G'$ in $<^{< \omega} \omega$: For $x \in ^{< \omega} \omega$, let $S_I(x)$ and $S_{II}(x)$ be defined as in Exercise 1.4.1. If $S_I(x)$ is not the set of sentences true in an $\omega$-model of $T$ then I loses. Otherwise II loses unless $S_{II}(x)$ is also the set of sentences true in an $\omega$-model of $T$. If neither player has lost for this reason, then let $\mathcal{M}_I$ and $\mathcal{M}_{II}$ be $\omega$-models isomorphic to the term models of $S_I(x)$ and $S_{II}(x)$ respectively. Player I wins just in case one of the following holds:

1. The model $\mathcal{M}_I$ is isomorphic to an initial segment of $\mathcal{M}_{II}$ not of the form $L^{|\mathcal{M}_{II}|}_b$ for $b$ an ordinal of $\mathcal{M}_{II}$.

2. There is an ordinal $a$ of $\mathcal{M}_I$ such that $L^{|\mathcal{M}_I|}_a$ is isomorphic to an initial segment of $\mathcal{M}_{II}$ but $L^{|\mathcal{M}_I|}_{a+1}$ is not.

Prove that $G'$ is not a win for I. To do so, assume that $\sigma$ is a winning strategy for I. By absoluteness, you may assume that $\sigma \in L$. Let II play
against \( \sigma \) the set of sentences true in some \( L_\alpha \in Y \) such that \( \sigma \in L_\alpha \). It is easily seen that \( \Pi \)'s part of the resulting play \( x \) belongs to \( L_{\alpha+2} \). It follows that \( x \) belongs to \( L_{\alpha+2} \). Thus \( S_1(x) \in L_{\alpha+2} \), and so \( P(\omega) \cap M_\Pi \subseteq L_{\alpha+2} \). Let \( \beta \) be the least ordinal not belonging to \( \text{WFP}(M_\Pi) \). The wellfoundedness of \( M_\Pi \) implies that (1) can hold only if \( M_\Pi \cong M_\Pi \), i.e., if \( M_\Pi = L_\alpha \). But this is impossible, for \( \sigma \) belongs to \( L_\alpha \), and \( \Pi \) is simply copying as long as \( \Pi \) is playing the set of sentences true in \( L_\alpha \). Therefore (2) holds. This can happen only if \( \beta > \alpha \). But \( \beta \) is admissible, so \( \beta > \alpha + 2 \). Therefore \( L_{\alpha+3} \subseteq M_\Pi \). Because \( \beta \) does not exist, there is an \( a \subseteq \omega \) such that \( a \in L_{\alpha+3} \setminus L_{\alpha+2} \). This gives the contradiction that \( a \) both belongs and does not belong to \( M_\Pi \).

Now prove that the game \( G' \) is not a win for \( \Pi \). For this assume that \( \tau \in L \) is a winning strategy for \( \Pi \). Let \( \Pi \) play against \( \tau \) the set of sentences true in some \( L_\alpha \in Y \) such that \( \tau \in L_\alpha \). It follows that \( P(\omega) \cap M_\Pi \subseteq L_{\alpha+2} \). Let \( \beta \) be the least ordinal not belonging to \( \text{WFP}(M_\Pi) \). Since (1) fails, it is impossible that \( \beta = \alpha \). Since (2) fails, it is also impossible that \( \beta < \alpha \). Thus \( \beta > \alpha \). By an argument like that of the last paragraph, this gives a contradiction.

Derive a contradiction by showing that \( G' \) is \( \Sigma^0_4 \) and therefore, by hypothesis, determined.

Remark. In [Montalban and Shore, ], a strengthening of the result of Exercise 1.4.2 is proved. In [Montalban and Shore, ], they improve this result by showing that the consistency \( \text{ZFC}^- \) is implied by the statement that, for all \( n \in \omega \), all \( n \)-\( \Pi^0_3 \) games are determined. (The latter statement is what they call \( \omega \)-\( \Pi^0_3 \) determinacy.) This follows by an ultraproduct construction from their theorem that, for all \( n \geq 1 \), \( \text{KP} \vdash \Pi^0_{n+2} \) Determinacy \( \rightarrow \) “There is a wellfounded model of \( \text{KP} + \Delta^0_{n+2} \) Comprehension.” The theorem is proved using games similar to—but more complex than—the game used in the proof of Exercise 1.4.2.

**Exercise 1.4.3.** Let \( D \) be the set of all degrees of unsolvability, i.e., of all Turing degrees. A *cone* of Turing degrees is the set of all degrees greater than or equal to some fixed degree, which is called the *vertex* of the cone. For classes \( \Gamma \) of sets, \( \Gamma \) *Turing determinacy* is the assertion that, for all \( \mathcal{A} \subseteq D \) such that \( \bigcup \mathcal{A} \in \Gamma \), either \( \mathcal{A} \) or \( D \setminus \mathcal{A} \) contains a cone.

Work in \( \text{ZC}^- + \Sigma^1_1 \) Replacement. Assume \( \Sigma^0_6 \) Turing determinacy and prove that \( \beta_0 \) exists. This is another strengthening by the author of a result of [Friedman, 1971], with \( \Sigma^0_6 \) replacing the \( \Sigma^0_6 \) of [Friedman, 1971].
1.4. $\Delta^0_4$ GAMES

**Hint.** For elements $y$ and $z$ of $\omega_1$, let $y \oplus z \in \omega_1$ be such that $(y \oplus z)(2n) = y(n)$ and $(y \oplus z)(2n + 1) = z(n)$ for all $n \in \omega$. Let $B$ be the set of all $(y, z)$ such that I wins the play $y \oplus z$ of the game $G'$ of Exercise 1.4.2. Let

$$A = \{ a \in D \mid (\exists z \in \omega_1)(d(z) \leq a \land (\forall y \in \omega_1)(d(y) \leq a \rightarrow (y, z) \notin B)) \}. $$

Note that $\bigcup A \in \Sigma^0_5$.

Assume that $\beta_0$ does not exist. Let $b \in D \cap L$. Show that the cone with vertex $b$ is not contained in $A$. To do so, let $\alpha$ belong to the set $Y$ defined in the hint to Exercise 1.4.2 and let $b \in L_{\alpha}$. Let $S$ be the set of sentences true in $L_{\alpha}$. Let $y$ be the characteristic function of the set of Gödel numbers of members of $S$. Let $a = d(y)$. Show that

$$(\forall z)(d(z) \leq a \rightarrow (y, z) \in B).$$

(This is more than what is needed for $a \notin A$.) Similarly show that the cone with vertex $b$ is not disjoint from $A$.

**Exercise 1.4.4.** Work in $ZC^- + \Sigma_1$ Replacement. Let $D$ be as in Exercise 1.4.3.

(a) Let $A \subseteq D$. Prove that $G(\bigcup A; \omega)$ is determined if and only if either $A$ or $D \setminus A$ contains a cone. This observation is from [Martin, 1968]. It implies that, for all classes $\Gamma$, $\Gamma$ Turing determinacy follows from the hypothesis that all $\Gamma$ games are determined.

(b) Let $\alpha$ be a countable ordinal. Assume that all $\Sigma^0_\alpha$ games in $\omega$ are determined. Let $A \subseteq \omega_1$ with $A \in \Sigma^0_{\alpha+1}$. Show that if $A$ has members of arbitrarily large degree then $A$ meets each member of some cone of degrees. This result appears in [Harrington and Kechris, 1975]; the authors report that it was independently proved by Ramez Sami.

(c) Prove that, for every countable ordinal $\alpha$, $\Delta^0_{\alpha+2}$ Turing determinacy follows from the determinacy of all $\Sigma^0_\alpha$ games in $\omega$. This consequence of (b) is due to the author.

(d) Prove $\Delta^0_5$ Turing determinacy. (By Exercise 1.4.3, this is an optimal result.)

**Hint.** For (b), let $A \in \Sigma^0_{\alpha+1}$ and let $B_n, n \in \omega$, be $\Pi^0_\alpha$ sets such that be such that $A = \bigcup_{n \in \omega} B_n$. Let $B = \{ (n) \rightarrow x \mid x \in B_n \}$. Consider the game $G$ in which I plays $x$, II plays $y$, and I wins if $x \in B$ and $d(y) \leq d(x)$. Show that $G$ is a win for I and use this to show that $A$ meets all sufficiently large degrees.
For (c), use Theorem 1.4.2.

Remark. The result of Exercise 1.4.1 and those of parts (c) and (d) of Exercise 1.4.4 were proved in 1974, and the proofs were circulated in unpublished manuscripts. The same is true of the theorem that \( \Sigma_0^0 \) Turing determinacy is false \( L_{\beta_0} \). The result of Exercise 1.4.2 was proved somewhat later, and that of Exercise 1.4.3 was noticed only in 1995.

**Exercise 1.4.5.** There are various ways to generalize Friedman-style results on Borel games to the case of uncountable trees. Here is one generalization of Exercise 1.4.1.

Let \( \rho \) be any ordinal. Let \( \beta(\rho) \) be the least ordinal \( \beta > \rho \) such that \( L_\beta \models \text{ZFC}^- \). Prove that \( L_{\beta(\rho)} \) does not satisfy the determinacy of all \( \Sigma_4^0 \) games in \( <\omega \rho \).

*Hint.* Relativize to arbitrary elements of \( \omega^\omega \) the argument of the hint for Exercise 1.4.1. Now collapse \( \rho \) by forcing.

**Exercise 1.4.6.** Let \( \text{ZFCRec}^- \) be the theory \( \text{Rec}(\text{ZC}^- + \Sigma_1 \text{ Replacement}) \) introduced on page 47. Let \( \text{ZFCRec}^- \) be the result of adding, to the axioms of \( \text{Rec}(\text{ZC}^- + \Sigma_1 \text{ Replacement}) \), the Replacement Schema for all formulas of the language \( L_{\text{rec}} \). Show that the determinacy of all \( \Sigma_4^0 \) games in countable trees is not provable in \( \text{ZFCRec}^- \).

*Hint.* Replace the \( L_\alpha \) hierarchy by the \( L_{\text{rec}}^\alpha \) hierarchy, where \( L_{\alpha+1}^{\text{rec}} \) is the set of subsets of \( L_\alpha^{\text{rec}} \) defined over \( L_\alpha^{\text{rec}} \) by formulas of \( L_{\text{rec}} \). It is easy to see that \( L_\lambda^{\text{rec}} = L_\lambda \) for all limit ordinals \( \lambda \).

Let \( \beta_0^{\text{rec}} \) be the least ordinal \( \beta \) such that \( L_\beta^{\text{rec}} \) satisfies \( \text{ZFCRec}^- \). Show that \( L_{\omega_1}^{\text{rec}} \) is such an ordinal. Show that Condensation holds for the \( L_{\alpha}^{\text{rec}} \) hierarchy as it does for the \( L_\alpha \) hierarchy, and so \( \beta_0^{\text{rec}} \) is a countable ordinal. Show also that \( \beta_0^{\text{rec}} \) is the least ordinal \( \beta \) such that no \( a \subseteq \omega \) belongs to \( L_{\beta+1}^{\text{rec}} \setminus L_\beta^{\text{rec}} \).

Continue to imitate the proof for Exercise 1.4.1.
Chapter 2

General Borel Games

In this chapter we introduce the technical concept of a covering of a game tree, and we use this concept to prove the determinacy of all Borel games and—in uncountable trees—the determinacy of all games in a larger class.

Borel determinacy is proved in §2.1. In countable trees, the Borel sets are the same as the the $\Delta^1_1$ sets (to be defined in §2.2). In general, however, $\Delta^1_1$ is a larger class, the class of what we will call quasi-Borel sets. In §2.2 we prove this and also prove that all quasi-Borel games are determined. §2.1 and §2.2 depend only on §1.1 and §1.2 (and not on the rest of Chapter 1).

Readers interested only in main results may confine themselves to §2.1 (though §2.2 should present no extra difficulties).

In §2.3 we work again in the weak set theory of §§1.3–1.4. We use the proofs of §2.1 and the results of §1.4 to get $\Sigma^0_\alpha$ determinacy with the minimal possible amount of Power Set and Replacement (allowed by refinements—given in the exercises—of results of Harvey Friedman).

In §2.4 we consider a class of infinite games of imperfect information called Blackwell games after David Blackwell, who initiated their study. We introduce the basic theory of imperfect information games, and then we prove the determinacy of Borel Blackwell games by showing that it follows from ordinary Borel determinacy. This is done by proving a general theorem reducing the the determinacy of Blackwell games of any reasonably closed class to the determinacy of ordinary games of that class. Thus all our determinacy results in later chapters will imply corresponding determinacy results for Blackwell games.
2.1 Borel Determinacy

Almost all the determinacy results in the remainder of this book will be proved by the technique of auxiliary games: To prove $G(A; T)$ determined, we will associate with $G(A; T)$ another game $G(A^*; T^*)$. This auxiliary game we will know to be determined. Moreover the two games will be so related that the determinacy of $G(A; T)$ will follow from that of $G(A^*; T^*)$. In a sense we have already seen this technique. To prove Theorem 1.3.1, for example, we made use of the closed games $G(C; T)$ occurring in the proof of Lemma 1.3.2. Such games were used also in proving Theorems 1.3.3, 1.4.10, and 1.4.22. The auxiliary game technique as we will use it later differs from these examples in two important ways: (1) The determinacy of the given game $G(A; T)$ will be reduced to the determinacy of a single game $G(A^*; T^*)$. (2) $T^*$ will be larger than $T$, whereas the auxiliary game trees in the earlier examples were all subtrees of the given $T$. Indeed the results of Friedman [1971] show that (for, e.g., $T = \omega_1$) some use of existence principles for sets larger than $T$ is necessary to prove the determinacy of Borel games in $T$. (See Exercises 1.4.1–1.4.5 and Exercises 2.3.2–2.3.12.)

In using the auxiliary game technique, one can think of moves in the auxiliary tree as being moves in $T$ together with extra components. In later chapters the extra components will be elements of measure spaces. Winning strategies for the main game will be derived from winning strategies for the auxiliary game by integration. In this chapter the extra components of moves in the auxiliary tree will be, in the basic case, (a) subtrees of $T$ and (b) decisions about whether the element of $\lceil T \rceil$ being produced will belong to certain subsets of $\lceil T \rceil$. Exercise 2.1.2 illustrates this technique, reproving Theorem 1.3.1 with the help of an auxiliary game. However, components of the form (b) do not appear in this example. In more general cases, auxiliary trees will come from iterations of the process that gives the basic case.

Remark. The first proof of $\Sigma^0_4$ determinacy, that in [Paris, 1972], used an auxiliary game technique modeled on the one we will present in Chapter 4. James Baumgartner had earlier found, adapting the method of Chapter 4, a new proof of $\Sigma^0_3$ determinacy.

Our proof of Borel determinacy will be like that in [Martin, 1985] in that we will prove inductively that all Borel sets have a certain property, the property of being reducible in a certain way to a clopen set of plays in a different tree. The determinacy of a set with this property will follow
2.1. BOREL DETERMINACY

easily from the determinacy of a set related to the clopen set. In the details there we will be several differences between the proof in [Martin, 1985] and the proof as we will present it below. Our presentation will be similar to that in [Hurkens, 1993]. This similarity is partly coincidental and partly by choice. When the first draft of this section was written around 1990, it was influenced by an idea of Moschovakis (found in the proof of Theorem 6F.1 of [Moschovakis, 1980]). Moschovakis’ idea eliminates from the original proof of Borel determinacy (the proof in [Martin, 1975]) part of its use of quasistrategies and subsidiary games. In writing the present chapter, the author wished to go further: (a) to combine Moschovakis’ idea with the purely inductive proof in [Martin, 1985] and (b) to eliminate from the proof every vestige of the use of quasistrategies. To accomplish these aims, the author introduced game trees with taboos, game trees in which each terminal position is automatically lost for one player or the other—is taboo for one player or the other—indeed of the payoff set. (In the first draft of the section, non-taboo terminal positions were also permitted.) Hurkens, who explicitly had aim (a), produced a proof that has essentially all the ingredients in the author’s draft (which Hurkens had not seen). Hurkens’ proof introduces one additional idea, an idea that both simplifies and helps motivate the main construction of the proof. Although the author had in his possession a copy of [Hurkens, 1993], he learned about this idea only indirectly, in conversation with Marco Vervoort. Afterwards he actually consulted [Hurkens, 1993] and discovered the similarities between Hurkens’ proof and his own. Hurkens’ additional idea seemed too valuable to omit, so the author has revised his draft to incorporate that idea (and to make some other modifications). In the course of giving the proof, we will explain Hurkens’ idea and we will comment on relations between the two proofs.

A game tree with taboos is a triple $T = (T, T_I, T_{II})$, where

1. $T$ is a game tree;
2. $T_I$ and $T_{II}$ are disjoint sets of terminal positions in $T$;
3. every terminal position in $T$ belongs to $T_I$ or to $T_{II}$.

Recall that terminal positions in $T$ are members of $T$ that are also finite plays in $T$. Infinite plays are not positions, and so are not terminal positions.

Convention. We always use boldface letters, perhaps with other markings, for game trees with taboos. For the underlying game trees, we use the corresponding italic lightface letters, with the same markings; for the
other two components, we use the corresponding calligraphic letters, with the same markings and with subscripts “I” and “II.” For example, $\mathcal{T}^i$ will be $\langle \tilde{T}^i, \tilde{T}^i_I, \tilde{T}^i_{II} \rangle$.

If $\mathcal{T}$ is a game tree with taboos, then positions, moves, plays, strategies, etc. in $\mathcal{T}$ are just positions, moves, plays, strategies, etc. in $T$. If $p \in T$, then $\mathcal{T}_p$ is the game tree with taboos $\langle T_p, \mathcal{T}_I \cap T_p, \mathcal{T}_{II} \cap T_p \rangle$.

For any game tree $T$, we let $[T]$ be the set of all infinite plays in $T$. Note that $[T]$ is a closed subset of $\lceil T \rceil$.

Let $\mathcal{T}$ be a game tree with taboos. Plays belonging to $\mathcal{T}_I$ are taboo for I in $\mathcal{T}$, and plays belonging to $\mathcal{T}_{II}$ are taboo for II in $\mathcal{T}$. Hence $[T]$ is the set of all plays that are not taboo for either player in $\mathcal{T}$, i.e., that are not taboo in $\mathcal{T}$. For $A \subseteq [T]$, we define the game $G(A; \mathcal{T})$ as follows: A finite play of $G(A; \mathcal{T})$ is lost by the player for whom it is taboo. A play $x \in [T]$ is won by I if and only if $x \in A$. Thus $G(A; \mathcal{T})$ is the same game as $G((A \cup \mathcal{T}_{II}) \setminus \mathcal{T}_I; T)$. The notions, for $G(A; \mathcal{T})$, of winning strategy and being determined are the same as those for $G((A \cup \mathcal{T}_{II}) \setminus \mathcal{T}_I; T)$.

Remark. [Hurkens, 1993] does not have game trees with taboos, but it has a device that does the same work. It has a move function of the sort we discussed on page 2. The move function is defined even in terminal positions, and whichever player has the impossible task of moving in a terminal position loses the that play of the game.

We could have omitted clause (3) from the definition of game trees with taboos, i.e., we could have permitted the existence of finite non-taboo plays. Indeed, this would have been the more natural definition, since we permitted finite plays throughout Chapter 1. The reason why we include clause (3) is that without it many of our definitions and proofs would have been more complicated, since we would have had to worry about whether any given finite play was taboo or not.

Remark. While Hurkens’ use of a move function does all the work done by game trees with taboos, it would not in a straightforward way do the work of game trees with taboos in the more liberal sense just discussed.

It is important to make sure that proving determinacy results only for game trees with taboos in our restricted sense involves no loss of generality. First note that, for ordinary game trees (without taboos), nothing is lost by considering only trees without finite plays. To see this, let $T$ be a game tree.
Consider the tree 

\[ T' = T \cup \{ p^{-} \langle 0, \ldots, 0 \rangle \mid n \in \omega \land p \text{ is terminal in } T \}. \]

The tree \( T' \) has no terminal postions. The obvious bijection \( f : [T'] \rightarrow [T] \) is a homeomorphism such that, for each \( A \subseteq [T] \), \( G(A; T) \) is determined if and only if \( G(f^{-1}(A); T') \) is determined. Similarly, let \( T \) be an game tree with taboos in the unrestricted sense (possibly not satisfying clause (3)). Set 

\[ T' = T \cup \{ p^{-} \langle 0, \ldots, 0 \rangle \mid n \in \omega \land p \text{ is terminal in } T \text{ and not taboo in } T \}. \]

Let \( T' = \langle T', T_I, T_{II} \rangle \). Then \( T' \) is a game tree with taboos. Furthermore, the obvious homeomorphism \( f : [T'] \rightarrow [T] \) restricts to a homeomorphism (in the sense of the definition below, adapted to allow for game trees with taboos in the unrestricted sense) from \([T']\) to the set of all non-taboo plays in \( T \). Moreover, for any set \( A \) of non-taboo plays in \( T \), \( G(A; T) \) is determined if and only if \( G(f^{-1}(A); T') \) is determined (under the obvious definition).

We give \([T]\) the relative topology: A subset \( A \) of \([T]\) is open just in case there is an open \( B \subseteq [T] \) such that \( A = B \cap [T] \). We will construe our topological definitions as making sense even in the case \([T]\) is empty, so that the unique subset \( \emptyset \) of \([T]\) is open, Borel, etc. The following easy lemma will allow us usually not to worry about the distinction between the Borel hierarchy on \([T]\) and that on \([T]\).

**Lemma 2.1.1.** Let \( T \) be a game tree with taboos. For all ordinals \( \alpha \geq 1 \) and all subsets \( A \) of \([T]\), \( A \) belongs to \( \Pi_\alpha^0 \) as a subset of \([T]\) if and only if \( A \) belongs to \( \Pi_\alpha^0 \) as a subset of \([T]\). For all ordinals \( \alpha > 1 \) and all subsets \( A \) of \([T]\), \( A \) belongs to \( \Sigma_\alpha^0 \) as a subset of \([T]\) if and only if \( A \) belongs to \( \Sigma_\alpha^0 \) as a subset of \([T]\).

**Proof.** We prove the lemma by induction on \( \alpha \geq 1 \).

By definition of the relative topology, every subset of \([T]\) closed as a subset of \([T]\) is closed as a subset of \([T]\). Since \([T]\) is closed as a subset of \([T]\), every subset of \([T]\) closed as a subset of \([T]\) is closed as a subset of \([T]\).

Let \( \alpha > 1 \) and assume that the lemma holds for all \( \beta < \alpha \). The fact that any subset of \([T]\) belongs to \( \Sigma_\alpha^0 \) as subset of \([T]\) if and only if it belongs to \( \Sigma_\alpha^0 \) as a subset of \([T]\) follows directly from the definition of \( \Sigma_\alpha^0 \) and our induction.
hypothesis. Suppose that $A \in \Pi^0_{\alpha}$ as a subset of $[T]$. Thus $[T] \setminus A \in \Sigma^0_{\alpha}$ as a subset of $[T]$. $[T] \setminus [T] \in \Sigma^0_{\alpha}$ (by Lemma 1.1.1) $\Sigma^0_{\alpha}$. By Lemma 1.1.1 again, $[T] \setminus A = ([T] \setminus A) \cup ([T] \setminus [T])$ belongs to $\Sigma^0_{\alpha}$. By the definition of $\Pi^0_{\alpha}$, $A \in \Pi^0_{\alpha}$ as a subset of $[T]$. Suppose now that $A \subseteq [T]$ and that $A \in \Pi^0_{\alpha}$ as a subset of $[T]$. Thus $[T] \setminus A \in \Sigma^0_{\alpha}$. By Lemma 1.1.1 again, $[T] \setminus A = ([T] \setminus A) \cup ([T] \setminus [T])$ belongs to $\Sigma^0_{\alpha}$ as a subset of $[T]$. Thus $[T] \setminus A \in \Sigma^0_{\alpha}$ as a subset of $[T]$. $\square$

There is another way to characterize the topology on $[T]$. Note that $[T] = [\bar{T}]$, where $\bar{T} = \{ p \in T \mid (\exists x \supset p) x \in [T] \}$. If $[T]$ is nonempty, then $\bar{T}$ is a game tree, and our topology for $[T]$ is the same as the topology it has as $[\bar{T}]$. Thus Lemma 1.1.1 holds for the Borel hierarchy on $[T]$. (One can also see this using Lemma 2.1.1.)

Let us now show that determinacy for games in game trees with taboos is level by level equivalent to determinacy for games in ordinary game trees. By the remark above, determinacy in ordinary game trees is equivalent level by level to determinacy in ordinary game trees that have no terminal positions, so we need only consider the latter. In one direction, note that any game tree without terminal nodes can be considered a game tree with taboos by setting $T_I = T_{II} = \emptyset$. In the other direction, let $T$ be a game tree with taboos. If $G([T] \setminus T_I; T)$ is a win for $\Pi$, then all games in $T$ are wins for $\Pi$. Assume otherwise and let $R$ be $I$’s non-losing quasistrategy for $G([T] \setminus T_I; T)$. If $G(T_{II}; R)$ is a win for $I$, then all games in $T$ are wins for $I$. Assume otherwise and let $S$ be $\Pi$’s non-losing quasistrategy for $G(T_{II}; R)$. The game subtree $S$ of $T$ satisfies $S \subseteq [T]$. Moreover, for any $A \subseteq [T]$, the games $G(A \cap [S]; S)$ and $G(A; T)$ are completely equivalent; in particular, the latter is determined if the former is. Finally, we have that $A \cap [S]$ is as simple topologically as $A$. One consequence of this is that our previous determinacy results hold also for game trees with taboos:

**Lemma 2.1.2.** Theorems 1.2.4, 1.3.1, 1.3.3, 1.4.10, and 1.4.22 and Corollaries 1.2.3, 1.4.15, and 1.4.23, hold for games in game trees with taboos.

**Proof.** The argument given in the paragraph preceding the statement of the lemma goes through in $\mathbf{ZC}^{-} + \Sigma_1$ Replacement. Thus the Theorems listed in the statement of the lemma holds for games in trees with taboos. Corollary 1.2.3 follows from Theorem 1.2.4. To see that Corollaries 1.4.15
and 1.4.23 follow, it is suffices to show that Theorems 1.4.2 and 1.4.21 hold in each $[T]$. This in turn follows from the original Theorems 1.4.2 and 1.4.21 for $[\bar{T}]$, where $\bar{T}$ is as above.

\[\square\]

**Remark.** Since games in $T$ are equivalent to games in the $S$ defined above, we could avoid dealing with game trees with taboos by replacing each $T$ with the corresponding $S$. In a sense, that is what is done in [Martin, 1975] and [Martin, 1985]. Here, however, we are interested in avoiding the nuisance of quasistrategies, and so we put up with the nuisance of taboos.

If $\tilde{T}$ and $T$ are game trees with taboos, we write $\pi : \tilde{T} \Rightarrow T$ to mean that

(i) $\pi : \tilde{T} \to T$;
(ii) $\tilde{p} \subseteq \tilde{q} \to \pi(\tilde{p}) \subseteq \pi(\tilde{q})$ for all $\tilde{p}$ and $\tilde{q}$ belonging to $\tilde{T}$;
(iii) $\ell h(\pi(\tilde{p})) = \ell h(\tilde{p})$ for all $\tilde{p} \in \tilde{T}$.
(iv) $\pi(\tilde{p}) \in T_1 \to \tilde{p} \in \tilde{T}_1$ for all $\tilde{p} \in \tilde{T}$;
(v) $\pi(\tilde{p}) \in T_{II} \to \tilde{p} \in \tilde{T}_{II}$ for all $\tilde{p} \in \tilde{T}$;

Note that it is allowed that $\tilde{p}$ be terminal in $\tilde{T}$ (and so taboo in $\tilde{T}$) even though $\pi(\tilde{p})$ is not terminal in $T$.

Let $\pi : T \Rightarrow T$. If $\tilde{x}$ is a play in $\tilde{T}$, then clause (ii) implies that $\bigcup_{\tilde{p} \subseteq \tilde{x}} \pi(\tilde{p})$ is either a position or a play in $T$. If $\tilde{x}$ is finite, then $\bigcup_{\tilde{p} \subseteq \tilde{x}} \pi(\tilde{p}) = \pi(\tilde{x})$. Thus we can extend $\pi$ to a function, which we also denote by “$\pi$,” from $\tilde{T} \cup [\bar{T}]$ to $T \cup [T]$. By clause (iii), $\ell h(\pi(\tilde{x})) = \ell h(\tilde{x})$ for all plays $\tilde{x}$, where we recall that $\ell h(\tilde{x}) = \omega$ if $x$ is infinite. If $\tilde{x}$ is an infinite play in $\tilde{T}$, then $\pi(\tilde{x})$ is an infinite play in $T$. Thus $\pi$ induces a function

$$\pi : [\tilde{T}] \to [T].$$

The function $\pi$ is continuous and satisfies a “Lipschitz condition,” i.e. $\pi(\tilde{x}) \upharpoonright n$ depends only on $\tilde{x} \upharpoonright n$.

If $\tilde{T}$ and $T$ are game trees with taboos, we write $\phi : \tilde{T} \Rightarrow \tilde{T}$ to mean that

(i) $\phi : S(\tilde{T}) \to S(T)$;
(ii) each $\phi(\tilde{\sigma})$ is a strategy for the same player as is $\tilde{\sigma}$;
(iii) for each $n \in \omega$, the restriction of $\phi(\tilde{\sigma})$ to positions of length $< n$ depends only on the restriction of $\tilde{\sigma}$ to positions of length $< n$. 

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If $T$ is a game tree and $k \in \omega$, let

$$\kappa T = \{ p \in T \mid \ell h(p) \leq k \}.$$  

By clause (iii) of the definition, we can think of a $\phi : \tilde{T} \xrightarrow{S} T$ as acting on $\bigcup_{k \in \omega} S(k\tilde{T})$ so that, for each $k$, $\phi \upharpoonright S(k\tilde{T}) : S(k\tilde{T}) \to S(kT)$.

We are now ready to give the main technical definition of this chapter. If $T$ is a game tree with taboos, then a covering of $T$ is a quadruple $\langle \tilde{T}, \pi, \phi, \Psi \rangle$ such that

(a) $\tilde{T}$ is a game tree with taboos;

(b) $\pi : \tilde{T} \Rightarrow T$;

(c) $\phi : \tilde{T} \xrightarrow{S} T$;

(d) $\Psi : \{ \langle \tilde{\sigma}, x \rangle \mid \tilde{\sigma} \in S(\tilde{T}) \land x \in [T] \land x \text{ is consistent with } \phi(\tilde{\sigma}) \} \to [\tilde{T}]$, and, for all $\langle \tilde{\sigma}, x \rangle \in \text{domain (}\Psi\text{)},$

(i) $\Psi(\tilde{\sigma}, x)$ is consistent with $\tilde{\sigma}$;

(ii) $\pi(\Psi(\tilde{\sigma}, x)) \subseteq x$;

(iii) either $\pi(\Psi(\tilde{\sigma}, x)) = x$ or $\Psi(\tilde{\sigma}, x)$ is taboo for the player for whom $\tilde{\sigma}$ is a strategy.

With regard to clause (d)(iii), note that $\pi(\Psi(\tilde{\sigma}, x)) = x$ implies $\ell h(\Psi(\tilde{\sigma}, x)) = \ell h(x)$; and this in turn implies that $\Psi(\tilde{\sigma}, x)$ and $x$ are both finite or both infinite. Note also that if both are finite then, by clauses (iv) and (v) of the definition of $\pi : \tilde{T} \Rightarrow T$, both are taboo for the same player.

Remarks:

(a) A variant definition, and one that has some advantages which we will point out later, would replace the quadruple $\langle \tilde{T}, \pi, \phi, \Psi \rangle$ by the triple $\langle \tilde{T}, \pi, \phi, \rangle$ and replace clause (d) by

(d’) if $\tilde{\sigma} \in S(\tilde{T})$ and $x$ is consistent with $\tilde{\sigma}$, then there is an $\tilde{x} \in [\tilde{T}]$ such that

(i) $\tilde{x}$ is consistent with $\tilde{\sigma}$;

(ii) $\pi(\tilde{x}) \subseteq x$;

(iii) either $\pi(\tilde{x}) = x$ or $\tilde{x}$ is taboo for the player for whom $\tilde{\sigma}$ is a strategy.
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(b) Although the fact will not be directly used by us, the $\pi$ of a covering is a surjection. Indeed, every play in $T$ is in the range of the extended $\pi$. (Exercise 2.1.4). For an example and an almost-example of coverings, see Exercises 2.1.3 and 2.1.5.

We say that a covering $\langle \tilde{T}, \pi, \phi, \Psi \rangle$ unravels a subset $A$ of $[T]$ if the preimage $\pi^{-1}(A)$ is a clopen subset of $[\tilde{T}]$.

We prove at once the basic lemma connecting coverings and unraveling with determinacy:

**Lemma 2.1.3.** Let $T$ be a game tree with taboos. If there is a covering of $T$ that unravels $A \subseteq [T]$, then $G(A; T)$ is determined.

**Proof.** Let $\langle \tilde{T}, \pi, \phi, \Psi \rangle$ be a covering of $T$ that unravels $A \subseteq [T]$. By Lemma 2.1.2 (as applied to Corollary 1.2.3), $G(\pi^{-1}(A); \tilde{T})$ is determined. Let us call the player for whom $G(\pi^{-1}(A); \tilde{T})$ is a win the good player and let us call the other player the bad player. Let $\tilde{\sigma}$ be a winning strategy for the good player for $G(\pi^{-1}(A); \tilde{T})$. We show that $\phi(\tilde{\sigma})$ is a winning strategy for the good player for $G(A; T)$. Let $x$ be a play in $T$ consistent with $\phi(\tilde{\sigma})$. We must prove that $x$ is a win for the good player in $G(A; T)$. We may assume that $x$ is not taboo for the bad player.

It is enough to show that $\pi(\Psi(\tilde{\sigma}, x)) = x$ and that $\Psi(\tilde{\sigma}, x)$ is infinite. If this is true then, since $\pi = \pi \upharpoonright [\tilde{T}]$ and $\pi : [\tilde{T}] \to [T]$,

$$\Psi(\tilde{\sigma}, x) \in \pi^{-1}(A) \leftrightarrow \pi(\Psi(\tilde{\sigma}, x)) \in A \leftrightarrow x \in A.$$ 

Because $\Psi(\tilde{\sigma}, x)$ is a win for the good player in $G(\pi^{-1}(A); \tilde{T})$, it follows that $x$ is a win for the good player in $G(A; T)$.

By clause (d)(i) in the definition of a covering, $\Psi(\tilde{\sigma}, x)$ is a play in $\tilde{T}$ that is consistent with $\tilde{\sigma}$. Since $\tilde{\sigma}$ is a winning strategy, $\Psi(\tilde{\sigma}, x)$ cannot be taboo for the good player. Thus clause (d)(iii) gives that $\pi(\Psi(\tilde{\sigma}, x)) = x$. By the observations after the definition of a covering, $x$ and $\Psi(\tilde{\sigma}, x)$ are both finite or both taboo for the same player. They cannot both be taboo for the same player, for $\Psi(\tilde{\sigma}, x)$ is not taboo for the good player, and we are assuming that $x$ is not taboo for the bad player.

In the proof of Lemma 2.1.3, the fact that $\pi^{-1}(A)$ is clopen was used only to get that $G(\pi^{-1}(A); \tilde{T})$ is determined. Thus we have the following generalization of that Lemma.
Lemma 2.1.4. Let $T$ be a game tree with taboos, and let $A \subseteq [T]$. If there is a covering $\langle T, \pi, \phi, \Psi \rangle$ of $T$ such that $G(\pi^{-1}(A); T)$ is determined, then $G(A; T)$ is determined.

Borel determinacy will be proved if we can show that every Borel set is unraveled by a covering. To do this, we need to do two things: (i) We must show that every open set can be unraveled. (ii) We must find some operations on coverings corresponding to the operations that generate the Borel sets from the open sets. (i) is the heart of the proof. We begin with the more routine (ii).

Let $T$ be a game tree with taboos and let $C = \langle \tilde{T}, \pi, \phi, \Psi \rangle$ be a covering of $T$. For $k \in \omega$, $C$ is a $k$-covering of $T$ if

(i) $k\tilde{T} = kT$, $k\tilde{T} \cap \tilde{T}_I = kT \cap T_I$, and $k\tilde{T} \cap \tilde{T}_{II} = kT \cap T_{II}$;
(ii) $\pi \upharpoonright k\tilde{T}$ is the identity;
(iii) $\phi \upharpoonright S(k\tilde{T})$ is the identity.

Suppose that $C_1 = \langle T_1, \pi_1, \phi_1, \Psi_1 \rangle$ is a covering of $T_0$ and that $C_2 = \langle T_2, \pi_2, \phi_2, \Psi_2 \rangle$ is a covering of $T_1$. We define the composition $C_1 \circ C_2$ of $C_1$ and $C_2$ to be

$$\langle T_2, \pi_1 \circ \pi_2, \phi_1 \circ \phi_2, \Psi \rangle,$$

where $\Psi(\sigma, x) = \Psi_2(\sigma, \Psi_1(\phi_2(\sigma), x))$. We omit the routine proof of the following lemma.

Lemma 2.1.5. The composition of coverings is a covering. For natural numbers $k_1$ and $k_2$, the composition of a $k_1$-covering and a $k_2$-covering is a $\min\{k_1, k_2\}$-covering.

The next lemma gives us a sufficient condition that the limit of a sequence of $k$-coverings exist and be a $k$-covering. It is for constructing such limits that the concept of $k$-covering was introduced.

Lemma 2.1.6. Let $k \in \omega$, let $T_i$, $i \in \omega$, be game trees with taboos, and let $\langle k_{j,i}, \pi_{j,i}, \phi_{j,i}, \Psi_{i,j} \mid i \leq j \in \omega \rangle$ be such that

1. if $i \leq j \in \omega$ then $C_{j,i} = \langle T_j, \pi_{j,i}, \phi_{j,i}, \Psi_{i,j} \rangle$ is a $k_{j,i}$-covering of $T_i$;
2. if $i_1 \leq i_2 \leq i_3 \in \omega$ then $C_{i_3,i_1} = C_{i_2,i_1} \circ C_{i_3,i_2}$;
3. $\inf_{i \leq j \in \omega} k_{j,i} \geq k$;
(4) \( \lim_{j \in \omega} \inf_{j' \geq j} k_{j', j} = \infty \); i.e., for all \( n \in \omega \) there is an \( i \in \omega \) such that
\( k_{j', j} \geq n \) for all \( j' \geq j \geq i \).

Then there is a \( T_\omega \) with \( |T_\omega| \leq \sum_{i \in \omega} |T_i| \) and there is a system
\( \langle \pi_{\infty, i}, \phi_{\infty, i}, \Psi_{i, \infty} \mid i \in \omega \rangle \)
such that each \( C_{\infty, i} = \langle T_\omega, \pi_{\infty, i}, \phi_{\infty, i}, \Psi_{i, \infty} \rangle \) is a \( k \)-covering of \( T_i \) and such that, for \( i \leq j \in \omega \), \( C_{\infty, i} = C_{j, i} \circ C_{\infty, j} \).

**Proof.** The idea is that, because of (4), what is in essence the inverse limit exists. For \( n \in \omega \), let \( i_n \) be the least number \( i \) such that, for all \( j' \geq j \geq i \), \( k_{j', j} \geq n \). Thus \( nT_j, nT_j \cap (T_j)_I \), and \( nT_j \cap (T_j)_II \), are independent of \( j \) for \( j \geq i_n \). For any finite sequence \( p \), let
\[
\begin{align*}
p \in T_\omega & \iff p \in T_{i_n(p)}; \\
p \in (T_\omega)_I & \iff p \in (T_{i_n(p)})_I; \\
p \in (T_\omega)_II & \iff p \in (T_{i_n(p)})_II.
\end{align*}
\]
Clearly \( T_\omega \) is a game tree with taboos and \( |T_\omega| \leq \sum_{i \in \omega} |T_i| \). Since (3) gives that \( i_n = 0 \) for \( n \leq k \), we have that \( kT_\omega = kT_j, kT_\omega \cap (T_\omega)_I = kT_j \cap (T_j)_I \), and \( kT_\omega \cap (T_\omega)_II = kT_j \cap (T_j)_II \) for each \( j \), as required by clause (i) of the definition of a \( k \)-covering.

For \( p \in nT_\omega \), we let
\[
\begin{align*}
\pi_{\infty, j}(p) & = \begin{cases} 
p & \text{if } j \geq i_n; \\
\pi_{i_n, j}(p) & \text{if } j < i_n.
\end{cases}
\end{align*}
\]
It is routine to check that each \( \pi_{\infty, j} \) is well-defined, that \( \pi_{\infty, j} : T_\omega \Rightarrow T_j \), and that \( \pi_{\infty, j} = \pi_{j', j} \circ \pi_{\infty, j'} \) whenever \( j \leq j' \in \omega \). Clearly \( \pi_{\infty, j} \mid nT_\omega \) is the identity for each \( j \geq i_n \), and so the fact that \( i_n = 0 \) for \( n \leq k \) guarantees that every \( \pi_{\infty, j} \mid kT_\omega \) is the identity, as required by clause (ii) of the definition of a \( k \)-covering.

Similarly, for \( \sigma \in S(nT_\omega) \), we let
\[
\phi_{\infty, j}(\sigma) = \begin{cases} \sigma & \text{if } j \geq i_n; \\
\phi_{i_n, j}(\sigma) & \text{if } j < i_n.
\end{cases}
\]
We omit the verifications that each \( \phi_{\infty, j} \) is well-defined, that each \( \phi_{\infty, j} : T_\omega \Rightarrow T_j \), and that \( \phi_{\infty, j} = \phi_{j', j} \circ \phi_{\infty, j'} \) for all \( j \leq j' \in \omega \). Since \( \phi_{\infty, j} \mid S(nT_\omega) \)
is the identity whenever \( j \geq i_n \), the fact that \( i_n = 0 \) for \( n \leq k \) guarantees that clause (iii) of the definition of a \( k \)-covering holds.

It remains to define the \( \Psi^{j,\infty} \) and to verify clause (d) in the definition of a covering.

First note that we always have \((\Psi^{j,j'}(\sigma,x)) | k_{j,j'} = x | k_{j,j'}; \) for \((\Psi^{j,j'}(\sigma,x)) | k_{j,j'} = \pi_{j,j'}(\Psi^{j,j'}(\sigma,x)) \subseteq x \| k_{j,j'} \), and \((\Psi^{j,j'}(\sigma,x)) | k_{j,j'} \subseteq x \| k_{j,j'} \) is impossible. Let \( j \in \omega \) and let \( \sigma \in \mathcal{S}(T_{\infty}) \). For \( x \in [T_j] \) and \( x \) consistent with \( \phi_{\infty,j}(\sigma) \), we can set

\[
\Psi^{j,\infty}(\sigma,x) = \lim_{j' \to \infty} \Psi^{j,j'}(\phi_{\infty,j'}(\sigma), x),
\]

since, for each \( n \in \omega \),

\[
\lim_{j' \to \infty}((\Psi^{j,j'}(\phi_{\infty,j'}(\sigma), x)) | n) = \begin{cases} x | n & \text{if } j \geq i_n; \\ (\Psi^{j,i_n}(\phi_{\infty,i_n}(\sigma), x)) | n & \text{if } j < i_n. \end{cases}
\]

If some \((\Psi^{j,\infty}(\sigma,x)) | n\) is not consistent with \( \sigma \), then, for any \( j' \) such that \( j \leq j' \) and \( i_n \leq j' \), the same position \((\Psi^{j,j'}(\phi_{\infty,j'}(\sigma), x)) | n\) is not consistent with \( \phi_{\infty,j'}(\sigma) \), which agrees with \( \sigma \) on positions of length \( n \). This contradicts property (d)(i) of the covering \( C_{j,j'} \), so and property (d)(i) is verified for \( C_{\infty,j} \).

For (d)(ii) and (d)(iii), note that we have, for each \( n \in \omega \), for each \( j' \) such that \( j \leq j' \) and \( j' \geq i_n \), for each \( \sigma \in \mathcal{S}(T_{\infty}) \), and for each \( x \in [T_j] \) consistent with \( \phi_{\infty,j}(\sigma) \), that

\[
(\pi_{\infty,j}(\Psi^{j,\infty}(\sigma,x))) | n = (\pi_{j,j'}(\Psi^{j,j'}(\phi_{\infty,j'}(\sigma), x))) | n.
\]

Property (d)(ii) for \( C_{\infty,j} \) thus follows from property (d)(ii) for \( C_{j,j'} \). Moreover, since \( j' \geq i_n \) implies that \((\Psi^{j,\infty}(\sigma,x)) | n = (\Psi^{j,j'}(\phi_{\infty,j'}(\sigma), x)) | n\), property (d)(iii) for \( C_{\infty,j} \) also follows from property (d)(iii) for \( C_{j,j'} \). We omit the verification that \( \Psi^{j,\infty}(\sigma,x) = \Psi^{j,j'}(\phi_{\infty,j'}(\sigma), x) \) for all \( j \leq j' \) and all \( \langle \sigma,x \rangle \) in domain \((\Psi^{j,\infty})\).

Remark. One advantage of adopting the alternative definition of covering considered in remark (a) on page 66 would be that the construction of the proof of Lemma 2.1.6 would literally be the construction of the inverse limit of the given system of coverings.

Lemma 2.1.7. Let \( T \) be a game tree with taboos. If \( A \subseteq [T] \) is open or closed and \( k \in \omega \), then there is a \( k \)-covering of \( T \) that unravels \( A \).
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Proof. Since any covering that unravels a set also unravels its complement, it is enough to prove that every closed subset of \([T]\) is, for each \(k \in \omega\), unraveled by some \(k\)-covering of \(T\). Let then \(A \subseteq [T]\) be closed. Recall that \(A\) is also closed as a subset of \([T]\). Let \(k \in \omega\) and, since every \((k+1)\)-covering is also a \(k\)-covering, assume without loss of generality that \(k\) is even.

We will define \(C = \langle \tilde{T}, \pi, \phi, \Psi \rangle\) and show that \(C\) is a \(k\)-covering and that \(C\) unravels \(A\).

We begin with \(\tilde{T}\). Because we have to make \(C\) a \(k\)-covering, we let \(k \tilde{T} = \tilde{T}\), \(k \tilde{T} \cap \tilde{T} = k T \cap T\), and \(k \tilde{T} \cap \tilde{T}_I = k T \cap T_I\). All moves in \(\tilde{T}\) will be moves in \(T\), except for move \(k\) and move \(k+1\). Each of these two moves will consist of a move in \(T\) together with one or two extra components.

To describe move \(k\), let \(p \in \tilde{T}\) with \(\ell h(p) = k\). Thus \(p \in T\) also. If \(p\) is terminal in \(T\)—and so taboo in \(T\)—then \(p\) is taboo in \(\tilde{T}\)—and so terminal in \(\tilde{T}\); and hence there is no move \(k\). Assume therefore that \(p\) is not terminal in \(T\). Since \(k\) is even, it is I's turn to move at \(p\). We stipulate that I's move at \(p\) in \(\tilde{T}\) must be of the form \(\langle a, X \rangle\),

where \(a\) is a move legal in \(T\) at \(p\) and \(X\) is a subset of the set \(Z\) of all \(q \in T\) satisfying the following conditions:

(i) \(p^- \langle a \rangle \not\subset q\).
(ii) \(q\) is not terminal in \(T\).
(iii) \([T_q] \cap A = \emptyset\).
(iv) \((\forall r)(p^- \langle a \rangle \not\subset r \subset q \rightarrow [T_r] \cap A \neq \emptyset\).

Remark. Here is how to think of the move \(X\). Suppose that the players are considering playing some game \(G(B; T)\). Player I is asserting that he can win \(G(B; T_q)\) for every \(q \in X\) and is conceding that II can win \(G(B; T_q)\) for every \(q \in Z \setminus X\). If \(x \in [T_{p^- \langle a \rangle}]\) then \(x \not\in A\) if and only if \(x\) extends some \(q \in Z\). I is proposing that if and when a position \(q \in Z\) is reached the play be terminated immediately, with I declared the winner if \(q \in X\) and II declared the winner otherwise. In proposing this, I is proposing that the players should play out an infinite play only when that play belongs to \(A\). The idea of having I play subsets of \(Z\), rather than quasistrategies for I in \(T_{p^- \langle a \rangle}\) or subtrees of \(T_{p^- \langle a \rangle}\) is the idea of Hurkens mentioned on page 61.

If \(p^- \langle a \rangle\) is taboo in \(T\), then we must let \(p^- \langle \langle a, X \rangle \rangle\) be taboo for the same player in \(T\). Suppose that \(p^- \langle a \rangle\) is not taboo in \(T\), and so is not terminal,
in $T$. Then we make $p^\sim\langle\langle a, X \rangle\rangle$ not terminal in $\tilde{T}$. We allow II, in principle, two options for move $k + 1$ in $\tilde{T}$, though the second option is available only if $X \neq \emptyset$.

Option (1). II may accept $X$. If II accepts $X$, then II’s move in $\tilde{T}$ at $p^\sim\langle\langle a, X \rangle\rangle$ must be of the form

$$\langle 1, b \rangle,$$

where $b$ is a legal move for II in $T$ at $p^\sim\langle a \rangle$. We stipulate that the positions in $\tilde{T}$ that extend the resulting position $p^\sim\langle\langle a, X \rangle\rangle^\sim\langle\langle 1, b \rangle\rangle$ are an initial segment of the finite sequences of the form

$$p^\sim\langle\langle a, X \rangle\rangle^\sim\langle\langle 1, b \rangle\rangle^\sim s$$

with $p^\sim\langle a \rangle^\sim\langle b \rangle^\sim s \in T$. A position of this form is to be terminal in $\tilde{T}$ if and only if one of the following holds.

(i) $p^\sim\langle a \rangle^\sim\langle b \rangle^\sim s$ is terminal in $T$.

(ii) $p^\sim\langle a \rangle^\sim\langle b \rangle^\sim s \in Z$.

If (i) holds, then we make $p^\sim\langle\langle a, X \rangle\rangle^\sim\langle\langle 1, b \rangle\rangle^\sim s$ is taboo in $T$ for the player for whom $p^\sim\langle a \rangle^\sim\langle b \rangle^\sim s$ is taboo in $\tilde{T}$. If (ii) holds, then we let $p^\sim\langle\langle a, X \rangle\rangle^\sim\langle\langle 1, b \rangle\rangle$ be taboo in $\tilde{T}$ for II if $p^\sim\langle a \rangle^\sim\langle b \rangle \in X$ and for I otherwise. Note that (i) and (ii) cannot both hold, and note that either might hold for $s = \emptyset$.

Option (2). II may challenge $X$. If II challenges $X$, then II’s move in $\tilde{T}$ at $p^\sim\langle\langle a, X \rangle\rangle$ must be of the form

$$\langle 2, r, b \rangle,$$

where $r \in X$ and $b = r(k + 1)$ (so that $p^\sim\langle a \rangle^\sim\langle b \rangle \in T_r$). The positions in $\tilde{T}$ that extend $p^\sim\langle\langle a, X \rangle\rangle^\sim\langle\langle 2, r, b \rangle\rangle$ are to be precisely those of the form

$$p^\sim\langle\langle a, X \rangle\rangle^\sim\langle\langle 2, r, b \rangle\rangle^\sim s$$

with $p^\sim\langle a \rangle^\sim\langle b \rangle^\sim s \in T_r$. Such a position in $\tilde{T}$ is taboo for a player in $\tilde{T}$ if and only if $p^\sim\langle a \rangle^\sim\langle b \rangle^\sim s$ is taboo for that player in $T_r$.

Remark. Here is the way to think about II’s two options. If II accepts $X$, then II accepts the proposal of I that was described in the remark on page 71. If II challenges $X$ and makes the move $\langle 2, r, b \rangle$, then II is denying
I’s contention that I can win the game $G(B; T_r)$. The players then play that
game to decide who is right. (Remember, of course, that the set $B$ is entirely
imaginary. We imagine it only to motivate the definition of $\tilde{T}$.)

The definition of $\pi$ is the obvious one:

$$
(\pi(\tilde{p}))(i) = \begin{cases} 
\tilde{p}(i) & \text{if } i \neq k \text{ and } i \neq k + 1; \\
 a & \text{if } i = k \text{ and } \tilde{p}(k) = \langle a, X \rangle; \\
b & \text{if } i = k + 1 \text{ and } \tilde{p}(k + 1) = \langle 1, b \rangle; \\
b & \text{if } i = k + 1 \text{ and } \tilde{p}(k + 1) = \langle 2, r, b \rangle.
\end{cases}
$$

In other words, $\pi(\tilde{p})$ is obtained from $\tilde{p}$ by deleting the components
$X$, 1, 2, and $r$ that occur in $\tilde{p}$.

Before defining the rest of our covering, let us pause to verify that $\pi^{-1}(A)$
is clopen, so that our covering will unravel $A$. If a $\tilde{x}$ is an infinite play in $\tilde{T}$
in which II accepts I’s $X$, then no position in $\pi(\tilde{x})$ belongs to the associated
$Z$. Hence $[T]_q \cap A \neq \emptyset$ for all $q \subseteq \pi(\tilde{x})$. Since $A$ is closed, $\pi(\tilde{x}) \in A$. If $\tilde{x}$ is
any play in $\tilde{T}$ of length $> k + 1$ in which II challenges I’s $X$, then $\pi(\tilde{x}) \notin A$, for
$\pi(\tilde{x})$ must extend the $r$ played by II at move $k + 1$, and this $r$ belongs to
$X$ and so to $Z$. Define $\tilde{A} \subseteq [\tilde{T}]$ by stipulating, for $\tilde{x} \in [\tilde{T}]$, that

$$
\tilde{x} \in \tilde{A} \leftrightarrow (\ell h(\tilde{x}) > k + 1 \land \text{II accepts I’s } X).
$$

Clearly $\tilde{A}$ is clopen. Moreover $\tilde{A} \cap [\tilde{T}] = \pi^{-1}(A)$, as required for unraveling.

Next we define $\phi$ and $\Psi$ simultaneously. It will be clear from the definitions
that clauses (c) and (d) in the definition of a covering and clause (iii)
in the definition of a $k$-covering are satisfied.

First let $\tilde{\sigma} \in S_I(\tilde{T})$. Here is the idea: The strategy $\tilde{\sigma}$ supplies us with
values of $(\phi(\tilde{\sigma}))(p)$ for $\ell h(p) \leq k$. Furthermore $\tilde{\sigma}$ supplies us with an $X$, and
thus we have a clear choice for $\Psi(\tilde{\sigma}, x) \triangleright k + 1$. As long as no position is reached
that belongs to $X$, we get subsequent values of $\phi(\tilde{\sigma})$ from values of $\tilde{\sigma}$ gotten by
assuming that II accepts $X$. If no position belonging to $X$ is ever reached,
then this assumption gives us $\Psi(\tilde{\sigma}, x)$ also. Suppose we reach a position
$r \in X$. If we were to define $\Psi(\tilde{\sigma}, x)$ using the assumption that II accepts $X$,
then we would make $\Psi(\tilde{\sigma}, x)$ taboo for II, in violation of clause (d)(iii) in the
definition of a covering. But we can avoid such a violation, for $\langle 2, r, r(k + 1) \rangle$
is a legal move $k + 1$ in $\tilde{T}$ in the position $\Psi(\tilde{\sigma}, x) \triangleright k + 1$. We get subsequent
values of $\phi(\tilde{\sigma})$, and we get $\Psi(\tilde{\sigma}, x)$, by assuming that this move is made.

Here are the formal details. We describe $\phi(\tilde{\sigma}) = \sigma$ by describing an
arbitrary play $x$ consistent with $\sigma$. We thus omit the definition of $\sigma(p)$ for $p$
inconsistent with \( \sigma \). Such values can be assigned arbitrarily, except for the easily met constraints from clause (iii) in the definition of \( \phi : \bar{T} \not\models T \) and clause (iii) in the definition of a \( k \)-covering.

At each position \( p \subseteq x \), either we will have a guess \( \psi(p) \) for \( \Psi(\bar{q}, x) \mid \text{lh}(p) \) or else there will be a \( q \subseteq p \) such that \( \psi(q) \) is taboo for I in \( \bar{T} \) and we will have already set \( \Psi(\bar{q}, x) = \psi(q) \). Each \( \psi(p) \) will be such that \( \psi(p) \in \bar{T} \), \( \psi(p) \) is consistent with \( \bar{\sigma} \), and \( \pi(\psi(p)) = p \). At most once during the construction we will contradict our previous guesses: for at most one \( p \subseteq x \), \( \psi(p) \) will be defined but will not be an extension of the \( \psi(p \mid i) \) for \( i < \text{lh}(p) \).

We will arrange that \( \psi(p) \) is taboo for II in \( \bar{T} \) only if \( p \) is taboo for II in \( T \). If we reach a \( p \) such that \( \psi(p) \) is terminal in \( \bar{T} \), then we set \( \Psi(\bar{\sigma}, x) = \psi(p) \).

To begin, we let \( \bar{\sigma} \) agree with some fixed (independent of \( x \)) strategy \( \sigma_p \) in \( T_p \).

As long as no position is reached that belongs to \( X \), we proceed as follows. For positions \( q = p^-(a)^-b^-s \), let \( \bar{q} = p^-(\langle a, X \rangle)^-(\langle 1, b \rangle)^-s \). If \( \bar{q} \in \bar{T} \), then let \( \psi(q) = \bar{q} \) and, if \( \bar{q} \) is non-terminal and of even length, let \( \sigma(q) = \bar{\sigma}(\bar{q}) \). If there is a last \( q \subseteq x \) such that the associated \( \bar{q} \) belongs to \( \bar{T} \), then there are two possibilities for this last \( q \).

1. \( q \) is terminal. Then \( q = x \) and we let \( \Psi(\bar{\sigma}, x) = \psi(q) \).
2. \( q \in Z \setminus X \). Then \( \psi(q) \) is taboo for I and we let \( \Psi(\bar{\sigma}, x) = \psi(q) \).

If there is no last \( q \) such that the associated \( \bar{q} \in \bar{T} \), then the play \( x \) is infinite. In this case we set \( \Psi(\bar{\sigma}, x) = \bigcup_{q \subseteq x} \psi(q) \).

Suppose that there is a position \( r \subseteq x \) that belongs to \( X \). For some \( s \), \( r = p^-(a)^-b^-s \). We let \( \bar{r} = p^-(\langle a, X \rangle)^-(\langle 2, r, b \rangle)^-s \). Note that \( \bar{r} \) is a legal postion in \( \bar{T} \). Note also that \( \bar{r}^-t \in \bar{T} \) for any \( t \) such that \( r^-t \in T \). For positions \( r^-t \), we set \( \psi(r^-t) = \bar{r}^-t \) and, for \( r^-t \) of even length and not terminal, we let \( \sigma(r^-t) = \bar{\sigma}(\bar{r}^-t) \). If the play \( x \) is infinite, we let \( \Psi(\bar{\sigma}, x) = \bigcup_{n \geq \text{lh}(r)} \psi(x \mid n) \).

Next let \( \bar{\tau} \in S_II(\bar{T}) \). Here is the idea: When we reach a position \( p^-\langle a \rangle \) in \( T \) of length \( k + 1 \), there is a subset \( Y \) of the \( Z \) associated with \( p^-\langle a \rangle \) such that

1. \( \tau \) calls for II to accept \( Y \);
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(ii) for any \( r \in Z \setminus Y \), there is an \( X \subseteq Z \) such that \( \tau(p^-\langle \langle a, X \rangle \rangle) = \langle 2, r, r(k+1) \rangle \).

As long as no position is reached that belongs to \( Z \setminus Y \), we get subsequent values of \( \phi(\hat{\tau}) \) from values of \( \hat{\tau} \) gotten by assuming that I plays \( \langle a, Y \rangle \). If no position belonging to \( Z \setminus Y \) is ever reached, then this assumption gives us \( \Psi(\hat{\tau}, x) \) also. Suppose we reach a position \( r \in Z \setminus Y \). If we were to define \( \Psi(\hat{\tau}, x) \) using the assumption that I plays \( \langle a, Y \rangle \), then we would make \( \Psi(\hat{\tau}, x) \) taboo for I, in violation of clause (d)(iii) in the definition of a covering. We can avoid such a violation by using property (ii) of \( X \). If \( X \) is as given by (ii), then we get subsequent values of \( \phi(\hat{\tau}) \), and we get \( \Psi(\hat{\tau}, x) \), by assuming that the moves \( \langle a, X \rangle \) and \( \langle 2, r, r(k+1) \rangle \) are made.

Now we give the formal details. As in the preceding case, we describe \( \phi(\hat{\tau}) = \tau \) by describing an arbitrary play \( x \) consistent with \( \tau \). At each position \( p \subseteq x \), either we will have a guess \( \psi(p) \) for \( \Psi(\hat{\tau}, x) \upharpoonright \ell h(p) \) or else there will be a \( q \nsubseteq p \) such that \( \psi(q) \) is taboo for II in \( T \) and we will have already set \( \Psi(\hat{\tau}, x) = \psi(q) \). Each \( \psi(p) \) will be such that \( \psi(p) \in \tilde{T} \), \( \psi(p) \) is consistent with \( \hat{\tau} \), and \( \pi(\psi(p)) = p \). As before, there will be at most one \( p \subseteq x \) such that \( \psi(p) \) is defined but is not an extension of the \( \psi(p \upharpoonright i) \) for \( i < \ell h(p) \).

We will arrange that \( \psi(p) \) is not taboo for I in \( \tilde{T} \) unless \( p \) is taboo for I in \( T \). If we reach a \( p \) such that \( \psi(p) \) is terminal, then we set \( \Psi(\hat{\tau}, x) = \psi(p) \).

We use the same method as we used before for \( \sigma \) to define \( \tau \) on extensions of \( p \) when \( \psi(p) \) is terminal in \( \tilde{T} \) but \( p \) is not terminal in \( T \).

To begin, we let \( \tau \) agree with \( \hat{\tau} \) and \( \psi(p) = p \) until (if ever) we have reached a position \( p \) of length \( k \). For this \( p \) also, we let \( \psi(p) = p \). If \( p \) is not terminal, let \( a \) be I's move at \( p \). Let \( Z \) be the set associated with \( p^-\langle a \rangle \), the set of which the second component of move \( k \) must be a subset. Let

\[
Y = \{ r \in Z \mid (\forall X \subseteq Z) \: \tilde{\tau}(p^-\langle \langle a, X \rangle \rangle) \neq \langle 2, r, r(k+1) \rangle \}.
\]

The move \( \langle a, Y \rangle \) is legal for I in \( \tilde{T} \) at \( p \), and so we can let \( \psi(p^-\langle a \rangle) = p^-\langle \langle a, Y \rangle \rangle \). Assume that \( p^-\langle a \rangle \) is not terminal in \( T \). Then \( p^-\langle \langle a, Y \rangle \rangle \) is not terminal in \( \tilde{T} \).

It is obvious from the definition of \( Y \) that \( Y \) has property (ii) above. Let us show that \( Y \) has property (i), i.e., that \( \tilde{\tau} \) cannot call for II to challenge \( Y \) at \( p^-\langle \langle a, Y \rangle \rangle \). Assume the contrary and let \( \tilde{\tau}(p^-\langle \langle a, Y \rangle \rangle) = \langle 2, r, r(k+1) \rangle \).

By the definition of \( Y \), we have that \( r \notin Y \). But challenging \( Y \) requires that \( r \in Y \), so we have a contradiction.
Thus \( \tau(p^{-\langle\langle a, Y\rangle\rangle}) = (1, b) \) for some \( b \) with \( p^{-\langle a\rangle} \sim b \) in \( T \). We let \( \tau(p^{-\langle a\rangle}) = b \).

As long as no position is reached that belongs to \( Z \setminus Y \), we proceed as follows. For positions \( q = p^{-\langle a\rangle} \sim b \), let \( q = p^{-\langle\langle a, Y\rangle\rangle} \sim (1, b) \). If \( \tilde{q} \in \hat{T} \), then let \( \psi(q) = \tilde{q} \) and, if \( \tilde{q} \) is non-terminal and of odd length, let \( \tau(q) = \hat{\tau}(\tilde{q}) \). If there is a last \( q \subseteq x \) such that the associated \( \tilde{q} \) belongs to \( \hat{T} \), then there are two possibilities for this last \( q \).

(a) \( q \) is terminal. Then \( q = x \) and we let \( \Psi(\tilde{r}, x) = \psi(q) \).

(b) \( q \in Y \). Then \( \psi(q) \) is taboo for \( II \) and we let \( \Psi(\tilde{r}, x) = \psi(q) \).

If there is no last \( q \) such that the associated \( \tilde{q} \in \hat{T} \), then the play \( x \) is infinite. In this case we set \( \Psi(\tilde{r}, x) = \bigcup_{q \subseteq x} \psi(q) \).

Suppose that there is a position \( r \subseteq x \) that belongs to \( Z \setminus Y \). By property (ii) of \( Y \), let \( X \subseteq Z \) be such that \( \hat{\tau}(p^{-\langle\langle a, X\rangle\rangle}) = (2, r, r(k + 1)) \). For some \( s \), \( r = p^{-\langle a\rangle} \sim b \). We let \( \hat{\tau} = p^{-\langle\langle a, X\rangle\rangle} \sim (2, r, b) \). Note that \( \hat{\tau} \) is a legal position in \( \hat{T} \). Note also that \( \hat{\tau} \sim t \in \hat{T} \) for any \( t \) such that \( r \sim t \in T \). For positions \( r \sim t \), we set \( \psi(r \sim t) = \hat{\tau} \sim t \) and, for \( r \sim t \) of odd length and not terminal, we let \( \tau(r \sim t) = \hat{\tau}(r \sim t) \). If the play \( x \) is infinite, we let \( \Psi(\tilde{r}, x) = \bigcup_{n \geq n(r)} \psi(x \uparrow n) \). \( \square \)

**Theorem 2.1.8.** ([Martin, 1985]) Let \( T \) be a game tree with taboos. If \( A \subseteq [T] \) is Borel and \( k \in \omega \), then there is a \( k \)-covering of \( T \) that unravels \( A \).

**Proof.** By induction on countable ordinals \( \alpha \geq 1 \), we prove

\[(\dagger) \quad \text{For all } T, \text{ for all } A \subseteq [T] \text{ such that } A \in \Sigma^0_{\alpha'}, \text{ and for all } k \in \omega, \text{ there is a } k \text{-covering of } T \text{ that unravels } A.\]

\[(\dagger)_1 \quad \text{is equivalent with Lemma 2.1.7. Assume then that } \alpha > 1 \text{ and that } (\dagger)_1 \text{ holds for all } \beta \text{ with } 1 < \beta < \alpha. \text{ Let } k \in \omega \text{ and let } A \subseteq [T] \text{ with } A \in \Sigma^0_{\alpha'} \text{. By the definition of } \Sigma^0_{\alpha'}, \text{ there are } B_n, n \in \omega, \text{ such that each } B_n \text{ belongs to } \Pi^0_{\alpha_n} \text{ for some } \beta_n < \alpha \text{ and such that } A = \bigcup_{n \in \omega} B_n. \text{ Let } T_0 = T. \text{ By induction on } j' \in \omega, \text{ we define } T_{j'} \text{ and } \]

\[C_{j', j} = \langle T_{j'}, \pi_{j', j}, \phi_{j', j}, \Psi^{j'} \rangle \]

for \( j \leq j' \) such that \( C_{j', j} = C_{j', 0} \circ C_{j', j} \) for all \( i \leq j \leq j' \). We do this in such a way that each \( C_{j', j} \) is a \((k + j)\)-covering of \( T_j \) and \( C_{j', 0} \) unravels \( B_j \) for each
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Note that $C_{j',j'}$ must be the trivial covering, with $\pi_{j',j'}$ and $\phi_{j',j'}$ the identities and $\Psi_{j',j'}(\sigma, x) = x$ for all $\sigma$ and $x$.

Suppose that we have defined $T_{j'}$ and the $C_{j',j}$ for all $j' \leq n$. By the continuity of $\pi_{n,0}$, we have that $\pi_{n,0}^{-1}(B_n) \in \Pi^0_3$. By $(\dagger)_n$, let $C_n = \langle T_n, \pi, \phi, \Psi \rangle$ be a $(k+n)$-covering of $T_n$ that unravels $\pi_{n,0}^{-1}([T] \setminus B_n)$ and so unravels $\pi_{n,0}^{-1}(B_n)$. Let $T_{n+1} = T$. For $j \leq n$, let $C_{n+1,j} = C_{n,j} \circ C_n$; let $C_{n+1,n+1}$ be the trivial covering. The required properties of the $C_{n+1,j}$ follow directly from Lemma 2.1.5 and the continuity of the $\pi_{n,j}$.

If we let $k_{j,i} = k + i$, then the hypotheses of Lemma 2.1.6 hold. Let $T_\infty$ and, for $i \in \omega$, $C_{\infty,i} = \langle T_\infty, \pi_{\infty,i}, \phi_{\infty,i}, \Psi_{i,\infty} \rangle$ be given by that lemma. For each $n$, $\pi_{\infty,0}^{-1}(B_n)$ is clopen, by the continuity of $\pi_{\infty,n+1}$. Thus $\pi_{\infty,0}^{-1}(A)$ is open. By Lemma 2.1.7, let $C$ be a $k$-covering of $T_\infty$ that unravels $\pi_{\infty,0}^{-1}(A)$. $C_{\infty,0} \circ C$ is a $k$-covering of $T$ that unravels $A$. □

Theorem 2.1.9. ([Martin, 1975]) All Borel games are determined.

Proof. The theorem follows immediately from Lemma 2.1.3 and Theorem 2.1.8. □

Exercise 2.1.1. Consider the following two strengthenings of AD.

1. $\text{AD}_R$, the assertion that all games in $^{<\omega}(\omega, \omega)$ are determined;
2. $\text{AD}(\omega^2)$, the assertion that all games of length $\omega^2$ with moves in $\omega$ are determined.

Prove that $\text{AD}_R$ and $\text{AD}(\omega^2)$ are equivalent.

Hint. In the non-trivial direction, consider a game of length $\omega$ in which I’s individual moves are strategies for games in $^{<\omega}\omega$ and II’s moves are plays consistent with these strategies.

Remarks:

(a) This result was proved independently by Andreas Blass and Jan Mycielski. (See [Blass, 1975].) Until the author learned of it in 1974, his and others’ attempts to prove Borel determinacy involved auxiliary games with individual moves that were ordinal numbers. (See [Paris, 1972] for a partial success.) The Blass–Mycielski proof suggested trying games with individual moves that were strategies (or something similar). In [Martin, 1975] and [Martin, 1985], there are individual moves that are quasi-strategies. In the
version of the proof we have just presented, however, the quasistrategies have disappeared.

(b) Oddly enough, the determinacy of all games of countable length, with real or natural number moves, follows from AD$_R$. This fact is a consequence of a theorem independently proved by Hugh Woodin and the author, together with another theorem of Woodin. See [Martin, 2015].

**Exercise 2.1.2.** Let $A \subseteq [T]$ and suppose that $A = \bigcup_{i \in \omega} A_i$, with each $A_i$ closed. Consider the following game $G^* = G(A^*; T^*)$. I begins by picking a strategy $\sigma_0$ for I in $T$. II then chooses a position $p_0 \in T$ consistent with $\sigma_0$. If the position in $T^*$ is not terminal (as defined below), I next picks a strategy $\sigma_1$ for I in $T_{p_0}$; II picks $p_1 \in T_{p_0}$ consistent with $\sigma_1$ such that $p_1 \supseteq p_0$; etc. If some $[T_{p_i}]$ is not disjoint from $A_i$, then the position just after II has picked $p_i$ is terminal. This is the only way terminal positions in $G^*$ arise. A play of $G^*$ is a win for I if and only if the play is finite. Prove using $G^*$ and Theorem 1.2.4 that $G(A; T)$ is determined.

**Exercise 2.1.3.** Modify the $T^*$ of Exercise 2.1.2 to get a covering of $T = \langle T, \emptyset, \emptyset \rangle$ that unravels the $A$ of Exercise 2.1.2.

**Exercise 2.1.4.** Let $\langle \tilde{T}, \pi, \phi, \Psi \rangle$ be a covering of $T$. Show that the extended $\pi : \tilde{T} \cup [\tilde{T}] \rightarrow T \cup [T]$ is a surjection.

*Hint.* Let $x \in [T]$. Consider the game in $\tilde{T}$ that I wins unless someone makes a Move $\tilde{p}$ such that $\pi(\tilde{p}) \not\subseteq x$ and I is the first player to do so. Prove that this game is a win for I. Prove that the analogous game with the roles of the players reversed is a win for II.

**Exercise 2.1.5.** Work in ZF and assume AD. Let $T = <\omega, \omega>$. Let games in $\tilde{T}$ be played as follows:

I $\langle \sigma, n_0 \rangle$
II $\langle x, n_1 \rangle$

$n_2 \quad n_4 \quad \ldots$
$n_3 \quad n_5 \quad \ldots$

Here $\sigma$ must be a strategy for I in $T$ with $\sigma(\emptyset) = n_0$, and $x$ must be a play in $T$ consistent with $\sigma$. Each $n_i$ must be $x(i)$. (Thus only $\sigma$ and $x$ matter.) Use $\tilde{T}$ to get a $(T, \pi, \phi, \Psi)$ that fails to be a covering of $T = \langle T, \emptyset, \emptyset \rangle$ unraveling every subset of $\omega$ only in that $\phi$ is not single-valued.

In [Martin, 1985] it is asserted that a certain uniformization hypothesis permits one to get a single-valued $\phi$. The hypothesis is mistated in
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[Martin, 1985], but the intended one does not work. In [Neeman, 2000] it is shown that every $\Pi^1_1$ subset of $^{\omega}\omega$ can be unraveled by a covering of $\langle^{\omega}\omega, \emptyset, \emptyset\rangle$.

**Exercise 2.1.6.** Under the hypotheses of Lemma 2.1.6, let $T_\infty$ and $\langle C_{\infty,i} \mid i \in \omega \rangle$ be the tree and sequence of coverings constructed in the proof of that lemma. Suppose that $T'$ and $\langle C'_{\infty,i} \mid i \in \omega \rangle$ are such that each $C'_{\infty,i}$ is a $k$-covering of $T_i$ with first component $T'$ and such that, for $i \leq j \in \omega$, $C'_{\infty,i} = C'_{j,i} \circ C_{\infty,j}$. Show that there are $\pi', \phi'$, and $\Psi'$ such that $C' = \langle T', \pi', \phi', \Psi' \rangle$ is a $k$-covering of $T_\infty$ and, for each $i \in \omega$, $C'_{\infty,i} = C_{\infty,i} \circ C'$.

2.2 Uncountable Trees

The Souslin Theorem (see Theorem 2E.2 of [Moschovakis, 1980]) asserts that, in countable trees, the Borel sets are the same as the $\Delta^1_1$ sets (which will be defined below). For uncountable trees, the $\Delta^1_1$ sets form a larger class than the Borel sets. In this section, we will define the class of quasi-Borel sets. We will prove that the quasi-Borel subsets of $[T]$ are the same as the $\Delta^1_1$ subsets of $[T]$ for every $T$. This is the special case for spaces of the form $[T]$ of a theorem of R.W. Hansell ([Hansell, 1973a] and [Hansell, 1973b]). We will prove general $\Delta^1_1$ determinacy by proving that all quasi-Borel games are determined. This determinacy result is from [Martin, 1990].

**Remark.** In [Martin, 1990], the author credited the concept of quasi-Borel sets to himself. After the publication of [Martin, 1990], Alberto Marcone pointed out to the author that the concept had been introduced by R.W. Hansell in Hansell [1972]. In [Hansell, 1973a] and [Hansell, 1973b], what we call quasi-Borel sets were called extended Borel sets. In [Martin, 1990] the author also wrongly credited to himself the fact that the quasi-Borel sets are the same as the $\Delta^1_1$ sets.

The definition of the quasi-Borel sets is like that of the Borel sets, except that an additional operation, besides those of forming countable unions and complements, is required to generate them from the open sets. Our proof of quasi-Borel determinacy will be a minor modification of our proof of Borel determinacy, with extra lemmas to take care of the extra operation. This result is relevant even for games in countable trees: we will use it later §5.2 in getting as strong a determinacy result as possible from the assumption that a measurable cardinal exists.
We begin by defining the quasi-Borel sets and studying their properties. To do so we must define the extra operation needed to generate them. Let $T$ be a game tree. If $A$ and $B_j$, $j \in J$, are all subsets of $\lceil T \rceil$, then $A$ comes from $\{B_j \mid j \in J\}$ by the operation of open-separated union, or, equivalently, $A$ is the open-separated union of $\{B_j \mid j \in J\}$, if

(a) $A = \bigcup_{j \in J} B_j$;

(b) there are disjoint open sets $D_j$, $j \in J$, such that $B_j \subseteq D_j$ for each $j \in J$.

A set is quasi-Borel if it belongs to the smallest class containing the open sets and closed under countable unions, open-separated unions, and complements.

There is no clearly best way to define a quasi-Borel hierarchy. The one in [Martin, 1990] is different from the one we are about to give here.

Recall from §1.4 that $A$ is the fully open-separated union of $\{B_j \mid j \in J\}$ if some $\{D_j \mid j \in J\}$ witnessing that $A$ is the open-separated union of $\{B_j \mid j \in J\}$ satisfies $\bigcup_{j \in J} D_j = \lceil T \rceil$. Recall also that if $\{D_j \mid j \in J\}$ witnesses that $A$ is the fully open-separated union of $\{B_j \mid j \in J\}$, then each $D_j$ is clopen.

If $\alpha$ is a limit ordinal, then $\cof(\alpha)$, the cofinality of $\alpha$, is the least ordinal $\rho$ such that some $f : \rho \to \alpha$ has unbounded range. The cofinality of $\alpha$ is always a regular cardinal $\leq \alpha$.

We define the quasi-Borel hierarchy (of subsets of $\lceil T \rceil$) as follows:

1. $\Sigma^*_1$ is the class of open sets;
2. $\Pi^*_\alpha$ is the set of complements of members of $\Sigma^*_\alpha$;
3. if $\alpha > 1$ is a successor ordinal or if $\alpha$ is a limit ordinal and $\cof(\alpha) = \omega$, then $\Sigma^*_\alpha$ is the set of countable unions of members of $\bigcup_{\beta < \alpha} \Pi^*_\beta$;
4. if $\alpha$ is a limit ordinal and $\cof(\alpha) \neq \omega$ (so $\cof(\alpha)$ is uncountable), then $\Sigma^*_\alpha$ is the set of all fully open-separated unions of members of $\bigcup_{\beta < \alpha} \Pi^*_\beta$.

The following lemma gives some properties of the quasi-Borel hierarchy and of quasi-Borel sets. The important ones for us are (1)(f) and (2). The latter is non-trivial, because our definition of the $\Sigma^*_\alpha$ uses fully open-separated unions.

**Lemma 2.2.1.** (1) The following assertions hold for every $\alpha \geq 1$:
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(a) \((\forall \beta > \alpha) \Sigma^*_\alpha \cup \Pi^*_\alpha \subseteq \Delta^*_\beta\).

(b) If \(\alpha\) is a limit ordinal of uncountable cofinality, then \(\Sigma^*_\alpha = \Pi^*_\alpha\), i.e. \(\Sigma^*_\alpha\) is closed under complements.

(c) If \(\alpha\) is a successor ordinal or a limit ordinal of cofinality \(\omega\), then \(\Sigma^*_\alpha\) is closed under countable unions.

(d) If \(\alpha\) is a successor ordinal or a limit ordinal of cofinality \(\omega\), then \(\Pi^*_\alpha\) is closed under countable intersections.

(e) \(\Sigma^*_\alpha\) is closed under finite unions and finite intersections.

(f) If \(\alpha\) is a limit ordinal of uncountable cofinality and \(A\) is any subset of \([T]\), then \(A \in \Sigma^*_\alpha\) if and only if there is a set \(D \subseteq T\) and there are \(B_d \subseteq [T_d]\), such that each \(B_d \subseteq [T_d]\), such that each \(B_d \in \bigcup_{\beta < \alpha} \Pi^*_\beta\), and such that \(A = \bigcup_{d \in D} B_d\).

In other words, we can replace “fully open-separated unions” by “fully basic-open-separated unions” in clause (4) of the definition of the quasi-Borel hierarchy.

(2) If \(T\) is infinite, then a subset of \([T]\) is quasi-Borel if and only if it belongs to \(\bigcup_{\alpha < |T|^+} \Sigma^*_\alpha\). Thus, in particular, the Borel sets and the quasi-Borel sets are the same for countable \(T\).

Proof. (1)(a). If \(A \in \Pi^*_\alpha\), then \(A = \bigcup \{A\}\). Since this trivial union is both countable and fully open-separated, it follows that \(A \in \Sigma^*_\beta\) for every \(\beta > \alpha\). Thus \(\Pi^*_\alpha \subseteq \Sigma^*_\beta\) whenever \(1 \leq \alpha < \beta\). From this we have also that \(\Sigma^*_\alpha \subseteq \Pi^*_\beta\) whenever \(1 \leq \alpha < \beta\). If \(1 \leq \alpha < \beta\) and \(\beta > \alpha + 1\), then we have that

\[\Sigma^*_\alpha \subseteq \Pi^*_\alpha+1 \subseteq \Sigma^*_\beta.\]

It remains then only to show that \(\Sigma^*_\alpha \subseteq \Sigma^*_\alpha+1\) for all \(\alpha \geq 1\). The proof of (1)(a) of Lemma 1.1.1 showed that that every element of \(\Sigma^*_1\) is a countable union of \(\Pi^*_1\) sets, and so that \(\Sigma^*_1 \subseteq \Sigma^*_2\). If \(\alpha\) is a successor or has cofinality \(\omega\), then it is immediate from clause (3) of the definition that \(\Sigma^*_\alpha \subseteq \Sigma^*_\alpha+1\). If \(\alpha\) is a limit ordinal of uncountable cofinality, then it follows from (1)(b)—which is proved below using only the part of (1)(a) already proved—that \(\Sigma^*_\alpha = \Pi^*_\alpha \subseteq \Sigma^*_\alpha+1\).

(1)(b). Let \(\alpha\) be a limit ordinal of uncountable cofinality and suppose that \(\{D_j \mid j \in J\}\) witnesses that \(A\) is the fully open-separated union of
\{B_j \mid j \in J\}$, where each $B_j \in \bigcup_{\beta < \alpha} \Pi^*_\beta$. Then \(\{D_j \mid j \in J\}\) also witnesses that $\neg A$ is the fully open-separated union of $\{D_j \setminus B_j \mid j \in J\}$. Moreover each $\{D_j \setminus B_j\} \in \bigcup_{\beta < \alpha} \Sigma^*_\beta \subseteq \bigcup_{\beta < \alpha} \Pi^*_\beta \supseteq \bigcup_{\beta < \alpha} \Pi^*_\beta$.

(1)(c) is immediate as in Lemma 1.1.1, and (1)(d) follows directly from (1)(c).

(1)(e). We prove by induction on $\alpha$ that $\Sigma^*_\alpha$ is closed under finite intersections. (1)(e) then follows by (1)(b) and (1)(c). The case $\alpha = 1$ is immediate. For $\alpha$ a successor or a limit of cofinality $\omega$, we argue as in the proof of part (1)(b) of Lemma 1.1.1, except that the last step of the argument now comes by our induction hypothesis. Assume then that $\alpha$ is a limit of uncountable cofinality and suppose that, for each $i < n \in \omega$, \(\{D^i_j \mid j \in J_i\}\) witnesses that $A^i$ is the fully open-separated union of \(\{B^i_j \mid j \in J_i\}\), with each $B^i_j \in \bigcup_{\beta < \alpha} \Pi^*_\beta$. Then

\[\bigcap_{i < n} D^i_s(i) \mid s \in \prod_{i < n} J_i\]

witnesses that $\bigcap_{i < n} A^i$ is the fully open-separated union of

\[\bigcap_{i < n} B^i_s(i) \mid s \in \prod_{i < n} J_i\].

(1)(a) and our induction hypothesis give that each $\bigcap_{i < n} B^i_s(i) \in \bigcup_{\beta < \alpha} \Pi^*_\beta$.

(1)(f). The “if” direction is trivial, so we prove only the other direction. Let $\alpha$ be a limit ordinal of uncountable cofinality. Let \(\{D^i_j \mid j \in J\}\) witness that $A$ is the fully open-separated union of \(\{B^i_j \mid j \in J\}\), where each $B^i_j \in \bigcup_{\beta < \alpha} \Pi^*_\beta$. For each $j \in J$, let $D_j$ be the set of all $p \in T$ such that $[T_p] \subseteq D^i_j$ but $[\forall q \subseteq p][T_q] \not\subseteq D^i_j$. Clearly $\bigcup_{p \in D_j} [T_p] \subseteq D^i_j$. To see that the reverse inclusion also holds, suppose that $x \in D^i_j$. Since $D^i_j$ is open, there is an $n \in \omega$ such that $[T_x[\omega]] \subseteq D^i_j$. For the least such $n$, $x \upharpoonright n \in D_j$. Let $D = \bigcup_{j \in J} D_j$. By the definition of the $D_j$ and by the disjointness of the $D^i_j$, any two elements of $D$ are incomparable with respect to $\subseteq$. Furthermore, $\bigcup_{d \in D} [T_d] = \bigcup_{j \in J} [D^i_j] = [T]$. For $j \in J$ and $d \in D_j$, let $B_d = B^i_j \cap [T_d]$. Then

\[A = \bigcup_{j \in J} B^i_j = \bigcup_{j \in J} \bigcup_{d \in D_j} B_d = \bigcup_{d \in D} B_d.\]

Each $B_d$ is the intersection of a clopen set with a member of $\bigcup_{\beta < \alpha} \Pi^*_\beta$, and so (1)(e) gives that each $B_d \in \bigcup_{\beta < \alpha} \Pi^*_\beta$. 

(2). We may suppose that $T$ is infinite, since otherwise every subset of $[T]$ is clopen and so belongs to $\Sigma^*_1$. As in the proof of Lemma 1.1.1, we get that $\bigcup_{\alpha<|T|} \Sigma^*_\alpha$ is a class containing the open sets and closed under countable unions and complements. To see that this class is closed under open-separated unions as well, and so that every quasi-Borel set belongs to it, suppose that $\{D'_j \mid j \in J\}$ witnesses that $A$ is the open-separated union of $\{B'_j \mid j \in J\} \subseteq \bigcup_{\alpha<|T|} \Sigma^*_\alpha$. We may assume that $J$ is uncountable, since otherwise $A \in \bigcup_{\alpha<|T|} \Sigma^*_\alpha$, by closure under countable unions. For each $j \in J$, define $D_j$ as in the proof above of (1)(f). Similarly define $D$ and $B_d$, $d \in D$. We have that any two elements of $D$ are incomparable with respect to $\subseteq$, that $A = \bigcup_{d \in D} B_d$, and that each $B_d \in \bigcup_{\alpha<|T|} \Sigma^*_\alpha$. Since $D \subseteq T$ and $\text{cf}(|T|) > |T| \geq |D| \geq |J| > \aleph_0$, there is limit ordinal $\alpha < |T|$ such that $\text{cf}(\alpha) > \omega$ and such that each $B_d$ belongs to $\bigcup_{\beta<\alpha} \Pi^*_\beta$.

Our problem is that we may not have that $\bigcup_{d \in D} \lceil T_d \rceil = \lceil T \rceil$. To deal with this problem, let $D^n = \{d \in D \mid \ell h(d) = n\}$. Let $A^n = \bigcup_{d \in D^n} B_d$. Let $D^n_+ = \{p \in T \mid \ell h(p) = n\}$. For each $n$, $\{[T_d] \mid d \in D^n\}$ witnesses that $A^n$ is the fully open-separated union of $\{B_d \mid d \in D^n\} = \{B_d \mid d \in D^n\} \cup \emptyset$. Hence each $A^n \in \Sigma^*_\alpha$. Since $A = \bigcup_{n \in \omega} A^n$, we get that $A \in \Sigma^*_\alpha+1$.

The fact that every member of $\bigcup_{\alpha<|T|} \Sigma^*_\alpha$ is quasi-Borel is proved by an easy induction on $\alpha$. □

Remarks:

(a) In general, the quasi-Borel sets form a larger class than the Borel sets. For example, let $T = \{\langle \alpha \rangle \dashv p \mid p \in \langle \omega \omega \rangle \wedge \alpha < \omega_1\}$. For $\alpha < \omega_1$, let $B_\alpha \subseteq \omega_\omega$ with $B_\alpha \in \Pi^*_\alpha \setminus \Sigma^*_\alpha$. Let $A = \{\langle \alpha \rangle \dashv y \mid y \in B_\alpha\}$. $A$ is quasi-Borel but not Borel. See Exercise 2.2.1.

(b) Parts (1)(f) and (2) of Lemma 2.2.1 shows that, in the definition of quasi-Borel, we could replace “open-separated union” by “(fully) basic-open-separated union.” What if we made replacements in the other direction, broadening rather than narrowing the class of allowable separating sets? Unfortunately, this would trivialize the concept: All points in $[T]$ are closed, so every subset of $[T]$ is a closed-separated union of closed sets.

(c) If $T$ is a game tree with taboos and $[T]$ is nonempty, then, as we remarked in §2.1, the topological space $[T]$ is the same as the space $[\bar{T}]$, where $\bar{T}$ is the set of all $p \in T$ such that some infinite play in $T$ extends
Thus Lemma 2.2.1 applies to \([T]\). In addition, we have the following generalization of Lemma 2.1.1.

**Lemma 2.2.2.** Let \(T\) be a game tree with taboos. For all ordinals \(\alpha\), a subset \(A\) of \([T]\) belongs to \(\Pi^*_\alpha\) if and only if \(A \in \Pi^*_\alpha\) as a subset of \([T]\). For all ordinals \(\alpha > 1\), a subset \(A\) of \([T]\) belongs to \(\Sigma^*_\alpha\) if and only if \(A \in \Sigma^*_\alpha\) as a subset of \([T]\).

**Proof.** We prove the lemma by induction on \(\alpha\). The cases other than that of \(\alpha\) a limit ordinal of uncountable cofinality are handled as in the proof of Lemma 2.1.1. Assume then that \(\alpha\) is a limit ordinal and that \(\text{cf}(\alpha) > \omega\). Suppose first that \(A \subseteq [T]\) belongs to \(\Sigma^*_\alpha\) as a subset of \([T]\). Let \(\{D_j \mid j \in J\}\) witness that \(A\) is the fully open-separated union of \(\{B_j \mid j \in J\}\), with each \(B_j \in \bigcup_{\beta < \alpha} \Pi^*_\beta\) as a subset of \([T]\) and so, by induction, as a subset of \([T]\). Then \(\{D_j \cap [T] \mid j \in J\}\) witnesses for the space \([T]\) that \(A\) is the fully open-separated union of \(\{B_j \mid j \in J\}\). Thus \(A \in \Sigma^*_\alpha\) as a subset of \([T]\). Now suppose that \(A \in \Sigma^*_\alpha\) as a subset of \([T]\). Let \(\{D_j \mid j \in J\}\) witness that \(A\) is the fully open-separated union of \(\{B_j \mid j \in J\}\), with each \(B_j \in \bigcup_{\beta < \alpha} \Pi^*_\beta\). For each \(j \in J\) let \(D'_j\) be open in \([T]\) with \(D_j = D'_j \cap [T]\). Let \(J' = J \cup \{j'\}\), where \(j' \notin J\), and let \(D' = [T] \setminus \bigcup_{j \in J} D'_j\). Since all members of \(D'_j\) are finite, \(D'_j\) is open. Let \(B_j = \emptyset\). Then \(\{D_j \mid j \in J'\}\) witnesses that \(A\) is the fully open-separated union of \(\{B_j \mid j \in J'\}\). Hence \(A \in \Sigma^*_\alpha\) as a subset of \([T]\). \(\square\)

If \(T\) is a game tree and \(A \subseteq [T]\), then \(A \in \Sigma^1_1\) if and only if there is a closed \(C \subseteq [T] \times \omega\) \((= [T] \times [\omega \omega])\) such that

\[ (\forall x \in [T])(x \in A \iff (\exists y \in \omega \omega) \langle x, y \rangle \in C). \]

If \(A \subseteq [T]\) then \(A \in \Pi^1_1\) if and only if \([T] \setminus A \in \Sigma^1_1\). We let \(\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1\). (In Part 1 of Rogers et al. [1980], elements of \(\Sigma^1_1\) are called *Souslin* sets.)

The following theorem generalizes the Souslin Theorem.

**Theorem 2.2.3.** ([Hansell, 1973a] and [Hansell, 1973b]) For every game tree \(T\), the class of quasi-Borel sets coincides with \(\Delta^1_1\).

**Proof.** In the proof of Lemma 1.1.1, we showed that every open set is a countable union of clopen sets: If \(A\) is open then \(A = \bigcup_{n \in \omega} A_n\), where

\[ A_n = \bigcup \{[T_p] \mid p \in T \land \ell(h(p)) = n \land [T_p] \subseteq A\}. \]
Thus the quasi-Borel sets form the smallest class that contains the clopen sets and is closed under complements, countable unions, and open-separated unions. To prove that all quasi-Borel sets belong to $\Delta^1_1$, it then suffices to show (a) that every clopen set belongs $\Sigma^1_1$ (and so that every clopen set belongs to $\Pi^1_1$) and (b) that both $\Sigma^1_1$ and $\Pi^1_1$ are closed under (i) countable unions and (ii) open-separated unions.

(a). If $A$ is clopen (or even just closed), then let $C = A \times \omega$. $C$ witnesses that $A \in \Sigma^1_1$.

(b)(i). Suppose that, for each $n \in \omega$, $C_n$ witnesses that $A_n \in \Sigma^1_1$. Hence each $C_n$ is closed, and $A_n = \{x \mid (\exists y \in \omega) <x, y> \in C_n\}$. Let $C$ be defined by

$$<x, (n)\neg y> \in C \iff <x, y> \in C_n,$$

where $(<n>\neg y)(0) = n$ and, for each $i$, $(<n>\neg y)(i + 1) = y(i)$. It is easy to see that $C$ witnesses that $\bigcup_{n \in \omega} A_n \in \Sigma^1_1$.

Now suppose that, for each $n \in \omega$, $C_n$ witnesses that $\neg A_n \in \Sigma^1_1$. For $y \in \omega$ and $n \in \omega$, let $(y)_n \in \omega$ be defined by

$$(y)_n(k) = y(p_n^k),$$

where $(p_n \mid n \in \omega)$ is the sequence of all prime numbers in increasing order. Let

$$<x, y> \in C \iff (\forall n \in \omega) <x, (y)_n> \in C_n.$$

$C$ witnesses that $\bigcap_{n \in \omega} \neg A_n \in \Sigma^1_1$, and so that $\bigcup_{n \in \omega} A_n \in \Pi^1_1$.

For (b)(ii), first assume that $\{D_j \mid j \in J\}$ witnesses that $A$ is the open-separated union of $\{B_j \mid j \in J\}$ with each $B_j \in \Sigma^1_1$. Let $B_j = \{x \mid (\exists y \in \omega) <x, y> \in C_j\}$, with each $C_j$ closed. Let

$$<x, y> \in C \iff (\forall j \in J) (x \in D_j \rightarrow <x, y> \in C_j).$$

$C$ witnesses that $A \cup ([T] \setminus \bigcup_{j \in J} D_j) \in \Sigma^1_1$. (a) and (b)(i) imply that $A \in \Sigma^1_1$. (We could also have applied parts (1)(f) and (2) of Lemma 2.2.1 to get our $\{D_j \mid j \in J\}$ such that $\bigcup_{j \in J} D_j = [T]$.)

Next assume that $\{D_j \mid j \in J\}$ witnesses that $A$ is the open-separated union of $\{B_j \mid j \in J\}$, with each $B_j \in \Pi^1_1$. Let $[T] \setminus B_j = \{x \mid (\exists y \in \omega) <x, y> \in C_j\}$, with each $C_j$ closed. Let

$$<x, y> \in C \iff (\forall j \in J) (x \in D_j \rightarrow <x, y> \in C_j).$$

$C$ witnesses that $A \in \Pi^1_1$. 


For the other half of the theorem we repeat the proof of the result of
for the case of countable $T$, that any two disjoint $\Sigma^1_1$ sets can be separated
by a Borel set; we just replace “Borel” by “quasi-Borel.” Let $A = \{ x \mid (\exists y \in \omega \omega) (x, y) \in C \}$, with $C$ closed, and let $A' = \{ x \mid (\exists y \in \omega \omega) (x, y) \in C' \}$, with
$C'$ closed. Assume that $A$ and $A'$ are not separated by any quasi-Borel set,
i.e. assume that there is no quasi-Borel $B$ such that $A \subseteq B$ and $A' \cap B = \emptyset$.
We will prove that $A \cap A' \neq \emptyset$. For $q \in T$, $r \in \omega \omega$, and $r' \in \omega \omega$, let
\[ A_{q,r} = [T_q] \cap \{ x \mid (\exists y \in \omega \omega)(r \subseteq y \land (x, y) \in C) \} \]
\[ A'_{q,r'} = [T_q] \cap \{ x \mid (\exists y \in \omega \omega)(r' \subseteq y \land (x, y) \in C') \} \]
Assume inductively that $n \in \omega$ and that we have defined $q_n$, $r_n$, and $r_n'$, all
of length $n$, such that $A_{q_n,r_n}$ and $A'_{q_n,r_n'}$ are not separated by any quasi-Borel set. First note that there are $k$ and $k'$ such that $A_{q_n,r_n} \cap \langle k \rangle$ and $A'_{q_n,r_n'} \cap \langle k' \rangle$ are
not separated by any quasi-Borel set, since if sets $B_{k,k'; k', k' \in \omega}$, contradict
this then
\[ \bigcup_{k \in \omega} \bigcap_{k' \in \omega} B_{k,k'} \]
separates $A_{q_n,r_n}$ and $A'_{q_n,r_n'}$. Choose such $k$ and $k'$ and let $r_{n+1}$ and $r'_{n+1}$, be
$r_n \cap \langle k \rangle$ and $r_n' \cap \langle k' \rangle$ respectively. Now $A_{q_n,r_{n+1}}$ is the open-separated union
of $\{ A_{s,r_{n+1}} \mid q_n \subseteq s \land \ell h(s) = \ell h(q_n) + 1 \}$ and $A'_{q_n,r_{n+1}}$ is the open-separated union
of $\{ A'_{s',r_{n+1}'} \mid q_n \subseteq s \land \ell h(s) = \ell h(q_n) + 1 \}$. If for each $s \supseteq q_n$ with
$\ell h(s) = \ell h(q_n) + 1$ there were a quasi-Borel set $B_s$ separating $A_{s,r_{n+1}}$ and
$A'_{s',r_{n+1}'}$, then $\bigcup_s (B_s \cap [T_s])$ would be a quasi-Borel set separating $A_{q_n,r_{n+1}}$
and $A'_{q_n,r_{n+1}'}$. Thus we can let $q_{n+1}$ be some $s \supseteq q_n$ with $\ell h(s) = \ell h(q_n) + 1$
such that no quasi-Borel set separates $A_{s,r_{n+1}}$ and $A'_{s',r_{n+1}'}$. This completes
the induction step. Now let $x = \bigcup_n q_n$, let $y = \bigcup_n r_n$, and let $y' = \bigcup_n r_n'$. Then $y$ and $y'$ witness that $x \in A \cap B$. \[ \square \]

Remarks:

(a) Of course, Hansell’s theorem is not about spaces of the form $[T]$ but
about a wider class that includes these spaces.

(b) The separation theorem, which is what the second half of our proof
of Theorem 2.2.3 actually proves, is in [Hansell, 1973a] and [Hansell, 1973b].

(c) If $T$ is a game tree with taboos, it is easy to see that any subset $A$ of
$[T]$ belongs to $\Sigma^1_1$ as a subset of $[T]$ if and only if it belongs to $\Sigma^1_1$ as a subset
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of $[T]$. By the first part of Theorem 2.2.3 and by the closure properties of $\Sigma^1_1$ and $\Pi^1_1$ demonstrated in the proof, this also holds for $\Pi^1_1$ and $\Delta^1_1$.

Our proof of quasi-Borel determinacy will parallel that of Borel determinacy. For the analogue of Lemma 2.1.8, we will prove the analogue $(\dagger)_\alpha$ of $(\dagger)_\alpha$ for all ordinals $\alpha$. For $\alpha$ of uncountable cofinality, where $\Sigma^*_\alpha$ is gotten by the new operation of open-separated union, we will need an additional method of combining coverings.

If we are considering $T$, a game tree with taboos, and $S$ a subtree of $T$ or $S = \emptyset$, let us denote by $S$ the triple $\langle S, T \cap S, T_\Pi \cap S \rangle$. If $S$ is a game subtree of $T$, then $S$ is a game tree with taboos.

Suppose that $T$ is a game tree with taboos and that $p \in T$. Let

$$(p)T = \{ q \in T \mid \neg (p \subseteq q) \}.$$  

Let $T$ be a game tree with taboos and let $C = \langle \tilde{T}, \pi, \phi, \Psi \rangle$ be a covering of $T$. For $p \in T$, $C$ is a $(p)$-covering of $T$ if

(i) $\langle p \rangle \tilde{T} = \langle p \rangle T$;

(ii) $\pi \restriction \langle p \rangle \tilde{T}$ is the identity;

(iii) $(\phi(\tilde{\sigma}))(q) = \tilde{\sigma}(q)$, for all $\tilde{\sigma}$ and all $q \supseteq p$;

(iv) $\phi(\tilde{\sigma}) \restriction \{ q \in T \mid q \supseteq p \}$ depends only on $\tilde{\sigma} \restriction \{ \tilde{q} \in \tilde{T} \mid \tilde{q} \supseteq p \}$.

**Lemma 2.2.4.** Let $T$ be a game tree with taboos and let $p \in T$. Every $\ell h(p)$-covering of $T_p$ induces a unique $(p)$-covering of $T$; i.e., if $\langle \tilde{T}', \pi', \phi', \Psi' \rangle$ is a $\ell h(p)$-covering of $T_p$, then there is a unique $(p)$-covering $\langle T, \pi, \phi, \Psi \rangle$ of $T$ such that $T_p = \tilde{T}'$, $\pi \restriction \tilde{T}' = \pi'$, $\phi(\tilde{\sigma}) \restriction \{ q \in T \mid q \supseteq p \} = (\phi'(\tilde{\sigma}')) \restriction \{ q \in T \mid q \supseteq p \}$ for all $\tilde{\sigma} \in S(\tilde{T})$, and $\Psi(\tilde{\sigma}, x) = \Psi'(\tilde{\sigma}', x)$ for all $\langle \tilde{\sigma}, x \rangle \in \text{domain}(\Psi)$ with $p \subseteq x$, where in the last two clauses $\tilde{\sigma}'$ is the unique element of $S(\tilde{T}')$ agreeing with $\tilde{\sigma}$ on $\{ q \in \tilde{T} \mid q \supseteq p \}$.

**Proof.** The proof is quite routine, so we verify only the uniqueness of $\Psi$. Suppose that $x \in \langle (p)T \rangle$ is consistent with $\phi(\tilde{\sigma})$, with $\tilde{\sigma} \in S(\tilde{T})$. We show that we must have $\Psi(\tilde{\sigma}, x) = x$. First note that $\Psi(\tilde{\sigma}, x) \in \langle (p)T \rangle$, since otherwise clause (d)(ii) of the definition of a covering and (ii) above imply that $x \supseteq \pi(\Psi(\tilde{\sigma}, x)) \supseteq p$. Hence (d)(ii) and (ii) give that $x \supseteq \pi(\Psi(\tilde{\sigma}, x)) = \Psi(\tilde{\sigma}, x)$. But then (i) implies that $\Psi(\tilde{\sigma}, x) = x$. □

**Remark.** It is also true that every $(p)$-covering of $T$ induces a unique $\ell h(p)$-covering of $T_p$. (See Exercise 2.2.3.) Thus a $(p)$-covering of $T$ is essentially the same thing as a $\ell h(p)$-covering of $T_p$. 
Lemma 2.2.5. Let $T$ be a game tree with taboos and let $D \subseteq T$ be such that $\bigcup_{d \in D} [T_d] = [T]$ and such that any two distinct elements of $D$ are incomparable with respect to $\subseteq$. Suppose that $k \in \omega$ and, for each $d \in D$, that $C_d = (T^d, \pi_d, \phi_d, \Psi_d)$ is both a $k$-covering and a $(d)$-covering of $T$. Then there is a $k$-covering $C = (\tilde{T}, \pi, \phi, \Psi)$ of $T$ and, for each $d \in D$, there are $\tilde{\pi}_d$, $\tilde{\phi}_d$, and $\tilde{\Psi}_d$ such that

(i) $\tilde{C}_d = (\tilde{T}, \tilde{\pi}_d, \tilde{\phi}_d, \tilde{\Psi}_d)$ is a $k$-covering of $T^d$;

(ii) $C = C_d \circ \tilde{C}_d$.

Proof. We get $\tilde{T}$ from $T$ by replacing each $T_d$, $d \in D$, by $T^d_d$:

\[
\begin{align*}
\tilde{\pi}(\tilde{\pi}) & = \left\{ \begin{array}{ll}
\pi_d(\tilde{\pi}) & \text{if } d \in D \land d \subseteq \tilde{\pi}; \\
\tilde{\pi} & \text{if } (\forall d \in D) \land \tilde{\pi} \not\subseteq \tilde{\pi}; \\
\tilde{\phi}_d(\tilde{\phi}) & = \left\{ \begin{array}{ll}
\pi_d'(\tilde{\phi}) & \text{if } d' \in (D \setminus \{d\}) \land d' \subseteq \tilde{\phi}; \\
\tilde{\phi} & \text{if } (\forall d' \in (D \setminus \{d\})) \land d' \not\subseteq \tilde{\phi}.
\end{array} \right. \\
\phi & \end{align*}
\]

In the notation introduced on page 87, clause (i) in the definition of a $k$-covering says that $kT = kT$. That this is true follows from the fact that the $C_d$ are $k$-coverings.

We define $\pi$ and $\tilde{\pi}_d$, for $d \in D$, by

\[
\begin{align*}
\pi(\tilde{\pi}) & = \left\{ \begin{array}{ll}
\pi_d(\tilde{\pi}) & \text{if } d \in D \land d \subseteq \tilde{\pi}; \\
\tilde{\pi} & \text{if } (\forall d \in D) \land \tilde{\pi} \not\subseteq \tilde{\pi};
\end{array} \right. \\
\tilde{\phi}_d(\tilde{\phi}) & = \left\{ \begin{array}{ll}
\pi_d'(\tilde{\phi}) & \text{if } d' \in (D \setminus \{d\}) \land d' \subseteq \tilde{\phi}; \\
\tilde{\phi} & \text{if } (\forall d' \in (D \setminus \{d\})) \land d' \not\subseteq \tilde{\phi}.
\end{array} \right.
\]

It is easy to check that $\pi : T \Rightarrow T$ and that each $\tilde{\pi}_d : T \Rightarrow T^d$. The fact that $\pi$ and the $\tilde{\pi}_d$ are the identity on $kT = kT$ follows from the fact that the $C_d$ are $k$-coverings. To verify that $\tilde{\pi}_d = \pi_d \circ \tilde{\pi}_d$ for each $d \in D$, let $\tilde{\pi} \in \tilde{T}$ and $d \in D$. If $(\forall d' \in D) \land d' \not\subseteq \tilde{\pi}$, then $\pi(\tilde{\pi}) = \pi_d(\tilde{\pi}_d(\tilde{\pi})) = \tilde{\pi}$. So assume that $d' \in D$ and $d' \subseteq \tilde{\pi}$. By definition, $\pi_\tilde{\phi}(\tilde{\phi}) = \pi_d'(\tilde{\phi})$. Assume first that $d' = d$. By definition, $\tilde{\pi}_d(\tilde{\phi}) = \tilde{\phi}$. Thus $\pi_d(\tilde{\phi}_d(\tilde{\phi})) = \pi_\phi(\tilde{\phi}) = \pi(\tilde{\phi})$. Assume now that $d' \neq d$. We have that $\tilde{\pi}_d(\tilde{\phi}) = \pi_d'(\tilde{\phi})$ and, since $\pi_d'(\tilde{\phi}) \supseteq d' \neq d$, that $\pi_d(\pi_d'(\tilde{\phi})) = \pi_d(\tilde{\phi})$. Hence $\pi_\tilde{\phi}(\tilde{\phi}) = \pi_\tilde{\phi}(\tilde{\phi}) = \pi(\tilde{\phi})$. 
2.2. UNCOUNTABLE TREES

For \( \tilde{\sigma} \in S(\tilde{T}) \) and \( d \in \mathcal{D} \), let \( \tilde{\sigma}_d \) be any element of \( S(T^d) \) that agrees with \( \tilde{\sigma} \) on \( \{ q \in \tilde{T} \mid q \supseteq d \} \). Clause (iv) in the definition of a \((p)\)-covering guarantees that the following definitions of \( \phi \) and \( \phi_d \), for \( d \in \mathcal{D} \), are independent of the choices of the \( \tilde{\sigma}_d \).

\[
(\phi(\tilde{\sigma}))(p) = \begin{cases} 
(\phi_d(\tilde{\sigma}_d))(p) & \text{if } d \in \mathcal{D} \land d \subseteq p; \\
\tilde{\sigma}(p) & \text{if } (\forall d \in \mathcal{D}) d \nsubseteq p; 
\end{cases}
\]

\[
(\tilde{\phi}_d(\tilde{\sigma}))(p) = \begin{cases} 
(\phi_d(\tilde{\sigma}_d))(p) & \text{if } d' \in (\mathcal{D} \setminus \{d\}) \land d' \subseteq p; \\
\tilde{\sigma}(p) & \text{if } (\forall d' \in (\mathcal{D} \setminus \{d\})) d' \nsubseteq p. 
\end{cases}
\]

It is easy to verify that \( \phi : \tilde{T} \xrightarrow{S} T \). The fact that \( \phi \) and the \( \tilde{\phi}_d \) are the identity on \( kS(\tilde{T}) \) follows from the fact that the \( \mathcal{C}_d \) are \( k \)-coverings. The proof that \( \phi_d \circ \tilde{\phi}_d = \phi \) for every \( d \in \mathcal{D} \) is like the proof above that \( \pi_d \circ \pi_d = \pi \), and we omit it.

We define \( \Psi \) and \( \tilde{\Psi}_d \), for \( d \in \mathcal{D} \), as follows:

\[
\Psi(\tilde{\sigma}, x) = \Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x), \quad \text{where } d \in \mathcal{D} \land d \subseteq x;
\]

\[
\tilde{\Psi}_d(\tilde{\sigma}, x) = \begin{cases} 
\Psi_d'(\tilde{\phi}_d'(\tilde{\sigma}), x) & \text{if } d' \in \mathcal{D} \setminus \{d\} \land d' \subseteq x; \\
x & \text{if } d \subseteq x.
\end{cases}
\]

Let us check clause (d) in the definition of a covering for \( \mathcal{C} \). (Clause (d) for the \( \tilde{\mathcal{C}}_d \) has a similar proof.) Let \( \tilde{\sigma} \in S(\tilde{T}) \) and let \( x \in [\tilde{T}] \) be consistent with \( \tilde{\sigma} \). Let \( d \in \mathcal{D} \) be such that \( d \subseteq x \). Since \( x \) is consistent with \( \phi(\tilde{\sigma}) = \phi_d(\tilde{\phi}_d(\tilde{\sigma})) \), it follows that \( \Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x) \) is consistent with \( \tilde{\phi}_d(\tilde{\sigma}) \). Now \( \Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x) \supseteq d \), since otherwise we would have \( x \supseteq \pi_d(\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)) = \Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x) \supsetneq d \). But \( \tilde{\sigma} \) and \( \tilde{\phi}_d(\tilde{\sigma}) \) agree on \( \tilde{T}_d \), so \( \Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x) \) is consistent with \( \tilde{\sigma} \). The fact that \( \Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x) \supseteq d \) implies that \( \pi(\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)) = \pi_d(\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)) \), and so we have that \( \pi(\Psi(\tilde{\sigma}, x)) = \pi(\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)) = \pi_d(\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)) \subseteq x \). For clause (d)(iii), suppose for definiteness that \( \tilde{\sigma} \) is a strategy for \( I \). If \( \Psi(\tilde{\sigma}, x) \) is not taboo for \( I \) in \( T \), then this same play \( \Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x) \) is not taboo for \( I \) in \( T^d \). By clause (d)(iii) for \( \mathcal{C}_d \), \( \pi_d(\Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)) = x \). Thus \( \pi(\Psi(\tilde{\sigma}, x)) = x \).

Finally, we must verify that \( \Psi(\tilde{\sigma}, x) = \tilde{\Psi}_d(\tilde{\sigma}, \Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)) \).

If \( d \subseteq x \), then \( \Psi(\tilde{\sigma}, x) = \Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x) \) (since \( \Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x) \supseteq d \) \( \tilde{\Psi}_d(\tilde{\sigma}, \Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)) \).

If \( x \supseteq d' \neq d \), then \( \Psi(\tilde{\sigma}, x) = \Psi_d'(\tilde{\phi}_d'(\tilde{\sigma}), x) = \tilde{\Psi}_d(\tilde{\sigma}, x) \) (since \( \mathcal{C}_d \) is a \( d \)-covering) \( \tilde{\Psi}_d(\tilde{\sigma}, \Psi_d(\tilde{\phi}_d(\tilde{\sigma}), x)) \). \( \Box \)
Remark. Lemma 2.2.5 is the basic new step in the proof of quasi-Borel determinacy. Its proof turns on the fact that the non-trivial parts of the $C_d$ are separated, and so these coverings can be combined without interference. We have given most of the details of the proof, but the proof really should be obvious. The significance of the lemma is that all subsets of $[T]$ unraveled by any of the given coverings are simultaneously unraveled by $C$.

**Theorem 2.2.6.** ([Martin, 1990]) Let $T$ be a game tree with taboos. If $A$ is a quasi-Borel subset of $[T]$ and $k \in \omega$, then there is a $k$-covering of $T$ that unravels $A$.

**Proof.** By induction of ordinals $\alpha \geq 1$, we prove

$$(\dagger)_\alpha^* \text{ For all } T, \text{ for all } A \subseteq [T] \text{ such that } A \in \Sigma^*_\alpha, \text{ and for all } k \in \omega, \text{ there is a } k\text{-covering of } T \text{ that unravels } A.$$

Since $\Sigma^*_\alpha = \Sigma^*_1$, $(\dagger)_1^*$ is equivalent with Lemma 2.1.7. Assume then that $\alpha > 1$ and that $(\dagger)_\beta$ holds for all $\beta$ with $1 \leq \beta < \alpha$. If $\alpha$ is a successor ordinal or if $\text{cf}(\alpha) = \omega$, then the proof of Lemma 2.1.8 gives $(\dagger)_\alpha^*$. We may then assume that $\alpha$ has uncountable cofinality. Let $k \in \omega$ and let $A \subseteq [T]$ with $A \in \Sigma^*_\alpha$. By Lemma 2.2.2 and part $(1)(f)$ of Lemma 2.2.1 there is a set $D \subseteq T$ such that $\bigcup_{d \in D} [T_d] = [T]$ and any two elements of $D$ are incomparable with respect to $\subseteq$, and there are $B_d, d \in D$, such that each $B_d \subseteq [T_d]$, such that each $B_d \in \bigcup_{\beta < \alpha} \Pi^*_\beta$, and such that $A = \bigcup_{d \in D} B_d$. By our induction hypothesis, for each $d \in D$ there is a max $\{k, \ell \text{h}(d)\}$-covering $C'_d$ of $T_d$ that unravels $B_d$. For $d \in D$, let $C_d = \langle T_d, \pi_d, \phi_d, \Psi'_d \rangle$ be the $(d)$-covering of $T$ given by Lemma 2.2.4. Each $C_d$ is a $k$-covering and unravels $B_d$. Let $C$ and $C_d$, $d \in D$, be given by Lemma 2.2.5. Since $C = C_d \circ \tilde{C}_d$ for each $d \in D$, it follows that $C$ unravels each $B_d$. Thus $\pi^{-1}(A)$ is open. Let $\tilde{C}$ be a $k$-covering of $T$ that unravels $\pi^{-1}(A)$. Then $C \circ \tilde{C}$ is a $k$-covering of $T$ that unravels $A$. □

**Theorem 2.2.7.** ([Martin, 1990]) All quasi-Borel games are determined.

**Proof.** The theorem follows immediately from Lemma 2.1.3 and Theorem 2.2.6 □

**Theorem 2.2.8.** All $\Delta^1_1$ games are determined.
2.3. OPTIMAL HYPOTHESES

Proof. The theorem follows immediately from Theorem 2.2.3 and Theorem 2.2.7

Exercise 2.2.1. Show that the set $A$ defined in Remark (a) following the proof of Lemma 2.2.1 is quasi-Borel but not Borel.

Exercise 2.2.2. Prove the remark following the proof of Theorem 2.2.3.

Exercise 2.2.3. Let $\langle \hat{T}, \pi, \phi, \Psi \rangle$ be a $(p)$-covering of $T$. Show that there is a unique $\ell h(p)$-covering $\langle T', \pi', \phi', \Psi' \rangle$ of $T_p$ such that the conditions of Lemma 2.2.4 are met.

2.3 Optimal Hypotheses

Results of [Friedman, 1971] show that more and more of the strength of the Power Set and Replacement Axioms is needed to prove $\Sigma_0^\alpha$ determinacy for larger and larger countable $\alpha$. (See Exercises 2.3.2–2.3.5.) Our aim in this section is to show that $\Sigma_0^3$ determinacy follows from essentially the weakest Power Set and Replacement assumptions permitted by slight refinements of Friedman’s theorems. Throughout the section, we work again in ZC$^- + \Sigma_1$ Replacement.

Note first that the proof of Lemma 2.1.6 goes through in our weak set theory, provided we take the given $i \mapsto T_i$ to be a genuine function (i.e. a set) rather than just a (class) operation.

The proofs of Lemmas 2.1.3 and 2.1.4 also go through in the weak set theory. Here are some results that come from combining Lemma 2.1.4 with facts proved in Chapter 1.

Lemma 2.3.1. (ZC$^- + \Sigma_1$ Replacement) Let $T$ be a game tree with taboos and let $A \subseteq [T]$. If there is a covering $\langle \hat{T}, \pi, \phi, \Psi \rangle$ of $T$ such that $\pi^{-1}(A) \in \Sigma_0^3$, then $G(A; T)$ is determined.

Proof. This follows from Lemmas 2.1.4 and Corollary 1.3.4.

Using Theorem 1.4.9, the Montalban-Shore theorem, we can get a stronger result, at least in countable trees:

Lemma 2.3.2. For all $k \in \omega$, ZC$^- + \Sigma_1$ Replacement $\vdash$ “For all countable game trees with taboos $T$, and for all $A \subseteq [T]$, if there is a covering $\langle \hat{T}, \pi, \phi, \Psi \rangle$ of $T$ such that $\pi^{-1}(A) \in k-\Pi_3^1$, then $G(A; T)$ is determined.”
If we strengthen $\text{ZC}^- + \Sigma_1$ Replacement to $\text{Rec}(\text{ZC}^- + \Sigma_1 \text{ Replacement})$, then Corollary 1.4.23 lets us strengthen the conclusion to $\Delta^0_4$.

**Lemma 2.3.3.** (Rec($\text{ZC}^- + \Sigma_1 \text{ Replacement}$)) Let $T$ be a game tree with taboos and let $A \subseteq [T]$. If there is a covering $\langle \tilde{T}, \pi, \phi, \Psi \rangle$ of $T$ such that $\pi^{-1}(A) \in \Delta^0_4$, then $G(A; T)$ is determined.

We will mainly use the first of the three lemmas just stated, but we will occasionally mention the consequences of the others. From Lemma 2.1.6 and the proof of Lemma 2.1.7 we can extract the following fact.

**Lemma 2.3.4.** (ZC$^- + \Sigma_1$ Replacement) Let $T$ be a game tree with taboos. If $k \in \omega$, if $A$ is a countable set of open or closed subsets of $[T]$, and if $\mathcal{P}(T)$ (the power set of $T$) exists, then there is a $k$-covering $C$ of $T$ that unravels every member of $A$ and is such that if $T$ is infinite then

$$|\tilde{T}| \leq |\mathcal{P}(T)|.$$

**Proof.** The proof of Lemma 2.1.7 gives an operation

$$\langle T, A, k \rangle \mapsto C(T, A, k),$$

defined on triples consisting of (1) a game tree with taboos $T$ such that the power set of $T$ exists, (2) a closed subset $A$ of $[T]$, (3) and an even $k \in \omega$. Let $T(T, A, k)$ be the first component of $C(T, A, k)$ and let $\pi(T, A, k)$ be its second component. The main properties of this operation are the following, where we suppress $(T, A, k)$:

(i) $C$ is a $k$-covering of $T$ that unravels $A$;

(ii) if $T$ is infinite, then $|\tilde{T}| \leq |\mathcal{P}(T)|$;

(iii) if $\tilde{p} \in \tilde{T}$ and $\ell h(\tilde{p}) \geq k + 2$, then every move in $\tilde{T}$ at $\tilde{p}$ is a move in $T$ at $\pi(\tilde{p})$.

(i) and (iii) are clear. (ii) holds because the two extra components (other than the numbers 1 and 2) of moves in $\tilde{T}$ are subsets or members of $T$. Because we are working in the weak set theory, it will simplify matters if we change $\tilde{T}$ so that we have
(iv) if $T$ is infinite, if $\tilde{p} \in \tilde{T}$, and if $\ell h(\tilde{p}) \in \{k, k + 1\}$, then every move in $\tilde{T}$ at $\tilde{p}$ is a subset of $T_{\pi(\tilde{p})}$.

For example, we can make move $k$ be a subset of $T_{\pi(\tilde{p})} = T_p$ by having $I$ play $\{p \langle a \rangle \cup X \}$ instead of $\langle a, X \rangle$. We leave it to the reader the problem of finding an appropriate modification of the rules for move $k + 1$.

Assume that $P(T)$ exists and, without loss of generality, assume that $T$ is infinite. Let $k$ and $A$ be as in the statement of the lemma. We may assume that $k$ is even, and we may assume that all members of $A$ are closed. Let then $A = \{A_i | i \in \omega\}$, with each $A_i$ closed. Let $T_0 = T$. Inductively define $C_i = \langle T_i, \pi_1 \circ \cdots \circ \pi_i, \phi_{i+1}, \Psi_{i+1} \rangle$ by $C_i = C(T_i, (\pi_1 \circ \cdots \circ \pi_i)^{-1}(A_i), k + 2i)$.

For $i < j \in \omega$, let $C_{j,i} = C_{i+1} \circ \cdots \circ C_j$.

For $j \in \omega$, let $C_{j,j}$ be the trivial covering of $T_j$. It follows by induction using (iii) and (iv) that, for all $i \in \omega$, (iii) holds with “$T_i$” replacing “$\tilde{T}$” and “$k + 2i$” replacing “$k + 2$,” and

$$(\forall \tilde{p} \in T_i)(\forall m < \ell h(\tilde{p}))(k \leq m < k + 2i \rightarrow \tilde{p}(m) \in P(T)).$$

Thus we can set $k_{j,i} = k + 2i$, and the hypotheses of Lemma 2.1.6 will be satisfied. Applying Lemma 2.1.6, we get, in particular, a covering $C_{\infty,0} = \langle T_{\infty}, \pi_{\infty,0}, \phi_{\infty,0}, \Psi_{0,\infty} \rangle$, a $(k + 2i)$-covering of $T$ that unravels all the $A_i$, and is such that $|T_\infty| \leq \sum_{i \in \omega} |T_i| \leq |\omega \cup P(T)| = |P(T)|$. □

We next prove a standard fact about Borel sets that will be useful in deriving a fact related to Theorem 2.1.8 as Lemma 2.3.4 is related to Lemma 2.1.7.

Let us call a set $A$ of Borel subsets of $[T]$ self-sufficient if, whenever $\beta > 1$ and $A \in A \cap (\Sigma_0^\beta \setminus \bigcup_{\gamma < \beta} \Sigma_0^\gamma)$, there are $A_i, i \in \omega$, with each $A_i \in A \cap \bigcup_{\gamma < \beta} \Sigma_0^\gamma$ and with $A = \bigcup_{i \in \omega} A_i$.

**Lemma 2.3.5.** (ZC $^-$ + $\Sigma_1$ Replacement) Every countable set of Borel subsets of $[T]$ can be extended to a countable, self-sufficient set.

**Proof.** For every countable set $A$ of Borel sets, there is a countable ordinal $\alpha$ such that $A \subseteq \Sigma_0^\alpha$. Thus we may assume inductively that $A$ is a countable subset of $\Sigma_0^\alpha$ with $\alpha$ countable and $\geq 1$, and that for each $\beta < \alpha$ every
countable subset of $\Sigma_\beta^0$ can be extended to a countable self-sufficient set. The case $\alpha = 1$ is trivial, so assume that $\alpha > 1$. For each $A \in A \setminus \bigcup_{\beta < \alpha} \Sigma_\beta^0$, let $\langle B_{i,A} \mid i \in \omega \rangle$ be such that each $B_{i,A} \in \bigcup_{\beta < \alpha} \Sigma_\beta^0$ and $A = \bigcup_{i \in \omega} \neg B_{i,A}$. By induction, for each $\beta < \alpha$ let $B_\beta$ be a countable self-sufficient set extending $\{B_{i,A} \mid i \in \omega \land A \in A \setminus \bigcup_{\beta < \alpha} \Sigma_\beta^0 \land B_{i,A} \in \Sigma_\beta^0\} \cup \{A \in A \mid A \in \Sigma_\beta^0\}$. Let $B = A \cup \bigcup_{\beta < \alpha} B_\beta$.

It is easy to see that $B$ is self-sufficient. □

For sets $X$ and ordinals $\alpha$, we define $P^\alpha(X)$ inductively as follows:

- $P^0(X) = X$;
- $P^{\alpha+1}(X) = P(P^\alpha(X)) \cup P^\alpha(X)$;
- $P^\lambda(X) = \bigcup_{\beta < \lambda} P^\beta(X)$ for $\beta$ a limit ordinal.

Of course, it does not follow in our weak set theory that $P^\alpha(X)$ always exists, even for $\alpha = 1$.

**Lemma 2.3.6.** (ZC$^-$ + $\Sigma_1$ Replacement) Let $\alpha$ be any countable ordinal $\geq 1$. Let

\[
\alpha^* = \begin{cases} 
\alpha - 1 & \text{if } \alpha \text{ is finite;} \\
\alpha & \text{if } \alpha \text{ is infinite.}
\end{cases}
\]

Let $k \in \omega$, and let $T$ be a game tree with taboos. Let $A$ be a countable set of subsets of $[T]$ such that $A \subseteq \bigcup_{1 \leq \beta < \alpha} \Sigma_\beta^0$. If $P^{\alpha^*}(T)$ exists then there is a $k$-covering $C = \langle \tilde{T}, \pi, \phi, \Psi \rangle$ of $T$ such that $C$ unravels every member of $A$ and such that if $T$ is infinite then $|\tilde{T}| \leq |P^{\alpha^*}(T)|$.

**Proof.** We prove the lemma by induction on $\alpha$. The case $\alpha = 1$ is trivial. Let $\alpha > 1$ and assume that the lemma holds for all non-zero ordinals smaller than $\alpha$. Fix $T$ and assume that $P^{\alpha^*}(T)$ exists. Let $k \in \omega$ and let $A$ be a countable set of subsets of $[T]$ such that $A \subseteq \bigcup_{\beta < \alpha} \Sigma_\beta^0$. By Lemma 2.3.5, we may assume that $A$ is self-sufficient. Clearly we may assume that $T$ is infinite.

First suppose that $\alpha = \beta + 1$ for some $\beta$. If $\beta > 1$, then by induction let $C' = \langle T', \pi', \psi', \phi' \rangle$ be a $k$-covering of $T'$ that unravels every member of
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Let \( A \cap \bigcup_{\gamma < \beta} \Sigma^0_{\gamma} \) be a countable subset of \( \Sigma^0_{\beta} \), and is such that \(|T'| \leq |\mathcal{P}^{\beta^*}(T)|\). If \( \beta = 1 \), let \( \mathcal{C}' \) be the trivial covering, with \( T' = T \), etc. Note that in this case \(|T'| = |T| = |\mathcal{P}^0(T)| = |\mathcal{P}^{\beta^*}(T)|\).

Let \( A \in \mathcal{A} \setminus \bigcup_{\gamma < \beta} \Sigma^0_{\gamma} \). If \( \beta > 1 \), then, since \( A \in \Sigma^0_{\beta} \) and \( A \) is self-sufficient, there are \( A_i, i \in \omega \), such that \( A = \bigcup_{i \in \omega} \neg A_i \) and each \( A_i \in A \cap \bigcup_{\gamma < \beta} \Sigma^0_{\gamma} \). Hence \( \pi^{i-1}(A) = \bigcup_{i \in \omega} \pi^{i-1}(\neg A_i) \), and therefore \( \pi^{i-1}(A) \) is open. If \( \beta = 1 \) then \( \pi^{i-1}(A) = A \), which is \( \Sigma^0_1 \), i.e. open.

Since \( \mathcal{P}^{\alpha^*}(T) = \mathcal{P}(\mathcal{P}^{\beta^*}(T)) \cup \mathcal{P}^{\beta^*}(T) \), we have the existence of \( \mathcal{P}(T') \). Applying Lemma 2.3.4 to \( T' \) and \( \mathcal{A}' = \{ \pi^{i-1}(A) \mid A \in \mathcal{A} \} \), we get a covering \( \hat{C} = \langle T, \pi, \phi, \Psi \rangle \) of \( T' \) that unravels every member of \( \mathcal{A}' \) and satisfies

\[
|\hat{T}| \leq |\mathcal{P}(T')| \leq |\mathcal{P}^{\alpha^*}(T)|.
\]

Let \( \mathcal{C} = \mathcal{C}' \circ \hat{C} \).

Now suppose that \( \alpha \) is a limit ordinal. Let \( \langle \beta_n \mid n \in \omega \rangle \) be an increasing sequence of ordinals \( < \alpha \) such that \( \sup_{n \in \omega} \beta_n = \alpha \) and such that \( \beta_0 = 1 \).

Inductively we define \( T_j, j \in \omega \), and \( C_{j,i} = \langle T_j, \pi_j, \phi_j, \Psi_j \rangle, i \leq j \in \omega \), so that

(i) the hypotheses of Lemma 2.1.6 are satisfied with \( k_{j,i} = k + i \).

(ii) \( T_0 = T \).

(iii) for all \( n \in \omega \), every move in \( T_n \) belongs to \( \mathcal{P}^{\beta_n}(T) \).

(iv) for all \( n \in \omega \), \( C_{n,0} \) unravels every element of \( \mathcal{A}_n = \mathcal{A} \cap \bigcup_{\gamma < \beta_n} \Sigma^0_{\gamma} \).

Let \( \mathcal{A}_n = \{ \pi_m^{i-1}(A) \mid A \in \mathcal{A}_n \} \). Clause (iv) says that each \( \mathcal{A}_n \) is a set of clopen sets.

Assume that the \( T_j \) and the \( C_{j,i} \) are defined for \( i \leq j \leq n \) and have the stated properties. (This is trivial for \( n = 0 \).)

Let \( \gamma \) be such that \( \beta_n + \gamma = \beta_{n+1} \). Now \( \mathcal{A}_{n+1}^\gamma \) is readily seen to be self-sufficient, and \( \mathcal{A}_{n+1}^\gamma \cap \{ \pi_m^{i-1}(A) \mid A \in \bigcup_{\delta < \beta_n} \Sigma^0_{\delta} \} = \mathcal{A}_n^\gamma \), a set of clopen sets.

It follows by an easy inductive argument that \( \mathcal{A}_{n+1}^\gamma \in \bigcup_{\delta < \gamma} \Sigma^0_{\delta} \). Now \( \mathcal{P}^{\gamma^*}(\mathcal{P}^{\beta_n}(T)) = \mathcal{P}^{\beta_{n+1}}(T) \). It follows by (iii) that \( \mathcal{P}^{\gamma^*}(T_n) \) exists. Since \( \mathcal{A}_{n+1}^\gamma \) is a countable subset of \( [T_n] \cap \bigcup_{\delta < \gamma} \Sigma^0_{\delta} \) and \( \gamma \leq \beta_{n+1} < \alpha \), it follows by induction that there is a \( (k + n) \)-covering \( \mathcal{C}' = \langle T', \pi', \phi', \Psi' \rangle \) of \( T_n \) that unravels every member of \( \mathcal{A}_{n+1}^\gamma \) and satisfies \(|T'| \leq |\mathcal{P}^{\gamma^*}(T_n)| \leq |\mathcal{P}^{\beta_{n+1}}(T)| \).

Modifying \( T' \) to make clause (iii) hold, we get our \( C_{n+1,n} \); we get the \( C_{n+1,j} \) for \( j < n \) by composition.
Lemma 2.1.6 yields, in particular, a $k$-covering
$$C_\infty = \langle T_\infty, \pi_\infty, 0, \phi_\infty, 0, \Psi_\infty, 0 \rangle$$
of $T$ that unravels every member of $A$ and satisfies $|T_\infty| \leq |P^\alpha(T)|$. Since $\alpha^* = \alpha$, we can let $C = C_{\infty, 0}$. □

**Theorem 2.3.7.** (ZC$^-$ + $\Sigma_1$ Replacement) Let $T$ be a game tree with taboos.
(a) If $n \in \omega$ and $P^n(T)$ exists, then all $\Sigma^0_{n+3}$ games in $T$ are determined.
(b) If $\alpha$ is an infinite countable ordinal and $P^\alpha(T)$ exists, then all $\Sigma^0_{\alpha+2}$ games in $T$ are determined.

**Proof.** (a) Assume that $P^n(T)$ exists and let $A \subseteq [T]$ with $A \in \Sigma^0_{n+3}$. By Lemma 2.3.5, let $B$ be countable and self-sufficient with $A \in B$. By Lemma 2.3.6, let $\langle T, \pi, \phi, \Psi \rangle$ be a covering of $T$ that unravels every member of $B \cap \Sigma^0_n$. We have that $\pi^{-1}(A) \in \Sigma^0_n$. By Lemma 2.3.1, $G(A; T)$ is determined.

The proof of (b) is similar to that of (a), and we omit it. □

**Corollary 2.3.8.** (ZC$^-$ + $\Sigma_1$ Replacement) If $P^\alpha(T)$ exists for every countable $\alpha$, then all Borel games in $T$ are determined.

**Corollary 2.3.9.** (ZC + $\Sigma_1$ Replacement) For all $n \in \omega$, every $\Sigma^0_n$ game is determined.

**Proof.** Since Zermelo Set Theory (ZC) gives the existence of $P^\alpha(T)$ for every $n \in \omega$, the Corollary follows by Theorem 2.3.7. □

**Remarks:**

(i) “$\Sigma_1$ Replacement” can be dropped from the statements of Corollaries 2.3.8 and 2.3.9. Though we cannot then use (von Neumann) ordinal numbers, we can replace them by wellordered sets, making use of Zermelo’s theorem that every set can be wellordered. If we restrict ourselves to, say, the tree $\omega^\omega$, then “$+$ $\Sigma_1$ Replacement” can be dropped from the statements of all our other results as well, though—since we don’t have in general the existence of cartesian products—we must exercise some care in formulating these results.

(ii) For countable trees and for any fixed $k \in \omega$, “$\Sigma^0_{n+3}$” can be replaced in Theorem 2.3.7 by “$k$-$\Pi^0_3$,” and “$\Sigma^0_{\alpha+2}$” can be replaced there by “$k$-$\Pi^0_{\alpha+2}$.” This follows by Lemma 2.3.2.
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(iii) If “$\text{ZC}^{-} + \Sigma_1$ Replacement” is replaced by in Theorem 2.3.7 by “Rec($\text{ZC}^{-} + \Sigma_1$ Replacement),” then “$\Sigma_{n+3}^0$” and “$\Sigma_{\alpha+2}^0$” can be replaced by “$\Delta_{n+4}^0$” and “$\Delta_{\alpha+3}^0$” respectively. This follows by Lemma 2.3.3.

For elements $x$ of $\omega$ and countable ordinals $\alpha$, let $\beta^x_\alpha$ be the least ordinal $\beta$ such that $L_\beta[x] = \text{ZC}^{-} + \Sigma_1$ Replacement + “$\mathcal{P}^\alpha(\omega)$ exists” (provided, of course, that such a $\beta$ exists).

**Theorem 2.3.10.** ($\text{ZC}^{-} + \Sigma_1$ Replacement)

(a) If $n \in \omega$ and $\beta^x_\alpha$ exists for every $x \in \omega$, then all $\Delta_{n+4}^0$ games in $<\omega$, $<\omega$ are determined.

(b) If $\alpha$ is an infinite countable ordinal and $\beta^x_\alpha$ exists for every $x \in \omega$, then all $\Delta_{\alpha+3}^0$ games in $<\omega$, $<\omega$ are determined.

**Proof.** (a) Assume that $\beta^x_\alpha$ exists for every $x \in \omega$. Let $A$ be a $\Delta_{n+4}^0$ subset of $\omega$. Let $x$ be such that $A$ is $\Delta_{n+4}^0$ in $x$. Fix a definition witnessing that $A$ is $\Delta_{n+4}^0$ in $x$.

During this paragraph, we work in $L_{\beta^x_\alpha}[x]$ and we write “$A$” for the set satisfying in $L_{\beta^x_\alpha}[x]$ our chosen definition of $A$. By Lemma 2.3.5, let $\mathcal{B}$ be countable and self-sufficient with $A \in \mathcal{B}$. By 2.3.6, let $\langle \tilde{T}, \pi, \phi, \Psi \rangle$ be a covering of $\mathcal{T}$ that unravels every member of $\mathcal{B} \cap \Sigma_0^n$. The set $\pi^{-1}(A)$ belongs to $\Delta_3^0$. By Theorem 1.4.2 (which holds in $L_{\beta^x_\alpha}[x]$), $\pi^{-1}(A) \in \text{Diff}(\Pi_3^0)$.

We have just shown that $L_{\beta^x_\alpha}[x]$ satisfies “$\pi^{-1}(A) \in \text{Diff}(\Pi_3^0)$.” By Theorem 1.4.16, $L_{\beta^x_\alpha}[x] \models \text{"All Diff}(\Pi_3^0)$ games in $\mathcal{T}$ are determined.” Thus $L_{\beta^x_\alpha}[x] \models \text{"G}(\pi^{-1}(A); \mathcal{T})$ is determined.” By Lemma 2.1.4, $L_{\beta^x_\alpha}[x] \models \text{"G}(A; <\omega, \omega)$ is determined.” By absoluteness, $G(A; <\omega, \omega)$ is determined.

The proof of (b) is similar.

As in the exercises at the end of §1.4, in the exercises below we will use “$\omega$-model” to mean a model $(M; E)$ such that $\omega \in \text{WFP}(M; E)$ and the restriction of $E$ to $\text{WFP}(M; E)$ is the membership relation.

**Exercise 2.3.1.** This exercise extends the results of the the present section to the quasi-Borel hierarchy introduced in §2.2.

Work in $\text{ZC}^{-} + \Sigma_1$ Replacement. Let $\alpha$ be any infinite ordinal. Prove that, for every game tree with taboos $\mathcal{T}$, for every countable set $\mathcal{A}$ of subsets of $[\mathcal{T}]$ such that $\mathcal{A} \subseteq \bigcup_{1 \leq \beta < \alpha} \Sigma^\alpha_\beta$, and for every $k \in \omega$, if $\mathcal{P}^\alpha(T)$ exists then there is a $k$-covering $\mathcal{C} = \langle \mathcal{T}, \pi, \phi, \Psi \rangle$ of $\mathcal{T}$ such that $\mathcal{C}$ unravels every member of $\mathcal{A}$ and such that if $T$ is infinite then $|\tilde{T}| \leq |\mathcal{P}^\alpha(T)|$. 

Deduce that, if $T$ is a game tree with taboos and $\mathcal{P}^\alpha(T)$ exists, then all $\Sigma^*_{\alpha+2}$ games in $T$ are determined.

Hint. Adapt the proof of Lemma 2.3.6, using the proof of Lemma 2.2.5 to handle the case that $\alpha$ is the successor of an ordinal of uncountable cofinality.

Exercise 2.3.2. This exercise and the four that follow it are, like Exercises 1.4.1 and 1.4.2, refinements by the author of results of [Friedman, 1971].

Show that, for each ordinal $\alpha < \omega_1$, there is a model $(M; \in)$ of $\text{ZFC}^- + \Sigma_1$ Replacement such that $M$ is a transitive set, $\alpha \in M$, and $(M; \in) \models "\mathcal{P}^\alpha(\omega)"$ exists,” and (a) $\Sigma^0_{\alpha+4}$ determinacy for games in $\prec\omega \omega$ fails in $(M; \in)$ if $\alpha$ is finite and (b) $\Sigma^0_{\alpha+3}$ determinacy for games in $\prec\omega \omega$ fails for $\alpha$ infinite.

Hint. Proceed as with Exercise 1.4.1, except—for $\alpha < \omega_1$—replace $\beta_0$ by $\beta_\alpha$, where $\beta_\alpha$ is the least ordinal number $\beta$ such that $L_\beta \models \text{ZFC}^- + \Sigma_1$ Replacement + “$\mathcal{P}^\alpha(\omega)$ exists.” A key fact about $\beta_\alpha$ for $\alpha < \omega_1$ is that it is the least ordinal $\beta$ such that there is no $a \subseteq \mathcal{P}^\alpha(\omega)$ such that $a \in L_{\beta_\alpha+1} \setminus L_{\beta_\alpha}$.

Exercise 2.3.3. Work in $\text{ZFC}^- + \Sigma_1$ Replacement. Let $\alpha$ be a small enough countable ordinal that $\alpha$ is definable and the lightface class $\Sigma^0_\alpha$ makes sense (e.g., let $\alpha < \omega_1^{CK}$). Assume that (a) all $\Sigma^0_{\alpha+4}$ games in $\prec\omega \omega$ are determined if $\alpha$ is finite and that all $\Sigma^0_{\alpha+3}$ games in $\prec\omega \omega$ are determined if $\alpha$ is infinite.

Prove that $\beta_\alpha$ exists. It follows that the consistency of $\text{ZFC}^- + "\mathcal{P}^\alpha(\omega)"$ exists” can be proved in $\text{ZFC}^- + \Sigma_1$ Replacement + “either all $\Sigma^0_{\alpha+4}$ games are determined or $\alpha$ is infinite and all $\Sigma^0_{\alpha+3}$ games are determined.”

Hint. Combine the hints to Exercises 1.4.2 and 2.3.2.

Exercise 2.3.4. Show that, for each limit ordinal $\lambda < \omega_1$, there is a model $(M; \in)$ of $\text{ZFC}^- + \Sigma_1$ Replacement such that $M$ is a transitive set, $(\forall \alpha < \lambda) \mathcal{P}^\alpha(\omega) \cap M \in M$, and $\Sigma^0_{\lambda+1}$ determinacy for games in countable trees fails in $(M; \in)$. Your $(M; \in)$ should also be a model of the Power Set Axiom, and so you can deduce that the determinacy of all $\Sigma^0_{\omega+1}$ games in $\prec\omega \omega$ is not provable in $\text{ZFC} + \Sigma_1$ Replacement.

Exercise 2.3.5. Work in $\text{ZFC}^- + \Sigma_1$ Replacement. Let $\lambda$ be a small enough countable limit ordinal that $\lambda$ is definable and the $\Sigma^0_\lambda$ makes sense. Assume that all $\Sigma^0_{\lambda+1}$ games in $\prec\omega \omega$ are determined. Let $\beta_\lambda$ be the least ordinal $\gamma$,
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if one exists, such that $L_\gamma \models ZC^- + \Sigma_1$ Replacement + “$(\forall \alpha < \lambda) P^\alpha(\omega)$ exists.” Prove that $\beta_\lambda$ exists.

In particular, this means that in $ZC^- + \Sigma_1$ Replacement the determinacy of all $\Sigma^0_{\omega+1}$ games in $<\omega_\omega$ implies the consistency of ZC.

**Exercise 2.3.6.** Work in $ZC^- + \Sigma_1$ Replacement. Prove the following generalization of the result of Exercise 1.4.3. Let $\alpha$ be a countable ordinal and assume that $\Sigma^0_{\alpha+5}$ Turing determinacy holds if $\alpha$ is finite and that $\Sigma^0_{\alpha+4}$ Turing determinacy holds if $\alpha$ is infinite. Show that $\beta^\alpha_x$ exists for every $x \in \omega$. 

*Hint.* Combine the hints for Exercises 2.3.3 and 1.4.3. In the game, require that the models satisfy $V = L[x]$ instead of $V = L$.

**Exercise 2.3.7.** Work in $ZC^- + \Sigma_1$ Replacement. Show that, for every countable ordinal $\alpha$, $\Sigma^0_{\alpha+5}$ Turing determinacy implies the determinacy of all $\Delta^0_{\alpha+4}$ games in $<\omega_\omega$. Note that a consequence of this is that Borel Turing determinacy implies the determinacy of all Borel games in $<\omega_\omega$.

*Hint.* Use Theorem 2.3.10.

**Exercise 2.3.8.** This and the following four exercises give a a proof due to Ramez Sami of a non-level-by-level form of of Friedman’s result on the strength of Borel determinacy. Sami’s proof has more in common with the proof in [Friedman, 1971] than with the proof sketched in our hints to earlier exercises. In particular, Friedman and Sami use Turing degrees in similar ways. But Sami’s proof has ingredients not in either Friedman’s proofs or ours, principally the result of the present exercise.

In order that the result of this exercise will be applicable to Exercise 2.3.12, work in the theory $ZFC^-$. Assume that all $\Delta^1_1$ games in $<\omega_\omega$ are determined. Let $A \in \Sigma^1_1$. Show that at least one of the following holds.

1. There is a strategy $\sigma$ for I such that, if $x \in \Delta^1_1(\sigma)$ is any play consistent with $\sigma$, then $x \in A$.
2. There is a winning strategy for II for $G(A; <\omega_\omega)$.

*Hint.* Assume that (1) fails and show that (1) fails for some $B \supseteq A$ with $B \in \Delta^1_1$. Then use Borel determinacy.

**Exercise 2.3.9.** Once again work in $ZFC^-$. Assume, say, that $\nu < \omega^\text{CK}_1$. Let $T_\nu$ be the theory $\text{KP} + V = L + \text{“there is no ordinal } \alpha \text{ such that } L_\alpha \models (ZFC^- + \mathcal{N}_\nu \text{ exists).”}$
Let $\mathcal{M}$ and $\mathcal{N}$ be $\omega$-models of $T_\nu$. Let $d$ be an ordinal of $M$ and let $e_1$ and $e_2$ be ordinals of $N$. Suppose that $f_1 : L^M_d \cong L^N_{e_1}$ and $f_2 : L^M_d \cong L^N_{e_2}$. Prove that $f_1 = f_2$.

*Hint.* It is enough to show that $f_1$ and $f_2$ agree on the ordinals of $M$ that are less than $d$. Show that the order type of the infinite cardinals of $L^M_d$ is $\leq \nu + 1$. Prove by induction on infinite cardinals $b$ of $M$ that $f_1(a) = f_2(a)$ for every ordinal $a$ of $M$ such that $|a|^M \leq b$.

**Exercise 2.3.10.** Once again work in the theory $\text{ZFC}^-$. Let $\nu$ and $T_\nu$ be as in Exercise 2.3.9. Assume $V = L$ and assume that there is no ordinal $\alpha$ such that $L_\alpha \models \text{ZFC}^- + \exists \aleph_\nu$. Prove that the set of Turing degrees of complete extensions of $T_\nu$ with wellfounded term models is unbounded.

*Hint.* Under the assumptions it is enough to prove that there are arbitrarily large countable ordinals $\alpha$ such that $L^M_\alpha \models \text{KP}$ and every member of $L^M_\alpha$ is definable in $L^M_\alpha$. See Exercise 1.4.2.

**Exercise 2.3.11.** Once again work in the theory $\text{ZFC}^-$. Let $\nu$ and $T_\nu$ be as in the preceding two exercises. Say that the term model $\mathcal{M}$ of a complete extension $S$ of $T_\nu$ is *pseudo-wellfounded* if every non-empty subset of the universe of $\mathcal{M}$ that is $\Delta^1_1$ in $S$ has an element that is minimal with respect to $\in_\mathcal{M}$. Let $\mathcal{S}_\nu$ be the set of all complete extensions $S$ of $T_\nu$ whose term models are pseudo-wellfounded. Note that $\mathcal{S}_\nu$ is $\Sigma^1_1$. Prove that two distinct members of $\mathcal{S}_\nu$ cannot be $\Delta^1_1$ in one another.

*Hint.* Assume this is false and let $\mathcal{M}$ and $\mathcal{N}$ be the term models of theories witnessing its falsity. For $d$ and $e$ ordinals of $\mathcal{M}$ and $\mathcal{N}$ respectively, say that $d \sim e$ if $L^M_d \cong L^N_e$. Use Exercise 2.3.9 to prove that $\sim$ is $\Delta^1_1$ in $\mathcal{M}$ and in $\mathcal{N}$. Deduce that either $\mathcal{M} \cong L^N_\alpha$ for some ordinal $\alpha$ of $\mathcal{N}$ or else $\mathcal{N} \cong L^M_d$ for some ordinal $d$ of $\mathcal{M}$. Get a contradiction as in the analogous parts of the proofs for Exercises 1.4.2, 2.3.3, and 2.3.5.

**Exercise 2.3.12.** Work again in $\text{ZFC}^-$. Let $\nu$ be as in the preceding three exercises. Prove that the determinacy of all $\Delta^1_1$ games in $\lt \omega \omega$ implies that there is an ordinal $\alpha$ such that $L_\alpha \models \text{ZFC}^- + \exists \aleph_\nu$.

*Hint.* By absoluteness, you may assume $V = L$. Assume that what you are trying to prove is false. Use Exercises 2.3.8 and 2.3.10 to prove that there is a Turing degree $d$ such that, for any $d' \geq d$ and $\Delta^1_1$ in $d$, there is a member of $\mathcal{S}_\nu$ of degree $d'$. Use Exercise 2.3.11 to obtain a contradiction.
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Exercise 2.3.13. If $A$ and $B$ are subsets of $\omega^\omega$, we say that $A$ is Wadge reducible to be, or $A \leq_w B$, if there is a continuous $f : \omega^\omega \to \omega^\omega$ such that $A = f^{-1}B$. If $A \leq_w B$ then, in a strong sense, $A$ is at least as simple a set as $B$. Prove that, for any Borel $A$ and $B$, either $A \leq_w B$ or $B \leq_w \neg A$.

*Hint.* Say that $A$ is Lipschitz reducible to $B$, or $A \leq_\ell B$, if there is a winning strategy $\tau$ for II for the game $G_w(A, B; <\omega^\omega) = G(C; <\omega^\omega)$, where $\langle a_i \mid i \in \omega \rangle \in C$ if and only if

$$
\langle a_{2i} \mid i \in \omega \rangle \in A \iff \langle a_{2i+1} \mid i \in \omega \rangle \in B.
$$

Note that $A \leq_\ell B \rightarrow A \leq_w B$. Use Borel determinacy.

*Remarks:*

(a) This fundamental result (announced in Wadge [1972]) was proved by William Wadge in about 1967. Wadge used determinacy as a hypothesis. Borel determinacy had not been proved at the time. In the papers [Louveau and Saint-Raymond, 1987] and [Louveau and Saint-Raymond, 1988], it is shown that the result, unlike Borel determinacy, can be proved in, say, $ZC^-$. They do this by proving in the weak theory that the Wadge game $G(A, B; <\omega^\omega)$ is determined for Borel sets $A$ and $B$.

(b) Clearly Wadge’s result still holds if we replace $\omega^\omega$ by $[T]$, where $T$ is any game tree, and if we replace “Borel” by “quasi-Borel.” Clearly also, determinacy hypotheses for larger classes imply Wadge’s result for larger classes. For example, the determinacy of all projective games in $<\omega^\omega$ implies that Wadge’s result holds for all projective subsets of $\omega^\omega$. (See Chapter 8 for the definition of “projective.”) Moreover $AD$ implies that Wadge’s result holds for all subsets of $\omega^\omega$.

Exercise 2.3.14. This exercise show that Wadge reducibility stratifies the Borel subsets of $\omega^\omega$ into a wellordered hierarchy.

If $A$ and $B$ are subsets of $\omega^\omega$, say that $A \sim_w B$ if both $A \leq_w B$ and $B \leq_w A$. Similarly define $A \sim_\ell B$. Exercise 2.3.13 and the argument of the hint show that $\leq_w$ and $\leq_\ell$ give linear orderings of the equivalence classes with respect to $\sim_w$ and $\sim_\ell$ respectively of Borel sets, except that the classes of a set and its complement may be incomparable. Show that these linear orderings are well-orderings.

*Hint.* Let $A$ be such that every continuous preimage of $A$ in $\omega^2$ is measurable with respect to the product of the measure on $\{0,1\}$ that gives each
point measure 1/2. Show that for any \( \langle A_i \mid i \in \omega \rangle \) such that \( A_0 = A \), there is an \( i \in \omega \) such that I does not have winning strategies for both \( G_w(A_i, A_{i+1}; <\omega \omega) \) and \( G_w(A_i, \neg A_{i+1}; <\omega \omega) \). To do this, assume for a contradiction that winning strategies \( \sigma_0^i \) for I for \( G_w(A_i, A_{i+1}; <\omega \omega) \) and \( \sigma_1^i \) for I for \( G_w(A_i, \neg A_{i+1}; <\omega \omega) \) exist. For \( z \in \omega^2 \), we get a sequence \( \langle x^i_z \mid i \in \omega \rangle \) of elements of \( \omega^\omega \) by letting I follow the strategies \( \sigma^i_{z(i)} \) to produce simultaneously the plays \( x^i_z(0), x^{i+1}_z(0), x^i_z(1), x^{i+1}_z(1), \ldots \). Consider the probability that \( x^i_z \in A_i \). Show that the 0–1 law is contradicted.

Remarks:
(a) This result is due to Martin, but the basic idea of using the 0–1 law was introduced by Leonard Monk, who proved a partial result.
(b) Remark (b) for Exercise 2.3.13 applies to the result of this exercise as well.

2.4 Blackwell Games

In [Blackwell, 1969], David Blackwell introduced a class of infinite games of imperfect information. These games differ in one basic way from the ones we have been studying. Instead of taking turns moving, the players make their \( n \)th moves simultaneously. The fact that moves are made simultaneously rules out, even for games where each player makes only a single move, determinacy theorems of the kind we have been studying. To make determinacy theorems possible, the strategies we have considered heretofore have to be replaced by mixed strategies, strategies that involve randomization. Moreover determinacy has to be defined in terms of having a value.

Another other difference between Blackwell’s games and the ones we have been studying is a restriction: in each position, each player can have only finitely many legal moves. We will explain the reason for this restriction later, and we will also weaken the restriction. A third difference from our perfect information games is that we will allow payoff functions, not just payoff sets. Payoff functions make sense for perfect information games also, but their introduction adds little to the theory or applications of such games.

The concepts (mixed strategies, values of games, and payoff functions) we have just mentioned and will shortly explain in detail are among the basic concepts of ordinary game theory. What [Blackwell, 1969] proposes is that these concepts be studied in the context of games of infinite length.
From the fundamental von Neumann Minimax Theorem of [von Neumann, 1928], it follows that all Blackwell games of finite length are determined. [Blackwell, 1969] proves that all \( \Sigma^0_2 \) Blackwell games with payoff sets are determined. [Orkin, 1972] extends this result to Boolean combinations of \( F_{\sigma} \)'s. Determinacy for what we will consider the general class of \( \Sigma^0_2 \) Blackwell games ("general" in that payoff functions are allowed) is proved in [Maitra and Sudderth, 1992]. [Vervoort, 1996] betters Blackwell's result by a whole level of the Borel hierarchy, proving the determinacy of all \( \Sigma^0_3 (G_{\delta\sigma}) \) Blackwell games with payoff sets.

In this section, we present a result from [Martin, 1998] showing that the determinacy of any given Blackwell game is implied by the determinacy of perfect information games of roughly the same complexity. This result yields, in particular, a level-by-level reduction of Borel Blackwell determinacy to ordinary Borel determinacy. Blackwell conjectured in [Blackwell, 1969] that all Borel Blackwell games are determined, and so his conjecture is confirmed. (Blackwell did not, of course, use the word "Blackwell.")

We now turn to the formal introduction of Blackwell games. A game tree \( T \) is a Blackwell game tree if

(a) the members of \( T \) are finite sequences of ordered pairs;
(b) if \( p \in T \) is non-terminal and has length \( i \), then there are non-empty sets \( X_p \) and \( Y_p \) such that

(i) at least one of \( X_p \) and \( Y_p \) is finite;
(ii) the length \( i + 1 \) extensions of \( p \) that belong to \( T \) are precisely the \( p^\downarrow \langle a, b \rangle \) with \( a \in X_p \) and \( b \in Y_p \).

If \( T \) is a Blackwell game tree, then Blackwell games in \( T \) are played as follows.

\[
\begin{array}{cccc}
I & a_0 & a_1 & a_2 & \ldots \\
II & b_0 & b_1 & b_2 & \ldots 
\end{array}
\]

In other words, for each \( i \) the moves \( a_i \) and \( b_i \) are made simultaneously. It is required that all positions \( \langle \langle a_i, b_i \rangle \mid i < n \rangle \) belong to \( T \).

Remark. If clause (b) is relaxed to allow both players to have countably infinitely many moves, then determinacy fails even for games in which each player makes only a single move. See Exercise 2.4.1.

Let \( T \) be a Blackwell game tree. A mixed strategy for player I or II in \( T \) is a function \( \sigma \) that assigns to each position in \( T \) a discrete probability
distribution on the set of legal moves for that player in the position. To see what we mean by this, assume for definiteness that $\sigma$ is a strategy for I. Let $X_p$ be as in the definition of Blackwell game trees. Then, for each $p \in T$,

(i) $\sigma(p) : X_p \rightarrow [0, 1]$;
(ii) $\sum_{a \in X_p}(\sigma(p))(a) = 1$.

Note that (i) and (ii) imply that the set of $a \in X_p$ such that $(\sigma(p))(a) > 0$ is countable. We are not, then, considering the more general case in which the sum of (ii) is replaced by an integral.

Let $T$ be a Blackwell game tree. Let $\sigma$ and $\tau$ be mixed strategies for I and II respectively in $T$. The strategies $\sigma$ and $\tau$ give, in the following manner, a probability measure $\mu_{\sigma,\tau}$ on the set of all plays in $T$. If $p = \langle \langle a_i, b_i \rangle \mid i < n \rangle$ is a position in $T$ then set

$$\mu_{\sigma,\tau}(\lceil T_p \rceil) = \prod_{i<n}(\sigma(p \upharpoonright i))(a_i) \cdot (\tau(p \upharpoonright i))(b_i).$$

By a standard construction and argument, there is a unique probability measure defined on the Borel subsets of $\lceil T \rceil$ that has the specified values on the $\lceil T_p \rceil$. The $\mu_{\sigma,\tau}$-measurable sets are defined in the usual way, as the set of all $A \subseteq \lceil T \rceil$ such that $A$ the symmetric difference of $A$ and some Borel set is contained in a Borel set $B$ with $\mu_{\sigma,\tau}(B) = 0$. A function $f : \lceil T \rceil \rightarrow \mathbb{R}$ is $\mu_{\sigma,\tau}$-measurable if the $f$-preimage of each open set is $\mu_{\sigma,\tau}$-measurable.

A payoff function for a game tree $T$ is a function $f$ from the set of all plays in $T$ into a bounded subset of the real numbers. For each Blackwell game tree $T$ and each payoff function $f$ for $T$, there is a Blackwell game which we call $\Gamma(f; T)$.

If $\Gamma(f; T)$ is a Blackwell game and $f$ is $\mu_{\sigma,\tau}$-measurable, then set

$$E_{\sigma,\tau}(f) = \int f \, d\mu_{\sigma,\tau}.$$ 

For for Blackwell games $\Gamma(f; T)$ with arbitrary payoff functions $f$, set

$$E^{-}_{\sigma,\tau}(f) = \sup \{ E_{\sigma,\tau}(g) \mid g \text{ is Borel measurable } \land (\forall x) g(x) \leq f(x) \};$$
$$E^{+}_{\sigma,\tau}(f) = \inf \{ E_{\sigma,\tau}(g) \mid g \text{ is Borel measurable } \land (\forall x) g(x) \geq f(x) \}.$$

If $f$ is $\mu_{\sigma,\tau}$-measurable, then $E^{-}_{\sigma,\tau}(f) = E^{+}_{\sigma,\tau}(f) = E_{\sigma,\tau}(f)$. 
2.4. BLACKWELL GAMES

Let $\Gamma(f;T)$ be a Blackwell game. If $\sigma$ is a mixed strategy for I in $T$ then the value of $\sigma$ in $\Gamma(f;T)$ is

$$\inf \{ E_{\sigma,\tau}^{-}(f) | \tau \text{ is a mixed strategy for II} \}.$$ 

If $\tau$ is a mixed strategy for II in $T$ then the value of $\tau$ in $\Gamma(f;T)$ is

$$\sup \{ E_{\sigma,\tau}^{+}(f) | \sigma \text{ is a mixed strategy for I} \}.$$ 

Let $\text{val}_{\downarrow}(\Gamma(f;T))$ be the supremum over all mixed strategies $\sigma$ for I in $T$ of the value of $\sigma$ in $\Gamma(f;T)$ and let $\text{val}_{\uparrow}(\Gamma(f;T))$ be the infimum over all mixed strategies $\tau$ for II in $T$ of the value of $\tau$ in $\Gamma(f)$. The game $\Gamma(f;T)$ is determined if

$$\text{val}_{\downarrow}(\Gamma(f;T)) = \text{val}_{\uparrow}(\Gamma(f;T)).$$

If $\Gamma(f;T)$ is determined, then the value of $\Gamma(f;T) = \text{val}(\Gamma(f;T)) = \text{val}_{\downarrow}(\Gamma(f;T)) = \text{val}_{\uparrow}(\Gamma(f;T))$.

Remarks:

(a) We are using the term “Blackwell games” to cover a rather wide class. It might be more accurate to reserve the term “Blackwell games” for infinite length games. Moreover Blackwell considered only measurable payoff functions. The definitions for the non-measurable case are from [Vervoort, 1996].

Somewhat artificially, we say that a Blackwell game $\Gamma(f;T)$ is open, closed, Borel, etc., if for all rationals $y$ the set of all plays $x$ such that $y \leq f(x)$ is open, closed, Borel, etc. Note that a Blackwell game is Borel just in case its payoff function is Borel measurable, i.e., just in case the $f$-preimage of each Borel set is Borel.

Suppose that $T$ is a Blackwell game tree and that $A$ is a set of plays in $T$. Let $\chi(A)$ be the characteristic function of $A$, the function $f$ such that $f(x) = 1$ for $x \in A$ and $f(x) = 0$ for $x \notin A$. According to the definition of the preceding paragraph, $\Gamma(\chi(A);T)$ is open, closed, Borel, etc., just in case $A$ is open, closed, Borel, etc.

Say that a mixed strategy $\sigma$ for I for a Blackwell game $\Gamma(f;T)$ is an optimal strategy if the value of $\sigma$ in $\Gamma(f;T)$ is $\text{val}_{\downarrow}(\Gamma(f;T))$. Similarly say that a mixed strategy $\tau$ for II is an optimal strategy if the value of $\tau$ is $\text{val}_{\uparrow}(\Gamma(f;T))$. We say that $\Gamma(f;T)$ is determined in optimal strategies if $\Gamma(f;T)$ is determined and each player has an optimal strategy. Note that
this means that each player has a mixed strategy whose value is the value of \( \Gamma(f; T) \).

The basic form of the von Neumann Minimax Theorem of [von Neumann, 1928] is as follows.

**Theorem 2.4.1 (Minimax Theorem)** Let \( T \) be a Blackwell game tree in which all plays have length 1 and for which both \( X_\emptyset \) and \( Y_\emptyset \) are finite. Then all Blackwell games in \( T \) are determined in optimal strategies.

A proof of this theorem may be found in [Vervoort, 2000] (and, of course, in [von Neumann, 1928] and in many other places).

**Corollary 2.4.2.** Let \( T \) be a Blackwell game tree in which all plays have length \( \leq \) some fixed natural number \( n \). Assume that all the sets \( X_p \) and \( Y_p \) associated with \( T \) are finite. Then all Blackwell games in \( T \) are determined in optimal strategies.

**Proof.** We proceed by induction on \( n \). The case \( n = 0 \) is trivial. Let \( n \geq 0 \) and assume that the corollary holds for \( n \). Let \( T \) be a Blackwell game tree in which all plays have length \( \leq n + 1 \). For each non-terminal position \( p \) in \( T \) of length \( n \), let

\[
T^p = \{\emptyset\} \cup \{\langle w \rangle \mid p \vdash \langle w \rangle \in T\}.
\]

Let \( f^p \) be the payoff function for \( T^p \) given by

\[
f^p(\langle w \rangle) = f(p \vdash \langle w \rangle).
\]

By the theorem, each \( \Gamma(f^p; T^p) \) is determined in optimal strategies.

Let

\[
T' = \{p \in T \mid \ell h(p) \leq n\}.
\]

Define a payoff function \( f' \) for \( T' \) by

\[
f'(p) = \begin{cases} 
  f(p) & \text{if } p \text{ is terminal in } T; \\
  \text{val}(\Gamma(f^p; T^p)) & \text{otherwise}.
\end{cases}
\]

By our induction hypothesis, \( \Gamma(f'; T') \) is determined in optimal strategies.

It is easy to see that one gets optimal strategies witnessing that the corollary holds for \( \Gamma(f; T) \) by combining in the obvious way optimal strategies for \( \Gamma(f^p; T^p) \) with optimal strategies for the \( \Gamma(f^p; T^p) \). \( \square \)
Corollary 2.4.3. Let $T$ be a Blackwell game tree in which all plays have length $\leq$ some fixed natural number $n$. Then all Blackwell games in $T$ are determined.

Proof. First consider the case of a $T$ in which all plays have length 1. By symmetry, we may assume that $X_\emptyset$ is finite. For any finite subset $u$ of $Y_\emptyset$, let

$$T^u = \{\emptyset\} \cup \{\langle a, b \rangle \mid a \in X_\emptyset \land b \in u\}$$

and let $f^u = f \upharpoonright [T^u]$. By the theorem, for each $u$ let $\sigma^u$ and $\tau^u$ be optimal strategies for I and II respectively for $\Gamma(f^u; T^u)$. Let

$$v = \inf_u \text{val}(\Gamma(f^u; T^u)).$$

It is easy to see, using the $\tau^u$, that $\text{val}(\Gamma(f; T)) \leq v$. To finish the proof, we show that $\text{val}_{\Gamma}(\Gamma(f; T)) \geq v$. Let $\varepsilon > 0$. We will show that $\text{val}_{\Gamma}(\Gamma(f; T)) \geq v - \varepsilon$.

Since the range of a payoff function is required to be bounded, let $s \in \mathbb{R}$ with $s > |f(x)|$ for all plays $x$ in $T$.

First we prove that there are $r_a$, $a \in X_\emptyset$, such that each $r_a \in [0, 1]$, such that $\sum_{a \in X_\emptyset} r_a = 1$, and such that for every finite $u \subseteq Y_\emptyset$ there is a finite $u' \subseteq Y_\emptyset$ such that, for all $a \in X_\emptyset$,

$$u \subseteq u' \land |(\sigma^{u'}(\emptyset))(a) - r_a| \leq \frac{\varepsilon}{2s|X_\emptyset|}.$$  

Let $X_\emptyset = \{a_1, \ldots a_k\}$. Let $0 \leq j < k$ and assume that we have $\tilde{r}_a$, for $i < j$ such that

$$(\forall u)(\exists u' \supseteq u')((\forall i < j) \quad |(\sigma^{u'}(\emptyset))(a) - \tilde{r}_a| \leq \frac{\varepsilon}{4s|X_\emptyset|}).$$

If there were no $\tilde{r}_a$, that made this hold with “$j + 1$” replacing “$j$,” then we could generate an infinite sequence of elements of $[0, 1]$ any two of which would differ by more than $\frac{\varepsilon}{4s|X_\emptyset|}$, and there can be no such sequence. Since

$$\left|1 - \sum_{a \in X_\emptyset} \tilde{r}_a\right| \leq |X_\emptyset| \frac{\varepsilon}{4s|X_\emptyset|},$$

we can get our $r_a$ by adding to or subtracting from each $\tilde{r}_a$ some number $\leq \frac{\varepsilon}{4s|X_\emptyset|}$. 

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Define a mixed strategy $\sigma$ for I in $T$ by
\[
(\sigma(\emptyset))(a) = r_a.
\]
We show that the value of $\sigma$ in $\Gamma(f;T)$ is $\geq v - \varepsilon$. Let $\tau$ be any mixed strategy for II in $T$. There is a non-empty finite subset $u$ of $Y_\emptyset$ such that
\[
\sum_{b \in u} (\tau(\emptyset))(b) < \frac{\varepsilon}{2s}.
\]
Let $u' \supseteq u$ be given by the property of the $r_a$. Let $b_0 \in u'$. Let $\tau'$ be the mixed strategy for II in $T^{u'}$ given by
\[
\tau'(b) = \begin{cases} 
\tau(b) & \text{if } b \neq b_0; \\
\tau(b) + \sum_{b \in u'} (\tau(\emptyset))(b) & \text{if } b = b_0.
\end{cases}
\]
We have that
\[
E_{\sigma,\tau'}(f^{u'}) - E_{\sigma,\tau}(f) \leq \frac{\varepsilon}{2s} + \sum_{a \in X_\emptyset} \frac{\varepsilon}{2s|X_\emptyset|^s} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Since
\[
E_{\sigma,\tau'}(f^{u'}) \geq \text{val}(\Gamma(f^{u'}; T^{u'})) \geq v,
\]
this completes the proof of the determinacy of $\Gamma(f;T)$.

The corollary can now be proved by an induction similar to the proof of Corollary 2.4.2.

Remark. For simplicity’s sake, in proving determinacy theorems we will work mainly with Blackwell game trees that have no terminal nodes and with payoff functions whose range is a subset of $[0,1]$. Arguments like those given in §2.1 for the perfect information case show that Blackwell determinacy is level-by-level equivalent to Blackwell determinacy in trees without terminal nodes. It is easy to see that restricting to payoff functions bounded to $[0,1]$ similarly makes no difference for determinacy.

Announcement. Until further notice, let $T$ be a Blackwell game tree with no terminal positions and let $f$ be a payoff function for $T$ such that $0 \leq f(x) \leq 1$ for every $x \in [T]$. 

□
For each $v \in (0, 1]$, we define a perfect information game $G_v$.

Play in the game $G_v$ is as follows:

\[
\begin{array}{cccccc}
I & h_0 & h_1 & h_2 & \ldots \\
II & p_0 & p_1 & p_2 & p_3 & \ldots \\
\end{array}
\]

Set $p_0 = \emptyset$, the starting position in $T$. For $i \geq 1$, $p_i$ must a position in $T$ of length $i$. It is required that $p_0 \subseteq p_1 \subseteq \ldots$. For each $i$, $h_i$ must be a function into $[0, 1]$ from the set of positions in $T$ that are length $i + 1$ extensions of $p_i$. Let $v_0 = v$ and for $i \geq 0$ let

\[v_{i+1} = h_i(p_{i+1}).\]

For each $i$, let $T^i$ be the Blackwell game tree in which the players start at $p_i$ and simultaneously make one move legal in $T$. I’s move $h_i$ is a payoff function for $T^i$. By Corollary 2.4.3, the game $\Gamma(h_i; T^i)$ is determined. The final requirement on $h_i$ is that

\[\text{val}(\Gamma(h_i; T^i)) \geq v_i.\]

Note that I always has a legal move that fulfills this requirement. For example, I may set $h_i(q) = 1$ for all $q$. The final requirement on $p_{i+1}$ is that

\[v_{i+1} > 0.\]

By induction on $i$, we can see that II always has a legal move that fulfills this requirement; for if $h_i(q) = 0$ for all $q$, then $v_i \leq \text{val}(\Gamma(h_i; T^i)) = 0$.

For each position $p^*$ in $G_v$, let $\pi(p^*)$ be the union of all the moves made by II in arriving at $p^*$. (If $\ell h(p^*) \leq 1$, then $\pi(p^*) = \emptyset$; otherwise $\pi(p^*)$ is the last move made by II.) For any play $x^*$ of $G_v$, let $\pi(x^*) = \bigcup_i \pi(x^* \upharpoonright i)$, i.e., let $\pi(x^*)$ be the play of $\Gamma$ extending all the $p_i$. A play $x^*$ is a win for I if and only if

\[\limsup_i v_i \leq f(\pi(x^*)).\]

One way to think of the game $G_v$ is to imagine that player I is trying to show that $\text{val}(\Gamma(f; T)) \geq v$. This account takes I to be asserting, at the point when $p_i$ has been chosen, that $\text{val}(\Gamma(f; T_{p_i})) \geq v_i$. To substantiate this assertion, I chooses the $h_i(q)$. If $\text{val}(\Gamma(f; T_q)) \geq h_i(q)$ for each $q$, then the fact that $\text{val}(\Gamma(h_i; T^i)) \geq v_i$ shows that I’s assertion is correct. Player II is therefore required to choose some $q$ as $p_{i+1}$, thus asking I to show that $\text{val}(\Gamma(f; T_q)) \geq h_i(q)$.
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Remark. The motivation just given for $G_v$ might suggest that $G_v$ is a
win for I if and only if $\text{val}_v(\Gamma(f; T)) \geq v$. But the “if” part of this statement
is not true in general, even in when both games are determined. Suppose
that the $X_p$ and $Y_p$ for $T$ are all $\{0, 1\}$. Suppose also that $f(x) = 0$ if 1
is never played or if the two players first play 1 simultaneously and that
$f(x) = 1$ otherwise. This game has value 1, but II has a winning strategy
for $G_1$. ([Vervoort, 1996] uses this $\Gamma(f; T)$ to illustrate a different, though
related, point. There are other choices for $T$ and $f$ such that I has a winning
strategy for $G_{\text{val}(\Gamma(f; T))}$. The “only if” part of the statement is true, as is
the analogous assertion about II. Both will be proved below.

Theorem 2.4.4. If I has a winning strategy for $G_v$, then $\text{val}_v(\Gamma(f; T)) \geq v$.

Proof. Suppose that $\sigma^*$ is a winning strategy for I for $G_v$. Let $\delta > 0$. We
will prove that $\text{val}(\Gamma(f; T)) \geq v - \delta$.

We simultaneously define

(i) a mixed strategy $\sigma$ for I in $T$;
(ii) the notion of an acceptable position in $T$;
(iii) for each acceptable position $p$ in $T$, a position $\psi(p)$ in $G_v$ such that
$\ell h(\psi(p)) = 2\ell h(p) + 1$, $\psi(p)$ is consistent with $\sigma^*$, and $\pi(\psi(p)) = p$.

The function $p \mapsto \psi(p)$ will satisfy the condition

$$p \subseteq q \rightarrow \psi(p) \subseteq \psi(q).$$

Thus for each play $x$ in $T$ that contains only acceptable positions, we will
have a play $\Psi(x) = \bigcup_{p \subseteq x} \psi(p)$ of $G_v$ such that $\Psi(x)$ is consistent with $\sigma^*$
and $\pi(\Psi(x)) = x$.

The starting position $\emptyset$ is acceptable. Every position extending an unac-
ceptable position is unacceptable.

For unacceptable positions $p$, define $\sigma(p)$ in an arbitrary fashion.

Let $\psi(\emptyset) = \langle h_0 \rangle$, where $h_0$ is given by $\sigma^*$.

Suppose inductively that we are given an acceptable $p$ of length $i$ and
that either (a) $i = 0$ or else (b) $i > 0$ and we have defined

$$\psi(p) = \langle h_0, \ldots, p_i, h_i \rangle,$$

a position in $G_v$ consistent with $\sigma^*$ and with $p_i = p$. Let $v_i$ and $T^i$ be the $v_i$
and the $T_i$ associated with $\psi(p)$.
For positions $q$ of length $i + 1$ that extend $p$, define $q$ to be acceptable if and only if $h_i(q) > 0$.

Because $\text{val}( \Gamma(h_i; T^i)) \geq v_i$, there is a mixed strategy for $I$ in $T^i$ whose value in $\Gamma(h_i; T^i)$ is $\geq v_i - \delta/2^{i+1}$. Let $\sigma(p)$ be the probability distribution given by such a mixed strategy. Given any acceptable $q$ of length $i + 1$ that extends $p$, set $\psi(q) = \psi(p)^\sim(p_{i+1}, h_{i+1})$, where $p_{i+1} = q$ and where $h_{i+1}$ is given by $\sigma^*$.

For acceptable positions $p$ in $T$, let $h^p$ be the last move made in reaching the position $\psi(p)$, i.e., let $h^p$ be the $h_{\ell h(p)}$ of $\psi(p)$. For acceptable $p$, also let $T^p$ be the $T^{\ell h(p)}$ of $\psi(p)$ and let $v^p$ be the $v_{\ell h(p)}$ of $\psi(p)$. For unacceptable $p$, let $v^p = 0$.

**Lemma 2.4.5.** Let $\tau$ be a mixed strategy for $II$ in $T$ and let $\mu = \mu_{\sigma, \tau}$. Let $p \in T$ with $\ell h(p) = i$. Then

$$v^p \mu([T^p]) \leq \sum_{\ell h(q) = i + 1} (v^q + \delta/2^{i+1}) \mu([T^q]).$$

**Proof of Lemma.** Since $v^p = 0$ for unacceptable $p$, we may assume that $p$ is acceptable. Because $\sigma(p)$ is the probability distribution of a mixed strategy in $T^p$ whose value in $\Gamma(h^p; T^p)$ is $\geq v^p - \delta/2^{i+1}$, we have that

$$v^p - \delta/2^{i+1} \leq \sum_{\ell h(q) = i + 1} h^p(q) \frac{\mu([T^q])}{\mu([T^p])}.$$  

Using the facts that $h^p(q) = v^q$ and that $\mu([T^p]) = \sum_q \mu([T^q])$, we get the desired inequality. $\square$

For plays $x$ in $T$, set

$$g(x) = \limsup_i v^{x_i}.$$  

Note that $g$ is Borel measurable and that range($g$) $\subseteq [0, 1]$. Note also that $g(x) \leq f(x)$ for every play $x$ in $T$. This is trivially true for those $x$ such that $g(x) = 0$. For any other $x$, $\Psi(x)$ is a play consistent with the winning strategy $\sigma^*$, and so $g(x) \leq f(\pi(\Psi(x))) = f(x)$.

**Lemma 2.4.6.** For any mixed strategy $\tau$ for $II$ in $T$, $E_{\sigma, \tau}(g) \geq v - \delta$. 

Proof of Lemma. Given $\tau$, let $\mu = \mu_{\sigma, \tau}$. Assume that $E_{\sigma, \tau}(g) < v - \delta$. Thus $\int g \, d\mu < v - \delta$. Let $\varepsilon > 0$ be such that $\int g \, d\mu < v - \delta - \varepsilon$. Then $\int (1 - g) \, d\mu > 1 - v + \delta + \varepsilon$. There is a closed set $C$ such that $g$ is continuous on $C$ and $\int_C (1 - g) \, d\mu > 1 - v + \delta + \varepsilon$. (See Kechris [1994], Theorem 17.12.)

We will define a play $x$ in $T$ such that, for all $i$, $x \upharpoonright i$ is acceptable and

$$\int_{C \cap [T_{x \upharpoonright i}]} (1 - g) \, d\mu > (1 - v^{x \upharpoonright i} + \delta/2^i + \varepsilon) \mu([T_{x \upharpoonright i}]).$$

Suppose that $x \upharpoonright i$ has been defined so that $x \upharpoonright i$ is acceptable and the inequality just stated holds. If there is an acceptable $q$ of length $i + 1$ that extends $x \upharpoonright i$ and is such that $\int_{C \cap [T_q]} (1 - g) \, d\mu > (1 - v^q + \delta/2^{i+1} + \varepsilon) \mu([T_q])$, then let $x \upharpoonright i + 1$ be such a $q$. Assume, in order to derive a contradiction, that the inequality

$$\int_{C \cap [T_q]} (1 - g) \, d\mu \leq (1 - v^q + \delta/2^{i+1} + \varepsilon) \mu([T_q])$$

holds for every acceptable $q$ of length $i + 1$ that extends $x \upharpoonright i$. This inequality holds also for unacceptable $q$, since for them $v^q = 0$. Thus

$$\int_{C \cap [T_{x \upharpoonright i}]} (1 - g) \, d\mu = \sum_q \int_{C \cap [T_q]} (1 - g) \, d\mu \leq \sum_q (1 - v^q + \delta/2^{i+1} + \varepsilon) \mu([T_q]) \leq (1 - v^{x \upharpoonright i} + \delta/2^i + \varepsilon) \mu([T_{x \upharpoonright i}]),$$

where the last inequality is by Lemma 2.4.5. This contradicts our induction hypothesis for $i$.

We next observe that for any $i$ there is a $y_i \in C \cap [T_{x \upharpoonright i}]$ such that

$$g(y_i) < v^{x \upharpoonright i} - \delta/2^i - \varepsilon.$$

Assume to the contrary that $g(y) \geq v^{x \upharpoonright i} - \delta/2^i - \varepsilon$ for every $y \in C \cap [T_{x \upharpoonright i}]$. Then

$$\int_{C \cap [T_{x \upharpoonright i}]} (1 - g) \, d\mu \leq (1 - v^{x \upharpoonright i} + \delta/2^i + \varepsilon) \mu([T_{x \upharpoonright i}]),$$

contradicting what we have just proved by induction about our play $x$. 


Since \( x = \lim_i y_i \), we have that \( x \in C \) and so that \( g(x) = \lim_i g(y_i) \). Let \( j \) be such that
\[
(\forall i \geq j) \ |g(x) - g(y_i)| < \varepsilon/2 .
\]
Then
\[
(\forall i \geq j) \ g(x) < g(y_i) + \varepsilon/2 < v^x_i - \delta/2i - \varepsilon/2 .
\]
Therefore
\[
g(x) \leq \limsup_i v^x_i - \varepsilon/2 = g(x) - \varepsilon/2 .
\]
This contradiction completes the proof of the lemma. \( \square \)

**Lemma 2.4.7.** The value of \( \sigma \) in \( \Gamma(f;T) \) is \( \geq v - \delta \).

**Proof of Lemma.** By the fact that \( g(x) \leq f(x) \) for all \( x \), the value of \( \sigma \) in \( \Gamma(f;T) \) is \( \geq \) the value of \( \sigma \) in \( \Gamma(g;T) \), which is \( \geq v - \delta \) by Lemma 2.4.6. \( \square \)

Since \( \delta \) was an arbitrary positive real number, the theorem is proved. \( \square \)

**Theorem 2.4.8.** If \( \Pi \) has a winning strategy for \( G_v \), then \( \text{val}_v(\Gamma(f;T)) \leq v \).

**Proof.** Suppose that \( \tau^* \) is a winning strategy for \( \Pi \) for \( G_v \). Let \( \delta > 0 \). We will prove that \( \text{val}(\Gamma(f;T)) \leq v + \delta \).

We simultaneously define

(i) a mixed strategy \( \tau \) for \( \Pi \) in \( T \);

(ii) the notion of an acceptable position in \( T \);

(iii) for each acceptable position \( p \) in \( T \), a function \( u_p \) into \([0, 1]\) from the set of all \( q \) extending \( p \) such that \( \ell h(q) = \ell h(p) + 1 \);

(iv) for each acceptable position \( p \) in \( T \), a position \( \psi(p) \) in \( G_v \) such that
\[
\ell h(\psi(p)) = 2\ell h(p) , \quad \psi(p) \text{ is consistent with } \tau^* , \text{ and } \pi(\psi(p)) = p .
\]

The function \( p \mapsto \psi(p) \) will satisfy the condition
\[
p \subseteq q \Rightarrow \psi(p) \subseteq \psi(q) .
\]
Thus for each play \( x \) in \( T \) that contains only acceptable positions, we will have a play \( \Psi(x) = \bigcup_{p \leq x} \psi(p) \) of \( G_v \) such that \( \Psi(x) \) is consistent with \( \tau^* \) and \( \pi(\Psi(x)) = x \).

The starting position \( \emptyset \) is acceptable. Every position extending an unacceptable position is unacceptable.
For unacceptable positions \(p\), define \(\tau(p)\) in an arbitrary fashion.

Let \(\psi(\emptyset) = \emptyset\).

Suppose inductively that we are given an acceptable \(p\) of length \(i\) and that either (a) \(i = 0\) or else (b) \(i > 0\) and we have defined

\[
\psi(p) = \langle h_0, \ldots , p_i \rangle ,
\]
a position in \(G_v\) consistent with \(\tau^*\) and with \(p_i = p\). Let \(v_i\) and \(T^i\) be the \(v_i\) and the \(T^i\) associated with \(\psi(p)\).

For positions \(q\) of length \(i + 1\) that extend \(p\), define \(q\) to be acceptable if and only if there is a legal move \(h\) for I in \(G_v\) at \(\psi(p)\) such that \(\tau^*(\psi(p) - \langle h\rangle) = q\).

For acceptable \(q\), set

\[
u_p(q) = \inf \{ h(q) \mid h \text{ is legal in } G_v \text{ at } \psi(p) \land \tau^*(\psi(p) - \langle h\rangle) = q \} .
\]

For unacceptable \(q\), set \(u_p(q) = 1\).

**Lemma 2.4.9.** \(\text{val}(\Gamma(u_p; T^i)) \leq v_i\).

**Proof of Lemma.** Assume that \(\text{val}(\Gamma(u_p; T^i)) > v_i\). Let \(\varepsilon > 0\) be such that \(\text{val}(\Gamma(u_p; T^i)) \geq v_i + \varepsilon\). Define a function \(h\), with the same domain as \(u_p\), by

\[
h(q) = \begin{cases} 
u_p(q) - \varepsilon & \text{if } \nu_p(q) > \varepsilon; \\ 0 & \text{if } \nu_p(q) \leq \varepsilon. \end{cases}
\]

Then \(\text{val}(\Gamma(h; T^i)) \geq \text{val}(\Gamma(u_p; T^i)) - \varepsilon \geq v_i\), and therefore \(h\) is a is a legal move for I at the position \(\psi(p)\). Hence there is some \(q\) such that \(\tau^*(\psi(p) - \langle h\rangle) = q\). If \(u_p(q) \leq \varepsilon\) then \(h(q) = 0\), and so \(q\) is not a legal move. If \(u_p(q) > \varepsilon\) then \(h(q) < u_p(q)\), and this contradicts the definition of \(u_p(q)\).

Let \(\tau(p)\) be the probability distribution given by some mixed strategy for II in \(T^i\) whose value in \(\Gamma(u_p; T^i)\) is \(\leq v_i + \delta/2^{i+2}\).

For each acceptable \(q\) of length \(i + 1\) and extending \(p\), we define \(\psi(q)\) as follows. Pick a legal move \(h_i\) for I at \(\psi(p)\) such that \(h_i(q) \leq u_p(q) + \delta/2^{i+2}\) and such that \(\tau^*(\psi(p) - \langle h_i\rangle) = q\). Set \(\psi(q) = \psi(p) - \langle h_i, p_{i+1} \rangle\), where \(p_{i+1} = q\).

For acceptable positions \(p\) in \(T\) with \(h\ell(p) > 0\), let \(h^p\) be the next to last move made in reaching the position \(\psi(p)\), i.e., let \(h^p\) be the \(h_{\ell(h(p)) - 1}\) of \(\psi(p)\). For acceptable \(p \in T\) of any length, let \(T^p\) be the \(T^{\ell(h(p))}\) of \(\psi(p)\) and let \(v^p\) be the \(v_{\ell(h(p))}\) of \(\psi(p)\). If \(p\) is unacceptable, set \(v^p = 1\).
Lemma 2.4.10. Let $\sigma$ be a mixed strategy for $I$ in $T$ and let $\mu = \mu_{\sigma, \tau}$. Let $p \in T$ with $\ell h(p) = i$. Then

$$v^p \mu([T_p]) \geq \sum_{p \leq q \atop \ell h(q) = i + 1} (v^q - \delta/2^{i+1}) \mu([T_q]).$$

Proof of Lemma. Since $v^p = 1$ for unacceptable $p$, we may assume that $p$ is acceptable. Because $\tau(p)$ is the probability distribution of a mixed strategy in $T^p$ whose value in $\Gamma(u_p; T^p)$ is $\leq v^p + \delta/2^{i+2}$, we have that

$$v^p + \delta/2^{i+2} \geq \sum_{p \leq q \atop \ell h(q) = i + 1} u_p(q) \frac{\mu([T_q])}{\mu([T_p])}.$$

Since $\mu([T_p]) = \sum_q \mu([T_q])$, we get that

$$v^p \mu([T_p]) \geq \sum_{p \leq q \atop \ell h(q) = i + 1} (u_p(q) - \delta/2^{i+2}) \mu([T_q]).$$

For acceptable $q$, $v^q = h^q(q) \leq u_p(q) + \delta/2^{i+2}$. For unacceptable $q$, $v^q = 1 = u_p(q)$. In either case, $v^q - \delta/2^{i+2} \leq u_p(q)$, and this gives us the desired inequality. $\square$

For plays $x$ in $T$, set

$$g(x) = \limsup_i v^{x^i}.$$

Note that $g$ is Borel measurable and that range($g$) $\subseteq [0,1]$. Note also that $g(x) \geq f(x)$ for every play $x$ in $T$. This is trivially true for those $x$ such that $g(x) = 1$. For any other $x$, $\Psi(x)$ is a play consistent with the winning strategy $\tau^*$, and so $g(x) \leq f(\pi(\Psi(x))) = f(x)$.

Lemma 2.4.11. For any strategy $\sigma$ for $I$ for $\Gamma$,

$$E_{\sigma, \tau}(g) \leq v + \delta.$$

Proof of Lemma. Given $\sigma$, let $\mu = \mu_{\sigma, \tau}$. Assume that $E_{\sigma, \tau}(g) > v + \delta$. Let $\varepsilon > 0$ be such that $E_{\sigma, \tau}(g) > v + \delta + \varepsilon$. Then $\int g \, d\mu > v + \delta + \varepsilon$. Let $C$ be a closed set such that $g$ is continuous on $C$ and such that $\int_C g \, d\mu > v + \delta + \varepsilon$. 

\[\int_C g \, d\mu > v + \delta + \varepsilon.\]
We will define a play $x$ in $T$ such that, for all $i$, $x \upharpoonright i$ is acceptable and
\[ \int_{C \cap [T_x]} g \, d\mu > (v^{x\upharpoonright i} + \delta/2^i + \varepsilon) \mu([T_{x \upharpoonright i}]). \]
Suppose that $x \upharpoonright i$ has been defined so that $x \upharpoonright i$ is acceptable and the inequality just stated holds. If there is an acceptable $q$ such that $\int_{C \cap [T_q]} g \, d\mu > (v^q + \delta/2^{i+1} + \varepsilon) \mu([T_q])$, then let $x \upharpoonright i + 1$ be such a $q$. If, for every acceptable $q$,
\[ \int_{C \cap [T_q]} g \, d\mu \leq (v^q + \delta/2^{i+1} + \varepsilon) \mu([T_q]), \]
then
\[ \int_{C \cap [T_{x \upharpoonright i}]} g \, d\mu = \sum_q \int_{C \cap [T_q]} g \, d\mu \]
\[ \leq \sum_q (v^q + \delta/2^i + \varepsilon) \mu([T_q]) \]
\[ \leq (v^{x\upharpoonright i} + \delta/2^i + \varepsilon) \mu([T_{x \upharpoonright i}]), \]
where the last inequality is by Lemma 2.4.10. This contradicts our induction hypothesis for $i$.

We next observe that for any $i$ there is a $y_i \in C \cap [T_{x \upharpoonright i}]$ such that
\[ g(y_i) > v^{x\upharpoonright i} + \delta/2^i + \varepsilon. \]
Assume to the contrary that $g(y) \leq v^{x\upharpoonright i} + \delta/2^i + \varepsilon$ for every $y \in C \cap [T_{x \upharpoonright i}]$. Then
\[ \int_{C \cap [T_{x \upharpoonright i}]} g \, d\mu \leq (v^{x\upharpoonright i} + \delta/2^i + \varepsilon) \mu([T_{x \upharpoonright i}]), \]
contradicting what we have just proved by induction about our play $x$.

Since $x = \lim_i y_i$, we have that $x \in C$ and so that $g(x) = \lim_i g(y_i)$. Let $j$ be such that
\[ (\forall i \geq j) \, |g(x) - g(y_i)| < \varepsilon/2. \]
Then
\[ (\forall i \geq j) \, g(x) > g(y_i) - \varepsilon/2 > v^{x\upharpoonright i} + \delta/2^i + \varepsilon/2. \]
Therefore
\[ g(x) \geq \limsup_i v^{x\upharpoonright i} + \varepsilon/2 = g(x) + \varepsilon/2. \]
This contradiction completes the proof of the lemma. \qed
Lemma 2.4.12. The value of $\tau$ in $\Gamma(f; T)$ is $\leq v + \delta$.

Proof of Lemma. By the fact that $g(x) \geq f(x)$ for all $x$, the value of $\tau$ in $\Gamma(f; T)$ is $\leq$ the value of $\tau$ in $\Gamma(g; T)$, which is $\leq v + \delta$ by Lemma 2.4.11. $\square$

Since $\delta$ was an arbitrary positive real number, the theorem is proved. $\square$

Theorem 2.4.13. If $G_v$ is determined for every $v \in (0, 1]$, then $\Gamma(f; T)$ is determined.

Proof. Assume that all the $G_v$ are determined. Let $w$ be the least upper bound of all the numbers $v$ such that I has a winning strategy for $G_v$. By Lemma 2.4.4, $\text{val}_I(\Gamma(f; T)) \geq w$. By Lemma 2.4.8, $\text{val}^I(\Gamma(f; T)) \leq w$. Thus $\text{val}(\Gamma(f; T)) = w$. $\square$

Since the games $G_v$ are of the form $G(A^*; T^*)$, Theorem 2.4.13 allows us to show that Blackwell determinacy for any given class follows from ordinary determinacy for a related class. But Theorem 2.4.13 does not yield optimal results when $|T| < 2^{\omega_0}$. This is because the tree $T^*$ for $G_v$ has size $\geq 2^{\omega_0}$ for all non-trivial trees $T$. This problem is easily remedied, however, as we now explain.

For $v \in (0, 1]$, let $\bar{G}_v$ differ from $G_v$ only in an additional requirement that all values $h_i(q)$ be rational. It is easy to check that our proofs go through unchanged for $\bar{G}_v$ in place of $G_v$. We state this formally as the following theorem.

Theorem 2.4.14. If $\bar{G}_v$ is determined for every real (indeed, for every rational) $v \in (0, 1]$, then $\Gamma(f; T)$ is determined.

If $f$ is the characteristic function of a set, then there is a modification of the games $G_v$ that gives our results with simpler proofs.

Announcement. Until further notice, let $A$ be a subset of $[T]$.

For $v \in (0, 1]$ let $G'_v$ be played exactly as is $G_v$, but let a play $x^*$ of $G'_v$ be a win for I if and only if $\pi(x^*) \in A$.

Theorem 2.4.15. If I has a winning strategy for $G'_v$, then $\text{val}_I(\Gamma(\chi(A); T)) \geq v$. 


Suppose that \( \sigma^* \) is a winning strategy for I for \( G_v \). Let \( \delta > 0 \). We will prove that \( \text{val}(\Gamma(\chi(f); T)) \geq v - \delta \). Define \( \sigma \), acceptable positions, and \( \psi \), exactly as in the proof of Theorem 2.4.4. Define \( h^p \), \( T^p \), and \( v^p \) as before.

Lemma 2.4.5 holds as before.

Lemma 2.4.16. Let \( \tau \) be a mixed strategy for II in \( T \) and let \( \mu = \mu_{\sigma,\tau} \). For each \( i \in \omega \),

\[
v \leq \sum_{\ell(h(p)) = i} (v^p + \delta(1 - 1/2^i))\mu([T^p_i]).
\]

Let \( C_1 \) be the closed set of all plays of \( \Gamma \) that contain only acceptable positions. Since \( x = \pi(\psi(x)) \) for \( x \in C_1 \), \( C_1 \subseteq A \).

Lemma 2.4.17. For any strategy \( \tau \) for II for \( \Gamma \), \( \mu_{\sigma,\tau}(C_1) \geq v \).

Proof. Given \( \tau \), assume that \( \mu_{\sigma,\tau}(C_1) < v \). It follows that there is a closed set \( C \) disjoint from \( C_1 \) such that \( \mu_{\sigma,\tau}(C) > 1 - v \). By a construction like that in the proof of Lemma 2.4.6, there is an \( x \in C_1 \) such that, for all \( i \), \( \mu_{\sigma,\tau}(C \cap [T_{x|i}]) > 1 - v^x_i \). But this is a contradiction, for such an \( x \) must belong to \( C_1 \cap C \). \( \square \)

Lemma 2.4.18. The value of \( \sigma \) in \( \Gamma(\chi(A)) \) is \( \geq v \).

Proof. The lemma follows from Lemma 2.4.17.

Here is a direct proof of the lemma. For each \( i \), consider the game \( \Gamma^i \) which is played in the same way as \( \Gamma \) except that play terminates when the position \( p \) has length \( i \). For plays \( p \) of \( \Gamma^i \), let

\[
h^i(p) = \begin{cases} v^p_i & \text{if } p \text{ is acceptable;} \\ 0 & \text{otherwise.} \end{cases}
\]

It is easy to prove by induction on \( i \) that the value of the appropriate fragment \( \sigma^i \) of \( \sigma \) in \( \Gamma^i(h^i) \) is \( \geq v \). Thus the value of \( \sigma^i \) in \( \Gamma^i(\chi(C_1^i)) \) is \( \geq v \), where \( C_1^i \) is the set of all acceptable plays of \( \Gamma^i \). Thus the value of \( \sigma \) in \( \Gamma(\chi(C_1)) \) is \( \geq v \). \( \square \)

Dropping our assumption about \( \sigma^* \), we get the following.

Lemma 2.4.19. If I has a winning strategy for \( G_v' \), then \( \text{val}_I(\Gamma(\chi(A))) \geq v \).
2.4. BLACKWELL GAMES

Now assume that $\tau^*$ is a winning strategy for II for $G'_{v}$. Let $\delta > 0$. Define $\tau$, acceptable positions, and $\psi$, exactly as in §1.

Let $C_2$ be the closed set of all plays of $\Gamma$ that contain only acceptable positions. Since $x = \pi(\psi(x))$ for $x \in C_2$, $C_2 \cap A = \emptyset$.

Lemma 2.4.20. For any strategy $\sigma$ for I for $\Gamma$, $\mu_{\sigma,\tau}(C_2) \geq 1 - v - \delta$.

Lemma 2.4.21. The value of $\tau$ in $\Gamma(\chi(A))$ is $\leq v + \delta$.

Proof. The lemma follows from Lemma 2.4.20. There is also a direct proof of the lemma, analogous to the direct proof of Lemma 2.4.18. □

Dropping our assumption about $\tau^*$, we get the following.

Lemma 2.4.22. If II has a winning strategy for $G'_{v}$, then $\text{val}^\uparrow(\Gamma(\chi(A))) \leq v$.

Theorem 2.4.23. If $G'_{v}$ is determined for every $v \in (0, 1]$, then $\Gamma(\chi(A))$ is determined.

We next indicate how to extend our results to stochastic games. In doing so we are reporting an observation of Maitra and Sudderth.

Stochastic games $\tilde{\Gamma}$ are played like Blackwell games, except that each pair of moves of I and II is followed by a move of a third player, whom we will call Nature. We will restrict ourselves to the case that Nature has a countable set of legal moves in each position in which she must make a move. Payoff functions for $\tilde{\Gamma}$ are functions of the entire play, including Nature’s moves. The analogue of $\Gamma(f)$ is $\tilde{\Gamma}(\tilde{f}, \rho)$ where $\tilde{f}$ is a payoff function and $\rho$ is a mixed strategy for Nature. If $\sigma$ and $\tau$ are strategies for I and II respectively, then $\sigma$, $\tau$, and $\rho$ give a probability measure $\mu_{\sigma,\tau,\rho}$ on the set of plays of $\tilde{\Gamma}$.

Using this measure, we define $E_{\sigma,\tau,\rho}(\tilde{f})$, $E_{\sigma,\tau,\rho}^-(\tilde{f})$, $E_{\sigma,\tau,\rho}^+(\tilde{f})$, $\text{val}_i(\tilde{\Gamma}(\tilde{f}, \rho))$, $\text{val}^\uparrow(\tilde{\Gamma}(\tilde{f}, \rho))$, determinacy of $\tilde{\Gamma}(\tilde{f}, \rho)$, and the value of $\tilde{\Gamma}(\tilde{f}, \rho)$ in the obvious way.

Fix $\tilde{\Gamma}$ with no terminal positions. Fix $\tilde{f}$ and $\rho$. For $v \in (0, 1]$, let $\tilde{G}_v$ be the perfect information game played as follows. Set $p_0 = \emptyset$. I’s moves are functions $h_0, h_1, \ldots$ and II’s moves are positions $p_1, p_2, \ldots$ in $\tilde{\Gamma}$. For each $i$, $h_i$ is a function into $[0, 1]$ from the set of all length $2i + 2$ extensions of $p_i$, which has length $2i$. Let $v_0 = v$ and for $i \geq 0$ let $v_{i+1} = h_i(p_{i+1})$. For each $i$,
let \( \hat{\Delta}_i \) be the game in which, starting at \( p_i \), the two players and then Nature make legal moves in \( \hat{\Gamma} \). The final requirement on \( h_i \) is that

\[
\text{val}(\hat{\Delta}_i(h_i, \rho_{p_i})) \geq v_i,
\]

where \( \rho_{p_i} \) is the strategy for Nature for \( \hat{\Delta}_i \) that is given by \( \rho \). The final requirement on \( p_{i+1} \) is that \( v_{i+1} > 0 \).

For positions \( p^* \) in \( \hat{G}_v \), define \( \pi(p^*) \), a position in \( \hat{G} \) of length \( 2\ell(h(p^*)) \), in the obvious way. Also call \( \pi \) the function induced by \( \pi \) from plays of \( \hat{G}_v \) to plays of \( \hat{\Gamma} \). A play \( x^* \) is a win for I if and only if \( \limsup_i v_i \leq \tilde{f}(x^*) \).

The constructions, lemmas, and proofs of the earlier part of this section adapt in obvious ways to \( \hat{G}_v \) and \( \hat{\Gamma}(\tilde{f}, \rho) \). (The first draft of our paper had a slightly different definition of the function \( h \) on page 114. Maitra and Sudderth remarked that the current definition, unlike the original one, would work for stochastic games.) Thus we get the following theorem.

**Theorem 2.4.24.** If \( \hat{G}_v \) is determined for every \( v \in (0, 1] \), then \( \hat{\Gamma}(\tilde{f}, \rho) \) is determined.

For more details, see [Maitra and Sudderth, 1993]. There Maitra and Sudderth adapt our proof to demonstrate the determinacy of a wider class of stochastic games. They work in the context of finitely additive probability measures, removing the restrictions that I and II choose their moves from finite sets and that Nature's moves are chosen from countable sets.

The proof of Theorem 2.4.23 gives the following stronger result.

**Theorem 2.4.25.** Assume that all \( G'_v \) are determined. Then

\[
\text{val}(\Gamma(\chi(A))) = \sup\{\text{val}(\Gamma(\chi(C))) \mid C \text{ closed and } C \subseteq A\}.
\]

**Proof.** Let \( v < \text{val}(\Gamma(\chi(A))) \). Let \( \sigma^* \) be a winning strategy for I for \( G'_v \). Let \( \sigma \) be the strategy defined from \( \sigma^* \) as above. Let \( C \) be the set \( C_1 \) defined just before the statement of Lemma 2.4.17. The proofs of Lemma 2.4.18 both show that the value of \( \sigma \) in \( \Gamma(\chi(C)) \) is \( \geq v \). \( \Box \)

**Remarks:**

(a) For \( \mathbf{F}_{\sigma \delta} \) sets \( A \), Vervoort in [Vervoort, 1996] directly proves a strengthening of the conclusion of Theorem 2.4.25. He conjectures that the conclusion of Theorem 2.4.25 holds for all Borel sets \( A \). Since the hypothesis of Theorem 2.4.25 holds for Borel \( A \), his conjecture is confirmed.
(b) Applying Theorem 2.4.25 to the complement of $A$, we see that the theorem’s hypothesis implies that

$$\text{val}(\Gamma(\chi(A))) = \inf \{\text{val}(\Gamma(\chi(B))) \mid B \text{ open and } B \supseteq A\}.$$ 

One can also get this conclusion directly from the proofs of Lemma 2.4.21.

(d) For Borel sets $A$, Maitra, Purves, and Sudderth [Maitra et al., 1991] show that the determinacy of $\Gamma(\chi(A))$ implies the conclusion of Theorem 2.4.25. As mentioned in (a) above, their result follows \textit{a fortiori} from Theorem 2.4.25 and the determinacy of all Borel perfect information games. It is not true that for arbitrary $A$ the determinacy of $\Gamma(\chi(A))$ implies the conclusion of Theorem 2.4.25. For a counterexample, see page 126 below. The last paragraph of the paper also discusses issues related to the theorem of [Maitra et al., 1991].

Let $\tilde{G}'_v$ be like $G'_v$ except that all $h_i(q)$ must be rational.

**Theorem 2.4.26.** If $\tilde{G}'_v$ is determined for every rational $v \in (0, 1]$, then $\Gamma(f)$ is determined.

**Theorem 2.4.27.** Assume that $\tilde{G}'_v$ is determined for every rational $v \in (0, 1]$. Then

$$\text{val}(\Gamma(\chi(A))) = \sup \{\text{val}(\Gamma(\chi(C))) \mid C \text{ closed } \land C \subseteq A\}.$$ 

Combining the proof of Theorem 2.4.23 with a proof of Vervoort [Vervoort, 1996] that Blackwell determinacy implies that all sets are Lebesgue measurable, one gets [Martin, 2003] on eliminating the Blackwell game, a new way to deduce Lebesgue measurability from the determinacy of perfect information games.

For convenenience, we think of Lebesgue measure as being the coin-flipping measure on $2^\mathbb{N}$.

Until the end of the statement of Theorem 2.4.29, let $B \subseteq 2^\mathbb{N}$.

Let $H_v$ be played as follows:

```
I  h_0  h_1  h_2  ...
II  p_1  p_2  p_3  ...
```

Set $p_0 = \emptyset$. For $i \geq 1$, $p_i$ must a sequence of 0’s and 1’s of length $i$. It is required that $p_0 \subseteq p_1 \subseteq \ldots$. For each $i$, $h_i$ must be a function into $[0, 1]$ from $\{p_i \preceq 0, p_i \preceq 1\}$. Let $v_0 = v$ and for $i \geq 0$ let

$$v_{i+1} = h_i(p_{i+1}).$$
The final requirement on $h_i$ is that

$$\frac{1}{2}h_i(p_i\overset{0}{\rightarrow}) + \frac{1}{2}h_i(p_i\overset{1}{\rightarrow}) \geq v_i.$$ 

The final requirement on $p_{i+1}$ is that $v_{i+1} > 0$.

For any play $x^*$ of $H_v$, let $\pi(x^*)$ be the member of $2^\mathbb{N}$ extending all the $p_i$. The play $x^*$ is a win for I if and only if $\pi(x^*) \in B$.

**Theorem 2.4.28.** If $H_v$ is determined for every $v$, then $B$ is Lebesgue measurable.

**Proof.** Analogues of Lemmas 2.4.19 and 2.4.22 give that the inner measure of $B$ is $\geq v$ if I has an winning strategy for $H_v$ and that the outer measure of $B$ is $\leq v$ if II has a winning strategy for $H_v$. \(\square\)

Let $\bar{H}_v$ be like $H_v$ except that all $h_i(q)$ must be rational.

**Theorem 2.4.29.** If $\bar{H}_v$ is determined for every rational $v \in (0, 1]$, then $B$ is Lebesgue measurable.

Our definition of Blackwell games requires that each player has only finitely many legal moves in each position. We could relax this requirement by demanding only that, in each position, each player has only countably many legal moves and at least one of the players has only finitely many legal moves. All our determinacy proofs would still work for this more general concept. Some proofs would change in a very minor way, because 1-move games would no longer have optimal strategies. Of course, one could generalize further by allowing positions in which one or the other player makes a move alone, from a countable set of possibilities.

We have thus far dealt only with Blackwell games $\Gamma(f)$ such that all plays of $\Gamma$ are infinite and such that the range of $f$ is a subset of $[0, 1]$ (though we made no real use of the former hypothesis). It is clear that our proofs work with only trivial modifications for general Blackwell games. We will therefore cite the theorems we have proved as if they were their generalizations.

**Theorem 2.4.30.** All Borel Blackwell games are determined.

**Proof.** For Borel measurable $f$, the games $G_v$ and $\bar{G}_v$ have Borel payoff sets. By [Martin, 1975] or [Martin, 1985], Borel games of perfect information are determined. \(\square\)
Theorem 2.4.31. All Borel stochastic games (of the kind above) are determined.

Proof. This follows from Theorem 2.4.24 and Borel perfect information determinacy. □

As we said earlier, it was Maitra and Sudderth who noticed that our methods yield Theorem 2.4.31, and in [Maitra and Sudderth, 1993] they prove a more general version of it.

Borel perfect information determinacy for the case of countable game trees can be stated in, for example, formal second order number theory. The same is true of Borel Blackwell determinacy. Results of Friedman [Friedman, 1971] show, in a technical sense, that Borel perfect information determinacy cannot be proved without appealing to uncountably many uncountable cardinal numbers. Indeed, for every new level of the Borel hierarchy beyond the third level, a new cardinal number is needed. Thus it is of interest that the proof of Theorem 2.4.30 goes through in second order number theory and that the proof is “local,” i.e., Blackwell determinacy for a given Borel level needs only perfect information Borel determinacy for the same level.

Theorem 2.4.32. Work in formal second order number theory. Let \( \alpha \) be a countable ordinal. Assume that, for countable game trees, every \( \Pi^0_\alpha \) perfect information game is determined. Then every \( \Pi^0_\alpha \) Blackwell game is determined. (For what is we mean by a “\( \Pi^0_\alpha \) Blackwell game,” see page 105.)

Proof. If \( \Gamma(f) \) is \( \Pi^0_\alpha \), then the games \( \bar{G}_v \) are \( \Pi^0_\alpha \) as long as \( \alpha > 2 \). □

Remarks:
(a) For all \( \alpha \geq 1 \), the games \( G'_v \) and \( \bar{G}'_v \) are \( \Pi^0_\alpha \) if the set \( A \) is \( \Pi^0_\alpha \).
(b) Theorem 2.4.32 holds for the stochastic games defined in §1, since the proof of Theorem 2.4.31 is local in the same way as the proof of Theorem 2.4.30.

Going beyond the Borel sets, we can derive from the results of §1 and §2 local results for pretty much any natural classes. Here are just two examples. Projective perfect information determinacy for countable game trees implies projective Blackwell determinacy. For each positive integer \( n \), \( \Sigma^1_n \) perfect information determinacy implies \( \Sigma^1_n \) Blackwell determinacy.
As we have already said, the determinacy of many classes of perfect information games has been deduced from so-called large cardinal axioms. With the aid of our theorem, we get corresponding determinacy results for Blackwell games. For example, for all $n \geq 0$, $\Sigma^1_{n+1}$ Blackwell determinacy follows from the existence of $n$ Woodin cardinals with a measurable cardinal above them.

Vervoort in [Vervoort, 1996] introduces the Axiom of Determinacy for Blackwell Games (AD-Bl), the assertion that all Blackwell Games are determined. He shows that AD-Bl, like AD, contradicts the Axiom of Choice. He deduces from AD-Bl an important known consequence of AD: that all sets of reals are Lebesgue measurable.

Itay Neeman pointed out to us that there are several variants of AD-Bl that are not obviously equivalent to one another. One could restrict to games of the form $\Gamma(\chi(A))$. Whether or not one did this, one could require that each player has exactly 2 legal moves in each position, or one could replace 2 by some other number $n$. In the opposite direction, one could allow that in each position one of the players has countably infinitely many legal moves. We know of no simple argument that any two of the possible versions of AD-Bl are equivalent. Nonetheless, it can be shown that they are all equivalent. The games $\bar{G}_v$ of §2 can easily be turned into equivalent games in which only two legal moves are available to each player in each position. Our proofs adapt to show that the mixed strategy determinacy of these games is enough to yield the determinacy of the given game $\Gamma(f)$.

[Vervoort, 1996] asks whether either of AD and AD-Bl implies the other. Our results obviously give an implication in one direction.

Theorem 2.4.33. Work in ZF without the Axiom of Choice. AD implies AD-Bl.

What about the converse? Also, do forms of Blackwell determinacy consistent with the Axiom of Choice imply the corresponding forms of perfect information determinacy?

The main results on these questions are in [Martin et al., 2003]. Examples of pointclasses $\Gamma$ for which this paper proves that Blackwell determinacy implies determinacy are $\Delta^1_{2n}$ for $n \in \omega$, the class of projective sets, the class of sets in $L(\mathbb{R})$. The fact that Blackwell determinacy for sets in $L(\mathbb{R})$ implies determinacy for sets in $L(\mathbb{R})$ implies that AD is consistent if AD-Bl is consistent.
2.4. BLACKWELL GAMES

Here is a rough sketch of how such theorems are proved. By a \textit{perfect information game}, let us mean a game in $<\omega 2$ played as in all sections of the book prior to the present one. To show that $\Gamma$ Blackwell determinacy implies $\Gamma$ determinacy, it is enough to show that if all perfect information $\Gamma$ games are determined in the sense of mixed strategies, then all perfect information $\Gamma$ games are determined in the sense of pure strategies. A theorem of Vervoort shows that every perfect information game determined in mixed strategies has value 0 or 1 and is determined in optimal strategies. In other words, either player I has a strategy whose value is 1 or player II has a strategy whose value is 0. In §6E of [Moschovakis, 1980], a method is introduced for using perfect information determinacy to get optimal (in a different sense from the present one) pure winning strategies for $\Gamma$ games for certain pointclasses $\Gamma$. It turns out that the method applies when the given determinacy is not in pure strategies but just in mixed strategies, provided that the games have value 0 or 1. Even though the input is weakened to such mixed strategies, the output is still pure strategies. This yields Lemma 4.1 of [Martin et al., 2003]:

\textit{Let $\Gamma$ be a weakly scaled adequate pointclass. Let $\Delta$ be the intersection of $\Gamma$ and its dual. Then $\Delta$ perfect information Blackwell determinacy implies $\Delta$ determinacy.}

Here \textit{adequate} means closed under recursive substitutions, $\lor$, $\land$, and bounded number quantification, and $\Gamma$ is \textit{weakly scaled} if every set in $\Gamma$ has a scale such that each of the norms is a $\Gamma$ norm. The mentioned results for the projective hierarchy are proved using by bootstrapping, using Lemma 4.1 and Moschovakis’ method for getting scales for $\Sigma^1_{2n+2}$ from $\Delta^1_{2n}$ determinacy. The case of $L(\mathbb{R})$ is handled using a result of [Kechris and Woodin, 1983].

Here is a more indirect method that sometimes works for getting determinacy from Blackwell determinacy. Many of the proofs of consequences of perfect information determinacy still work if the existence of mixed strategies replaces that of pure strategies. Among the consequences of perfect information determinacy is the existence of good inner models for various large cardinal axioms. Many of the proofs of perfect information determinacy from large cardinal axioms need as hypotheses only the existence of good inner models of the large cardinal axioms. In this way one often gets the equivalence of forms of determinacy and the existence of good inner models of large cardinal axioms. These facts provide a method for proving perfect information determinacy from Blackwell determinacy. A sample theorem that can
be proved in this way is the following. \( \Sigma^1_1 \) Blackwell determinacy—even just for games of the form \( \Gamma(\chi(A)) \)—implies \( \Sigma^1_1 \) perfect information determinacy.

What about trying to show directly that Blackwell determinacy implies determinacy? The most direct way to proceed would be to show that any countable-tree perfect information game that is determined in the sense of mixed strategies is determined in the sense of pure strategies. Unfortunately, this is false, as the following example of Greg Hjorth shows. Let \( A \) be any uncountable subset of \( 2^\mathbb{N} \) such that \( \mu(A) = 0 \) for every atomless Borel probability measure \( \mu \). (For example, let the members of \( A \) code wellorderings, exactly one of the order type of each countable ordinal.) Consider the game \( G^*(A) \) defined on page 149 of [Kechris, 1994]. Player II has a mixed strategy whose value in \( G^*(A) \) is 0: in each position, assign 1/2 to each of the two legal moves. But II has no winning pure strategy. (See part (ii) of Theorem 21.1 of [Kechris, 1994].) This counterexample does not destroy all branches of the direct route. For example, Vervoort’s theorem lets one assume that all sets are Lebesgue measurable, and this rules out counterexamples of the kind described in parentheses above. Moreover, although mixed strategy determinacy for a perfect information game does not imply pure strategy determinacy, there are useful constraints on the values of perfect information games. We have been able to prove that the upper or lower value (in the mixed strategy sense) of a perfect information game in our sense (i.e., one whose winning condition is given by a set of plays) is either 0 or 1.

Theorems 2.4.25 and 2.4.27 and the related result for general payoff functions give a strong version of Blackwell determinacy. It is easy to see that this strong version implies, level by level, perfect information determinacy. Thus another route to our goal would be to show that Blackwell determinacy implies strong Blackwell determinacy. As we mentioned earlier, Maitra, Purves, and Sudderth [Maitra et al., 1991] have shown that, for Borel \( A \), the determinacy of \( \Gamma(\chi(A)) \) implies the strong determinacy of \( \Gamma(\chi(A)) \). Hjorth’s example given in the preceding paragraph shows that, under Choice, this is not true for arbitrary \( A \). Nevertheless, their proof may still be relevant. That proof uses the fact that Borel sets are universally measurable. The proof of the Lebesgue measurability result of [Vervoort, 1996] shows that the universal measurability of a set follows from the Blackwell determinacy of sets of about the same complexity. The additional fact about Borel sets used in the proof of [Maitra et al., 1991] is their universal capacitability. This does not generalize to all sets under AD-Bl, for there exist even \( \Pi^1_1 \) sets that are not universally capacitatable. But to prove that AD-Bl implies strong
AD-Bl it would be enough to prove from AD-Bl that all sets are capacitable with respect to the capacities of [Maitra et al., 1991]. See Section 30 and Exercises 36.22 and 39.14 of [Kechris, 1994] for some of the capacitability consequences of perfect information determinacy, consequences that are due independently to Busch, Shohat, and Mycielski.

**Exercise 2.4.1.** Let $T$ be the game tree in which every play has length 1 and for which $X_0 = Y_0 = \omega$. Blackwell games in $T$ are thus played by each player’s choosing a natural number. Let $A$ be the set of plays in $T$ such that I’s number is $\geq$ II’s number. Prove that $\text{val}_I(\Gamma(A;T)) = 0$ and $\text{val}^{\uparrow}(\Gamma(A;T)) = 1$.

**Exercise 2.4.2.** Verify that the proofs of Theorems 2.4.4 and 2.4.8, with trivial changes, still work if we replace “lim sup” by “lim inf” in stating the winning condition for $G_v$.

**Exercise 2.4.3.** (c) Prove the following strengthening of Theorem 2.4.13: If $G_v$ is determined for every $v \in (0, 1]$, then $\text{val}(\Gamma(f;T))$ is the supremum of the $\text{val}(\Gamma(g;T))$ for functions $g$ such that $(\forall x \in [T]) g(x) \leq f(x)$ and $g$ is the lim sup of a function defined on positions in $T$. This is the analogue of Theorem 2.4.25 in the context of the $G_v$ (instead of the $G'_v$).
Chapter 3

Measurable Cardinals

The results of Chapter 2 exhaust the determinacy that can be proved in ZFC, at least if we are talking of proving the determinacy of all games in natural topological or definability classes. The next natural class after $\Delta^1_1$ is $\Pi^1_1$ or its dual $\Sigma^1_1$. (See page 84 for the definitions of these classes.) By what is essentially a result of [Davis, 1964], the determinacy of all $\Pi^1_1$ games is not provable in ZFC. (See Exercise 4.1.1.) The rest of the determinacy results in this book will be proved with the help of large cardinal axioms. In the next chapter, we will prove the determinacy of $\Pi^1_1$ games in an arbitrary tree $T$ from the assumption that a measurable cardinal exists.

The purpose of this chapter is to introduce measurable cardinals and related notions and to prove the basic facts about them. In §3.1 we give the definition of measurable cardinals and establish the facts about them that we need in order to give the proof of $\Pi^1_1$ determinacy in §4.1. Sections 4.1 and 4.2 may be read independently of the rest of Chapter 3. In §3.2 we introduce ultrapowers and prove a characterization of measurable cardinals in terms of elementary embeddings. The elementary embedding definition of measurable cardinals is important not only because we will later make use of it but also because it is the elementary embedding versions of large cardinal axioms that (1) support most arguments for their plausibility and (2) allow one to see that the known large cardinal axioms are arranged in a coherent hierarchy. In §3.3 we extend the concepts and results of §3.2 to iterated ultrapowers and iterations of elementary embeddings. These notions will be used in §4.3 and throughout the later chapters. As always in this book, our aim is to prove determinacy results from the weakest possible large cardinal axioms. In §4.4, we show that $\Pi^1_1$ determinacy for games in countable trees...
can be proved from a consequence of measurable cardinals, the existence of sharps of elements of \( \omega \). §3.4 is devoted to an exposition of constructible sets, relative constructibility, and sharps. In §3.5 we study canonical inner models for ZFC + “there is a measurable cardinal.” We also study related models. These models will be used in the last three sections of Chapter 5.

Measurable cardinals were introduced in [Ulam, 1930]. Most of the material in this chapter dates, however, from the 1960’s, when there was a major revival in the study of large cardinals.

### 3.1 Basic Properties

For any nonempty set \( A \), a filter on \( A \) is a set \( \mathcal{F} \) of subsets of \( A \) such that

(a) \( A \in \mathcal{F} \) and \( \emptyset \notin \mathcal{F} \);
(b) \( (\forall X \in \mathcal{F})(\forall Y \in \mathcal{F}) X \cap Y \in \mathcal{F} \);
(c) \( (\forall X \in \mathcal{F})(\forall Y \subseteq A)(X \subseteq Y \rightarrow Y \in \mathcal{F}) \).

An ultrafilter on a set \( A \) is a filter \( \mathcal{U} \) on \( A \) such that

(d) \( (\forall X \subseteq A)(X \in \mathcal{U} \lor A \setminus X \in \mathcal{U}) \).

A filter \( \mathcal{F} \) on \( A \) is principal if there is a \( Y \subseteq A \) such that \( \mathcal{F} = \{ X \subseteq A \mid Y \subseteq X \} \).

**Lemma 3.1.1.** An ultrafilter \( \mathcal{U} \) on \( A \) is principal if and only if there is an \( a \in A \) such that \( \{a\} \in \mathcal{U} \).

**Proof.** Let \( \mathcal{U} \) be an ultrafilter on \( A \). If \( a \in A \) and \( \{a\} \in \mathcal{U} \), then clause (c) in the definition of a filter implies that \( \mathcal{U} \supseteq \{ X \subseteq A \mid a \in X \} \), and clauses (a) and (b) then imply that \( \mathcal{U} = \{ X \subseteq A \mid a \in X \} \). Suppose that \( Y \subseteq A \) and that \( \mathcal{U} = \{ X \subseteq A \mid Y \subseteq X \} \). By (a), \( Y \) is nonempty. Let \( a \in Y \). By (d), one of \( \{a\} \) and \( A \setminus \{a\} \) belongs to \( \mathcal{U} \) and so is a superset of \( Y \). This is possible only if \( Y = \{a\} \).

If \( \kappa \) is a cardinal number, a filter \( \mathcal{F} \) is \( \kappa \)-complete if every intersection of fewer than \( \kappa \) elements of \( \mathcal{F} \) belongs to \( \mathcal{F} \). Clause (b) in the definition of a filter is thus equivalent with the assertion that \( \mathcal{F} \) is \( \aleph_0 \)-complete. A filter \( \mathcal{F} \) is countably complete if \( \mathcal{F} \) is closed under countable intersections. Note
3.1. BASIC PROPERTIES

that countable completeness is equivalent with \( \aleph_1 \)-completeness, not with \( \aleph_0 \)-completeness.

A cardinal number \( \kappa \) is measurable if \( \kappa > \aleph_0 \) and there is a \( \kappa \)-complete, non-principal ultrafilter on \( \kappa \). (Recall that a cardinal number \( \kappa \) is identical with the set of all ordinals \( \alpha \) such that the cardinal number \( |\alpha| \) of \( \alpha \)—i.e., of the set of predecessors of \( \alpha \)—is smaller than \( \kappa \). Thus a cardinal number \( \kappa \) is a set and \( |\kappa| = \kappa \).)

The study of measurable cardinals arose out of [Banach, 1930] and especially [Ulam, 1930]. These papers dealt with the question of whether there can be a countably additive real-valued measure defined on all subsets of the unit interval and giving singletons measure 0 and the whole unit interval positive measure. An ultrafilter on \( A \) is essentially the same as a finitely additive \( \{0, 1\} \)-measure (a finitely additive measure taking only the values 0 and 1) defined on the whole power set \( \mathcal{P}(A) \) of \( A \) and giving the empty set measure 0 and \( A \) measure 1: If \( \mu \) is such a measure, let \( \mathcal{U} = \{ X \subseteq A \mid \mu(X) = 1 \} \). If \( \mathcal{U} \) is an ultrafilter on \( A \), let \( \mu : \mathcal{P}(A) \to \{0, 1\} \) be given by \( \mu(X) = 1 \iff X \in \mathcal{U} \). Thus the definition of a measurable cardinal can be given in terms of measures, and this fact explains the name. If we left out the conventional constraint that a measurable cardinal must be uncountable, then \( \aleph_0 \) would qualify as a measurable cardinal.

If a filter \( \mathcal{F} \) is non-principal, then \( \bigcap \mathcal{F} \notin \mathcal{F} \). We may then define the completeness of a non-principal filter \( \mathcal{F} \) to be the least cardinal \( \kappa \) such that some intersection of \( \kappa \) elements of \( \mathcal{F} \) does not belong to \( \mathcal{F} \). It is not hard to show that every non-principal filter on \( A \) has a completeness \( \leq |A| \). For ultrafilters this is immediate from Lemma 3.1.1 and clause (a) in the definition of a filter. By clause (b), the completeness of a filter is at least \( \aleph_0 \).

The following lemma gives a very useful method of constructing new filters and ultrafilters from old ones.

**Lemma 3.1.2.** Let \( A \) and \( B \) be sets, let \( \mathcal{F} \) be a filter on \( A \), and let \( f : A \to B \). Let \( \mathcal{G} \) be the set of all \( X \subseteq B \) such that \( f^{-1}(X) \in \mathcal{F} \). Then

1. \( \mathcal{G} \) is a filter on \( B \);
2. if \( \mathcal{F} \) is an ultrafilter then so is \( \mathcal{G} \);
3. the completeness of \( \mathcal{G} \) is at least as large as the completeness of \( \mathcal{F} \), where we think of the completeness of a principal filter as Ord.
4. \( \mathcal{G} \) is a principal ultrafilter if and only if \((\exists b \in B) f^{-1}(\{b\}) \in \mathcal{F} \).
Proof. (1) Since $f^{-1}(B) = A$ and $f^{-1}(\emptyset) = \emptyset$, the fact that $\mathcal{F}$ satisfies clause (a) in the definition of a filter implies that $\mathcal{G}$ satisfies (a). For clause (b), we note that $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$. Hence clause (b) for $\mathcal{F}$ implies clause (b) for $\mathcal{G}$. Similarly clause (c) for $\mathcal{G}$ follows from clause (c) for $\mathcal{F}$, because if $X \subseteq Y$ then $f^{-1}(X) \subseteq f^{-1}(Y)$.

(2) Assume that $\mathcal{F}$ is an ultrafilter. Since $f^{-1}(B \setminus X) = A \setminus f^{-1}(X)$, property (d) for $\mathcal{F}$ implies property (d) for $\mathcal{G}$.

(3) Let $\kappa$ be a cardinal number. Since

$$f^{-1}\left(\bigcap_{\alpha<\kappa} X_{\alpha}\right) = \bigcap_{\alpha<\kappa} f^{-1}(X_{\alpha}),$$

it follows that $\mathcal{G}$ is closed under intersections of size $\kappa$ if $\mathcal{F}$ is closed under intersections of size $\kappa$.

(4) This is an immediate consequence of Lemma 3.1.1 and the definition of $\mathcal{G}$.

We will see later that the existence of measurable cardinals cannot be demonstrated in ZFC. The following lemma shows that their existence is equivalent with the existence of a countably complete, non-principal ultrafilter on some set.

**Lemma 3.1.3.** If $\mathcal{U}$ is a countably complete, non-principal ultrafilter on $A$ and $\kappa$ is the completeness of $\mathcal{U}$, then $\kappa$ is a measurable cardinal.

**Proof.** Let $\kappa > \aleph_0$ be the completeness of a non-principal ultrafilter $\mathcal{U}$ on $A$. Let $\{X_\alpha \mid \alpha < \kappa\}$ witness that the completeness of $\mathcal{U}$ is no greater than $\kappa$. Thus each $X_\alpha \in \mathcal{U}$ but $\bigcap_{\alpha<\kappa} X_\alpha \notin \mathcal{U}$. Define $f : A \to \kappa$ by

$$f(a) = \begin{cases} \mu\gamma(a \notin X_\gamma) & \text{if } a \notin \bigcap_{\gamma<\kappa} X_\gamma; \\ 0 & \text{otherwise.} \end{cases}$$

Here, as usual, “$\mu$” means “the least.” Let $\mathcal{V} = \{X \subseteq \kappa \mid f^{-1}(X) \in \mathcal{U}\}$. By Lemma 3.1.2, we get that $\mathcal{V}$ is a $\kappa$-complete ultrafilter on $\kappa$. For each non-zero $\alpha < \kappa$, $f^{-1}(\{\alpha\})$ is disjoint from $X_\alpha$, and $f^{-1}(\{0\})$ is disjoint from $X_0 \setminus \bigcap_{\gamma<\kappa} X_\gamma$. Thus no $f^{-1}(\{\alpha\})$ belongs to $\mathcal{U}$, and so clause (4) of Lemma 3.1.2 implies that $\mathcal{V}$ is non-principal.

**Corollary 3.1.4.** If there is a cardinal $\kappa$ such that there is a countably complete, non-principal ultrafilter on $\kappa$, then there is a measurable cardinal $\leq \kappa$. 
In the next section, we will present techniques that give easy proofs that measurable cardinals are very large. Even without these techniques, it is not hard to show (Exercises 3.1.1 and 3.1.2) that every measurable cardinal is inaccessible, i.e. regular and a strong limit. An infinite cardinal $\kappa$ is regular if there is no ordinal $\lambda < \kappa$ such that some $f : \lambda \to \kappa$ has unbounded range; equivalently, $\kappa$ is regular if $\text{cf}(\kappa) = \kappa$. An infinite cardinal $\kappa$ is a strong limit if whenever $\lambda$ is a cardinal $< \kappa$ then $2^\lambda < \kappa$.

An ultrafilter $\mathcal{U}$ on an infinite cardinal $\kappa$ is normal if, for all functions $f : \kappa \to \kappa$, if $\{ \alpha < \kappa \mid f(\alpha) < \alpha \} \in \mathcal{U}$ then there is a $\beta < \kappa$ such that $\{ \alpha < \kappa \mid f(\alpha) = \beta \} \in \mathcal{U}$.

Let $\kappa$ be an infinite cardinal. No non-principal ultrafilter on $\kappa$ can be closed under all intersections of $\kappa$-many sets. If $\langle X_\beta \mid \beta < \kappa \rangle$ is a sequence of subsets of $\kappa$, then the diagonal intersection $\Delta_{\beta < \kappa} X_\beta$ is defined by

$$\Delta_{\beta < \kappa} X_\beta = \{ \alpha < \kappa \mid (\forall \beta < \alpha) \alpha \in X_\beta \}.$$

Lemma 3.1.5. (Dana Scott; see [Keisler and Tarski, 1964]) If $\mathcal{U}$ is an ultrafilter on an infinite cardinal $\kappa$, then $\mathcal{U}$ is normal if and only if $\mathcal{U}$ is closed under diagonal intersections.

Proof. Let $\mathcal{U}$ be an ultrafilter on $\kappa$, with $\kappa$ infinite. Assume first that $\mathcal{U}$ is normal. Let $\langle X_\beta \mid \beta < \kappa \rangle$ be such that each $X_\beta \in \mathcal{U}$. Suppose that $\Delta_{\beta < \kappa} X_\beta \notin \mathcal{U}$. Define $f : \kappa \to \kappa$ by

$$f(\alpha) = \begin{cases} \mu \beta(\beta < \alpha \land \alpha \notin X_\beta) & \text{if } (\exists \beta < \alpha) \alpha \notin X_\beta; \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Delta_{\beta < \kappa} X_\beta \notin \mathcal{U}$, the set of $\alpha$ for which the first clause in the definition of $f$ applies is a set in $\mathcal{U}$, i.e. $\{ \alpha \mid f(\alpha) < \alpha \land \alpha \notin X_{f(\alpha)} \} \in \mathcal{U}$. By normality, let $\beta < \kappa$ be such that $\{ \alpha < \kappa \mid f(\alpha) = \beta \} \in \mathcal{U}$. But then $\{ \alpha < \kappa \mid \alpha \notin X_\beta \} \in \mathcal{U}$, contrary to assumption.

Now assume that $\mathcal{U}$ is closed under diagonal intersections. Let $f : \kappa \to \kappa$ be such that $\{ \alpha < \kappa \mid f(\alpha) < \alpha \} \in \mathcal{U}$. For $\beta < \kappa$, let $X_\beta = \{ \alpha < \kappa \mid f(\alpha) = \beta \}$. We have that $\Delta_{\beta < \kappa} X_\beta \notin \mathcal{U}$. From closure under diagonal intersections, we get a $\beta < \kappa$ such that $X_\beta \notin \mathcal{U}$. But then $\{ \alpha < \kappa \mid f(\alpha) = \beta \} \in \mathcal{U}$. \qed

An ultrafilter $\mathcal{U}$ on an infinite cardinal $\kappa$ is uniform if every element of $\mathcal{U}$ has size $\kappa$; it is weakly uniform if for each $\delta < \kappa$ the set of all $\alpha < \kappa$ such that $\delta \leq \alpha$ belongs to $\mathcal{U}$.
Lemma 3.1.6. Let $\mathcal{U}$ be a normal ultrafilter on an infinite cardinal $\kappa$. Then the following are equivalent:

(a) $\mathcal{U}$ is $\kappa$-complete and non-principal.

(b) $\mathcal{U}$ is uniform.

(c) $\mathcal{U}$ is weakly uniform.

Proof. That (a) implies (b) and that (b) implies (c) follow directly from the definitions, and these implications do not depend on the hypothesis of normality.

To show that (c) implies (a) assume that $\mathcal{U}$ is weakly uniform. Obviously $\mathcal{U}$ is non-principal. For $\kappa$-completeness, let $\delta < \kappa$ and let $\langle X_\gamma \mid \gamma < \delta \rangle$ be a sequence of members of $\mathcal{U}$. For $\gamma \geq \delta$, let $X_\gamma = \kappa$. By Lemma 3.1.5, $\Delta_{\gamma<\kappa}X_\gamma \in \mathcal{U}$. Since $\mathcal{U}$ is uniform, $\{\alpha < \kappa \mid \delta \leq \alpha\} \in \mathcal{U}$. But then

$$\bigcap_{\gamma<\delta} X_\gamma \supseteq ((\kappa \setminus \delta) \cap \Delta_{\gamma<\kappa}X_\gamma) \in \mathcal{U}.$$ 

□

Lemma 3.1.7. (Dana Scott; see [Keisler and Tarski, 1964]) If $\kappa$ is a measurable cardinal, then there is a uniform normal ultrafilter on $\kappa$.

Proof. Let $\kappa$ be a measurable cardinal. Let $\mathcal{U}$ be a non-principal, $\kappa$-complete ultrafilter on $\kappa$.

We first show that there is an $f : \kappa \to \kappa$ such that

(i) $f$ is not constant on any member of $\mathcal{U}$; i.e., $(\forall \beta < \kappa)\{\alpha < \kappa \mid f(\alpha) = \beta\} \notin \mathcal{U}$;

(ii) for every $g : \kappa \to \kappa$, if $g$ is not constant on any member of $\mathcal{U}$, then $\{\alpha < \kappa \mid f(\alpha) \leq g(\alpha)\} \in \mathcal{U}$.

Assume that no $f$ satisfying (i) and (ii) exists. Let $f_0$ be the identity function on $\kappa$. Since $\mathcal{U}$ is non-principal, $f_0$ satisfies (i). Assume inductively that $f_0, f_1, \ldots, f_n$ all satisfy (i) and that $(\forall i < n)\{\alpha < \kappa \mid f_{i+1}(\alpha) < f_i(\alpha)\} \in \mathcal{U}$. By assumption, $f_n$ does not satisfy (ii). Let $f_{n+1}$ be a $g$ witnessing this fact. Our induction hypothesis thus holds for $n + 1$. Since $\mathcal{U}$ is $\kappa$-complete and $\kappa$ is uncountable, we have that $\mathcal{U}$ is countably complete. Hence

$$\bigcap_{n \in \omega} \{\alpha < \kappa \mid f_{n+1}(\alpha) < f_n(\alpha)\} \in \mathcal{U}.$$
By clause (a) in the definition of a filter, no element of $U$ can be empty. Let then $\alpha \in \bigcap_{n \in \omega} \{ \alpha < \kappa \mid f_{n+1}(\alpha) < f_n(\alpha) \}$. We have that

$$f_0(\alpha) > f_1(\alpha) > f_2(\alpha) > \cdots,$$

contradicting the fact that $\kappa$ is wellordered by $\prec$.

Let $f : \kappa \to \kappa$ satisfy (i) and (ii). Define an ultrafilter $V \subseteq \mathcal{P}(\kappa)$ by

$$X \in V \leftrightarrow f^{-1}(X) \in U.$$

Lemma 3.1.2 and (i) give that $V$ is a non-principal ultrafilter on $\kappa$. To prove the normality of $V$, suppose that $\{ \alpha < \kappa \mid h(\alpha) < \alpha \} \in V$. We must show that $(\exists \beta < \kappa) \{ \alpha < \kappa \mid h(\alpha) = \beta \} \in V$. Define $g : \kappa \to \kappa$ by

$$g(\alpha) = h(f(\alpha)).$$

Since $\{ \alpha < \kappa \mid h(\alpha) < \alpha \} \in V$, it follows from the definition of $V$ that $\{ \alpha < \kappa \mid h(f(\alpha)) < f(\alpha) \} \in U$. Hence $\{ \alpha < \kappa \mid g(\alpha) < f(\alpha) \} \in U$. But $f$ satisfies (ii); so $g$ cannot satisfy (i). Thus we get a $\beta < \kappa$ such that $\{ \alpha < \kappa \mid g(\alpha) = \beta \} \in U$. By the definitions of $g$ and $V$, this implies that $\{ \alpha < \kappa \mid h(\alpha) = \beta \} \in V$. □

The proof of Lemma 3.1.7 uses the uncountability of measurable cardinals. This use is necessary: no non-principal ultrafilter on $\omega$ is normal. (See Exercise 3.1.3.)

For any set $z$ and any cardinal number $\lambda$, $[z]^\lambda$ is the set of all subsets $w$ of $z$ such that $|w| = \lambda$. One reason that normal ultrafilters are useful is the following result of Frederick Rowbottom, which shows that a $\kappa$-complete normal ultrafilter on $\kappa$ generates $\kappa$-complete ultrafilters on $[\kappa]^n$ for all $n \in \omega$.

**Lemma 3.1.8.** ([Rowbottom, 1964]) *Let $n \in \omega$ and let $U$ be a normal ultrafilter on a cardinal $\kappa$. If $Z \subseteq [\kappa]^n$, there is an $X \in U$ such that either $[X]^n \subseteq Z$ or $[X]^n \cap Z = \emptyset$.***

**Proof.** We prove the lemma by induction on $n$.

The case $n = 0$ is trivial, since $[\kappa]^0$ has only one member, $\emptyset$.

Assume that the lemma holds for $n \geq 0$. Let $Z \subseteq [\kappa]^{n+1}$. For each $\beta < \kappa$, let

$$Z_\beta = \{ u \in [\kappa \setminus \{ \beta \}]^n \mid \{ \beta \} \cup u \in Z \}.$$
By our induction hypothesis, we have for each \( \beta \) an \( X_\beta \in U \) such that either \([X_\beta]^n \subseteq Z_\beta\) or \([X_\beta]^n \cap Z_\beta = \emptyset\). Let \( Y = \{ \beta \mid [X_\beta]^n \subseteq Z_\beta \}\) if that set belongs to \( U \) and \( \{ \beta \mid [X_\beta]^n \cap Z_\beta = \emptyset \}\) otherwise. Let

\[
X = Y \cap \Delta_{\beta<\kappa} X_\beta.
\]

By Lemma 3.1.5, we have that \( X \in U \). Assume first that \( Y = \{ \beta \mid [X_\beta]^n \subseteq Z_\beta \}\). Let \( t \in [X]^{n+1} \). Let \( \beta \) be the least element of \( t \). Let \( u = t \setminus \{ \beta \} \). We have that \( u \in [X]^n \subseteq [\Delta_{\eta<\kappa} X_\eta]^n \), so \( u \in [X_\beta]^n \subseteq Z_\beta \). By the definitions of \( t \) and \( Z_\beta \), we get that \( t \in Z \). If we now assume that \( Y = \{ \beta \mid [X_\beta]^n \cap Z_\beta = \emptyset \}\), then a similar argument shows that no \( t \in [X]^{n+1} \) belongs to \( Z \).

Suppose that \( U \) is a uniform normal ultrafilter on a cardinal \( \kappa \). For \( n \in \omega \), we define the Rowbottom ultrafilter \( U^{[n]} \) on \( [\kappa]^n \) by

\[
Z \in U^{[n]} \leftrightarrow (\exists X \in U) [X]^n \subseteq Z.
\]

The fact that \( U \) is \( \kappa \)-complete and non-principal implies that \( U^{[n]} \) is a \( \kappa \)-complete filter. (We need that \( U \) is non-principal in order to show that \( \emptyset \notin U^{[n]} \).) Lemma 3.1.8 implies that \( U^{[n]} \) is an ultrafilter. It is clear that \( U^{[n]} \) is non-principal if \( n > 0 \). The Rowbottom ultrafilter on \( [\kappa]^n \) is essentially the same as the iterated product ultrafilter on \( \kappa^n \). (See Exercise 3.1.7.)

**Exercise 3.1.1.** Prove that every measurable cardinal is regular. (Ulam [1930].)

**Exercise 3.1.2.** Prove that every measurable cardinal is a strong limit. (This is due to Alfred Tarski and Ulam independently. See [Ulam, 1930].)

*Hint.* Assume that \( \lambda < \kappa \) and that \( U \) is a \( \kappa \)-complete ultrafilter on a subset of \( \lambda^2 \). Prove that \( U \) is principal.

**Exercise 3.1.3.** Show that every normal ultrafilter on \( \omega \) is principal.

**Exercise 3.1.4.** Let \( U \) be a normal ultrafilter on \( \kappa \). Let \( f : \kappa \to \kappa \) be such that \((\forall \alpha < \kappa) f^{-1}(\{\alpha\}) \notin U \). Prove that there is a set belonging to \( U \) on which \( f \) is one-one.

**Exercise 3.1.5.** If \( F \) and \( G \) are filters on \( A \) and \( B \) respectively, then \( F \) and \( G \) are isomorphic \((F \cong G)\) if there is a bijection \( f : A \to B \) such that \( G = \{ X \subseteq B \mid f^{-1}(X) \in F \} \).
3.2. ULTRAPOWERS AND ELEMENTARY EMBEDDINGS

Prove that a $\kappa$-complete non-principal ultrafilter $U$ on a measurable cardinal $\kappa$ is isomorphic to a normal ultrafilter if and only if $U$ satisfies Lemma 3.1.8, i.e. if and only if

$$(\forall n \in \omega)(\forall Z \subseteq [\kappa]^n)(\exists X \in U)([X]^n \subseteq Z \vee [X]^n \cap Z = \emptyset).$$

(This result is probably due to Dana Scott.)

**Hint.** For the non-trivial direction, consider a function $f$ satisfying (i) and (ii) in the proof of Lemma 3.1.7.

**Exercise 3.1.6.** Prove that not every $\kappa$-complete non-principal ultrafilter on a measurable cardinal $\kappa$ is isomorphic to a normal ultrafilter.

**Hint.** Prove that the function $f : [\kappa]^2 \to \kappa$ given by $f(u) = \min(u)$ is not one-one on any set belonging to $U[2]$.

**Exercise 3.1.7.** If $U$ is a filter on a set $A$, then for the iterated product filter $U^n$ on $^nA$ is defined by letting $U^0$ be the unique filter on $^0A = \{\emptyset\}$ and inductively setting

$$W \in U^{n+1} \leftrightarrow \{a \in A \mid \{s \in ^nA \mid \langle a \rangle \downarrow s \in W\} \in U^n\} \in U.$$

Let $U$ be a uniform normal ultrafilter on a cardinal $\kappa$ and, for $n \in \omega$, let the injection $g : [\kappa]^n \to ^n\kappa$ be given by letting each $g(u)$ enumerate $u$ in increasing order. Prove that, for all $Z \in [\kappa]^n$, $Z \in U^n$ if and only if $g(Z) \in U^n$.

3.2 Ultrapowers and Elementary Embeddings

We now have developed enough of the theory of measurable cardinals to prove the main theorem of Chapter 4: that the determinacy of all $\Pi_1^1$ games in a tree $T$ follows from the existence of a measurable cardinal larger than $|T|$. Section 4.1, which contains the proof of this theorem, and also Section 4.2 can be read without reading the rest of Chapter 3.

For the determinacy proofs of the later chapters, however, we need a further technical tool: the ultrapower construction.

**Convention.** Except where we explicitly state otherwise, we mean by a *model* a model for the language of set theory: a model $M = (M; E)$, where $M$ is a nonempty set and $E$ is a binary relation in $M$ (a subset of $M \times M$).
CHAPTER 3. MEASURABLE CARDINALS

Let \( U \) be an ultrafilter on a set \( A \). Let \( \mathcal{M} = (M; E) \) be a model. The ultrafilter \( U \) gives rise to an equivalence relation \( \sim_{U,M} \) on \( A^M \): If \( f : A \to M \) and \( g : A \to M \), then

\[
f \sim_{U,M} g \iff \{ a \in A \mid f(a) = g(a) \} \in U.\]

We will sometimes suppress the subscript \( U, M \) and write simply \( \sim \) when there is no ambiguity about \( U \) or \( M \).

For \( f \in A^M \), we write \([f]_{U,M}\) or just \([f]\) for the equivalence class of \( f \) with respect to \( \sim_{U,M} \). We denote the set of all the equivalence classes by \( A^M/U \).

We define a binary relation \( E_{U,M} \) in \( A^M/U \) as follows:

\[
[f]_{U,M} E_{U,M} [g] \iff \{ a \in A \mid f(a) E g(a) \} \in U.\]

It is easy to see that \( E_{U,M} \) is well-defined.

The ultrapower of \( \mathcal{M} \) with respect to \( U \) is the model

\[
\prod_U \mathcal{M} = (A^M/U; E_{U,M}).
\]

Note that \( \prod_U \mathcal{M} \) is, like \( \mathcal{M} \), a model for the language of set theory.

Remark. We have defined \( \prod_U \mathcal{M} \) only for models of one particular similarity type, but the definition can easily be extended to arbitrary models. \( \prod_U \mathcal{M} \) is always a model of the same similarity type as \( \mathcal{M} \). The proof of Theorem 3.2.1 below works also for ultrapowers in this more general sense.

Ultrapowers were introduced in [Loś, 1955], where the following theorem essentially appears.

**Theorem 3.2.1.** ([Loś, 1955]) Let \( U \) be an ultrafilter on \( A \) and let \( \mathcal{M} = (M; E) \) be a model. Let \( \varphi(v_1, \ldots, v_n) \) be any formula of the language of set theory. Let \( f_1, \ldots, f_n \) be elements of \( A^M \). Then

\[
\prod_U \mathcal{M} \models \varphi([f_1], \ldots, [f_n]) \iff \{ a \in A \mid \mathcal{M} \models \varphi(f_1(a), \ldots, f_n(a)) \} \in U.
\]

**Proof.** We may assume that the only connectives in \( \varphi \) are \( \land \) and \( \neg \) and that the only quantifier in \( \varphi \) is \( \exists \). We prove the theorem by induction on the complexity of the formula \( \varphi \).
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For \( \varphi \) atomic, the theorem holds of \( \varphi \) by the definitions of \( \sim_{U,M} \) and \( E_{U,M} \).

If \( \varphi \) is \( \psi \land \chi \), then
\[
\prod_{U,M} \mathcal{M} \models \varphi[[f_1], \ldots, [f_n]] \leftrightarrow 
\prod_{U,M} \mathcal{M} \models \psi[[f_1], \ldots, [f_n]] \land \prod_{U,M} \mathcal{M} \models \chi[[f_1], \ldots, [f_n]] 
\leftrightarrow 
\left( \{a \in A \mid \mathcal{M} \models \psi[f_1(a), \ldots, f_n(a)] \} \in \mathcal{U} \land \{a \in A \mid \mathcal{M} \models \chi[f_1(a), \ldots, f_n(a)] \} \in \mathcal{U} \right) 
\leftrightarrow 
\{a \in A \mid \mathcal{M} \models \varphi[f_1(a), \ldots, f_n(a)] \} \in \mathcal{U}.
\]

Here we have used induction to get the second equivalence, and we have used clauses (b) and (c) in the definition of a filter to get the last equivalence.

If \( \varphi \) is \( \neg \psi \), then
\[
\prod_{U,M} \mathcal{M} \models \varphi[[f_1], \ldots, [f_n]] \leftrightarrow 
\neg (\prod_{U,M} \mathcal{M} \models \psi[[f_1], \ldots, [f_n]]) 
\leftrightarrow 
\{a \in A \mid \mathcal{M} \models \psi[f_1(a), \ldots, f_n(a)] \} \notin \mathcal{U} 
\leftrightarrow 
\{a \in A \mid \mathcal{M} \models \varphi[f_1(a), \ldots, f_n(a)] \} \in \mathcal{U}.
\]

The last line follows from the preceding line by clause (d) in the definition of an ultrafilter. This is the only place in the proof where we use the fact that \( \mathcal{U} \) is an ultrafilter rather than just a filter.

If \( \varphi \) is \( (\exists v_0) \psi \), then
\[
\prod_{U,M} \mathcal{M} \models \varphi[[f_1], \ldots, [f_n]] \leftrightarrow 
(\exists f_0 \in A^M) \prod_{U,M} \mathcal{M} \models \psi[[f_0], [f_1], \ldots, [f_n]] 
\leftrightarrow 
(\exists f_0 \in A^M) (\{a \in A \mid \mathcal{M} \models \psi[f_0(a), f_1(a), \ldots, f_n(a)] \} \in \mathcal{U}) 
\leftrightarrow 
\{a \in A \mid (\exists b \in M) \mathcal{M} \models \psi[b, f_1(a), \ldots, f_n(a)] \} \in \mathcal{U} 
\leftrightarrow 
\{a \in A \mid \mathcal{M} \models \varphi[f_1(a), \ldots, f_n(a)] \} \in \mathcal{U}.
\]

Note that the Axiom of Choice is used to deduce that the fourth line implies the third. \( \square \)

If \( \mathcal{M} = (M; E) \) and \( \mathcal{N} = (N; F) \) are models, an elementary embedding of \( \mathcal{M} \) into \( \mathcal{N} \) is a function \( j : M \rightarrow N \) such that, for any formula \( \varphi(v_1, \ldots, v_n) \) of the language of set theory and for any \( n \)-tuple \( \langle b_1, \ldots, b_n \rangle \) of elements of \( M \),
\[
\mathcal{M} \models \varphi[b_1, \ldots, b_n] \leftrightarrow \mathcal{N} \models \varphi[j(b_1), \ldots, j(b_n)].
\]

We write \( j : \mathcal{M} \prec \mathcal{N} \) to mean that \( j \) is an elementary embedding of \( \mathcal{M} \) into \( \mathcal{N} \).
Let $\mathcal{M} = (M; E)$ be a model and let $\mathcal{U}$ be an ultrafilter on a set $A$. We define $j : M \to A^{M/\mathcal{U}}$ by

$$j(b) = [c_b],$$

where $c_b : A \to M$ is the constant function with value $b$.

**Corollary 3.2.2.** $j : \mathcal{M} \prec \prod_{\mathcal{U}} \mathcal{M}$.

**Proof.** For any $\varphi(v_1, \ldots, v_n)$ and $(b_1, \ldots, b_n)$, we have by Theorem 3.2.1 that

$$\prod_{\mathcal{U}} \mathcal{M} \models \varphi[[c_{b_1}], \ldots, [c_{b_n}]] \leftrightarrow \{a \in A \mid \mathcal{M} \models \varphi[c_{b_1}(a), \ldots, c_{b_n}(a)]\} \in \mathcal{U}.$$

But the left-hand side is equivalent with $\prod_{\mathcal{U}} \mathcal{M} \models \varphi[j(b_1), \ldots, j(b_n)]$, and the right-hand side just says that $\{a \in A \mid \mathcal{M} \models \varphi[b_1, \ldots, b_n]\} \in \mathcal{U}$, i.e. that $\mathcal{M} \models \varphi[b_1, \ldots, b_n]$. \qed

A model $(M; E)$ is **wellfounded** if the relation $E$ is wellfounded, i.e. if every nonempty subset of $M$ has an $E$-minimal element. This is equivalent (using Choice) with the non-existence of an infinite sequence $(b_i \mid i \in \omega)$ such that $b_{i+1} E b_i$ for each $i \in \omega$. In [Keisler, 1962b] ultrapowers of wellfounded structures were first used to get results about measurable cardinals. The next lemma is fundamental for the method.

**Lemma 3.2.3.** Let $\mathcal{U}$ be a countably complete ultrafilter on the set $A$ and and let $\mathcal{M} = (M; E)$ be a wellfounded model. Then $\prod_{\mathcal{U}} \mathcal{M}$ is also wellfounded.

**Proof.** Suppose that $\langle f_i \mid i \in \omega \rangle$ is a counterexample to the wellfoundedness of $\prod_{\mathcal{U}} \mathcal{M}$; that is suppose that

$$\cdots E_{\mathcal{U}, \mathcal{M}} [f_2] E_{\mathcal{U}, \mathcal{M}} [f_1] E_{\mathcal{U}, \mathcal{M}} [f_0].$$

By the definition of $E_{\mathcal{U}, \mathcal{M}}$,

$$(\forall i \in \omega)\{a \in A \mid f_{i+1}(a) E f_i(a)\} \in \mathcal{U}.$$

By the countable completeness of $\mathcal{U}$, $\bigcap_{i \in \omega} \{a \in A \mid f_{i+1}(a) E f_i(a)\} \in \mathcal{U}$. Let $a$ belong to this set. Then

$$\cdots E f_2(a) E f_1(a) E f_0(a),$$
We mainly want to apply the ultrapower construction to the case that $\mathcal{M}$ is a model of ZFC and that the relation $E$ is the restriction of the membership relation to $M$, i.e. $\mathcal{M} = (M; \in \cap (M \times M))$. For simplicity we will write $(M; \in)$ instead of $\mathcal{M} = (M; \in \cap (M \times M))$. The Axiom of Foundation asserts that such models are always wellfounded. The following lemma of Andrzej Mostowski implies that wellfounded models of ZFC are all isomorphic to such models.

Recall that a set $x$ is transitive if every member of a member of $x$ belongs to $x$.

**Lemma 3.2.4.** ([Mostowski, 1949]) Let $(M; E)$ be a wellfounded model of the Axiom of Extensionality. Then there is a unique transitive set $N$ such that $(M; E) \cong (N; \in)$, and the isomorphism $\pi : (M; E) \cong (N; \in)$ is unique.

**Proof.** We define $\pi(x)$ by transfinite recursion on the wellfounded relation $E$:

$$\pi(x) = \{\pi(y) \mid y E x\}.$$ 

Note that this must be $\pi(x)$ if $\pi$ is to be an isomorphism. Let, as we must, $N = \{\pi(x) \mid x \in M\}$. It is immediate that $N$ is transitive. It is immediate from the definition that $$(\forall x \in M)(\forall y \in M)(y E x \to \pi(y) \in \pi(x)).$$

If $\pi$ is one-one, then it also follows that $$(\forall x \in M)(\forall y \in M)(\pi(y) \in \pi(x) \to y E x),$$

and so that $\pi$ is an isomorphism. We prove by induction on $E$ that for every $x \in M$ there is no $x' \in M$ such that $x' \neq x$ and $\pi(x') = \pi(x)$. Assume then that $x' \neq x$. Since $(M; E)$ satisfies Extensionality, there is a $z \in M$ that bears $E$ to exactly one of $x'$ and $x$. Assume for definiteness that $z E x'$ but that not $z E x$; the other case is similar. Then by induction there is no $w E x$ such that $\pi(z) = \pi(w)$. But this means that $\pi(z) \in \pi(x') \setminus \pi(x)$ and so that $\pi(x') \neq \pi(x)$. □

Suppose that $(M; \in)$ is a model and that $\mathcal{U}$ is a countably complete ultrafilter on a set $A$. Let $j : (M; \in) \prec \prod_\mathcal{U}(M; \in)$ be the canonical elementary embedding as defined on page 140. By Lemma 3.2.3, $\prod_\mathcal{U}(M; \in)$ is wellfounded. Let $\pi : \prod_\mathcal{U}(M; \in) \cong (N; \in)$ be given by Lemma 3.2.4. We have then that

$$\pi \circ j : (M; \in) \prec (N; \in).$$
We want to study such embeddings arising from a uniform normal $\mathcal{U}$ on a measurable cardinal $\kappa$. However, we want to replace $M$ by a proper class, in particular by the set-theoretic universe $V$, and consequently to replace $N$ also by a proper class. For this we must first check that the results we have derived so far hold for ultrapowers of proper class models.

Since we are officially working in ZFC, we can’t literally talk about proper classes. What we mean by a class is something of the form

$$\{x \mid \varphi(x, y_1, \ldots, y_n)\},$$

where $y_1, \ldots, y_n$ are sets and $\varphi$ is a formula of the language of (ZFC) set theory. Hence each class is determined by a formula and a finite sequence of sets. We cannot in our language make general statements about classes; thus most of the theorems in the rest of this section should be construed as theorem schemata. See pages 23–24 of [Kunen, 1980] for a discussion of this. If the reader prefers to construe our talk of classes literally, he can mostly take us to be working in von Neumann–Bernays–Gödel set theory.

Warning. We will be very casual in dealing with proper classes. The advantage of doing so is that ideas are less likely to be obscured by technical details. The disadvantage is that it will sometimes be a non-trivial problem for the careful reader to see how our discussion could be formalized in ZFC or even in von Neumann–Bernays–Gödel set theory.

As is usual, we will identify a non-proper class with the corresponding set.

Except where we explicitly state otherwise, we mean by a class model something of the form $(M; E)$ where $M$ is a nonempty class and $E$ is a subclass of $M \times M$. We haven’t actually indicated what kind of set-theoretic object an ordinary model is (we haven’t used ordered pair notation $(M, E)$), so we may blithely keep the same ambiguity as to what specific object a class model is.

Let $\mathcal{M} = (M; E)$ be a class model and let $\mathcal{U}$ be an ultrafilter on a set $A$. We want to define an ultrapower as in the set model case. As before, for $f : A \to M$ and $g : A \to M$ we can let $f \sim_{\mathcal{U}, M} g$ just in case $\{a \in A \mid f(a) = g(a)\} \in \mathcal{U}$. The first problem comes when we try to define the equivalence class $[f]_{\mathcal{U}, M}$. The genuine equivalence class, $\{g \mid f \sim_{\mathcal{U}, M} g\}$, is a proper class if $M$ is a proper class, unless $|A| = 1$. Since all our classes are to be classes of sets, using these classes for the $[f]_{\mathcal{U}, M}$ would render us unable to define $^M A/\mathcal{U}$. We could try picking a representative from each equivalence
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class, but this would require a global form of the Axiom of Choice, and we do not want to make such an assumption. A satisfactory solution comes from [Scott, 1955], and we will make use of it below.

First let us recall the cumulative hierarchy of sets. Inductively we define for each ordinal \(\alpha\) a set \(V_\alpha\):

(i) \(V_0 = \emptyset\);
(ii) \(V_{\alpha + 1} = V_\alpha \cup \mathcal{P}(V_\alpha)\);
(iii) \(V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha\) if \(\lambda\) is a limit ordinal.

It is easy to show by induction that each \(V_\alpha\) is transitive and so that the definition would be unaffected if we changed clause (ii) to set \(V_{\alpha + 1} = \mathcal{P}(V_\alpha)\).

The Axiom of Foundation implies that every set belongs to the class \(V = \bigcup_{\alpha \in \text{Ord}} V_\alpha\). (See III, §4 of [Kunen, 1980].) Thus we can define the rank of any set \(x\) by

\[
\text{rank}(x) = \mu \alpha \in V_{\alpha + 1}.
\]

Let us say that \(f \sim_U g\) if \(f \sim_U V\), i.e. if \(f\) and \(g\) are functions with domain \(A\) and \(\{a \in A \mid f(a) = g(a)\} \in U\). Following [Scott, 1961] we define \([f]_U\) for \(f : A \to V\) to be the set of all \(g\) of minimal rank such that \(f \sim_U g\), i.e.

\[
[f]_U = \{ g \mid f \sim_U g \land (\forall h)(f \sim_U h \rightarrow \text{rank}(g) \leq \text{rank}(h)) \}\).

With this definition, \([f]_U \subseteq V_{\alpha}\) for some \(\alpha \leq \text{rank}(f) + 1\). Thus \([f]_U\) is a set. When there is no ambiguity, we may write “\([f]\)” for “\([f]_U\)”.

Remark. We have chosen, since Scott’s trick makes it possible, to use “equivalence classes” \([f]_U\) that are independent of \(M\).

Continuing with our class model \(\mathcal{M} = (M; E)\) and our ultrafilter \(U\) on the set \(A\), we denote as in the set model case the class of all \([f]_U\) for \(f : A \to M\) by \(A^M/U\). The class \(A^M/U\) is a proper class if (and only if) \(M\) is a proper class. Also as in the set model case we let \([f] E_U [g]\) hold if and only if \(\{a \in A \mid f(a) \in g(a)\} \in U\) and we let \(\prod_U \mathcal{M}\) be the class model \((A^M/U; E_U)\).

Remark. Since a set model is also a class model, there is an ambiguity in our definitions of \(A^M/U\) and \(\prod_U \mathcal{M}\) when applied to set models. Let us officially adopt the new definition in all cases, though nothing important will turn on this.
Theorem 3.2.5. Let $\mathcal{U}$ be an ultrafilter on $A$ and let $\mathcal{M} = (M; E)$ be a class model. Let $\varphi(v_1, \ldots, v_n)$ be any formula of the language of set theory. Let $f_1, \ldots, f_n$ be elements of $^A M$. Then

$$\prod_{\mathcal{U}} \mathcal{M} \models \varphi[[f_1], \ldots, [f_n]] \leftrightarrow \{a \in A \mid \mathcal{M} \models \varphi[f_1(a), \ldots, f_n(a)]\} \in \mathcal{U}.$$ 

The proof of Theorem 3.2.5 is just like that of Theorem 3.2.1.

Remark. Theorem 3.2.5 is a theorem schema both because it is about arbitrary class models and because it is about arbitrary formulas. Since we cannot in ZFC talk in general about the satisfaction relation even for a fixed class model, $\varphi$ as well as $\mathcal{M}$ must be treated schematically. Thus we are not proving a fixed sentence by induction but rather are inductively showing how to prove all sentences of a certain form.

We define elementary embeddings for class models just as we defined them for set models. We can define a function $j : M \rightarrow {}^A M/\mathcal{U}$ just as on page 140. The proof of elementarity of $j$ in the set case works in the class case.

Corollary 3.2.6. $j : \mathcal{M} \prec \prod_{\mathcal{U}} \mathcal{M}$.

Wellfoundedness is defined for class models as for set models. The proof of Lemma 3.2.3 also gives the following lemma.

Lemma 3.2.7. Let $\mathcal{U}$ be a countably complete ultrafilter on the set $A$ and let $\mathcal{M}$ be a wellfounded class model. Then $\prod_{\mathcal{U}} \mathcal{M}$ is also wellfounded.

Mostowski’s Lemma (Lemma 3.2.4) does not hold in general for class models. The point is that wellfounded class models can be longer than the ordinals, and so need not be isomorphic to class models $(N; \in)$. (See Exercise 3.2.1.) To rule out this possibility we define (more or less following [Kunen, 1980]) a class model $(M; E)$ to be set-like if for all $x \in M$ the class $\{y \in M \mid y E x\}$ is a set.

Lemma 3.2.8. If $(M; E)$ is a wellfounded set-like model of the Axiom of Extensionality, then there is a unique transitive class $N$ such that $(M; E) \cong (N; \in)$, and the isomorphism $\pi : (M; E) \cong (N; \in)$ is unique.
The proof of 3.2.8 is just like that of 3.2.4. The assumption that \((M; E)\) is set-like justifies the inductive definition of \(\pi\), in particular it guarantees that \(\pi(x) = \{\pi(y) \mid y E x\}\) is a set.

**Remark.** Since the \(N\) and the \(\pi\) of Lemma 3.2.8 are proper classes, a word is in order about the significance of the “there is” in the statement of the lemma. The point is that our proof defines \(N\) and \(\pi\) from \(M\) and \(E\), and hence we show how to construct formulas determining \(N\) and \(\pi\) from formulas determining \(M\) and \(E\).

The following lemma guarantees that the class models we are interested in are set-like.

**Lemma 3.2.9.** Let \(\mathcal{M} = (M; \in)\). Let \(\mathcal{U}\) be an ultrafilter on a set \(A\). Then \(\prod \mathcal{U} \mathcal{M}\) is set-like.

**Proof.** Let \(f : A \to M\). We must prove that \(\{[g] \in A M/\mathcal{U} \mid [g] \in \mathcal{U} [f]\}\) is a set. Let \(g : A \to M\) be such that \([g] \in \mathcal{U} [f]\). By definition, this means that \(\{a \in A \mid g(a) \in f(a)\} \in \mathcal{U}\). Define \(g' : A \to M\) by

\[
g'(a) = \begin{cases} g(a) & \text{if } g(a) \in f(a); \\ f(a) & \text{otherwise.} \end{cases}
\]

Clearly \(g' \sim g\) and \(\text{rank}(g') \leq \text{rank}(f)\). By the definition of \([g]\), it follows that every member of \([g]\) has rank no greater than rank \((f)\). Let \(\alpha = \text{rank}(f)\). We have shown that whenever \([g] \in \mathcal{U} [f]\) then \([g] \subseteq V_{\alpha+1}\) and so \([g] \in V_{\alpha+2}\). But then \(\{[g] \mid [g] \in \mathcal{U} [f]\} \subseteq V_{\alpha+2}\) and is therefore a set. \(\square\)

Let \(\mathcal{U}\) be a countably complete ultrafilter on a set \(A\). Let \(i_\mathcal{U}'\) be the embedding \(j : (V; \in) \prec (A V/\mathcal{U}; \in_\mathcal{U})\) defined on page 140. By Lemmas 3.2.7 and 3.2.9, \((A V/\mathcal{U}; \in_\mathcal{U})\) is a wellfounded set-like class model. Let \(\pi_\mathcal{U} : (A V/\mathcal{U}; \in_\mathcal{U}) \cong (N; \in)\) be given by Lemma 3.2.8. Note that \((N; \in)\) is a class model of ZFC, since \(V\) is such a model. By \(\text{Ult}(V; \mathcal{U})\) we mean the class \(N\). We denote by \(i_\mathcal{U}\) the embedding \(\pi \circ i_\mathcal{U}'\).

**Convention.** We will often attribute properties of models \((M; \in)\) to the corresponding sets \(M\). Thus we will say, e.g. that \(V \models \text{ZFC}\) and that, for the \(i_\mathcal{U}\) and \(N\) of the last paragraph, that \(i_\mathcal{U} : V \prec N\).

If \(M\) and \(N\) are classes, if \(h : M \to N\), and if there is an ordinal \(\alpha \in M\) such that \(h(\alpha) \neq \alpha\), then we let \(\text{crit}(h)\) be the least such \(\alpha\) and we call it the **critical point** of \(h\). We will mainly use this terminology when \(h : M \prec N\).
Lemma 3.2.10. Let $\mathcal{U}$ be a countably complete ultrafilter on a set $A$. If $\mathcal{U}$ is principal, then $\text{Ult}(V;\mathcal{U}) = V$ and $i_{\mathcal{U}}$ is identity. If $\mathcal{U}$ is non-principal, then $i_{\mathcal{U}}$ is the identity on $V_\kappa$, where $\kappa$ is the completeness of $\mathcal{U}$, but $i_{\mathcal{U}}$ is not the identity and $\kappa = \text{crit}(i_{\mathcal{U}})$.

Proof. Let us construe the completeness of $\mathcal{U}$ to be $\text{Ord}$ if $\mathcal{U}$ is principal, since then $\mathcal{U}$ is closed under arbitrary intersections. (But $\mathcal{U}$ is closed under intersections of size $\text{Ord}$, so this convention is not completely natural.) Let $\kappa$ be the completeness of $\mathcal{U}$.

We first show that $i_{\mathcal{U}}$ is the identity on $\kappa$. To do this we prove by induction that $i_{\mathcal{U}}(\alpha) = \alpha$ for all $\alpha < \kappa$. Suppose then that $\alpha < \kappa$ and that $i_{\mathcal{U}}(\beta) = \beta$ for all $\beta < \alpha$. For each $\beta < \alpha$, $i_{\mathcal{U}}(\beta) \in i_{\mathcal{U}}(\alpha)$, by the elementarity of $i_{\mathcal{U}}$. Suppose that $\pi([g]) \in i_{\mathcal{U}}(\alpha)$, where $\pi = \pi_{\mathcal{U}} : \prod_{\mathcal{U}}(V; \in) \cong (\text{Ult}(V;\mathcal{U}); \in)$. Then $[g] \in [c_\alpha]$, and so $\{a \in A \mid g(a) \in \alpha\} \in \mathcal{U}$. But $\alpha < \kappa$ and so the $\kappa$-completeness of $\mathcal{U}$ implies that there is a $\beta < \alpha$ such that $\{a \in A \mid g(a) = \beta\} \in \mathcal{U}$. But then $g \sim c_\beta$, and so $\pi([g]) = \pi([c_\beta]) = i_{\mathcal{U}}(\beta) = \beta$.

This completes the inductive proof that $i_{\mathcal{U}}$ is the identity on $\kappa$.

Next we prove by induction on $\alpha < \kappa$ that $i_{\mathcal{U}}$ is the identity on $V_\alpha$. The only non-trivial case of the induction is that of successor $\alpha$. Assume then that $\alpha = \beta + 1$ and $i_{\mathcal{U}}| V_\beta$ is the identity. Let $x \in V_\alpha$. By the elementarity of $i_{\mathcal{U}}$,

$$i_{\mathcal{U}}(x) \in V_{i_{\mathcal{U}}(\alpha)} = V_\alpha.$$ 

Thus every member of $i_{\mathcal{U}}(x)$ belongs to $V_\beta$. If $y \in V_\beta$, then the elementarity of $i_{\mathcal{U}}$ and our induction hypothesis give that

$$y \in x \iff i_{\mathcal{U}}(y) \in i_{\mathcal{U}}(x) \iff y \in i_{\mathcal{U}}(x).$$

We have shown that $i_{\mathcal{U}}(x)$ and $x$ have the same members and hence that $i_{\mathcal{U}}(x) = x$.

It only remains to prove that if $\mathcal{U}$ is non-principal then $i_{\mathcal{U}}(\kappa) \neq \kappa$. Assume that $\mathcal{U}$ is non-principal. Let $\langle X_\alpha \mid \alpha < \kappa \rangle$ be such that each $X_\alpha \in \mathcal{U}$ but $\cap_{\alpha < \kappa} X_\alpha \notin \mathcal{U}$. Let $f : A \to V$ be given by

$$f(a) = \begin{cases} 
\mu \alpha a \notin X_\alpha & \text{if } a \notin \cap_{\alpha < \kappa} X_\alpha; \\
0 & \text{if } a \in \cap_{\alpha < \kappa} X_\alpha.
\end{cases}$$
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Now, for each \( \alpha < \kappa \), \( c_\alpha(a) = \alpha < f(a) \) for every \( a \in \bigcap_{\beta < \alpha} X_\beta \setminus \bigcap_{\alpha < \kappa} X_\alpha \). Since \( \bigcap_{\beta < \alpha} X_\beta \in \mathcal{U} \) and \( \bigcap_{\alpha < \kappa} X_\alpha \notin \mathcal{U} \), we have that \([c_\alpha] \prec [f] \) for each \( \alpha < \kappa \). But then

\[
\alpha = i_\mathcal{U}(\alpha) = \pi([c_\alpha]) \prec \pi([f]),
\]

for each \( \alpha < \kappa \). Thus \( \pi([f]) \) is an ordinal \( \geq \kappa \). But we also have that \( f(a) < \kappa \) for every \( a \in A \). Hence \([f] \prec [c_\kappa] \), and so \( \kappa \leq \pi([f]) \prec \pi([c_\kappa]) \), and so \( \kappa \leq \pi([f]) \prec \pi([c_\kappa]) = i_\mathcal{U}(\kappa) \). □

**Remark.** Note that the proof that \( i_\mathcal{U} \) is the identity on \( V_\kappa \) used only that \( i_\mathcal{U} : V \prec M \) for some \( M \) and that \( i_\mathcal{U} \) is the identity on \( \kappa \). Thus any \( j : V \prec M \) is the identity on \( V_{\text{crit}(j)} \).

If \( M \) is a class model of ZFC (or a large enough fragment of ZFC), then by \( V_\alpha^M \) we mean the \( \alpha \)-th stage of the rank hierarchy as defined in \( M \). If \( M \) is transitive, this is just \( V_\alpha \cap M \). If \( \mathcal{U} \) is a non-principal ultrafilter, then Lemma 3.2.10 shows that \( \text{Ult}(V; \mathcal{U}) \) and \( V \) agree to the completeness \( \kappa \) of \( \mathcal{U} \), i.e. \( V_\kappa^M = V_\kappa \). The following lemma shows that they agree one level further.

**Lemma 3.2.11.** Let \( \kappa \) be the completeness of a non-principal ultrafilter \( \mathcal{U} \) on a set \( A \). Then \( V_{\kappa+1}^{\text{Ult}(V; \mathcal{U})} = V_{\kappa+1} \). Indeed, \( \kappa(\text{Ult}(V; \mathcal{U})) \subseteq \text{Ult}(V; \mathcal{U}) \).

**Proof.** The second assertion actually implies the first; for, by Lemma 3.1.3 and either Exercises 3.1.1 and 3.1.2 or Lemma 3.2.15, the cardinal \( \kappa \) is inaccessible and so \( |V_\kappa| = \kappa \). But the first assertion has a simpler proof, so we give that proof separately: Let \( x \in V_{\kappa+1} \). Thus \( x \subseteq V_\kappa \). If \( y \in V_\kappa \), then

\[
y \in x \iff i_\mathcal{U}(y) \in i_\mathcal{U}(x) \iff y \in i_\mathcal{U}(x).
\]

Thus \( i_\mathcal{U}(x) \cap V_\kappa = x \). Since \( V_\kappa \) belongs to the transitive \( \text{Ult}(V; \mathcal{U}) \), it follows that \( x \in \text{Ult}(V; \mathcal{U}) \).

For the second assertion, let \( h : \kappa \to \text{Ult}(V; \mathcal{U}) \). For each \( \alpha < \kappa \), let \( h(\alpha) = \pi([f_\alpha]) \), with \( \pi = \pi_\mathcal{U} \). Let \( g : A \to \kappa \) be given by

\[
(g(a))(\alpha) = f_\alpha(a).
\]

Now \( \pi([g]) : i_\mathcal{U}(\kappa) \to \text{Ult}(V; \mathcal{U}) \) and for \( \alpha < \kappa \) we have that \( (\pi([g]))(\alpha) = (\pi([g]))(i_\mathcal{U}(\alpha)) = (\pi([g]))(\pi([c_\alpha])) = \pi([f_\alpha]) = h(\alpha) \). Thus \( h = \pi([g]) \upharpoonright \kappa \in \text{Ult}(V; \mathcal{U}) \). □
Theorem 3.2.12. ([Scott, 1961], [Keisler, 1962a]) Let \( \kappa \) be an ordinal number. The following are equivalent:

(a) \( \kappa \) is a measurable cardinal.

(b) There are a transitive class \( M \) and an embedding \( j : V \prec M \) such that \( \text{crit}(j) = \kappa \).

(c) There is transitive set \( N \) and an embedding \( k : V_{\kappa+1} \prec N \) such that \( \text{crit}(k) = \kappa \).

Proof. (a) \( \Rightarrow \) (b): Assume (a) and let \( U \) be a \( \kappa \)-complete non-principal ultrafilter on \( \kappa \). Since the completeness of \( U \) cannot be greater than \( |\kappa| = \kappa \), we can apply Lemma 3.2.10 with \( A = \kappa \). Thus (b) holds with \( M = \text{Ult}(V; U) \) and \( j = i_U \).

(b) \( \Rightarrow \) (c): Assume that \( M \) and \( j \) witness (b). Then \( N = V_{j(\kappa)+1} \cap M \) and \( k = j | V_{\kappa+1} \) witness (c).

(c) \( \Rightarrow \) (a): Assume that \( N \) and \( k \) witness (c). Let \( U = \{ X \subseteq \kappa \mid \kappa \in k(X) \} \).

Since \( \kappa \) is the critical point of \( k \), \( \kappa < k(\kappa) \), i.e. \( \kappa \in k(\kappa) \). By the elementarity of \( k \), we have that \( k(\emptyset) = \emptyset \) and so that \( \kappa \notin k(\emptyset) \). Thus \( U \) satisfies clause (a) in the definition of a filter. The elementarity of \( k \) also gives that \( k(X \cap Y) = k(X) \cap k(Y) \), that \( X \subseteq Y \rightarrow k(X) \subseteq k(Y) \), and that \( k(\kappa \setminus X) = k(\kappa) \setminus k(X) \); therefore \( U \) satisfies clauses (b), (c), and (d) in the definition of an ultrafilter. To verify the \( \kappa \)-completeness of \( U \), let \( \delta < \kappa \) and let \( X = \langle X_\gamma \mid \gamma < \delta \rangle \) be a sequence of elements of \( U \). The elementarity of \( k \) and the fact that \( \delta < \text{crit}(k) \) yield that 
\[
k(\bigcap_{\gamma < \delta} X_\gamma) = \bigcap_{\gamma < k(\delta)} (k(X))_\gamma = \bigcap_{\gamma < \delta} k(X_\gamma).
\]
But \( \kappa \in \bigcap_{\gamma < \delta} k(X_\gamma) \), so \( \bigcap_{\gamma < \delta} X_\gamma \in U \). \( U \) is non-principal, since for \( \alpha < \kappa \) we have that \( \kappa \notin \{ \alpha \} = k(\{ \alpha \}) \).

Remark. We included (c) in the statement of Theorem 3.2.12 to show that (b), which involves proper classes, has an equivalent version that involves only sets. Obviously (b) and (c) are also equivalent to each of the intermediate propositions gotten by replacing \( V_{\kappa+1} \) in (c) by \( V_{\kappa+\alpha} \) for ordinals \( \alpha > 1 \).

It is of interest that the ultrafilter \( U \) defined in the proof of (a) from (c) is actually normal:
Lemma 3.2.13. Let \( j : V \prec M \) with \( M \) transitive or let \( j : V_{\kappa+\alpha} \prec N \) with \( N \) transitive and \( \alpha \geq 1 \). Assume that \( \kappa = \text{crit}(j) \). Let \( U = \{ X \subseteq \kappa \mid \kappa \in j(X) \} \). Then \( U \) is a normal ultrafilter on \( \kappa \).

Proof. Suppose that \( \langle X_\beta \mid \beta < \kappa \rangle \) is a sequence of elements of \( U \). Then \( \kappa \in j(X_\beta) \) for each \( \beta < \kappa \). Thus \( \kappa \) belongs to the diagonal intersection of \( j(\langle X_\beta \mid \beta < \kappa \rangle) \). Thus \( \Delta_{\beta<\kappa} X_\beta \in U \). \( \square \)

Many large cardinal properties of a cardinal \( \kappa \) can be expressed in the form:

There is an elementary embedding \( j : V \prec M \) with \( M \) transitive, with \( \text{crit}(j) = \kappa \), and with \( M \) like \( V \) in respect \( R \).

For the property of being measurable, nothing like the last clause appears in (b) of Theorem 3.2.12. But such a clause could be added, as Lemma 3.2.11 shows. Thus we could strengthen (b) by adding “and with \( V_{\kappa+1} \subseteq M \)” or even “and with “\( M \subseteq M \)”.” In fact, the proof of the first assertion of Lemma 3.2.11 uses nothing special about \( i_U \) and \( \text{Ult}(V;U) \), so we have:

Lemma 3.2.14. If \( j : V \prec M \) with \( M \) transitive and \( \text{crit}(j) = \kappa \), then \( V_{\kappa+1} \subseteq M \).

The proof of the second part of Lemma 3.2.11 (that \( "(\text{Ult}(V;U)) \) \subseteq \text{Ult}(V;U) \), or—as we will say—that \( \text{Ult}(V;U) \) is \( \kappa \)-closed) depended on specific properties of \( \text{Ult}(V;U) \). Indeed the analogue of Lemma 3.2.14 fails: If \( \kappa \) is a measurable cardinal, then there is an embedding \( j : V \prec M \) with \( M \) transitive and \( \text{crit}(j) = \kappa \) such that \( M \) is not even countably closed. (See Exercise 3.3.2.)

Various properties of measurable cardinals can be proved rather easily from the elementary embedding version of measurability. For example, the fact that there is a normal ultrafilter on each measurable cardinal (Lemma 3.1.7) follows from Lemma 3.2.13. Another example is the following lemma. The original proof of its first assertion is in [Ulam, 1930]. (See Exercises 3.1.1 and 3.1.2.) The original proof of the second assertion is in [Hanf, 1964] and [Tarski, 1962]. The first part of the proof below is essentially from [Keisler, 1962b].

Lemma 3.2.15. Let \( \kappa \) be a measurable cardinal. Then \( \kappa \) is inaccessible. In fact \( \kappa \) is the \( \kappa \)th inaccessible cardinal.
Prove that there is no class 

Exercise 3.2.1.

Let \( V < M \) with \( M \) transitive and \( \text{crit}(j) = \kappa \).

To prove that \( \kappa \) is regular, let \( \delta < \kappa \) and let \( f: \delta \to \kappa \). Elementarity gives that \( j(f) : j(\delta) \to j(\kappa) \) and so that \( j(f) : \delta \to j(\kappa) \). Moreover for all \( \gamma < \delta \) we have that \( (j(f))(\gamma) = (j(f))(j(\gamma)) = j(f(\gamma)) = f(\gamma) \) (since \( f(\gamma) < \kappa \)). Hence \( j(f) = f \). But then the range of \( j(f) \) is not unbounded in \( j(\kappa) \), since it is bounded by \( \kappa < j(\kappa) \). By elementarity, the range of \( f \) is bounded in \( \kappa \).

To show \( \kappa \) is a strong limit cardinal, let \( \delta < \kappa \). If \( x \subseteq \delta \), then \( j(x) = x \). Moreover \( j(\mathcal{P}(\delta)) = \mathcal{P}(\delta) \). Let \( \lambda = |\mathcal{P}(\delta)| \). Let \( h: \mathcal{P}(\delta) \to \lambda \) be a bijection. Then \( j(h): \mathcal{P}(\delta) \to j(\lambda) \) is a bijection. For each \( x \in \mathcal{P}(\delta) \), \( (j(h))(x) = (j(h))(j(x)) = j(h(x)) \). But then every ordinal smaller than \( j(\lambda) \) belongs to the range of \( j \). Since \( \kappa \notin \text{range}(j) \), it follows that \( j(\lambda) \leq \kappa \) and so that \( \lambda < \kappa \). Thus we have shown that \( 2^\delta < \kappa \).

Now \( \kappa \) is inaccessible in \( M \), since any witness that \( \kappa \) is not inaccessible in \( M \) would also be a witness that \( \kappa \) is not inaccessible in \( V \). If \( \alpha < \kappa \), then \( M \models (\exists \beta)(\alpha < \beta < j(\kappa) \land \beta \text{ is inaccessible}) \). (Take \( \kappa \) for \( \beta \).) Hence in \( V \models (\exists \beta)(\alpha < \beta < \kappa \land \beta \text{ is inaccessible}) \). We have shown that the there are unboundedly many inaccessible cardinals smaller than \( \kappa \). Since \( \kappa \) is regular, this means that there are \( \kappa \) inaccessible cardinals smaller than \( \kappa \).

Exercise 3.2.7 is another example of this sort.

Suppose that we start with a uniform normal ultrafilter \( \mathcal{U} \) on a measurable cardinal \( \kappa \), that we form the elementary embedding \( i_{\mathcal{U}} \), and that we then construct a normal measure \( \mathcal{V} \) on \( \kappa \) from \( i_{\mathcal{U}} \) by letting \( \mathcal{V} = \{X \subseteq \kappa \mid \kappa \in i_{\mathcal{U}}(X)\} \). Then \( \mathcal{V} = \mathcal{U} \) (Exercise 3.2.3). On the other hand, if we start with \( j: V < M \) with \( M \) transitive and \( \text{crit}(j) = \kappa \) and if we then form \( \mathcal{V} = \{X \subseteq \kappa \mid \kappa \in j(X)\} \), it need not be true that \( i_{\mathcal{V}} = j \). (See Exercise 3.2.4.)

Exercise 3.2.1. Let \( M = \text{Ord} \). Define a relation \( E \) in \( M \) by

\[
\alpha E \beta \iff \begin{cases} 
(\alpha < \beta \text{ and } \alpha \text{ and } \beta \text{ are even}) \lor \\
(\alpha < \beta \text{ and } \alpha \text{ and } \beta \text{ are odd}) \lor \\
(\alpha \text{ is even and } \beta \text{ is odd})
\end{cases}
\]

Prove that there is no class \( N \) such that \( (M; E) \cong (N; \in) \).

Exercise 3.2.2. Let \( \mathcal{U} \) be a \( \kappa \)-complete, non-principal ultrafilter on a measurable cardinal \( \kappa \). Let \( \text{id} : \kappa \to \kappa \) be the identity. Prove that \( \mathcal{U} \) is normal if and only if \( \pi_{\mathcal{U}}([\text{id}]_{\mathcal{U}}) = \kappa \).
Exercise 3.2.3. Let \( \mathcal{U} \) be a uniform normal ultrafilter on a cardinal \( \kappa \). Let \( \mathcal{V} = \{ X \subseteq \kappa \mid \kappa \in i_{\mathcal{U}}(X) \} \). Show that \( \mathcal{V} = \mathcal{U} \).

Exercise 3.2.4. Let \( \mathcal{U} \) be a uniform normal ultrafilter on a cardinal \( \kappa \). Let \( \mathcal{V} = \{ X \subseteq \kappa \mid \kappa \in i_{\mathcal{U}^{[2]}}(X) \} \), where \( \mathcal{U}^{[2]} \) is the Rowbottom measure defined on page 136. Prove that \( \mathcal{V} = \mathcal{U} \) and that \( i_{\mathcal{V}}(\kappa) < i_{\mathcal{U}^{[2]}}(\kappa) \).

Exercise 3.2.5. Let \( \mathcal{U} \) be a uniform normal ultrafilter on \( \kappa \). Let \( f : \kappa \to V \). Show that

\[ (i_{\mathcal{U}}(f))(\kappa) = \pi_{\mathcal{U}}(\lfloor f \rfloor_{\mathcal{U}}). \]

Hint. Use Exercise 3.2.2.

Exercise 3.2.6. (a) Let \( j : V \prec M \) with \( M \) transitive and \( \text{crit}(j) = \kappa \). Let \( \mathcal{U} = \{ x \subseteq \kappa \mid \kappa \in j(X) \} \). Prove that there is a unique \( k : \text{Ult}(V; \mathcal{U}) \prec M \) such that \( k \circ i_{\mathcal{U}} = j \) and \( k \upharpoonright \kappa + 1 \) is the identity.

(b) Show that if \( \kappa \) is a measurable cardinal then there is a \( j : V \prec M \) with \( M \) transitive and \( \text{crit}(j) = \kappa \) such that, with \( \mathcal{U} \) as in (a), there is more than one \( k : \text{Ult}(V; \mathcal{U}) \prec M \) with \( k \circ i_{\mathcal{U}} = j \).

Hint. For (a), define \( k \) by setting \( k(\pi_{\mathcal{U}}(\lfloor f \rfloor_{\mathcal{U}})) = (j(f))(\kappa) \). For (b), let \( j = i_{\mathcal{U}^{[2]}} \) and let \( k' = i_{\mathcal{U} \upharpoonright \text{Ult}(V; \mathcal{U})} \).

Exercise 3.2.7. If \( \lambda \) is an ordinal number, then a subset \( C \) of \( \lambda \) is closed in \( \lambda \) if it is closed in the order topology or, equivalently, if \( \alpha \in C \) whenever \( \alpha < \lambda \) is a limit ordinal and \( C \) is unbounded in \( \alpha \). If \( \lambda \) is a limit ordinal, then a subset \( X \) of \( \lambda \) is stationary in \( \lambda \) if \( X \) meets every closed, unbounded subset of \( \lambda \). Note that the only stationary subsets of an ordinal \( \lambda \) of cofinality \( \omega \) are the complements in \( \lambda \) of bounded sets. A cardinal \( \kappa \) is Mahlo if \( \kappa \) is a strong limit cardinal and the set of all regular \( \alpha < \kappa \) is stationary in \( \kappa \). Clearly every Mahlo cardinal has uncountable cofinality. If \( \text{cf}(\kappa) > \omega \) and \( f : \delta \to \kappa \) witnesses that \( \kappa \) is not regular, then the set \( C \) of limit points of \( \text{range}(f) \) which are greater than \( \delta \) witnesses that \( \kappa \) is not Mahlo. Thus every Mahlo cardinal is inaccessible. Prove that every Mahlo cardinal \( \kappa \) is the \( \kappa \)th inaccessible cardinal. Prove that every measurable cardinal \( \kappa \) is the \( \kappa \)th Mahlo cardinal. This last result is a consequence of theorems in [Tarski, 1962] and [Hanf, 1964].
3.3 Iterated Ultrapowers

In this section we show how to iterate the ultrapower construction to get a sequence

\[ V = M_0 \overset{j_0}{\rightarrow} M_1 \overset{j_1}{\rightarrow} M_2 \overset{j_2}{\rightarrow} \ldots \]

defining class models and elementary embeddings. This system will be used in §4.3 and in determinacy proofs in Chapter 5. We also show how to extend the sequence into the transfinite. The machinery of iterated ultrapowers was introduced by Haim Gaifman and used by him to obtain the results of [Gaifman, 1964]. The machinery, in generalized form, is presented in [Gaifman, 1974].

If \( U \) is a countably complete ultrafilter on \( A \), then

\[ \text{Ult}(V; U) \models \text{ZFC} + \chi[i_U(U), i_U(A)], \]

where \( \chi(v_1, v_2) \) says "\( v_1 \) is a countably complete ultrafilter on \( v_2 \)." Thus we can, within \( \text{Ult}(V; U) \), form the ultrapower of the universe of sets with respect to \( i_U(U) \). The elements of this ultrapower are the "equivalence classes" of functions \( f : i_U(A) \rightarrow \text{Ult}(V; U) \) with \( f \in \text{Ult}(V; U) \); in other words, each element of the ultrapower is, for some such \( f \), the set of all \( g : i_U(A) \rightarrow \text{Ult}(V; U) \) of minimal rank such that \( g \in \text{Ult}(V; U) \) and \( \{ a \in i_U(A) \mid f(a) = g(a) \} \in i_U(U) \). We denote these classes by

\[ [f]^{\text{Ult}(V; U)}_{i_U(U)}. \]

Note that this need not be a true ultrapower (in the full universe \( V \)), since (1) \( i_U(U) \) may not be (and, unless \( U \) is principal, is in fact not) an ultrafilter in \( V \), and (2) we are using only functions in \( \text{Ult}(V; U) \). The class model \( \text{Ult}(V; U) \) satisfies the formula saying that this ultrapower is wellfounded and set-like. Since \( \text{Ult}(V; U) \) also satisfies (the relevant instance of) Lemma 3.2.8, this ultrapower is isomorphic to a transitive class. We denote this transitive class by \( \text{Ult}(\text{Ult}(V; U); i_U(U)) \), and we denote the canonical elementary embedding of \( \text{Ult}(V; U) \) into \( \text{Ult}(\text{Ult}(V; U); i_U(U)) \) by \( i_{\text{Ult}(V; U)} \).

In general, suppose that \( M \) is a transitive class satisfying ZFC, that \( A \) and \( V \in M \), and that \( M \models "V \) is an ultrafilter on \( A." \) We will denote the element represented by \( f \) in the ultrapower taken inside \( M \) of \( M \) with respect to \( V \) by \( [f]^M_{i_U(U)} \). We will denote the transitive class isomorphic to this ultrapower by \( \text{Ult}(M; V) \), and we will denote the canonical elementary embedding of \( M \)
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into \( \text{Ult}(M; \mathcal{V}) \) by \( i^M_\mathcal{V} \). Later we will generalize these notions further, allowing \( M \) to be a model of a fragment of ZFC and not requiring that \( \mathcal{V} \) belong to \( M \). Even in these more general cases, we will use the same notation, with the “M” in \([f]_\mathcal{V}^M\), in \( i^M_\mathcal{V} \), and in \( \text{Ult}(M; \mathcal{V}) \) signifying that the ultrapower is the ultrapower of \( M \) using the functions in \( M \).

We can think of an elementary embedding \( j : M \prec N \), for \( M \) and \( N \) (with \( \in \)) transitive class models, as acting on subclasses \( Y \) of \( M \) which satisfy \((\forall \alpha \in \text{Ord} \cap M) Y \cap V_\alpha \in M \); for we can let

\[
j(Y) = \bigcup_{\alpha \in \text{Ord} \cap M} j(Y \cap V_\alpha).
\]

Thus

\[
\text{Ult}(\mathcal{V}; \mathcal{U}) = i_\mathcal{U}(\mathcal{V});
\]
\[
\text{Ult}(\text{Ult}(\mathcal{V}; \mathcal{U}); i_\mathcal{U}(\mathcal{U})) = i_\mathcal{U}(\text{Ult}(\mathcal{V}; \mathcal{U}));
\]
\[
i_{\text{Ult}(\mathcal{V}; \mathcal{U})} = i_\mathcal{U}(i_\mathcal{U}).
\]

Moreover, for any transitive \( M \) and \( j : V \prec M \) we can define \( j(V) (= M) \) and \( j(j) : M \prec j(M) \). Suppose that \( j : M \prec N \) with \( M \) and \( N \) transitive and with \( M \) satisfying, say, ZFC. If \( j \subseteq M \) and is definable in \( M \) from elements of \( M \), i.e. if \( j \) is a class in \( M \), then we have \( j(M) = N \) and \( j(j) : N \rightarrow j(N) \).

Let \( M \) be a transitive class model of ZFC and let \( j : M \prec N \) with \( N \) transitive and \( j \) a class in \( M \). We define inductively, for \( n \in \omega \), transitive classes \( M^j_n \) and embeddings \( j_n : M^j_n \prec M^j_{n+1} \) as follows:

(a) \( M^j_0 = M \);
(b) \( j_0 = j \);
(c) \( M^j_{n+1} = j_n(M^j_n) \);
(d) \( j_{n+1} = j_n(j_n) \).

We can also define \( j_{m,n} : M^j_m \prec M^j_n \), for \( m \leq n \in \omega \) by composition:

(i) \( j_{m,m} \) is the identity;
(ii) \( j_{m,n+1} = j_n \circ j_m \).

Note that each \( j_{m,n} \) is a class in \( M^j_m \). Exercise 3.3.1 concerns some properties of the \( j_{m,n} \).
Let $M$ be a transitive class model of ZFC and let $U \in M$ be, in $M$, a countably complete ultrafilter on $A \in M$. For $n \in \omega$ we let

$$\text{Ult}_n(M; U) = M^U_n.$$  

**Lemma 3.3.1.** Let $M$ and $U$ be as in the preceding paragraph and let $i = i_U$. For each $n \in \omega$, $\mathcal{U}_n = i_{0,n}(U)$ is, in $\text{Ult}_n(M; U)$, a countably complete ultrafilter on $i_{0,n}(A)$. Moreover each $\text{Ult}_{n+1}(M; U) = \text{Ult}(\text{Ult}_n(M; U); \mathcal{U}_n)$ and each $i_n = i_{\text{Ult}_n(M; U)}$.

The proof of the lemma is routine, and we omit it.

**Lemma 3.3.2.** Let $\kappa$ be the completeness of a countably complete, non-principal ultrafilter $U$ on a set $A$. Then, for all $n \in \omega$, $V^{\text{Ult}_n(V; U)}_{i_0,n}(\kappa) = V^+=\kappa+1$. Indeed all $\text{Ult}_n(V; U)$ are $\kappa$-closed, i.e. $\kappa(\text{Ult}_n(V; U)) \subseteq \text{Ult}_n(V; U)$.

**Proof.** Let $i = i_U$. For each $m \in \omega$, applying Lemma 3.2.11 in $\text{Ult}_m(V; U)$ gives us that

$$V^{\text{Ult}_{m+1}(V; U)}_{i_{0,m}(\kappa)+1} = V^{\text{Ult}_m(V; U)}_{i_{0,m}(\kappa)+1}$$

and that

$$\text{Ult}_m(V; U) \cap i_{0,m}(\kappa)(\text{Ult}_{m+1}(V; U)) \subseteq \text{Ult}_{m+1}(V; U).$$

Since

$$\kappa < i_{0,1}(\kappa) < i_{0,2}(\kappa) < \cdots,$$

the lemma follows by induction. 

The first assertion of Lemma 3.3.2 follows also from the special case $n = 0$ and the fact that $\text{crit}(i_{1,n+1}) > \kappa$, and so, using Lemma 3.2.14:

**Lemma 3.3.3.** If $j : V \prec M$ with $M$ transitive and $\text{crit}(j) = \kappa$ then, for all $n \in \omega$, $V^{\kappa+1} \subseteq j_{0,n}(V)$.

Let $M$ be a transitive class model of ZFC and let $j : M \prec N$ with $N$ transitive and $j$ a class in $M$. The direct limit

$$(\mathcal{M}_i^j, \langle j_{m,\omega} \mid m \in \omega \rangle)$$

of $((\langle M_i^j \rangle \mid n \in \omega), \langle j_{m,n} \mid m \leq n \in \omega \rangle)$ is given as follows:
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For \( x \in M^j_m \) and \( y \in M^j_n \) and \( m \leq n \), we let

\[
\langle m, x \rangle \sim \langle n, y \rangle \iff \langle n, y \rangle \sim \langle m, x \rangle \iff j_{m,n}(x) = y.
\]

Let \([m, x]\) be the equivalence class of \( \langle m, x \rangle \) with respect to the equivalence relation \( \sim \). We set

\[\tilde{M}^j = \{ [m, x] \mid m \in \omega \land x \in M^j_m \}\]

For \( x \in M^j_m \) and \( y \in M^j_n \), we define

\[\{m, x\} \tilde{E}^j_{\omega} [n, y] \iff \begin{cases} 
j_{m,n}(x) \in y & \text{if } m \leq n; \\
x \in j_{n,m}(y) & \text{if } n < m.
\end{cases}\]

Let \( \tilde{M}^j = (\tilde{M}^j; \tilde{E}^j_\omega) \). Finally, we let \( j_{m,\omega}(x) = [m, x] \).

Remarks:

(a) We have used the natural notation “\( \langle j_{m,n} \mid m \leq n \in \omega \rangle \)” but this should not be construed literally, since we want the object to be a genuine class (with sets as members). A similar comment applies to, e.g., “\( (M^j_n; \in) \mid n \in \omega \).”

(b) Since each \( M^j_n = j_{0,n}(M) \), we could dispense with talk of \( \langle (M^j_n; \in) \mid n \in \omega \rangle \) and say that \( \langle j_{m,\omega} \mid m \in \omega \rangle \) is the direct limit of \( \langle j_{m,n} \mid m \leq n \in \omega \rangle \).

Lemma 3.3.4. Let \( M \) be a transitive class model of ZFC and let \( j : M \prec N \) with \( N \) transitive and \( j \) a class in \( M \). Then for all natural numbers \( m \) and \( n \) with \( m \leq n \), \( j_{m,\omega} = j_{n,\omega} \circ j_{m,n} \). Moreover \( j_{m,\omega} : (M^j_m; \in) \prec (M^j_n; \tilde{E}^j_\omega) \) for all \( n \in \omega \).

Proof. The proof of the first assertion is routine, and we omit it. The second assertion follows from the elementary chain theorem of Tarski–Vaught [1957]. Nevertheless, we give the proof: We proceed by induction on the complexity of formulas \( \varphi \). The only non-trivial case is that of a formula \( \varphi(v_1, \ldots, v_k) \) of the form \((\exists v_0) \psi(v_0, \ldots, v_k)\). Consider such a \( \varphi \) and \( \psi \). Let \( m \in \omega \) and let \( \langle x_1, \ldots, x_k \rangle \in M^j_m \). If \( M^j_m \models \varphi[x_1, \ldots, x_k] \), then there is an \( x_0 \in M^j_m \) such that \( M^j_m \models \psi[x_0, \ldots, x_k] \). By the induction hypothesis for \( \psi \), we get that \( M^j_m \models \psi[j_{m,\omega}(x_0), \ldots, j_{m,\omega}(x_k)] \) and so that \( M^j_m \models \varphi[j_{m,\omega}(x_1), \ldots, j_{m,\omega}(x_k)] \). Suppose then that \( M^j_m \models \varphi[j_{m,\omega}(x_1), \ldots, j_{m,\omega}(x_k)] \). Let \( \tilde{x} \in M^j_m \) be such that \( \tilde{x} =


$n, y]$. Assume, without loss of generality, that $n \geq m$. By induction and the first assertion of the lemma, we have that $M_n^j \models \psi[n, j_m(x_1), \ldots, j_m(x_k)]$. Thus $M_n^j \models \varphi[j_m, n(x_1), \ldots, j_m, n(x_k)]$. The elementarity of $j_m, n$ implies that $M_n^j \models \varphi[x_1, \ldots, x_k]$. \hfill \Box

**Lemma 3.3.5.** ([Gaifman, 1974]) Let $M$ and $j$ be as in the statement of Lemma 3.3.4. Then $\check{M}_\omega^j$ is wellfounded.

**Proof.** Assume that the lemma is false. There is an infinite sequence $\langle \check{y}_n \mid n \in \omega \rangle$ such that

$$\ldots \check{E}^j_\omega \check{y}_2 \check{E}^j_\omega \check{y}_1 \check{E}^j_\omega \check{y}_0.$$ 

Hence there is such a sequence with $\check{y}_0$ of the form $[n, y]$ and so of the form $\check{j}_{0, n}(y)$. If $x = V^M_{\text{rank}(y)+1}$, then $j_{0, n}(x) = V^M_{j_{0, n}(\text{rank}(y)+1)}$ and so $y \in j_{0, n}(x)$. Hence there is an $x \in M$ such that there is a sequence $\langle \check{y}_n \mid n \in \omega \rangle$ with

$$\ldots \check{E}^j_\omega \check{y}_2 \check{E}^j_\omega \check{y}_1 \check{E}^j_\omega \check{y}_0 = \check{j}_{0, \omega}(x).$$

Let $x \in M$ have minimal rank with this property. Choose such a sequence and let $n$ and $u \in M^j_n$ be such that $\check{y}_1 = \check{j}_{n, \omega}(u)$. By the elementarity of $j_{0, n}$, we have that $j_{0, n}(x)$ is, in $M^j_n$, a $\omega$ of minimal rank with the following property $P$: There is an infinite sequence $\langle \check{z}_i \mid i \in \omega \rangle$ such that

$$\ldots \check{E}^j_\omega \check{z}_2 \check{E}^j_\omega \check{z}_1 \check{E}^j_\omega \check{z}_0 = (\check{j}_n)_{0, \omega}(w).$$

But $M^j_n = M^j_n$ and $(j_n)_{k, m} = j_{n+k,n+m}$ for all $k \leq n \in \omega$. Thus $P(w)$ is equivalent with the existence of a sequence $\langle \check{z}_i \mid i \in \omega \rangle$ such that

$$\ldots \check{E}^j_\omega \check{z}_2 \check{E}^j_\omega \check{z}_1 \check{E}^j_\omega \check{z}_0 = \check{j}_{n, \omega}(w).$$

But this is a contradiction; for $u$ also has this property, and $u \in j_{0,n}(x)$ so $\text{rank}(u) < \text{rank}(j_{0,n}(x))$. \hfill \Box

**Lemma 3.3.6.** Let $M$ and $j$ be as in the statement of Lemma 3.3.4. Then $(\check{M}_\omega^j; \check{E})$ is set-like.

**Proof.** Let $\check{y} \in \check{M}_\omega^j$. Let $n \in \omega$ and $x \in M^j_n$ be such that $\check{y} = [n, x]$, so that $\check{y} = \check{j}_{n, \omega}(x)$. Suppose $\check{z} \check{E}^j_\omega \check{y}$. Let $m \in \omega$ and $u \in M^j_m$ be such that $\check{z} = j_{m, \omega}(u)$. If $k \geq \max\{m, n\}$, then $j_{k, \omega}(j_{m, k}(u)) = \check{z}$ and $j_{k, \omega}(j_{n, k}(x)) = \check{y}$ and so the elementarity of $j_{k, \omega}$ yields that $j_{m, k}(u) \in j_{n, k}(x)$. We have then shown that
3.3. ITERATED ULTRAPOWERS

for every \( \tilde{z} \tilde{E} j_{\omega} \tilde{y} \) there is a \( k \geq n \) and a \( v \in j_{n,k}(x) \) such that \( \tilde{z} = \tilde{j}_{k,\omega}(v) \). By the Axiom of Replacement, \( \{ \tilde{z} | \tilde{z} \tilde{E} j_{\omega} \tilde{y} \} \) is a set. \( \square \)

If \( M \) and \( j \) are as in the statement of Lemma 3.3.4, then Lemmas 3.3.5, 3.3.6, and 3.2.8 imply that there is a transitive class \( M^j \) such that \((M^j; \in)\) is isomorphic to \((M^j; \tilde{E} j_{\omega})\). Thus we can extend the \( M^j \) and \( j_{m,n} \) into the transfinite. Our next goal is to carry this out.

First we introduce some general terminology for direct limits. A \( \text{directed relation} \) on a set \( D \) is a transitive, reflexive relation \( R \) in \( D \) such that

\[
(\forall x \in D)(\forall y \in D)(\exists z \in D)(x R z \land y R z).
\]

A \( \text{directed system of homomorphisms} \) is something of the form

\[
(\langle M_d | d \in D \rangle; \langle j_{d,d'} | d \in D \land d' \in D \land d R d' \rangle),
\]

where \( R \) is a directed relation on \( D \), each \( M_d \) is a class model, each \( j_{d,d'} : M_d \rightarrow M_{d'} \) is a homomorphism, and

\[
(\forall d_1 \in D)(\forall d_2 \in D)(\forall d_3 \in D)(d_1 R d_2 R d_3 \rightarrow j_{d_1,d_3} = j_{d_2,d_3} \circ j_{d_1,d_2}).
\]

Such a directed system of homomorphisms is a \( \text{directed system of elementary embeddings} \) if each \( j_{d,d'} : M_d \prec M_{d'} \).

The \( \text{direct limit} \) of a directed system \((\langle M_d | d \in D \rangle; \langle j_{d,d'} | d \in D \land d' \in D \land d R d' \rangle)\) of homomorphisms is \((\tilde{M}; \tilde{E})\), where \( \tilde{M} = (\tilde{M}; \tilde{E}) \) and the \( \tilde{j}_d \) are defined as follows. For \( d \in D \) let \( \tilde{M}_d = (\tilde{M}_d; \tilde{E}_d) \). For \( x \in \tilde{M}_d \) and \( y \in \tilde{M}_{d'} \), define

\[
\langle d, x \rangle \sim \langle d', y \rangle \iff (\exists d'')(d R d'' \land d' R d'' \land j_{d,d''}(x) = j_{d',d''}(y)).
\]

Note that the defining condition on the right is equivalent with

\[
(\forall d'')(d R d'' \land d' R d'') \rightarrow j_{d,d''}(x) = j_{d',d''}(y).
\]

Let \([d, x]\) be the equivalence class of \( \langle d, x \rangle \) with respect to the equivalence relation \( \sim \). We set

\[
\tilde{M} = \{ [d, x] | d \in D \land x \in M_d \}.
\]

For \( x \in \tilde{M}_d \) and \( y \in \tilde{M}_{d'} \), we define

\[
[d, x] \tilde{E} [d', y] \iff (\exists d'')(d R d'' \land d' R d'' \land j_{d,d''}(x) \tilde{E}_{d''} j_{d',d''}(y)).
\]

Finally, we let \( \tilde{j}_d(x) = [d, x] \).
Lemma 3.3.7. Let \((\mathcal{M}_d \mid d \in D); (j_{d,d'} \mid d \in D \wedge d' \in D \wedge dRd')\) be a directed system of elementary embeddings. Let \((\tilde{\mathcal{M}}; \tilde{j}_d \mid d \in D)\) be the direct limit of this directed system. Then \(\tilde{j}_d : \mathcal{M}_d \prec \tilde{\mathcal{M}}\) for all \(d \in D\). For all \(d\) and \(d'\) in \(D\), with \(dRd'\), \(\tilde{j}_d = \tilde{j}_d' \circ j_{d,d'}\).

Moreover \(\tilde{\mathcal{M}}\) is set-like if all the \(\mathcal{M}_d\) are set-like.

We omit the proof of Lemma 3.3.7. The proof of the first two assertions is similar to the proof of Lemma 3.3.4. The proof of the last assertion is like that of Lemma 3.3.6.

Let \(M\) be a transitive class model of ZFC and let \(j : M \prec N\) with \(N\) transitive and \(j\) a class in \(N\). We define inductively (1) \(\xi_j\), which will be either an ordinal number or \(\text{Ord}\), (2) for \(\alpha < \xi_j\), transitive class models \(M^j_{\alpha}\), and (3) for \(\alpha \leq \beta < \xi_j\), embeddings \(j_{\alpha,\beta} : M^j_{\alpha} \prec M^j_{\beta}\). (What we define inductively is, of course, not \(\xi_j\) but rather membership in \(\xi_j\), i.e. an ordinal’s being < \(\xi_j\).) Our inductive definition will guarantee that the \(j_{\alpha,\beta}\) commute: for \(\alpha \leq \beta \leq \gamma < \xi_j\), \(j_{\alpha,\gamma} = j_{\beta,\gamma} \circ j_{\alpha,\beta}\).

(i) \(0 < \xi_j\) and \(M^j_0 = M\).

(ii) If \(\alpha < \xi_j\) then \(j_{\alpha,\alpha} = \text{id}\).

(iii) If \(\alpha < \xi_j\) then \(\alpha + 1 < \xi_j\).

(iv) If \(\alpha < \xi_j\) then \(j_{\alpha,\alpha+1} = j_{\alpha,\alpha}(j)\) and \(M^j_{\alpha+1} = j_{\alpha,\alpha+1}(M^j_{\alpha})\). (This latter stipulation makes sense, as \(j_{\alpha,\alpha+1}\) is a class in \(M^j_{\alpha}\).)

(v) If \(\gamma < \alpha < \xi_j\) then \(j_{\gamma,\alpha+1} = j_{\alpha,\alpha+1} \circ j_{\gamma,\alpha}\).

(vi) If \(\alpha \leq \xi_j\) and \(\alpha\) is a limit ordinal, we define ((\(\tilde{\mathcal{M}}^j_\alpha; \tilde{E}_\alpha^j\), \(\{j_{\beta,\alpha} \mid \beta < \alpha\}\)) to be the direct limit of ((\(M^j_{\beta}; \in\) \(\mid \beta < \alpha\), \(\{j_{\beta,\gamma} \mid \beta \leq \gamma < \alpha\}\)). If \((\tilde{\mathcal{M}}^j_\alpha; \tilde{E}_\alpha^j)\) is not wellfounded, then \(\alpha = \xi_j\). If \((\tilde{\mathcal{M}}^j_\alpha; \tilde{E}_\alpha^j)\) is wellfounded, then, since Lemma 3.3.7 implies that it is set-like as well, we let \(\pi^j_\alpha : (\tilde{M}^j_\alpha; \tilde{E}^j_\alpha) \equiv (M^j_\alpha; \in)\) be given by Lemma 3.2.8. We set \(j_{\gamma,\alpha} = \pi^j_\alpha \circ \tilde{j}_{\gamma,\alpha}\).

Extending our notation from the case of finite \(\alpha\), let us set \(j_\alpha = j_{0,\alpha}(j)\) for \(\alpha < \xi_j\).

Lemma 3.3.8. ([Gaifman, 1974]) Let \(M\) be a transitive class model of ZFC and let \(j : M \prec N\) with \(N\) transitive and \(j\) a class in \(M\). Then \(\xi_j \geq \text{Ord} \cap M\).
Proof. The proof is similar to that of Lemma 3.3.5. Assume the lemma is false for \( j \) and \( M \). In \( M \) define \( x \) to have minimal rank such that there is an infinite sequence \( \langle y_i \mid i \in \omega \rangle \) such that

\[
\ldots \tilde{E}^j_{x_j} \tilde{y}_2 \tilde{E}^j_{x_j} \tilde{y}_1 \tilde{E}^j_{x_j} \tilde{y}_0 = \tilde{j}_0, x_j(\tilde{y}).
\]

Choose such a sequence and let \( \alpha < \xi \) and \( u \in M^j_\alpha \) be such that \( \tilde{y}_1 = \tilde{j}_0, x_j(u) \).

By the elementarity of \( j_{0,\alpha} \), we have that \( j_{0,\alpha}(x) \) is in \( M^j_\alpha \), a \( w \) of minimal rank such that there is an infinite sequence \( \langle z_i \mid i \in \omega \rangle \) such that

\[
\ldots \tilde{E}^{j_\alpha}_{x_{j_\alpha}} \tilde{z}_2 \tilde{E}^{j_\alpha}_{x_{j_\alpha}} \tilde{z}_1 \tilde{E}^{j_\alpha}_{x_{j_\alpha}} \tilde{z}_0 = (j_\alpha)_{0,j_{0,\alpha}}(w).
\]

But, since \( (M_\beta)^{j_\alpha} = M^{j_\alpha}_{\beta+\alpha} \) and \( (j_{0,\alpha})_{\beta,\gamma} = j_{\alpha+\beta,\alpha+\gamma} \) for all \( \beta \leq \gamma < \xi_{j_\alpha} \), we get a contradiction as in the proof of Lemma 3.3.5: \( u \) as well as \( j_{0,\alpha}(x) \) has this property, and \( u \in j_{0,\alpha}(x) \) so rank \( (u) < \text{rank}(j_{0,\alpha}(x)) \). \( \square \)

Let \( M \) be a transitive class model of ZFC and let \( \mathcal{U} \in M \) be in \( M \) a countably complete ultrafilter on \( A \in M \). For each \( \alpha < \xi_{\mathcal{U}} \), so in particular for each \( \alpha \leq \text{Ord} \cap M \), we let

\[ \text{Ult}_\alpha(M; \mathcal{U}) = M^{M_\alpha}_{\mathcal{U}}. \]

Lemma 3.3.9. Let \( M \) and \( \mathcal{U} \) be as in the preceding paragraph and let \( i = i^M_{\mathcal{U}} \).

For each \( \alpha < \xi_{\mathcal{U}} \), \( \mathcal{U}_\alpha = i_{0,\alpha}(\mathcal{U}) \) is, in \( \text{Ult}_\alpha(M; \mathcal{U}) \), a countably complete ultrafilter on \( i_{0,\alpha}(A) \). Moreover each \( \text{Ult}_{\alpha+1}(M; \mathcal{U}) = \text{Ult}(\text{Ult}_\alpha(M; \mathcal{U}); \mathcal{U}_\alpha) \)
and each \( i_\alpha = i_{\text{Ult}_\alpha(M; \mathcal{U})} \).

Lemma 3.3.10. Let \( M \) be a transitive class model of ZFC. Let \( \kappa \) be a cardinal of \( M \) and let \( \mathcal{U} \in M \) be such that \( M \models \mathcal{U} \) is a uniform normal ultrafilter on \( \kappa \).” Let \( i = i^M_{\mathcal{U}} \). Let \( \beta \) be a limit ordinal such that \( \beta < \xi_{\mathcal{U}} \). Then the set \( \{ i_{0,\gamma}(\kappa) \mid \gamma < \beta \} \) is unbounded in \( i_{0,\beta}(\kappa) \).

Proof. Let \( \eta < i_{0,\beta}(\kappa) \). Because \( \eta \) belongs to \( \text{Ult}_\beta(M; \mathcal{U}) \), there must be ordinals \( \gamma < \beta \) and \( \nu \) such that \( \eta = i_{\gamma,\beta}(\nu) \). If \( \nu \geq i_{0,\gamma}(\kappa) \), then \( \eta \geq i_{\gamma,\beta}(i_{0,\gamma}(\kappa)) = i_{0,\beta}(\kappa) \). Thus \( \nu < i_{0,\gamma}(\kappa) \). But \( \text{crit}(i_{\gamma,\beta}) = i_{0,\gamma}(\kappa) \), and so \( \eta = \nu \) and thus \( \eta < i_{0,\gamma}(\kappa) \). \( \square \)

Lemma 3.3.11. ([Kunen, 1968]) Let \( M \) be a transitive class model of ZFC. Let \( \kappa \) be a cardinal of \( M \) and let \( \mathcal{U} \in M \) be such that \( M \models \mathcal{U} \) is a uniform normal ultrafilter on \( \kappa \).” Let \( i = i^M_{\mathcal{U}} \). Let \( \alpha \) be a limit ordinal with \( \alpha < \xi_{\mathcal{U}} \). Let \( X \subseteq i_{0,\alpha}(\mathcal{U}) \) with \( X \in \text{Ult}_\alpha(M; \mathcal{U}) \). Then

\[ X \in i_{0,\alpha}(\mathcal{U}) \iff (\exists \beta < \alpha)(\forall \gamma)(\beta \leq \gamma < \alpha \rightarrow i_{0,\gamma}(\kappa) \in X). \]
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Proof. We begin by, in effect, doing Exercise 3.2.3 and half of Exercise 3.2.2. Let \( \pi = \pi^M_\mathcal{U} \) and, for \( f : \kappa \to M \) with \( f \in M \), let \( [f] = [f]_\mathcal{U}^M \).

First we show that \( \pi([\text{id}]) = \kappa \). Since \( \pi([c_\eta]) < \pi([\text{id}]) \) for every \( \eta < \kappa \) by the uniformity of \( \mathcal{U} \) in \( M \), we know that \( \kappa \leq \pi([\text{id}]) \). If \( \pi([f]) < \pi([\text{id}]) \), then the normality of \( \mathcal{U} \) in \( M \) implies that \( [f] = [c_\eta] \) for some \( \eta < \kappa \). Thus \( \pi([\text{id}]) \leq \kappa \).

Next we show that \( \mathcal{U} = \{ Y \subseteq \kappa \mid Y \in M \wedge \kappa \in i(Y) \} \).

This is because, for \( Y \subseteq \kappa \) with \( Y \in M \),

\[
Y \in \mathcal{U} \leftrightarrow \{ \eta \mid \text{id}(\eta) \in Y \} \in \mathcal{U} \leftrightarrow \pi([\text{id}]) \in i(Y) \leftrightarrow \kappa \in i(Y).
\]

Since \( \alpha \) is a limit ordinal, there exist \( \beta < \alpha \) and \( Y \in \text{Ult}_{\beta}(M; \mathcal{U}) \) such that \( X = i_{\beta,\alpha}(Y) \). By the elementarity of \( i_{0,\beta} \), we have that

\[
Y \in i_{0,\beta}(\mathcal{U}) \leftrightarrow i_{0,\beta}(\kappa) \in i_{\beta}(Y).
\]

Let \( \gamma \) be such that \( \beta \leq \gamma < \alpha \). Then \( X \in i_{0,\alpha}(\mathcal{U}) \leftrightarrow Y \in i_{0,\beta}(\mathcal{U}) \leftrightarrow i_{0,\beta}(\kappa) \in i_{\beta}(Y) \leftrightarrow i_{0,\gamma}(\kappa) \in i_{\beta,\gamma+1}(Y) \leftrightarrow i_{0,\gamma}(\kappa) \in i_{\beta,\alpha}(Y) \), where the last equivalence holds because \( \text{crit}(i_{\gamma+1,\alpha}) > i_{0,\gamma}(\kappa) \).

\[ \square \]

Lemma 3.3.12. ([Kunen, 1968]) Let \( M \) be a transitive class model of ZFC. Let \( \kappa \) be a cardinal number and let \( \mathcal{U} \) be a uniform normal ultrafilter on \( \kappa \). Let \( i = i^M_\mathcal{U} \). Let \( \alpha \) be a limit ordinal such that \( \alpha < \xi_i \) and \( \text{cf}(\alpha) > \omega \).

Then \( i_{0,\alpha}(\mathcal{U}) \) is the restriction to \( \text{Ult}_{\alpha}(M; \mathcal{U}) \) of the closed, unbounded filter on \( i_{0,\alpha}(\kappa) \); i.e., if \( X \in \text{Ult}_{\alpha}(M; \mathcal{U}) \) and \( X \subseteq i_{0,\alpha}(\kappa) \), then \( X \in i_{0,\alpha}(\mathcal{U}) \) if and only if \( X \) has a subset that is closed and unbounded in \( i_{0,\alpha}(\kappa) \).

Proof. By the preceding lemma, it suffices to prove that \( \{ i_{0,\gamma}(\kappa) \mid \gamma < \alpha \} \) is closed and unbounded in \( i_{0,\alpha}(\kappa) \). But this follows easily from Lemma 3.3.10. \[ \square \]

Lemma 3.3.13. Let \( \kappa \) be a cardinal number and let \( \mathcal{U} \) be a uniform normal ultrafilter on \( \kappa \). Let \( i = i_\mathcal{U} \).

(a) For all ordinals \( \eta \) and \( \alpha \) with \( \alpha > 0 \), \( |i_{0,\alpha}(\eta)| \leq \max\{|\eta|^\kappa, |\alpha|\} \).

(b) For every ordinal \( \alpha \geq 2^\kappa \), \( |i_{0,\alpha}(\kappa)| = |\alpha| \).
(c) For every cardinal \( \delta > 2^\kappa \), \( i_{0,\delta}(\kappa) = \delta \).

(d) For every cardinal \( \delta > 2^\kappa \) such that \( \text{cf}(\delta) > \kappa \) and for every ordinal \( \alpha < \delta \), \( i_{0,\alpha}(\delta) = \delta \).

**Proof.** We prove (a) by induction on \( \alpha \), simultaneously for all \( \eta \).

If \( \eta > 0 \) and \( [f]_U < [c_\eta]_U \), then there is a \( g : \kappa \to \eta \) such that \( g \sim_U f \).

This fact implies (a) for the case \( \alpha = 1 \).

Assume that (a) holds for \( \alpha \). For any \( \eta \), we have that

\[
|i_{0,\alpha+1}(\eta)| \leq |i_{0,\alpha}(\kappa) i_{0,\alpha}(\eta)|^{\text{Ult}(V, U)} \\
= i_{0,\alpha}(|\eta|^\kappa) \\
\leq \max\{|\eta|^\kappa, |\alpha|\} \\
= \max\{|\eta|^\kappa, |\alpha|\}.
\]

Here the first inequality is by the case \( \alpha = 1 \) and the elementarity of \( i_{0,\alpha} \).

Finally assume that (a) holds for all \( \alpha < \lambda \), where \( \lambda \) is a limit ordinal.

For any \( \eta \),

\[
|i_{0,\lambda}(\eta)| = \sup_{\alpha < \lambda} |i_{0,\alpha}(\eta)| \\
\leq \sup_{\alpha < \lambda} \max\{|\eta|^\kappa, |\alpha|\} \\
= \max\{|\eta|^\kappa, |\lambda|\}.
\]

(b) follows easily from (a).

For (c), let \( \delta \) be a cardinal larger than \( 2^\kappa \). Part (b) of the lemma implies that \( i_{0,\gamma}(\kappa) < \delta \) for all \( \gamma < \delta \). Lemma 3.3.10 then gives that \( i_{0,\delta}(\kappa) = \delta \).

For (d), let \( \delta > 2^\kappa \) be a cardinal with \( \text{cf}(\delta) > \kappa \). Let \( \alpha < \delta \) be such that (d) fails for \( \delta \) and \( \alpha \) but holds for \( \delta \) and all \( \beta < \alpha \). Obviously \( \alpha > 0 \).

Suppose that \( \alpha = 1 \). Let \( f \) be such that \( \delta \leq \pi_U([f]_U) < i(\delta) \). We may assume that \( f : \kappa \to \delta \). Since \( \text{cf}(\delta) > \kappa \), there is an \( \eta < \delta \) such that \( f : \kappa \to \eta \). But then \( \pi_U([f]_U) < i(\eta) \). But this contradicts part (a), which implies that \( i(\eta) < \delta \).

The fact that \( \alpha \neq 1 \) implies that, for any \( \beta \), \( i_\beta(i_{0,\beta}(\delta)) = i_{0,\beta}(\delta) \). This means that \( \alpha \) cannot be a successor ordinal \( \beta + 1 \).

Thus \( \alpha \) must be a limit ordinal. Let \( \nu \) be such that \( \delta < \nu < i_{0,\alpha}(\delta) \). Let \( \gamma < \alpha \) and \( \rho \) be such that \( \nu = i_{\gamma,\alpha}(\rho) \). Then \( \rho < i_{0,\gamma}(\delta) = \delta \). Hence \( \rho + 1 < \delta \).

But \( \nu < i_{\gamma,\alpha}(\rho + 1) \leq i_{0,\alpha}(\rho + 1) \), and this contradicts part (a), which implies that \( i_{0,\alpha}(\eta) < \delta \) for every \( \eta < \delta \). \( \square \)
Lemma 3.3.14. Let $\kappa$ and $\kappa'$ be a cardinal numbers with $\kappa < \kappa'$. Let $\mathcal{U}$ and $\mathcal{U}'$ be uniform normal ultrafilters on $\kappa$ and $\kappa'$ respectively. Let $i = i_{\mathcal{U}}$. For any $\alpha < \kappa'$, $i_{0,\alpha}(\mathcal{U}') = \mathcal{U}' \cap \Ult_\alpha(V;\mathcal{U})$.

**Proof.** By part (d) of Lemma 3.3.13, $i_{0,\alpha}(\kappa') = \kappa'$ for every $\alpha < \kappa'$. It also follows easily from part (d) of Lemma 3.3.13 that, for all $\eta < \kappa$, the set $W_\alpha$ of all $\eta < \kappa'$ such that $\eta$ is a fixed point of $i_{0,\alpha}$ belongs to $\mathcal{U}'$.

We prove the lemma by induction on $\alpha \geq 1$.

First consider the case $\alpha = 1$. Let $X = \pi_\mathcal{U}(\text{fix}(\mathcal{U}))$ belong to $i(\mathcal{U}')$. Then $Z \in \mathcal{U}$, where

$$Z = \{\gamma < \kappa \mid f(\gamma) \in \mathcal{U}'\}.$$ 

Let $Y = \bigcap_{\gamma \in Z} f(\gamma)$. The $\kappa'$-completeness of $\mathcal{U}'$ implies that $Y \in \mathcal{U}'$. Since $\{\gamma < \kappa \mid Y \subseteq f(\gamma)\}$ belongs to $\mathcal{U}$, it follows by the Los Theorem that $i(Y) \subseteq X$. Now $Y \cap W_1$ belongs to $\mathcal{U}'$ and is a subset of $i(Y)$, and so $i(Y) \in \mathcal{U}'$. Therefore $X \in \mathcal{U}'$.

Assume that the lemma holds for $\alpha$. By the elementarity of $i_{0,\alpha}$ and the case $\alpha = 1$ of the lemma, we have that $i_{0,\alpha+1}(\mathcal{U}) = i_{0,\alpha}(\mathcal{U}) \cap \Ult_{\alpha+1}(V;\mathcal{U})$. This fact and our induction hypothesis imply that the lemma holds for $\alpha + 1$.

Let $\lambda < \kappa'$ be a limit ordinal and assume that the lemma holds for all $\alpha < \lambda$. Let $X \in i_{0,\lambda}(\mathcal{U}')$. For some $\alpha < \lambda$ and some $Y \in \Ult_\alpha(V;\mathcal{U})$, $X = i_{\alpha,\lambda}(Y)$. By the elementarity of $i_{\alpha,\lambda}$ and our induction hypothesis, $Y \in \mathcal{U}'$. Hence $Y \cap W_\lambda \in \mathcal{U}'$. Since $X \subseteq Y \cap W_\lambda$, $X \in \mathcal{U}'$. \hfill $\square$

**Exercise 3.3.1.** Let $j : M \prec N$ with $N$ transitive and $j$ a class in $M$.

(a) Prove that $j \circ j = j_1 \circ j$.

(b) Let $x \in N$. Prove that $j(x) = j_1(x)$ if and only if $x \in \text{range}(j)$.

(c) Prove that $j_{n+1}(\alpha) \leq j_n(\alpha)$ for all $n \in \omega$ and all ordinals $\alpha \in M$.

**Hint.** For (c), let $\alpha$ be an ordinal of $M$ and let $\beta$ be the least ordinal $\gamma$ such that $j(\gamma) > \alpha$. By the elementarity of $j$, $j(\beta)$ is the least ordinal $\gamma$ such that $j_1(\gamma) > j(\alpha)$.

**Exercise 3.3.2.** Let $\mathcal{U}$ be a uniform normal ultrafilter on a cardinal $\kappa$. Prove that $\Ult_\omega(V;\mathcal{U})$ is not countably closed: Prove that there is an $f : \omega \rightarrow \Ult_\omega(V;\mathcal{U})$ that does not belong to $\Ult_\omega(V;\mathcal{U})$.

**Hint.** Prove that $\langle (i_\mathcal{U})_{0,n}(\kappa) \mid n \in \omega \rangle \notin \Ult_\omega(V;\mathcal{U})$.

**Exercise 3.3.3.** Let $\mathcal{U}$ be a uniform normal ultrafilter on a cardinal $\kappa$. Show that $i_{\mathcal{U}^{|\omega|}} = (i_{\mathcal{U}})_{0,n}$ for all $n \in \omega$.

**Exercise 3.3.4.** Prove that part (a) of Lemma 3.3.13 remains true for $\eta \geq \kappa$ if “$\leq$” is replaced by “$\equiv$” in its conclusion.
In order to prove $\Pi_1$ determinacy from a hypothesis that is actually equivalent with $\Pi_1$ determinacy, we need to introduce the notion of what are called sharps. The existence of these objects follows from measurable cardinals, but it is actually weaker. The simplest example of a sharp, $0^\#$, is a set of natural numbers that codes up the entire universe $L$ of constructible sets. To introduce sharps, we must then first introduce $L$. We carry out this latter task in a rather sketchy fashion, letting the reader consult other works, such as [Kunen, 1980], for a complete treatement.

Gödel’s constructible universe $L$ and hierarchy of constructible sets are defined as follows:

1. $L_0 = \emptyset$.
2. $L_{\alpha+1}$ is the collection of all subsets of $L_\alpha$ that are first order definable over $L_\alpha$ from elements of $L_\alpha$. In other words, a set $x$ belongs to $L_{\alpha+1}$ if and only if there is a formula $\varphi(v_0, \ldots, v_n)$ of the language of set theory and there are elements $y_1, \ldots, y_n$ of $L_\alpha$ such that
   $$x = \{y_0 \in L_\alpha \mid (L_\alpha; \in) \models \varphi[y_0, \ldots, y_n]\}.$$
3. If $\alpha$ is a limit ordinal, then $L_\alpha = \bigcup_{\beta < \alpha} L_\beta$.
4. $L = \bigcup_{\alpha \in \text{Ord}} L_\alpha$.

We now give the basic facts about $L$. Some of the proofs we outline and some of them we omit altogether. See [Kunen, 1980] for details.

Theorem 3.4.1. (Gödel [1939]) $L$ is a transitive class model of ZFC.

Proof. We briefly sketch the proof. If $x \in L_\alpha$ and $x \subseteq L_\alpha$, the formula $v_0 \in v_1$ witnesses that $x \in L_{\alpha+1}$. If $L_\alpha$ is transitive, it follows that $L_\alpha \subseteq L_{\alpha+1}$ and that $L_{\alpha+1}$ is transitive. By induction one easily shows that $L_\alpha \subseteq L_\beta$ whenever $\alpha < \beta$ and that each $L_\alpha$ is transitive. Thus $L$ is transitive. The Axiom of Foundation holds in any transitive class. One readily constructs formulas to show that, for limit ordinals $\alpha$, $L_\alpha$ is closed under pairing and union. Since the formulas expressing $v_0 = \{v_1, v_2\}$ and $v_0 = \bigcup v_1$ are absolute for $L$—and indeed for any transitive class—it follows that the Axioms of Pairing and Union hold in $L$. (A formula $\varphi(v_1, \ldots, v_n)$ is absolute for a class
N if whenever \( a_1, \ldots, a_n \) are elements of \( N \), then \( N \models \varphi[a_1, \ldots, a_n] \) if and only if \( V \models \varphi[a_1, \ldots, a_n] \).) By induction, using the formula “\( v_0 \) is an ordinal number,” one can show that each ordinal \( \alpha \) belongs to \( L_{\alpha+1} \). Hence \( \omega \) belongs to \( L_{\omega+1} \) and, by absoluteness, witnesses that \( L \) is a model of the Axiom of Infinity. If \( x \in L \), then \( \mathcal{P}(x) \cap L \) is, by Replacement in \( V \), a subset of some \( L_\alpha \) and so a member of some \( L_{\alpha+1} \). Thus, for each \( x \in L \), the set \( \mathcal{P}(x) \cap L \) witnesses that the Power Set Axiom holds in \( L \) for \( x \). If \( u \in L \) and

\[(\forall x \in u)(\exists! y \in L) L \models \varphi(x, y),\]

then Replacement in \( V \) gives an \( \alpha \) such that

\[(\forall x \in u)(\exists! y \in L_\alpha) L \models \varphi(x, y).\]

Thus Replacement holds in \( L \) if Comprehension does. By Replacement in \( V \), there are for each formula \( \varphi(v_1, \ldots, v_n) \) arbitrarily large ordinals \( \alpha \) such that

\[(\forall a_1 \in L_\alpha) \cdots (\forall a_n \in L_\alpha)(L_\alpha \models \varphi[a_1, \ldots, a_n] \leftrightarrow L \models \varphi[a_1, \ldots, a_n]).\]

(Such an instance of the schema called Reflection for \( L \) is proved by a kind of Löwenheim–Skolem argument applied to the finitely many subformulas of \( \varphi \).) If \( u \in L \) and we want to verify that \( (\exists v)(\forall x)(x \in v \leftrightarrow (x \in u \land L \models \varphi[x, u, y_1, \ldots, y_n])) \), then we apply Reflection to \( \varphi \) to get an \( \alpha \) with \( u \in L_\alpha \) and we deduce that the desired \( v \) belongs to \( L_{\alpha+1} \). The Axiom of Choice holds in \( L \) because there is a wellordering \( <_L \) of \( L \) definable in \( L \): First order by \( \text{rank}_L(x) = \mu \alpha(x \in L_{\alpha+1}) \); for \( \text{rank}_L(x) = \text{rank}_L(y) \), order inductively by the formulas and parameters from \( L_{\text{rank}_L(x)} \) witnessing \( x \) and \( y \) in \( L_{\text{rank}_L(x)+1} \). (We assume in the sequel that some such specific definition of \( <_L \) has been fixed.)

The Axiom of Constructibility asserts that \( V = L \).

**Theorem 3.4.2.** ([Gödel, 1939]) The Axiom of Constructibility holds in \( L \).

Theorem 3.4.2 is proved by showing that the formula \( v \in L \) is absolute for \( L \).

**Lemma 3.4.3.** ([Gödel, 1939]) For each infinite ordinal \( \alpha \), the cardinal number of \( L_\alpha \) is \( |\alpha| \).
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Proof. Since each $\alpha \in L_{\alpha+1}$, we have that $|L_\alpha| \geq |\alpha|$ for all infinite $\alpha$.

We prove by transfinite induction that $|L_\alpha| \leq |\alpha|$ for all infinite $\alpha$. It is easy to see that $L_{n+1} = \mathcal{P}(L_n)$ for $n \in \omega$. This implies that each $L_n$ is finite and so that $|L_\omega| = \aleph_0$. Assume that $\alpha$ is infinite and that $|L_\alpha| \leq |\alpha|$. Each member of $L_{\alpha+1}$ is determined by a formula and finitely many elements of $L_\alpha$. Thus $|L_{\alpha+1}| \leq \max\{\aleph_0, |L_\alpha|\} = |L_\alpha| \leq |\alpha|$. Now assume that $\alpha$ is a limit ordinal and that $|L_\beta| \leq |\beta|$ for each infinite $\beta < \alpha$. Then $|L_\alpha| = |\bigcup_{\beta<\alpha} L_\beta| \leq \sum_{\beta<\alpha} |\beta| \leq \sum_{\beta<\alpha} |\alpha| = |\alpha| \cdot |\alpha| = |\alpha|$. □

Theorem 3.4.4. ([Gödel, 1939]) The Generalized Continuum Hypothesis holds in $L$.

The proof, which we omit, proceeds by showing that

$$(\forall x \in L_\alpha) \mathcal{P}(x) \cap L \subseteq L_{\alpha^+},$$

where as usual $\alpha^+$ is the least cardinal greater than $\alpha$. The theorem then follows by Lemma 3.4.3.

Lemma 3.4.5. (a) $\text{ZFC}^- + V = L$ holds in $L_\gamma$ for every uncountable regular cardinal $\gamma$, where $\text{ZFC}^-$ is $\text{ZFC}$ without the Power Set Axiom.

(b) If $N$ is a transitive class model of $\text{ZFC}^- + V = L$, then either $N = L$ or $N = L_\alpha$ for some limit ordinal $\alpha$.

Part (a) of Lemma 3.4.5 is proved much as are Theorem 3.4.1 and Theorem 3.4.2. Part (b) is proved by showing that $V = L$ is absolute for transitive models of $\text{ZFC}^-$. For each limit $\alpha$, the formula defining the wellordering $<_L$ of $L$ described in the proof of Theorem 3.4.1 is absolute for $L_\alpha$. The proof that the formula in question defines a wellordering of $L$ goes through in $\text{ZFC}^-$. The following striking result of Dana Scott was the inspiration for all subsequent work about the impact of large cardinals on $L$.

Theorem 3.4.6. ([Scott, 1961]) If a measurable cardinal exists, then $V \neq L$.

Proof. Let $\kappa$ be the least measurable cardinal. By Theorem 3.2.12, let $j : V \prec M$ with $M$ transitive and $\text{crit}(j) = \kappa$. By part (b) of Lemma 3.4.5, $j(L) = L$. If $V = L$ then $M = j(L) = L = V$ and so $\kappa$ is the least measurable cardinal in $M$. But $j(\kappa) > \kappa$, and the least measurable cardinal in $M$ is $j(\kappa)$. 
We now deduce some much stronger consequences of the existence of a measurable cardinal for the relation of $L$ to $V$. These consequences that will give us the best possible hypotheses for the determinacy result of Chapter 4 and for some of the results of Chapter 5.

A class $U$ is a class of indiscernibles for a transitive class $M$ if

(a) $U \subseteq \text{Ord} \cap M$;

(b) if $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ are elements of $U$ and $\varphi(v_1, \ldots, v_n)$ is a formula of the language of set theory, then

$$M \models \varphi[\alpha_1, \ldots, \alpha_n] \iff M \models \varphi[\beta_1, \ldots, \beta_n].$$

Recall that a subset $X$ of a limit ordinal $\lambda$ is closed if $X$ is closed in the order topology. Equivalently, $X$ is closed if whenever $\alpha < \lambda$ and $X$ is unbounded in $\alpha$ then $\alpha \in X$.

Fix some recursive bijection $\varphi \mapsto n_\varphi$ from the set $\Phi$ of formulas of the language of set theory whose free variables are among $v_1, v_2, \ldots$ to the set $\omega$. If there is a closed unbounded subset $C$ of $\omega_1$ such that $C$ is a set of indiscernibles for $L_{\omega_1}$, then $0^#_{\varphi}$ is

$$\{n_{\varphi(v_1, \ldots, v_{i_{\varphi}})} \mid L_{\omega_1} \models \varphi[\alpha_1, \ldots, \alpha_{i_{\varphi}}]\}.$$

Here $i_{\varphi}$ is the greatest $i$ such that $v_i$ is free in $\varphi$ if $\varphi$ is not a sentence and 0 otherwise, and $\alpha_1 < \cdots < \alpha_n$ are members of $C$. Since the intersection of two closed unbounded sets is closed and unbounded, there is no dependence on the choice of $C$. If such a $C$ does not exist, then there is no $0^#_{\varphi}$.

**Remark.** “$0^#$” is pronounced as “0 sharp.” It would perhaps then be better if it were written “$0^\sharp$,” and it sometimes is. The notation with # has its origin in the fundamental [Solovay, 1967]. $0^#$ was introduced and most of the results of this section were proved, in Jack Silver’s dissertation, [Silver, 1966], published in abridged form as [Silver, 1971].

**Lemma 3.4.7.** ([Rowbottom, 1964]) If $\kappa$ is a measurable cardinal and $\mathcal{U}$ is a uniform normal ultrafilter on $\kappa$, then there is subset $X$ of $\kappa$ such that $X \in \mathcal{U}$ and such that $X$ is a set of indiscernibles for $L_{\kappa}$. 
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**Proof.** Let \( \kappa \) be a measurable cardinal and let \( \mathcal{U} \) be a uniform normal ultrafilter on \( \kappa \). For \( n \in \omega \) and \( q \in [\kappa]^n \), let \((q)_1 < \cdots < (q)_n\) be the elements of \( q \). For each formula \( \varphi \in \Phi \), let \( n \) be minimal such that \( \varphi \) is \( \varphi(v_1, \ldots, v_n) \), i.e. such that the free variables of \( \varphi \) are among \( v_1, \ldots, v_n \). Let

\[
Y_\varphi = \{ q \in [\kappa]^n \mid L_\kappa \models \varphi[(q)_1, \ldots, (q)_n] \}.
\]

Let \( Z_\varphi = Y_\varphi \) if \( Y_\varphi \in \mathcal{U}^{(n)} \) and let \( Z_\varphi = [\kappa]^n \setminus Y_\varphi \) otherwise. (The Rowbottom ultrafilter \( \mathcal{U}^{(n)} \) is defined on page 136.) By the definition of \( \mathcal{U}^{(n)} \), let \( X_\varphi \subseteq \kappa \) be such that \( X_\varphi \in \mathcal{U} \) and \([X_\varphi]^n \subseteq Z_\varphi \). Let \( X = \bigcap \varphi X_\varphi \). Clearly \( X \in \mathcal{U} \) and \( X \) is a set of indiscernibles for \( L_\kappa \).

If \( \varphi(v_0, v_1, \ldots, v_n) \) is a formula of the language of set theory, then let us define \( f_\varphi : {}^nL \to L \) by

\[
f_\varphi(x_1, \ldots, x_n) = \begin{cases} 
\mu x_0 L \models \varphi[x_0, x_1, \ldots, x_n] & \text{if } (\exists x_0) L \models \varphi[x_0, x_1, \ldots, x_n]; \\
\emptyset & \text{otherwise}.
\end{cases}
\]

Here the \( \mu \)-operator is being applied to the ordering \( <_L \). We also define \( f_\varphi^\alpha : {}^nL_\alpha \to L_\alpha \) by replacing "\( L \)" by "\( L_\alpha \)" in the definition of \( f_\varphi \).

For \( \alpha \) a limit ordinal and \( X \subseteq L_\alpha \), let \( \mathcal{H}(L_\alpha, X) \) be the closure of \( X \) under all the functions \( f_\varphi^\alpha \). Note that

\[
\mathcal{H}(L_\alpha, X) = \bigcup_{\varphi(v_0, \ldots, v_n)} \{ f_\varphi^\alpha(x_1, \ldots, x_n) \mid \langle x_1, \ldots, x_n \rangle \in {}^nX \}.
\]

This is because compositions of the \( f_\varphi^\alpha \) are also among the \( f_\varphi^\alpha \). Note also that \( \mathcal{H}(L_\alpha, X) \prec L_\alpha \), i.e. that \( \text{id} : \mathcal{H}(L_\alpha, X) \prec L_\alpha \). More generally, \( \mathcal{H}(L_\alpha, X) \prec \mathcal{H}(L_\alpha, Y) \) when \( X \subseteq Y \). Any class model \((M; E)\) of \( \text{ZFC}^- + V = L \) has its own internally defined version of the \( f_\varphi \). Let us call the closure of \( X \subseteq M \) under these functions \( \mathcal{H}((M; E), X) \). (Note that when \( M \) is a proper class, e.g. when \( M = L \), then we are going beyond the language of \( \text{ZFC} \) in making this definition.)

If \((M; E)\) is a class model and \( X \subseteq M \), let us say that \( X \) generates \((M; E)\) if for every \( a \in M \) there are \( n \in \omega \), \( f : {}^nM \to M \) and \( x_1, \ldots, x_n \) belonging to \( X \) such that \( f \) is definable (without parameters) in \((M; E)\) and \( a = f(x_1, \ldots, x_n) \). This is equivalent to saying the closure of \( X \) under the definable functions is all of \( M \). If \((M; E)\) is a class model of \( \text{ZFC}^- + V = L \) and \( X \subseteq M \), then it is easy to see that \( X \) generates \((M; E)\) if and only if \( \mathcal{H}((M; E), X) = M \). If \( \alpha \) is a limit ordinal and \( X \subseteq L_\alpha \), then \( X \) generates \( L_\alpha \) (i.e. \( X \) generates \((L_\alpha ; \in)\)) just in case \( \mathcal{H}(L_\alpha, X) = L_\alpha \).
Theorem 3.4.8. ([Silver, 1966]) The following are equivalent:

(i) $0^\#$ exists.

(ii) There is an uncountable regular cardinal $\gamma$ such that there is an unbounded subset of $\gamma$ that is a set of indiscernibles for $L_\gamma$.

(iii) There is a closed unbounded proper class $C$ such that $C$ is a class of indiscernibles for $L$ that generates $L$ and such that, for every uncountable cardinal $\eta$, $\mathcal{H}(L, C \cap \eta) = L_\eta$.

Proof. Clearly (i) implies (ii) and (iii) implies (i). We need then only show that (ii) implies (iii). Let $\gamma$ be an uncountable regular cardinal and let $X \subseteq \gamma$ be unbounded in $\gamma$ and a set of indiscernibles for $L_\gamma$. Since $\gamma$ is regular, $X$ has order type $\gamma$. Let $\alpha \mapsto x_\alpha$ be the order preserving bijection between $\gamma$ and $X$.

We may assume that $X$ has been chosen with the minimal possible value of $x_\omega$, i.e. that, for any unbounded subset $X'$ of $\gamma$ such that $X'$ is a set of indiscernibles for $L_\gamma$, the $\omega$th element of $X'$ is at least as large as $x_\omega$.

We next show that we can replace $X$ with a $Y$ that has, in addition to the properties of $X$, the additional property of generating $L_\gamma$.

By Lemma 3.2.4, let

$$\pi : \mathcal{H}(L_\gamma, X) \cong N,$$

with $N$ transitive. By part (a) of Lemma 3.4.5, $N \models \text{ZFC}^- + V = L$. By part (b) of the same lemma, $N = L_\alpha$ for some limit ordinal $\alpha$. Since $\pi(\beta) \leq \beta$ for each ordinal $\beta \in \mathcal{H}(L_\gamma, X)$, we must have $\alpha \leq \gamma$. But $|X| = \gamma$, so we get that $\alpha = \gamma$. Let

$$Y = \{\pi(x) \mid x \in X\}.$$

Clearly $Y$ is unbounded in $\gamma$. Since $\pi : \mathcal{H}(L_\gamma, X) \cong L_\gamma$ and $\mathcal{H}(L_\gamma, X) \prec L_\gamma$, we get that $Y$ is a set of indiscernibles for $L_\gamma$. Let $\alpha \mapsto y_\alpha$ be the order preserving bijection between $\gamma$ and $Y$. We have then that

(a) $Y$ is an unbounded subset of $\gamma$;

(b) $Y$ is a set of indiscernibles for $L_\gamma$;

(c) $Y$ has minimal $\omega$th element among sets with properties (a) and (b);

(d) $Y$ generates $L_\gamma$.

We next prove some useful facts about $Y$ and the $f^{\gamma}_{\varphi}$. 

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Thus we have \( j < \alpha \) every \( \alpha < \gamma \).

We have then shown that for any \( j < \alpha \) that 

\[
\langle \phi \rangle_{y_1, \ldots, y_n} = \phi_{1, \ldots, m}.
\]

Hence

\[
f(y_1, \ldots, y_j, y_{j+1}, \ldots, y_{j+m}) = f(y_1, \ldots, y_j, y_{a_1}, \ldots, y_{a_m})
\]

for any \( \alpha_1, \ldots, \alpha_m \) such that \( j < \alpha_1 < \cdots < \alpha_m < \gamma \).

If (1) fails, then indiscernibility gives that \( \text{rank}_L(f(y_1, \ldots, y_n)) \geq \alpha \) for every \( \alpha < \gamma \). Hence \( f(y_1, \ldots, y_n) \notin L_\gamma \).

Suppose that the hypothesis of (2) holds and that the conclusion fails.

Thus we have \( j < \alpha_1 < \cdots < \alpha_m < \gamma \) and

\[
y_{j+1} > f(y_1, \ldots, y_j, y_{j+1}, \ldots, y_{j+m}) \neq f(y_1, \ldots, y_j, y_{a_1}, \ldots, y_{a_m}).
\]

Let \( \alpha_m < \beta_1 < \cdots < \beta_m \). If

\[
f(y_1, \ldots, y_j, y_{a_1}, \ldots, y_{a_m}) = f(y_1, \ldots, y_j, y_{\beta_1}, \ldots, y_{\beta_m}),
\]

then indiscernibility gives the contradiction that

\[
f(y_1, \ldots, y_j, y_{j+1}, \ldots, y_{j+m}) = f(y_1, \ldots, y_j, y_{\beta_1}, \ldots, y_{\beta_m}).
\]

We have then shown that for any \( \alpha_1 < \cdots < \alpha_m \) and \( \beta_1 < \cdots < \beta_m \) such that \( j < \alpha_1, \alpha_m < \beta_1, \) and \( \beta_m < \gamma \),

\[
f(y_1, \ldots, y_j, y_{a_1}, \ldots, y_{a_m}) \neq f(y_1, \ldots, y_j, y_{\beta_1}, \ldots, y_{\beta_m}).
\]

By indiscernibility, we have that one of the following holds for all such \( \alpha_1, \ldots, \alpha_m \) and \( \beta_1, \ldots, \beta_m \):

\[
\begin{align*}
\langle \phi \rangle_{y_1, \ldots, y_j, y_{a_1}, \ldots, y_{a_m}} & < L \langle \phi \rangle_{y_1, \ldots, y_j, y_{\beta_1}, \ldots, y_{\beta_m}} \\
\langle \phi \rangle_{y_1, \ldots, y_j, y_{a_1}, \ldots, y_{a_m}} & > L \langle \phi \rangle_{y_1, \ldots, y_j, y_{\beta_1}, \ldots, y_{\beta_m}}
\end{align*}
\]

Let \( \alpha_{p,k} = j + mp + k \) for \( p < \gamma \) and \( 0 \leq k < m \). If it is the second inequality that holds, then \( \langle \phi \rangle_{y_1, \ldots, y_j, y_{a_1}, \ldots, y_{a_m}} \mid n \in \omega \) is an infinite descending sequence with respect to the wellordering \( <_L \). Thus it is the first inequality that holds. By indiscernibility and the fact that \( z <_L z' \) implies \( \text{rank}_L(z) \leq \text{rank}_L(z') \), we get that one of the following holds for all any \( \alpha_1 < \cdots < \alpha_m < \beta_1 < \cdots < \beta_m \) such that \( j < \alpha_1 \) and \( \beta_m < \gamma \):

\[
\begin{align*}
\text{rank}_L(\langle \phi \rangle_{y_1, \ldots, y_j, y_{a_1}, \ldots, y_{a_m}}) &= \text{rank}_L(\langle \phi \rangle_{y_1, \ldots, y_j, y_{\beta_1}, \ldots, y_{\beta_m}}) \\
\text{rank}_L(\langle \phi \rangle_{y_1, \ldots, y_j, y_{a_1}, \ldots, y_{a_m}}) &< \text{rank}_L(\langle \phi \rangle_{y_1, \ldots, y_j, y_{\beta_1}, \ldots, y_{\beta_m}})
\end{align*}
\]
If it is the equation that holds, then there is an \( \eta < \gamma \) such that
\[
\left\{ f^\gamma_\varphi(y_1, \ldots, y_j, y_{\alpha_{\rho,0}}, \ldots, y_{\alpha_{\rho,m-1}}) \mid \rho < \gamma \right\} \subseteq L_\eta.
\]
This is impossible, since \( |L_\eta| < \gamma \). Hence it is the inequality that holds. But then
\[
\{ \text{rank}_L(f^\gamma_\varphi(y_1, \ldots, y_j, y_{\alpha_{\rho,0}}, \ldots, y_{\alpha_{\rho,m-1}})) \mid \rho < \gamma \}
\]
is readily seen to be an unbounded subset of \( \gamma \) that is a set of indiscernibles for \( L_\gamma \). Moreover the \( \omega \)th element of this set is
\[
f^\gamma_\varphi(y_1, \ldots, y_j, y_\omega, \ldots, y_{\omega+m-1}) <_L y_\omega.
\]
This contradicts property (c) of \( Y \).

We use \( Y \) to generate a proper class model \( (M; E) \) as follows: Suppose that \( \varphi(v_0, \ldots, v_n) \) and \( \psi(v_0, \ldots, v_m) \) are formulas and \( \alpha_1 < \cdots < \alpha_n \) and \( \beta_1 < \cdots < \beta_m \) are ordinal numbers. Let
\[
q : \{ \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \} \to |\{ \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \}|
\]
be the order preserving bijection. Set
\[
\langle \varphi, \alpha_1, \ldots, \alpha_n \rangle \sim \langle \psi, \beta_1, \ldots, \beta_m \rangle \iff f^\gamma_\varphi(y_{q(\alpha_1)}, \ldots, y_{q(\alpha_n)}) = f^\gamma_\psi(y_{q(\beta_1)}, \ldots, y_{q(\beta_m)}).
\]
Let \([\varphi, \alpha_1, \ldots, \alpha_n]\) be the equivalence class of \( \langle \varphi, \alpha_1, \ldots, \alpha_n \rangle \) with respect to the equivalence relation \( \sim \), fixed up à la Scott to be a set. Let \( M \) be the class of all the \([\varphi, \alpha_1, \ldots, \alpha_n]\]. Define, for \( q \) as above,
\[
[\varphi, \alpha_1, \ldots, \alpha_n] E [\psi, \beta_1, \ldots, \beta_m] \iff f^\gamma_\varphi(y_{q(\alpha_1)}, \ldots, y_{q(\alpha_n)}) \in f^\gamma_\psi(y_{q(\beta_1)}, \ldots, y_{q(\beta_m)}).
\]
Suppose for a contradiction that \( (M; E) \) is not wellfounded. Then there exists a sequence \( \langle a_i \mid i \in \omega \rangle \) such that each \( a_{i+1} E a_i \). Let \( a_i = [\varphi_i, \delta_{i,1}, \ldots, \delta_{i,n_i}] \).

Let \( g \) be the order preserving bijection between a countable ordinal and \( \{ \delta_{i,j} \mid i \in \omega \wedge j \leq n_i \} \). For each \( i \), let \( e_i = f^\gamma_{\varphi_i}(y_{g^{-1}(\delta_{i,1})}, \ldots, y_{g^{-1}(\delta_{i,n_i})}) \). We have the contradiction that each \( e_{i+1} \in e_i \).

For a set \( A \) of ordinals, let \( \text{ot}(A) \) be the order type of \( A \), let \( \alpha \mapsto a_\alpha \) be the order isomorphism between \( \text{ot}(A) \) and \( A \), and let
\[
M_A = \{ [\varphi, a_{\alpha_1}, \ldots, a_{\alpha_n}] \mid \varphi \text{ a formula } \land \alpha_1 < \cdots < \alpha_n < \text{ot}(A) \}.
\]
Suppose that $\ot(A) \leq \gamma$. It is clear from the definition of $(M; E)$ that the correspondence given by

$$f^\gamma_\varphi(y_{\alpha_1}, \ldots, y_{\alpha_n}) \mapsto [\varphi, a_{\alpha_1}, \ldots, a_{\alpha_n}]$$

gives an isomorphism

$$h_A : (\mathcal{H}(L_\gamma, \{y_\alpha \mid \alpha < \ot(A)\}); \in) \cong (M_A; E).$$

For any ordinal $\delta$, let $z_\delta = [v_0 = v_1, \delta]$. Note that $h_A(y_\alpha) = z_{a_\alpha}$ for each $\alpha < \gamma$.

If $A \subseteq B \subseteq \text{Ord}$ and $\ot(B) \leq \gamma$, then the fact that

$$(\mathcal{H}(L_\gamma, \{y_\alpha \mid \alpha < \ot(A)\}); \in) \prec (\mathcal{H}(L_\gamma, \{y_\alpha \mid \alpha < \ot(B)\}); \in)$$

implies that

$$(M_A; E) \prec (M_B; E).$$

Thus, in particular,

$$\langle (M_A; E) \mid A \subseteq \text{Ord} \land A \text{ finite } \rangle,$$

together with the inclusions, forms a directed system of elementary embeddings. It follows from Lemma 3.3.7 that, for each finite $A$,

$$(M_A; E) \prec (\bigcup_{A \text{ finite}} M_A; E) = (M; E).$$

Hence, for every finite $A$,

$$h_A : (\mathcal{H}(L_\gamma, \{y_\alpha \mid \alpha < \ot(A)\}); \in) \prec (M; E).$$

We use properties (1) and (2) of $Y$ to show that $(M; E)$ is set-like. Let $b = [\varphi, \delta_1, \ldots, \delta_n] \in M$. Let $A = \{\delta_1, \ldots, \delta_n, \delta_n + 1\}$. By property (1) of $Y$ we have that

$$(M; E) \models \text{rank}_L(h_A(f^\gamma_\varphi(y_{\delta_1}, \ldots, y_{\delta_n}))) < h_A(y_{\delta_n + 1}),$$

i.e. that $(M; E) \models \text{rank}_L(b) < z_{\delta_n + 1}$. Suppose that $a E b$. Then $(M; E) \models \text{rank}_L(a) < z_{\delta_n + 1}$. Let $a = [\psi, \rho_1, \ldots, \rho_m]$. We may assume that there is a $j$ with $1 \leq j \leq m$ such that $\rho_{j+1} = \delta_n + 1$. Using $h_B$ with $B = \{\rho_1, \ldots, \rho_m, \xi_{j+1}, \ldots, \xi_m\}$ for arbitrary ordinals $\xi_{j+1}, \ldots, \xi_m$ and using property (2) of $Y$, we get that

$$a = [\psi, \rho_1, \ldots, \rho_j, \xi_{j+1}, \ldots, \xi_m]$$

for any $\xi_{j+1} < \cdots < \xi_m$ with $\rho_j < \xi_{j+1}$. Thus we have shown that each $a$ such that $a E b$ is determined by numbers $j$ and
\[m, \text{ a formula } \psi, \text{ and a } j\text{-tuple of ordinals smaller than } \delta_n + 1. \] The collection of such \( a \) thus forms a set.

By Lemma 3.2.8, \((M; E)\) is isomorphic to \((N; \in)\) for some transitive class \(N\). Since the \( z_\beta \) are distinct elements of \( M \), we have that \( M \) is a proper class and so that \( N \) also is a proper class. By part (b) of Lemma 3.4.5, \( N = L \). Let \( \pi : (M; E) \cong (L; \in) \). For ordinals \( \alpha \) let \( c_\alpha = \pi(z_\alpha) \). Let \( C = \{c_\alpha \mid \alpha \in \text{Ord}\} \).

For each finite set \( A = \{a_1, \ldots, a_n\} \) of ordinals with \( a_1 < \cdots < a_n \), let \( h^*_A = \pi \circ h_A \). We have that
\[
h^*_A : \mathcal{H}(L_\gamma, \{y_1, \ldots, y_n\}) \prec L.
\]
Moreover \( h^*_A(y_i) = c_{a_i} \) for each \( i \). Using such functions \( h^*_A \), it is easy to see that \( C \) is a class of indiscernibles for \( L \), that \( \[\[ \varphi, \alpha_1, \ldots, \alpha_n \] \] = f_\varphi(c_{\alpha_1}, \ldots, c_{\alpha_n}) \) and so that \( C \) generates \( L \)—and that \( C \) has properties (1) and (2), i.e. that
\[
(1) \text{ rank } L(f_\varphi(c_{\alpha_1}, \ldots, c_{\alpha_n})) < c_{n+1}.
\]
\[
(2) \text{ If rank } L(f_\varphi(c_1, \ldots, c_j, c_{j+1}, \ldots, c_{j+m})) < c_{j+1}, \text{ then }
\]
\[
f_\varphi(c_1, \ldots, c_j, c_{j+1}, \ldots, c_{j+m}) = f_\varphi(c_1, \ldots, c_j, c_{\alpha_1}, \ldots, c_{\alpha_m})
\]
for any \( \alpha_1, \ldots, \alpha_m \) such that \( j < \alpha_1 < \cdots < \alpha_m \).

We know that \( |\mathcal{H}(L, \{c_\beta \mid \beta < \eta\})| = |\eta| \) for each limit ordinal \( \eta \). By properties (1) and (2), we get that \( \mathcal{H}(L, \{c_\beta \mid \beta < \eta\}) = \mathcal{H}(L, C) \cap L_{\sup_{\beta < \eta} c_\beta} = L_{\sup_{\beta < \eta} c_\beta} \) for all limit ordinals \( \eta \). If \( \eta \) is an uncountable cardinal, these facts imply that
\[
L_\eta = \mathcal{H}(L, \{c_\beta \mid \beta < \eta\}) = \mathcal{H}(L, C \cap \eta).
\]

It remains only to show that \( C \) is closed. Suppose that \( \alpha \) is a limit ordinal and that \( \xi < c_\alpha \) (so that \( \xi \in L_{c_\alpha} \)). Let
\[
\xi = f_\varphi(c_{\beta_1}, \ldots, c_{\beta_j}, c_\alpha, c_{\delta_1}, \ldots, c_{\delta_{j+m}}),\]
where \( \beta_1 < \cdots < \beta_j < \alpha < \delta_1 < \cdots < \delta_{j+m} \). By property (2) of \( C \), \( \xi \in L_{c_{\beta_j+1}} \), i.e. \( \xi < c_{\beta_j+1} \). This shows that the \( c_\beta, \beta < \alpha \), are cofinal in \( c_\alpha \).

\[\square\]

**Corollary 3.4.9.** If a measurable cardinal exists, then \( 0^\# \) exists.

**Proof.** By Lemma 3.2.15 and Lemma 3.4.7, clause (ii) of the theorem holds for any measurable cardinal \( \gamma \).

\[\square\]
Corollary 3.4.10. Assume that \(0^\#\) exists. If \(\kappa\) and \(\lambda\) are uncountable cardinals with \(\kappa < \lambda\), then \(L_\kappa \prec L_\lambda \prec L\).

Proof. This follows directly from clause (iii) of the theorem. \(\square\)

It is easy to see that the class \(C\) of clause (iii) of Theorem 3.4.8 is unique. Indeed, the whole structure of \(H(L,C)\) is determined directly by \(0^\#\), which gives us the relations among the values \(f^\omega_\varphi(c_{\alpha_1},\ldots,c_{\alpha_n})\) and so, since \(L_\omega \prec L\), among the values \(f^\omega_\varphi(c_{\alpha_1},\ldots,c_{\alpha_n})\). We will refer to \(C\) as the \textit{Silver class of indiscernibles} for \(L\) and to the members of \(C\) as the \textit{Silver indiscernibles} for \(L\). Note that the uncountable cardinals are among the indiscernibles for \(L\).

The concepts and theorems of this section can easily be relativized.

Let \(a\) be any set. Let the language \(L\) be the result of adding to the language of set theory a one-place predicate symbol \(P\). We can expand any transitive class model \((M;\in)\) to a class model \((M;\in,a)\) for \(L\) by interpreting \(P\) by the property of belonging to \(a\). We define:

1. \(L_0[a] = \emptyset\).
2. \(L_{\alpha+1}[a]\) is the collection of all subsets \(x\) of \(L_\alpha[a]\) such that \(x\) is first order definable over \((L_\alpha;\in,a)\) from elements of \(L_\alpha[a]\).
3. \(L[a] = \bigcup_{\alpha \in \text{Ord}} L_\alpha[a]\).

In general, \(a\) need not belong to \(L[a]\). But it is always the case that \(a \cap L[a] \in L[a]\). Moreover \(L[a] = L[a \cap L[a]]\). If \(a \subseteq V_\omega\) or \(a\) is a set of ordinals, then \(a \in L[a]\). From large cardinal hypotheses it follows that \(\omega_\omega\) does not belong to \(L[\omega_\omega]\); this will be shown in Chapter 9.

We omit the proofs of the following results. These proofs are essentially the same as those of the corresponding unrelativized results.

**Theorem 3.4.11.** For every \(a\), \(L[a]\) is a transitive class model of ZFC.

**Theorem 3.4.12.** For every \(a\), \(L[a] \models V = L[a \cap L[a]];\) thus \(L[a] \models V = L[a]\) if \(a \in L[a]\). Furthermore, if \(b \cap L[a] = a\), then \(L[a] = L[b]\).

**Lemma 3.4.13.** For each \(a\) and each infinite ordinal \(\alpha\), the cardinal number of \(L_\alpha[a]\) is \(|\alpha|\).
Theorem 3.4.14. If $\nu$ is an ordinal and $a \subseteq \nu$, then for all cardinals $\kappa$ of $L[a]$ such that $\nu \leq \kappa$, $L[a] \models 2^\nu = \kappa^+$. For each $a \subseteq V_\omega$, the Generalized Continuum Hypothesis holds in $L[a]$.

Lemma 3.4.15. Let $a$ be any set.

(a) $\text{ZFC}^- + V = L[a \cap L[a]]$ holds in $L_\gamma[a]$ for every uncountable regular cardinal $\gamma$ such that $a \cap L[a] \in L_\gamma[a]$.

(b) If $a \in N$ and $N$ is a transitive class model of $\text{ZFC}^- + V = L[a]$, then either $N = L[a]$ or $N = L_\alpha[a]$ for some limit ordinal $\alpha$.

Theorem 3.4.16. If $\kappa$ is a measurable cardinal and $a \subseteq V_\kappa$, then $V \neq L[a]$.

For any set $a$, let $\text{tclos}(a)$ be the smallest transitive set of which $a$ is a member. In the special case when $a$ is a set of ordinals, $\text{tclos}(a) = \{a\} \cup \beta$, where $\beta$ is the least ordinal of which $a$ is a subset.

If $M$ is a transitive class and if $a \in M$, then a class $U$ is a class of indiscernibles for $M, a$ if

(a) $U \subseteq \text{Ord} \cap M$;
(b) if $b_1, \ldots, b_m$ are elements of $\text{tclos}(a)$, if $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ are elements of $U$, and if $\varphi(v_1, \ldots, v_n, v_{n+1}, \ldots, v_{m+n})$ is a formula of the language of set theory, then

$$M \models \varphi(b_1, \ldots, b_m, \alpha_1, \ldots, \alpha_n) \leftrightarrow M \models \varphi(b_1, \ldots, b_m, \beta_1, \ldots, \beta_n).$$

Let $\varphi \mapsto n_\varphi$ be as in the definition of $0^\#$. Let $a$ be a set such that $a \in L[a]$. Let $\nu$ be the least ordinal such that $a \in L_\nu[a]$. If there is a closed, unbounded subset $C$ of $\nu^+$ such that $C$ is a set of indiscernibles for $L_\nu[a], a$, then $a^#$ is the set of all

$$\langle n_{\varphi(v_1, \ldots, v_{m+n})}, \langle b_1, \ldots, b_m \rangle \rangle$$

such that

$$(\forall i)(1 \leq i \leq m \rightarrow b_i \in \text{tclos}(a)) \land L_\nu[a] \models \varphi[b_1, \ldots, b_m, \alpha_1, \ldots, \alpha_n],$$

where $\alpha_1 < \cdots < \alpha_n$ are members of $C$. If there is no such $C$, then there is no $a^#$.

Remarks:
(a) If $a \subseteq V_\omega$, then all members of $\text{tclos}(a)$ other than $a$ itself are definable in $L_{\omega_1}[a]$. Thus $a^\#$ is determined by $\{n \in \omega \mid \langle n, \langle a \rangle \rangle \in a^\#\}$, and so $a^\#$ is often defined to be this subset of $\omega$.

(b) We have not yet defined $a^\#$ or even what it means for $a^\#$ to exist when $a \not\in L[a]$. We will do so later in this section.

**Lemma 3.4.17.** Let $\kappa$ be a measurable cardinal, let $U$ be a uniform normal ultrafilter on $\kappa$, and let $a \in V_\kappa$ with $a \in L[a]$. Then $a \in L_\kappa[a]$ and there is subset $X$ of $\kappa$ such that $X \in U$ and such that $X$ is a set of indiscernibles for $L_\kappa[a], a$.

For any set $a$, $L[a]$ has a wellordering internally definable from $a \cap L[a]$. Using the definition of this ordering we can define $H(L[a], X)$ and $H(L_\alpha[a], X)$ for sets $X$ such that $a \cap L[a] \in X$, and we will have, e.g., $H(L_\alpha[a], X) \prec L_\alpha[a]$.

**Theorem 3.4.18.** Let $\nu$ be an infinite cardinal and let $a$ be such that $a \in V_\nu^+$ and $a \in L[a]$. The following are equivalent:

(i) $a^\#$ exists.

(ii) There is an uncountable regular cardinal $\gamma > \nu$ such that $a \in L_\gamma[a]$ and there is an unbounded subset of $\gamma$ that is a set of indiscernibles for $L_\gamma[a], a$.

(iii) $a \in L_\nu^+[a]$. Moreover there is a closed, unbounded proper class $C^a$ such that $C^a$ is a class of indiscernibles for $L[a], a$, such that $C^a \cup \text{tclos}(a)$ generates $L[a]$ and such that, for every uncountable cardinal $\eta > \nu$, $H(L[a]; (C^a \cap \eta) \cup \text{tclos}(a)) = L_\eta[a]$.

**Corollary 3.4.19.** If $\kappa$ is a measurable cardinal, then $a^\#$ exists for every $a \in V_\kappa$ such that $a \in L[a]$.

**Corollary 3.4.20.** Let $\nu$ be a cardinal, let $a \in V_\nu^+$, and assume that $a^\#$ exists. If $\kappa$ and $\lambda$ are uncountable cardinals with $\nu < \kappa < \lambda$, then $L_\kappa[a] \prec L_\lambda[a] \prec L[a]$.

The $C^a$ of clause (iii) of Theorem 3.4.18 is unique. We call it the Silver class of indiscernibles for $L[a], a$ and we call its members the Silver indiscernibles for $L[a], a$. It is the same as the Silver class of indiscernibles defined earlier in the special case $a \subseteq V_\omega$. 
When $a \notin L[a]$, we can still make sense of $a^\#$. This is done as follows.

Let $a$ be any set. Let the language $\mathcal{L}_a$ be the result of adding to the language of set theory a one-place predicate symbol $P_b$ for each $b \in \text{tclos}(a)$. We can expand any transitive class model $(M; \in)$ to a class model $(M; \in, b, \ldots)$ for $\mathcal{L}_a$ by interpreting each $P_b$ by the property of belonging to $b$. We define:

1. $L_0(a) = \emptyset$.
2. $L_{\alpha+1}(a)$ is the collection of all subsets $x$ of $L_\alpha(a)$ such that $x$ is first order definable over $(L_\alpha(a); \in, b, \ldots)$ from elements of $L_\alpha(a)$.
3. $L(a) = \bigcup_{\alpha \in \text{Ord}} L_\alpha(a)$.

Note that $a \in L(a)$ and moreover that $a \in L_\nu(a)$ if $a \in V_\nu$.

$L(a)$ need not satisfy the Axiom of Choice, but we still have the following fact.

**Theorem 3.4.21.** For every $a$, $L(a)$ is a transitive class model of ZF.

**Theorem 3.4.22.** For every $a$, $L(a) \models V = L(a)$.

**Lemma 3.4.23.** Let $a$ be any set.

(a) $\text{ZF}^- + V = L(a)$ holds in $L_\gamma(a)$ for every uncountable regular cardinal $\gamma$ such that $a \in V_\gamma$ and $\gamma > |\text{tclos}(a)|$.

(b) If $a \in N$ and $N$ is a transitive class model of $\text{ZFC}^- + V = L(a)$, then either $N = L(a)$ or $N = L_\alpha(a)$ for some limit ordinal $\alpha$.

Let $\varphi \mapsto n_\varphi$ be as in the definition of $0^\#$. Let $a$ be any set. Let $\nu$ be the least cardinal such that $a \in V_\nu^+$ and $|\text{tclos}(a)| \leq \nu$. If there is a closed, unbounded subset $C$ of $\nu^+$ such that $C$ is a set of indiscernibles for $L_\nu(a), a$, then $a^\#$ is the set of all

$$\langle n_{\varphi(v_1, \ldots, v_{m+n})}, \langle b_1, \ldots, b_m \rangle \rangle$$

such that

$$\forall i (1 \leq i \leq m \rightarrow b_i \in \text{tclos}(a)) \land L_{\nu^+}(a) \models \varphi(b_1, \ldots, b_m, \alpha_1, \ldots, \alpha_n),$$

where $\alpha_1 < \cdots < \alpha_n$ are members of $C$. If there is no such $C$, then there is no $a^\#$. 

Lemma 3.4.24. Let $\kappa$ be a measurable cardinal, let $U$ be a uniform normal ultrafilter on $\kappa$, and let $a \in V_\kappa$. Then $a \in L_\kappa(a)$ and there is a subset $X$ of $\kappa$ such that $X \in U$ and such that $X$ is a set of indiscernibles for $L_\kappa(a),a$.

Since $L(a)$ need not satisfy Choice, it certainly need not have an internally definable wellordering. Nevertheless, every element of any $L_\alpha(a)$ is definable in $L(a)$ from ordinals smaller than $\alpha$ and elements of $\text{tclos}(a)$ by a formula absolute for $L_\alpha(a)$. Thus we can define $\mathcal{H}(L(a),X)$ and $\mathcal{H}(L_\alpha(a),X)$ for any $X$ such that $\text{tclos}(a) \subseteq X$, and we have, e.g., that $\mathcal{H}(L_\alpha(a),X) \prec L_\alpha(a)$.

Theorem 3.4.25. Let $\nu$ be an infinite cardinal and let $a \in V_{\nu+}$ be such that $|\text{tclos}(a)| \leq \nu$. The following are equivalent:

(i) $a^#$ exists.

(ii) There is an uncountable regular cardinal $\gamma > \nu$ such that there is an unbounded subset of $\gamma$ that is a set of indiscernibles for $L_\gamma(a),a$.

(iii) There is a closed, unbounded proper class $C^a$ such that $C^a$ is a class of indiscernibles for $L(a),a$, such that $C^a \cup \text{tclos}(a)$ generates $L(a)$ and such that, for every uncountable cardinal $\eta > \nu$, $\mathcal{H}(L(a);(C^a \cap \eta) \cup \text{tclos}(a)) = L_\eta(a)$.

Corollary 3.4.26. If $\kappa$ is a measurable cardinal, then $a^#$ exists for every $a \in V_\kappa$.

Corollary 3.4.27. Let $\nu$ be a cardinal and let $a \in V_{\nu+}$ be such that $|\text{tclos}(a)| \leq \nu$. Assume that $a^#$ exists. If $\kappa$ and $\lambda$ are uncountable cardinals with $\nu < \kappa < \lambda$, then $L_\kappa(a) \prec L_\lambda(a) \prec L(a)$.

The $C^a$ of clause (iii) of Theorem 3.4.25 is unique. We call it the Silver class of indiscernibles for $L[a],a$ and we call its members the Silver indiscernibles for $L[a],a$.

Exercise 3.4.1. Assume $0^#$ exists. Prove that $(\omega_1)^L < \omega_1$. Indeed, prove that $\omega_1$ is inaccessible in $L$.

Hint. Use the fact that all the uncountable cardinals are Silver indiscernibles for $L$. 
CHAPTER 3. MEASURABLE CARDINALS

3.5 $L[U]$

Lemma 3.5.1. Let $U$ be a uniform normal ultrafilter on a measurable cardinal $\kappa$. Then $L[U] \models \text{ZFC + \ } \exists \mathcal{U} \cap L[U]$ is a uniform normal ultrafilter on $\kappa$.

Proof. That $L[U] \models \text{ZFC}$ follows from Lemma 3.4.11. It is easy to see that $L[U] \models \text{"\mathcal{U} \cap L[U] is a uniform ultrafilter on } \kappa\text{"}$, and any counterexample to normality in the model $L[U]$ would be a counterexample to normality in $V$. \hfill \Box

To state an easy generalization of Lemma 3.5.1, let us introduce some notation. For a function $\langle a_j \mid j \in J \rangle$, let us write

$$\langle a_j \mid j \in J \rangle$$

for

$$\{ (j, b) \mid j \in J \land b \in a_j \}.$$ 

Let us also write, for sets $a_1, a_2, \ldots, a_n$,

$$L[a_1, a_2, \ldots, a_n]$$

for

$$L[\langle a_i \mid 1 \leq i \leq n \rangle].$$

To see the point of this notation, note that $L[\langle a_1, a_2 \rangle] = L$ and that if $\mathcal{U}_\beta$, $\beta < \alpha$, are uniform normal ultrafilters then $L[\langle \mathcal{U}_\beta \mid \beta < \alpha \rangle] = L$. (See Exercise 3.5.1.)

Lemma 3.5.2. Let $\langle \kappa_\beta \mid \beta < \alpha \rangle$ and $\langle \mathcal{U}_\beta \mid \beta < \alpha \rangle$ be such that each $\mathcal{U}_\beta$ is a uniform normal ultrafilter on the measurable cardinal $\kappa_\beta$. Let $a$ be any set. Then $L[a, \langle \mathcal{U}_\beta \mid \beta < \alpha \rangle] \models \text{ZFC + \ "for all } \beta < \alpha, \mathcal{U}_\beta \cap L[a, \langle \mathcal{U}_\beta \mid \beta < \alpha \rangle] \text{ is a uniform normal ultrafilter on } \kappa_\beta\text{."}$

The next two theorems, which will be used in Chapter 5, are from [Kunen, 1968].

Theorem 3.5.3. Let $\kappa$ and $\mathcal{U}$ be such that $L[U] \models \text{"}\mathcal{U} \cap L[U] \text{ is a uniform normal ultrafilter on } \kappa\text{."}$ In $L[U]$, $\mathcal{U} \cap L[U]$ is the unique uniform normal ultrafilter on $\kappa$. 

Proof. Let us assume for notational simplicity that $\mathcal{U} = \mathcal{U} \cap L[\mathcal{U}]$. Assume that the conclusion of the Theorem fails. Let $X \subseteq \kappa$ be least in the canonical wellordering of $L[\mathcal{U}]$ such that $X \in \mathcal{U} \iff X \notin \mathcal{V}$ for some uniform normal ultrafilter $\mathcal{V}$ on $\kappa$ in $L[\mathcal{U}]$. Let $\mathcal{V}$ witness this fact. By Exercise 3.2.3, whose proof is given during the proof of Lemma 3.3.11, this means that

$$\kappa \in i_\mathcal{U}(X) \iff \kappa \notin i_\mathcal{V}(X),$$

where we write $i_\mathcal{U}$ and $i_\mathcal{V}$ for $i_{L[\mathcal{U}]}$ and $i_{L[\mathcal{U}]}$ respectively. Now let

$$\begin{align*}
M &= \text{Ult}(L[\mathcal{U}]; \mathcal{U}); i_\mathcal{U}(\mathcal{U}));
N &= \text{Ult}(L[\mathcal{U}]; \mathcal{V}); i_\mathcal{V}(\mathcal{U})).
\end{align*}$$

Since the canonical elementary embeddings $j : \text{Ult}(L[\mathcal{U}]; \mathcal{U}) \prec M$ and $k : \text{Ult}(L[\mathcal{U}]; \mathcal{V}) \prec N$ do not move $\kappa$, it follows that

$$\kappa \in j(i_\mathcal{U}(X)) \iff \kappa \notin k(i_\mathcal{V}(X)).$$

Part (c) of Lemma 3.3.13, applied in $L[\mathcal{U}]$ and in $\text{Ult}(L[\mathcal{U}]; \mathcal{V})$, yields that $j(i_\mathcal{U}(\kappa)) = (2^\kappa)^+ = k(i_\mathcal{V}(\kappa))$. Lemma 3.3.12 thus implies that $j(i_\mathcal{U}(\mathcal{U})) = \mathcal{F} \cap M$ and $k(i_\mathcal{V}(\mathcal{U})) = \mathcal{F} \cap N$, where $\mathcal{F}$ is the closed, unbounded filter on $(2^\kappa)^+$. The elementarity of $j \circ i_\mathcal{U}$ gives that $M = L[j(i_\mathcal{U}(\mathcal{U}))]$ and therefore that $M = L[\mathcal{F} \cap M] = L[\mathcal{F}]$, by Theorem 3.4.12. Similarly $N = L[\mathcal{F}]$. From this it follows that $j(i_\mathcal{U}(\mathcal{U})) = k(i_\mathcal{V}(\mathcal{U}))$ and that $M = N$. But $X$ is definable in $L[\mathcal{U}]$ from $\mathcal{U}$; thus $j(i_\mathcal{U}(X)) = k(i_\mathcal{V}(X))$, a contradiction.

Theorem 3.5.4. Let $\langle \kappa_\gamma \mid \gamma < \alpha \rangle$ be a strictly increasing sequence of ordinal numbers such that $\alpha < \kappa_0$. Let $a \in V_{\kappa_0}$. Let $\langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle$ be such that $L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle] \models "\mathcal{U}_\gamma \cap L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle] is a uniform normal ultrafilter on \kappa_\gamma,""$ for each $\gamma < \alpha$.

In $L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle]$, $\mathcal{U}_\gamma \cap L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle]$ is the unique uniform normal ultrafilter on $\kappa_\gamma$ for each $\gamma < \alpha$.

Proof. We may assume that $a = a \cap L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle]$ and that $\mathcal{U}_\gamma = \mathcal{U}_\gamma \cap L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle]$ for every $\gamma < \alpha$. As in the proof of Theorem 3.5.3, argue by contradiction. Let $\delta < \alpha$ and suppose that $X \subseteq \kappa_\delta$ is least in the canonical wellordering of $L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle]$ such that $X \in \mathcal{U}_\delta \iff X \notin \mathcal{V}$ for some $\mathcal{V}$ that in $L[a, \langle \mathcal{U}_\gamma \mid \gamma < \alpha \rangle]$ is a uniform normal ultrafilter on $\kappa_\delta$. 

Let $\mathcal V$ witness this fact. Taking iterated ultrapowers with respect to $\mathcal U_{\delta+1}$, if necessary, we may assume that $\kappa_{\delta+1} > (2^{\kappa_\delta})^+$. Let
\[
M = \text{Ult}(L[a, \langle \mathcal U_\gamma \mid \gamma < \alpha \rangle]; i_U(\mathcal U_\delta)); \\
N = \text{Ult}(L[a, \langle \mathcal U_\gamma \mid \gamma < \alpha \rangle]; i_V(\mathcal U_\delta)).
\]
Let $j : \text{Ult}(L[a, \langle \mathcal U_\gamma \mid \gamma < \alpha \rangle]; \mathcal U_\alpha) \prec M$ and $k : \text{Ult}(L[a, \langle \mathcal U_\gamma \mid \gamma < \alpha \rangle]; \mathcal V) \prec N$ be the canonical elementary embeddings. If we can show that $j(i_U(\mathcal U_\beta)) = k(i_V(\mathcal U_\beta))$ for all $\beta < \alpha$ such that $\beta \neq \delta$, then we can obtain a contradiction as in the proof of Theorem 3.5.3. (Note that, e.g., $k(i_V(-\langle \mathcal U_\gamma \mid \gamma < \alpha \rangle)) = -\langle k(i_V)(\mathcal U_\gamma) \mid \gamma < \alpha \rangle$, since $\alpha < \kappa_0$.) For $\beta < \delta$ it follows from the fact that both $\text{crit}(j \circ i_U)$ and $\text{crit}(k \circ i_V)$ are greater than $\kappa_\beta$. For $\beta > \delta$ it follows from Lemma 3.3.14 applied in $L[a, \langle \mathcal U_\gamma \mid \gamma < \alpha \rangle]$ and in $\text{Ult}(L[a, \langle \mathcal U_\gamma \mid \gamma < \alpha \rangle]; \mathcal V)$. □

Remark. In [Kunen, 1968], stronger uniqueness theorems are proved. For example, it is shown that if $L[\mathcal U] \models \text{"\mathcal U is a uniform normal ultrafilter on } \kappa\text{"}$ and $L[\mathcal V] \models \text{"\mathcal V is a uniform normal ultrafilter on } \kappa\text{"}$ then $\mathcal U = \mathcal V$.

Suppose that there is a $\mathcal U \in L[\mathcal U]$ such that $L[\mathcal U]$ satisfies “$\mathcal U$ is a uniform normal ultrafilter on $\kappa$” for some ordinal $\kappa$. If $\mathcal U^#$ exists, then the set of all $n \in \omega$ such that $\langle n, \mathcal U \rangle \in \mathcal U^#$ is called $0^\dagger$. (See page 174 for the definition of $\mathcal U^#$. The name “$0^\dagger$” is due to R. Solovay.) This definition of $0^\dagger$ might seem to depend upon the choice of $\mathcal U$, but it does not; see Exercise 3.5.2. If there are two measurable cardinals, then $0^\dagger$ exists. The existence of $0^\dagger$ is equivalent to the determinacy of a certain class of games. (See Exercise 5.3.5.) One can also define the notion of $a^\dagger$ and one can define analogues of $0^\dagger$ for models $L[\mathcal U_\gamma \mid \gamma < \alpha]$ as in Theorem 3.5.4. These generalizations also appear in determinacy results in Chapter 5.

Exercise 3.5.1. (a) Show that, for any finite set $a$, $L[a] = L$. (It follows that, with any of the standard definitions of $n$-tuples, $L[\langle a_1, \ldots, a_n \rangle] = L$ for any sets $a_1, \ldots, a_n$.)

(b) Let $\alpha$ be an ordinal number. For $\beta < \alpha$, let $\mathcal U_\beta$ be a uniform normal ultrafilter on a measurable cardinal $\kappa_\beta$. Show that $L[\mathcal U_\beta \mid \beta < \alpha] = L$. (Hint. No $\mathcal U_\beta$ belongs to $L$.)

Exercise 3.5.2. Prove that $0^\dagger$ is well-defined, i.e., that $0^\dagger$ as defined above does not depend on the choice of $\mathcal U$. (Hint. Use Lemma 3.3.12.)
Chapter 4

$\Pi_1^1$ Games

The classes $\Pi_1^1$ and $\Sigma_1^1$ were defined on page 84. (We will repeat these definitions in §4.1 below.) By Theorem 2.2.7, if $A \subseteq [T]$ belongs both to $\Pi_1^1$ and to $\Sigma_1^1$ (that is, if $A \in \Delta_1^1$), then $G(A; T)$ is determined. We would like to extend this determinacy result to sets that belong to only one of $\Pi_1^1$ and $\Sigma_1^1$. The use of a dummy first move, as in the proof of Theorem 1.2.4, shows that the determinacy of all $\Pi_1^1$ games is equivalent with the determinacy of all $\Sigma_1^1$ games. Hence we may restrict our attention to proving the former. Unfortunately, $\Pi_1^1$ determinacy, even in countable trees, is not provable in ZFC. (See Exercise 4.1.1.) If we are to prove $\Pi_1^1$ determinacy, we must then assume principles that go beyond the ZFC axioms.

In §4.1 we prove the determinacy of all $\Pi_1^1$ games in an arbitrary game tree $T$ from the hypothesis that there is a measurable cardinal larger than $|T|$. In the remainder of the chapter, we present three variants of this proof. In §4.2 we show that the proof of §4.1 may be organized in terms of the machinery of semicoverings, a machinery similar to that of Chapter 2. In Chapter 5 we will use semicoverings to get determinacy proofs for wider classes of games. The result of §4.2 will play in Chapter 5 a role analogous to the role that Lemma 2.1.7 played in Chapter 2. In §4.3 we reorganize the proof of §4.2 in terms of the machinery of homogeneous trees. The determinacy proofs of Chapters 8 and 9 will use this machinery. (Those of Chapter 5, however, will make no use of it.) In §4.4 we weaken the hypothesis of our $\Pi_1^1$ determinacy proof from the existence of a measurable cardinal larger than $|T|$ to the existence of $a^+$ for every subset $a$ of $|T|$. By a theorem of Leo Harrington, for countable $T$, this is equivalent with the determinacy of all $\Pi_1^1$ games in $T$. 181
CHAPTER 4. $\Pi_1^1$ GAMES

(See Exercises 4.4.1.) The only later section that depends upon §4.4 is §5.3.

4.1 $\Pi_1^1$ Determinacy

Recall that a subset $A$ of $[T]$ belongs to $\Pi_1^1$ if and only if there is a closed $C \subseteq [T] \times \omega \omega$ (i.e., $[T] \times [\omega \omega]$) such that

$$(\forall x \in [T])(x \in A \leftrightarrow (\forall y \in \omega \omega) \langle x, y \rangle \notin C).$$

Recall also that $A \subseteq [T]$ belongs to $\Sigma_1^1$ just in case $[T] \setminus A \in \Pi_1^1$.

The following lemma was proved in the course of proving Theorem 2.2.3. The two sentences of the lemma state the propositions (a) and (b) occurring in that proof. For the definition of open-separated union, see page 80.

Remark. The reader who has skipped §2.2 should now read the proof of Lemma 2.2.3. Such a reader may ignore the parts of that lemma and this one that are concerned with open-separated unions.

Lemma 4.1.1. Every clopen set belongs to $\Pi_1^1$ and to $\Sigma_1^1$. Both $\Pi_1^1$ and $\Sigma_1^1$ are closed under countable unions and open-separated unions.

Remark. It will be convenient to deal with game trees $T$ such that there are no terminal positions in $T$. To see that this is justified, let $T$ be a game tree. Consider the tree

$$T' = T \cup \{ p \prec \langle 0, \ldots, 0 \rangle \mid p \text{ is terminal in } T \}.$$

The obvious $f : T' \to T$ induces a homeomorphism from $[T']$ to $[T]$ such that, for each $A \subseteq [T]$, $G(A; T')$ is determined if and only if $G(f^{-1}(A); T')$ is determined.

We next prove two standard representations of $\Pi_1^1$ sets. The second of these will be useful for determinacy proofs. To avoid unnecessary details, we do not state the most general versions of these lemmas. For later applications, however, we state the lemmas in slightly more general form than we will need in this section.

If $T$ is a game tree, let us denote by $[T]$ the set of all infinite plays in $T$. If there are no terminal positions in $T$, then $[T] = [T]$. If $T$ is a game tree with taboos and every play normal in $T$ is infinite, then $[T] = [T]$. 


Lemmas 4.1.2 and 4.1.4 give ways to characterize the members of a $\Pi_1^1$ set that belong to $[T]$. With a little more complex characterizations, we could remove the qualification “that belong to $[T]$."

**Lemma 4.1.2.** Let $T$ be a game tree and let $A \subseteq [T]$. Then $A \in \Pi_1^1$ if and only if there is a relation $R \subseteq T \times \omega$, such that

(a) $R(\emptyset, \emptyset)$;

(b) $(\forall p \in T)(\forall s \in \omega)(R(p, s) \rightarrow \ell h(p) = \ell h(s))$;

(c) $(\forall p \in T)(\forall s \in \ell h(p) \omega)(\forall n < \ell h(p))(R(p, s) \rightarrow R(p \upharpoonright n, s \upharpoonright n))$.

(d) $(\forall x \in [T])(x \in A \leftrightarrow (\forall y \in \omega)(\exists n \in \omega) \neg R(x \upharpoonright n, y \upharpoonright n))$.

(Here we have written $R(p, s)$ for $\langle p, s \rangle \in R$.)

**Proof.** Suppose that $C$ witnesses that $A \in \Pi_1^1$. Let $R(p, s)$ hold if and only if $\ell h(p) = \ell h(s)$ and

$$p = s = \emptyset \lor (\exists x \in [T])(\exists y \in \omega)(p \subseteq x \land s \subseteq y \land \langle x, y \rangle \in C).$$

It is trivial that $R$ has properties (a), (b), and (c). Let $x \in [T]$. If $x \notin A$, then there is a $y \in \omega$ such that $\langle x, y \rangle \in C$; thus $R(x \upharpoonright n, y \upharpoonright n)$ holds for every $n \in \omega$. If $x \in A$, then the fact that $C$ is closed guarantees that for all $y \in \omega$ there is an $n$ such that $([T]_x \upharpoonright n) \times ([\omega]_y \upharpoonright n) \cap C = \emptyset$. Thus $R$ has property (d).

Suppose now that there is an $R$ with properties (a), (b), (c), and (d). Let

$$C = \{\langle x, y \rangle \mid x \in [T] \setminus [T] \lor (\forall n \in \omega) R(x \upharpoonright n, y \upharpoonright n)\}.$$

It is easy to see that $C$ witnesses that $A \cap [T] \in \Pi_1^1$. Since $A \setminus [T]$ is open, it follows by Lemma 4.1.1 that $A \in \Pi_1^1$. □

To prove what for us will be the most useful characterization of $\Pi_1^1$ sets, we need the definition and the lemma that follow.

The **Brouwer–Kleene ordering** $<^\text{BK}$ of $\omega$ is defined by

$$s <^\text{BK} t \leftrightarrow (s \supseteq t \lor (\exists n < \min\{\ell h(s), \ell h(t)\})(s \upharpoonright n = t \upharpoonright n \land s(n) < t(n))).$$

The Brouwer–Kleene ordering is a linear ordering of $\omega$. It agrees with the lexicographic ordering $<_\text{lex}$ except that when $s \supseteq t$ then $t <_\text{lex} s$ but $s <^\text{BK} t$. 
Lemma 4.1.3. Let $S$ be a subtree of $\omega^\omega$. Then $S$ is wellfounded (i.e., $[S] = \emptyset$) if and only if the restriction to $S$ of $<^{BK}$ is a wellordering.

Proof. Assume first that $S$ is not wellfounded. Let $y \in [S]$. Then $\langle y \upharpoonright n \mid n \in \omega \rangle$ is an infinite descending sequence with respect to $<^{BK}$, and so $<^{BK}$ is not a wellordering.

Now assume that $<^{BK} \upharpoonright S$ is not a wellordering. Since it is a linear ordering, it must not be wellfounded. Let $\langle t_i \mid i \in \omega \rangle$ be an infinite descending sequence with respect to $<^{BK}$ with each $t_i \in S$.

We prove by induction on $m \in \omega$:

(i) for all but finitely many $i \in \omega$, $\ell h(t_i) \geq m$;
(ii) $\lim_i (t_i \upharpoonright m)$ exists.

(i) and (ii) trivially hold for $m = 0$. Suppose that (i) and (ii) hold for $m$. Let $i_m$ be such that

$$(\forall i \geq i_m)(\ell h(t_i) \geq m \land t_i \upharpoonright m = t_{i_m} \upharpoonright m).$$

At most one of the $t_i$, $i \geq i_m$, can be $t_{i_m} \upharpoonright m$ and this one, if it exists, must be $t_{i_m}$. Therefore $\ell h(t_i) \geq m + 1$ for every $i > i_m$. By the definition of $<^{BK}$,

$$t_{i_m+1}(m) \geq t_{i_m+2}(m) \geq \cdots.$$ 

Thus $\lim_i t_i(m)$ exists.

Now let $y \in \omega^\omega$ be given by

$$y(m) = \lim_i t_i(m).$$

Since each $y \upharpoonright n$ is extended by some $t_i$, it follows that each $y \upharpoonright n$ belongs to $S$. Thus $y \in [S]$, and hence $S$ is not wellfounded. □

The following characterization of $\Pi^1_1$ sets is a well-known variant of those of [Kleene, 1955] and [Lusin and Sierpiński, 1923].

Lemma 4.1.4. Let $T$ be a game tree and let $A \subseteq [T]$. Then $A \in \Pi^1_1$ if and only if there is a function $p \mapsto <_p$ with domain $T$ such that

1. for all $p \in T$, $<_p$ is a linear ordering of $\ell h(p)$ with greatest element $0$ (if $\ell h(p) > 0$);
(2) for all \( p \subseteq q \in T \), \( <_p \) is the restriction of \( <_q \) to \( \ell h(p) \);

(3) \((\forall x \in [T])(x \in A \leftrightarrow <_x \text{ is a wellordering})\), where \( <_x \) is the relation \( \bigcup_{n \in \omega} <_{x|n} \).

**Proof.** Suppose first that \( A \in \Pi^1_1 \).

Let \( n \mapsto s_n \) be a bijection from \( \omega \) to \( \omega \) such that whenever \( s_m \subseteq s_n \) then \( m \leq n \), i.e. such that no sequence is enumerated before a sequence it properly extends. Let \( R \subseteq T \times \omega \) be given by Lemma 4.1.2.

For \( p \in T \) and \( m \) and \( n \) smaller than \( \ell h(p) \), we let \( m <_p n \) just in case one of the following holds:

(i) \( \neg R(p \upharpoonright \ell h(s_m), s_m) \land R(p \upharpoonright \ell h(s_n), s_n) \);
(ii) \( m < n \land \neg R(p \upharpoonright \ell h(s_m), s_m) \land \neg R(p \upharpoonright \ell h(s_n), s_n) \);
(iii) \( s_m <^{\text{BK}} s_n \land R(p \upharpoonright \ell h(s_m), s_m) \land R(p \upharpoonright \ell h(s_n), s_n) \).

In other words, we place all the \( n < \ell h(p) \) such that \( \neg R(p \upharpoonright \ell h(s_n), s_n) \) at the beginning of the ordering \( <_p \) in their natural order, and we then order the remaining numbers \( n < \ell h(p) \) according to the Brouwer–Kleene ordering of the corresponding sequences \( s_n \).

The fact that no \( s \) precedes in the enumeration any of its proper initial segments guarantees that \( \ell h(s_n) \leq n \) and so that \( <_p \) is well-defined. It is clear that \( <_p \) is a linear ordering of \( \ell h(p) \) for every \( p \in T \). Since \( s_0 = \emptyset \), property (a) of \( R \) and the fact that \( \emptyset \) is maximal with respect to \( <^{\text{BK}} \) imply that \( 0 \) is maximal with respect to every \( <_p \). Thus (1) holds.

Condition (2) is obvious from the definition.

To prove (3), fix \( x \in [T] \) and let

\( S = \{ t \in \omega : R(x \upharpoonright n, t) \} \).

Properties (b) and (c) of \( R \) imply that \( S \) is a subtree of \( \omega \). Property (d) implies that

\( x \in A \leftrightarrow [S] = \emptyset \).

By Lemma 4.1.3,

\( [S] = \emptyset \leftrightarrow <^{\text{BK}} [S] \) is a wellordering.

Since \( <_x \) is isomorphic to the natural ordering of \( \{ n \mid s_n \notin S \} \) followed by \( <^{\text{BK}} [S] \),

\( <^{\text{BK}} [S] \) is a wellordering \( \leftrightarrow <_x \) is a wellordering.
For the other half of the lemma, suppose there is a function \( p \mapsto <_p \) satisfying (1), (2), and (3). Define \( C \subseteq [T] \times \omega^\omega \) by
\[
\langle x, y \rangle \in C \iff (x \notin [T] \lor (\forall n \in \omega)(y(n + 1) <_x y(n))).
\]
Clearly we have that
\[
(\forall x \in [T])(x \in [T] \land <_x \text{ is a wellordering}) \iff (\forall y \in \omega^\omega)(\langle x, y \rangle \notin C).
\]
Thus \( C \) witnesses that \( A \cap [T] \in \Pi_1^1 \). Lemma 4.1.1 implies that \( A \in \Pi_1^1 \).

For later use, we also give a proof using Lemma 4.1.2 of this “if” half of the lemma: Let \(<_p \) have properties (1), (2), and (3). Define \( R \subseteq T \times \omega^\omega \) by letting \( R(p, s) \) hold if and only if \( \ell h(p) = \ell h(s) \) and, for all \( n \) such that \( n + 1 < \ell h(p) \),
\[
(s(n) < \ell h(p) \land s(n + 1) < \ell h(p)) \rightarrow s(n + 1) <_p s(n).
\]
It is easy to check that \( R \) satisfies (a)–(d) of Lemma 4.1.2. That lemma thus gives that \( x \in A \). \( \square \)

We will sometimes want to point out sharper lightface versions of our theorems. We make the following definitions explicitly only for subsets of \( \omega^\omega \), but the definitions extend in an obvious way to subsets of finite products of \( \omega \) and \( \omega^\omega \). A subset \( A \) of \( \omega^\omega \) belongs to \( \Pi_1^1 \) if there is a relation \( R \subseteq \omega^\omega \times \omega^\omega \) with properties (a)–(d) of Lemma 4.1.2 such that \( R \) is recursive. A set \( A \subseteq \omega^\omega \) belongs to \( \Sigma_1^1 \) if \( \omega^\omega \setminus A \in \Pi_1^1 \). For \( x \in \omega^\omega \), the classes \( \Pi_1^1(x) \) and \( \Sigma_1^1(x) \) are similarly defined, with “recursive in \( x \)” replacing “recursive.” It is easy to see that
\[
\Pi_1^1 = \bigcup_{x \in \omega^\omega} \Pi_1^1(x);
\]
\[
\Sigma_1^1 = \bigcup_{x \in \omega^\omega} \Sigma_1^1(x).
\]

Here is the lightface version of Lemma 4.1.4:

**Lemma 4.1.5.** Let \( A \subseteq \omega^\omega \). Then \( A \in \Pi_1^1 \) if and only if there is a recursive function \( p \mapsto <_p \) with domain \( \omega^\omega \) such that
\[
(1) \text{ for all } p \in \omega^\omega, <_p \text{ is a linear ordering of } \omega \text{ with greatest element } 0 \\
(\text{if } \ell h(p) > 0);
\]
4.1. \( \Pi_1^1 \) DETERMINACY

(2) for all \( p \subseteq q \in \omega^\omega \), \( <_p \) is the restriction of \( <_q \) to \( \ell \h(p) \);

(3) \( (\forall x \in \omega^\omega)(x \in A \leftrightarrow <_x \text{ is a wellordering}) \), where \( <_x \) is the relation \( \bigcup_{n \in \omega} <_{x|n} \).

The proof of the “only if” part of Lemma 4.1.5 is just like that of Lemma 4.1.4, except that the function \( n \mapsto s_n \) must be chosen to be recursive. The proof of the “if” part of Lemma 4.1.5 is just like the second proof of the “if” part of Lemma 4.1.4.

Lemma 4.1.4 and the Rowbottom ultrafilters introduced in §3.1 provide us with the tools for proving \( \Pi_1^1 \) determinacy from the existence of large enough measurable cardinals.

**Theorem 4.1.6. ([Martin, 1970])** Let \( T \) be a game tree. Assume there is a measurable cardinal larger than \( |T| \). Then all \( \Pi_1^1 \) games in \( T \) are determined.

**Proof.** As we argued on page 182, we may assume that there are no terminal positions in \( T \) and so that \( |T| = |T| \).

Let \( A \subseteq |T| \) with \( A \in \Pi_1^1 \). Let \( p \mapsto <_p \) and \( x \mapsto <_x \) be as given by Lemma 4.1.4.

Let \( \kappa \) be a measurable cardinal with \( |T| < \kappa \) and, by Lemma 3.1.7, let \( \mathcal{U} \) be a uniform normal ultrafilter on \( \kappa \).

We describe a game tree \( T^* \) by describing the legal plays in \( T^* \):

\[
\begin{align*}
\text{I} & \langle \langle a_0, \xi_0 \rangle, a_1 \rangle \langle a_2, \xi_1 \rangle \langle a_4, \xi_2 \rangle \ldots \\
\text{II} & a_3 \ldots
\end{align*}
\]

Each \( \langle a_i \mid i < n \rangle \) must be a legal position in \( T \). Each \( \xi_i \) must be an ordinal number smaller than \( \kappa \). Let \( \pi : T^* \to T \) be given by

\[
\pi(\langle \langle a_0, \xi_0 \rangle, a_1, \ldots, a_{2n-1}, \langle a_{2n}, \xi_{2n} \rangle \rangle) = \langle a_0, a_1, \ldots, a_{2n-1}, a_{2n} \rangle.
\]

\( \pi \) induces a continuous function, which we also call \( \pi \), from \( [T^*] \) to \( [T] \).

Consider the set \( A^* \subseteq [T^*] \) given by

\[
\langle \langle a_0, \xi_0 \rangle, a_1, \langle a_2, \xi_1 \rangle, a_3, \ldots \rangle \in A^* \leftrightarrow (\forall m \in \omega)(\forall n \in \omega)(m <_{(a_i| i \in \omega)} n \leftrightarrow \xi_m < \xi_n).
\]

I wins a play \( x^* \) of \( G(A^*; T^*) \) if his ordinal moves \( \xi_i \) give an embedding of \( (\omega; <_{\pi(x^*)}) \) into \( (\kappa; <) \). In particular this means that \( \pi(A^*) \subseteq A \). Thus I wins.
a play $x^*$ of $G(A^*; T^*)$ if he not only succeeds in making $\pi(x^*) \in A$ but also verifies that $\pi(x^*) \in A$ by embedding of $(\omega; <_{\pi(x^*)})$ into $(\kappa; <)$.

The set $A^*$ is closed; so, by Lemma 1.2.4, $G(A^*; T^*)$ is determined.

Suppose first that $G(A^*; T^*)$ is a win for I. Let $\sigma^*$ be a winning strategy for I for $G(A^*; T^*)$. Let $\sigma$ be a strategy for I in $T$ such that $\sigma(\pi(p^*))$ is the first component of $\pi^*(p^*)$ for every $p^*$ consistent with $\sigma$. (This condition fixes $\sigma$ on all positions consistent with $\sigma^*$.) A play consistent with $\sigma$ is thus the image under $\pi$ of a play consistent with $\sigma^*$. It follows that every play consistent with $\sigma$ belongs to $A$ and so that $\sigma$ is a winning strategy for $G(A; T)$.

Suppose now that $G(A^*; T^*)$ is a win for II. Let $\tau^*$ be a winning strategy for II for $G(A^*; T^*)$. We will define a strategy $\tau$ for II for $G(A; T)$.

Let $n \in \omega$, let $p = \langle a_0, a_1, \ldots, a_{2n} \rangle$ be a position in $T$, and let $v \in [\kappa]^{n+1}$. There is a unique $q^*(p, v) = \langle \langle a_0, \xi_0 \rangle, a_1, \ldots, \langle a_{2n}, \xi_n \rangle \rangle$ such that $\pi(q^*(p, v)) = p$ and $i \mapsto \xi_i$ embeds $(n+1; <_p)$ into $(v; <)$. Let

$$\tau(p) = a \iff \{ v \in [\kappa]^{n+1} \mid \tau^*(q^*(p, v)) = a \} \in \mathcal{U}^{[n+1]} ,$$

where $\mathcal{U}^{[n+1]}$ is the Rowbottom ultrafilter defined from $\mathcal{U}$ as on page 136. Since $|T| < \kappa$ and $\mathcal{U}^{[n+1]}$ is $\kappa$-complete, $\tau(p)$ is defined. Let

$$Z_p = \{ v \in [\kappa]^{n+1} \mid \tau(p) = \tau^*(q^*(p, v)) \} .$$

$Z_p$ belongs to $\mathcal{U}^{[n+1]}$.

Remark. Equivalently, we may define $\tau(p)$ by

$$\tau(p) = \int \tau^*(q^*(p, v)) d\mu^{[n+1]} ,$$

where $\mu^{[n+1]}$ is the measure on $[\kappa]^{n+1}$ given by

$$\mu^{[n+1]}(X) = \begin{cases} 1 & \text{if } X \in \mathcal{U}^{[n+1]} ; \\ 0 & \text{otherwise.} \end{cases}$$

Thus one might call the technique by which $\tau$ is obtained from $\tau^*$ integration.

Since each $Z_p$ belongs to $\mathcal{U}^{[n+1]}$, we may, by the definition of $\mathcal{U}^{[n+1]}$, let $X_p \subseteq \kappa$ be such that $X_p \in \mathcal{U}$ and $[X_p]^{n+1} \subseteq Z_p$. Let

$$X = \bigcap \{ X_p \mid p \in T \land \ellh(p) \text{ odd} \} .$$
Since $U$ is $\kappa$-complete and $|T| < \kappa$, we have that $X \in U$. Moreover, for every $n$ and every $p \in T$ of length $2n + 1$,

$$(\forall v \in [X]^{n+1}) \tau(p) = \tau^*(q^*(p, v)).$$

To show that $\tau$ is a winning strategy for $\Pi$ for $G(A; T)$, let $x \in [T]$ be consistent with $\tau$. Assume for a contradiction that $x \in A$. Since $|X| = \kappa > \aleph_0$, let $i \mapsto \xi_i$ embed the wellordering $(\omega; <_x)$ into $(X; <)$. Let $x^*$ be the play in $T^*$ with these values of the $\xi_i$ and with $\pi(x^*) = x$. Clearly $x^*$ is a win for I in $G(A^*; T^*)$. But, for each $p^* \subseteq x^*$ of odd length,

$$x^*(\ell h(p^*)) = \tau(\pi(p^*)) = \tau^*(q^*(\pi(p^*), \{\xi_i \mid 2i < \ell h(p^*)\})) = \tau^*(p^*).$$

Thus $x^*$ is a play consistent with the winning strategy $\tau^*$, and this contradicts the fact that $x^*$ is a win for I. $\square$

**Exercise 4.1.1.** Assume that all $\Pi^1_1$ games in countable trees are determined and prove that $\omega_1$ is inaccessible in $L$. Deduce that $\Pi^1_1$ determinacy is not provable in ZFC.

**Hint.** First show that every uncountable $\Pi^1_1$ subset of $\omega^2 (= [<\omega^2])$ has a perfect subset (a nonempty subset without isolated points). To do this, let $A \subseteq \omega_2$ with $A \in \Pi^1_1$ and let $R$ witness that $A \in \Pi^1_1$. Consider a game tree $T$ plays in which are as follows:

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>s_0</th>
<th>s_1</th>
<th>s_2</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td></td>
<td>e_0</td>
<td>e_1</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>II</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Each $e_i$ must belong to $2 (= \{0, 1\})$. Each $s_i$ must belong to $<\omega_2$, and each $s_{i+1}$ must satisfy $s_{i+1} \supseteq s_i \setminus \{e_i\}$.

Let $G(B; T)$ be the game that I wins if and only if $\bigcup_{i<\omega} s_i \in A$. Prove that $G(B; T)$ is a win for I if and only if $A$ has a perfect subset. Prove that $G(B; T)$ is a win for II if and only if $A$ is countable. (This argument, the same as alluded to in Exercises 1.1.2 and 1.1.4, is from Davis, 1964). See pages 295–297 of [Moschovakis, 1980].)

Now prove the result of Gödel [1938] that if $V = L$ then there is an uncountable subset of $\omega^2$ belonging to $\Pi^1_1$ and without a perfect subset. Note that the proof works under the weaker hypothesis that $(\omega_1)^L = \omega_1$. Relativize
the proof to get an uncountable set in $\Pi^1_1(x)$ without a perfect subset if $(\omega_1)^{L[x]} = \omega_1$ for any $x \in \omega\omega$. (See page 283 of [Moschovakis, 1980].)

The results of the preceding two paragraphs and the assumption of the exercise imply that $(\omega_1)^{L[x]}$ is countable for every $x \in \omega\omega$, and this implies that that $\omega_1$ is inaccessible in $L$.

The first part of the exercise implies that, under the assumption of the exercise, $L_{\omega_1} \models ZFC$. The last part of the exercise follows by the Second Incompleteness Theorem of Gödel. Of course, the last part follows simply from the fact that $\Pi^1_1$ determinacy is false in $L$.

**Exercise 4.1.2.** Prove that the determinacy of all $\Pi^1_1$ games in $\omega\omega$ is not provable in ZFC.

*Hint.* Show that $\Pi^1_1$ determinacy is false in $L$, by coding as games in $\omega\omega$ the games $G(B; T)$ of the hint to Exercise 4.1.1.

## 4.2 Semicoverings

The proof of Theorem 4.1.6 is reminiscent of those of Chapter 2. The function $\pi$ occurring in the former plays a role like that of the functions $\pi$ that are components of coverings. The construction of the strategies $\sigma$ and $\tau$ from the strategies $\sigma^*$ and $\tau^*$ respectively gives an operation like the $\phi$ component of a covering. Nevertheless, these operations do not in general give rise to a covering. The difficulty concerns the $\Psi$ component of a covering. In the proof of Theorem 4.1.6, we were given a play $x$ consistent with $\tau$ and we constructed an $x^*$ consistent with $\tau^*$ such that $\pi(x^*) = x$, but we were able do this only because we made the assumption that $x \in A$. This assumption provided us with the ordinals $\xi_i$ used to define $x^*$.

It is a theorem of Itay Neeman that, under a large cardinal hypothesis, every $\Pi^1_1$ set can be unraveled by a covering. We will discuss this theorem on page 307.

The proof of Theorem 4.1.6 does give rise to what we will call *semicoverings*. Semicoverings are enough like coverings that (1) any set $A$ unraveled by a what we will call an $A$ semicovering is determined and (2) certain operations on semicoverings yield semicoverings. In this section we will define semicoverings and use them to reprove $\Pi^1_1$ determinacy from measurable cardinals. In Chapter 5 we will use semicoverings to get further determinacy results.
As we did with coverings, we will define semicoverings in terms of game trees with taboos. If $T$ is a game tree with taboos, then a \textit{semicovering of $T$} is a quadruple $\langle \tilde{T}, \pi, \phi, \Psi \rangle$ such that

(a) $\tilde{T}$ is a game tree with taboos;
(b) $\pi : \tilde{T} \Rightarrow T$;
(c) $\phi : \tilde{T} \xrightarrow{\tilde{\gamma}} T$;
(d) $\Psi : \text{domain}(\Psi) \rightarrow [\tilde{T}]$, the domain of $\Psi$ is a subset of

$$\{\langle \tilde{\sigma}, x \rangle \mid \tilde{\sigma} \in S(\tilde{T}) \wedge x \in [T] \wedge x \text{ is consistent with } \phi(\tilde{\sigma})\},$$

and, for all $\langle \tilde{\sigma}, x \rangle \in \text{domain}(\Psi)$,

(i) $\Psi(\tilde{\sigma}, x)$ is consistent with $\tilde{\sigma}$;
(ii) $\pi(\Psi(\tilde{\sigma}, x)) \subseteq x$;
(iii) either (1) $\pi(\Psi(\tilde{\sigma}, x)) = x$ and $\Psi(\tilde{\sigma}, x)$ and $x$ are both normal or both taboo for the same player or (2) $\Psi(\tilde{\sigma}, x)$ is taboo for the player for whom $\tilde{\sigma}$ is a strategy.

The only difference between a covering of $T$ and a semicovering of $T$ is that clause (d) above, unlike the clause (d) on page 66, does not demand that $\Psi(\tilde{\sigma}, x)$ be defined for every pair $\langle \tilde{\sigma}, x \rangle$ such that $\tilde{\sigma} \in S(\tilde{T})$ and $x$ is a play in $T$ consistent with $\phi(\tilde{\sigma})$. On the one hand, this means that every covering of $T$ is a semicovering of $T$. On the other hand, it means that the notion of a semicovering is a very weak one. For example, domain($\Psi$) may be empty. Thus a semicovering will not be of much use to us unless the domain of $\Psi$ satisfies some further conditions.

We say that a semicovering $\langle \tilde{T}, \pi, \phi, \Psi \rangle$ of $T$ \textit{unravels} a subset $A$ of $[T]$ if $\pi^{-1}(A)$ is a clopen subset of $[\tilde{T}]$. Here $\pi : [\tilde{T}] \rightarrow [T]$ is defined as on page 65.

The existence of a mere semicovering of $T$ that unravels $A$ does not imply the determinacy of $G(A; T)$. (See Exercise 4.2.1.) For $A \subseteq [T]$, let us then define an \textit{$A$-semicovering} of $T$ to be a semicovering $\langle \tilde{T}, \pi, \phi, \Psi \rangle$ of $T$ such that

(e) If $\tilde{\sigma} \in S(\tilde{T})$ and $x$ witnesses that $\phi(\tilde{\sigma})$ is not a winning strategy for $G(A; T)$ (i.e., if $x$ is a play consistent with $\phi(\tilde{\sigma})$ that is a loss in $G(A; T)$ for the player for whom $\tilde{\sigma}$ is a strategy), then $\langle \tilde{\sigma}, x \rangle \in \text{domain}(\Psi)$. 

An $A$-semicovering is exactly what is needed to make the proof of Lemma 2.1.3 go through:

**Lemma 4.2.1.** If $A \subseteq [T]$ and there is an $A$-semicovering of $T$ that unravels $A$, then $G(A; T)$ is determined.

**Proof.** The proof of Lemma 2.1.3 works here too, since we may assume that the play $x$ occurring in that proof is a win for the bad player. $\square$

Just as we needed the notion of a $k$-covering, we will need in Chapter 5 the notion of a $k$-semicovering. Let $T$ be a game tree with taboos and let $C = \langle \tilde{T}, \pi, \phi, \Psi \rangle$ be a semicovering of $T$. For $k \in \omega$, $C$ is a $k$-semicovering of $T$ if

(i) $k \tilde{T} = kT$;
(ii) $\pi \restriction k\tilde{T}$ is the identity;
(iii) $\phi \restriction S(k\tilde{T})$ is the identity.

The proof of Theorem 4.1.6 adapts fairly easily to give an $A$-semicovering of $T$ that unravels $A$ (under the hypothesis that there is a measurable cardinal larger than $|T|$). But for our unraveling results in Chapter 5 for sets more complicated than $\Pi^1_1$ sets, we will need a stronger result for $\Pi^1_1$. Of course we will need $A_k$-semicoverings for arbitrary $k \in \omega$ and we will need a bound on the size of the $\tilde{T}$ of the semicovering, but we will need even more. To state this stronger result, we require another definition:

If $T$ is a game tree with taboos and if $A$ and $B$ are subsets of $[T]$, then an $(A, B)$-semicovering of $T$ is an $A$-semicovering $\langle \tilde{T}, \pi, \phi, \Psi \rangle$ of $T$ such that

(f) for every $\tilde{\sigma} \in S(\tilde{T})$ and for every $x \in B$ such that $x$ is consistent with $\phi(\tilde{\sigma})$, the pair $\langle \tilde{\sigma}, x \rangle$ belongs to the domain of $\Psi$;

(g) every normal play in $\tilde{T}$ belongs to $\pi^{-1}(B)$.

**Lemma 4.2.2.** Let $T$ be a game tree with taboos. Let $B \subseteq [T]$ with $B \in \Pi^1_1$. Let $k \in \omega$. Suppose that $\kappa$ is a measurable cardinal larger than $|T|$.

(i) There is a $(B, B)$-semicovering $\langle \tilde{T}, \pi, \phi, \Psi \rangle$ of $T$ such that $|\tilde{T}| \leq \kappa$.

(ii) There is a $(|[T] \setminus B, B)$-semicovering $\langle \tilde{T}, \pi, \phi, \Psi \rangle$ of $T$ such that $|\tilde{T}| \leq \kappa$. 

Proof. We give the proof of (i). The proof of (ii) is similar, with the roles of
the players reversed. Also, to keep the proof closer to that of Theorem 4.1.6,
we do the case $k = 0$. (We will indicate briefly how to handle the case $k > 0$.)

Let $p \mapsto <_p$ and $x \mapsto <_x$ be as given by Lemma 4.1.4. Let $\mathcal{U}$ be a uniform
normal ultrafilter on $\kappa$.

Define $\tilde{T}$ as follows. Plays in $\tilde{T}$ are of the form

\[\langle a_0, \xi_0 \rangle \langle a_2, \xi_1 \rangle \langle a_4, \xi_2 \rangle \ldots \]

Each $\langle a_i | i < n \rangle$ must be a legal position in $T$. Each $\xi_i$ must be an ordinal
number smaller than $\kappa$. So far the definition is like that of $T^*$ in the proof
of Theorem 4.1.6. But we impose a further restriction. We demand that

\[m < (a_0, \ldots, a_{2n}) m' \leftrightarrow \xi_m < \xi_{m'}\]

for all $m$ and $m'$ no greater than $n$. Thus all legal positions $\tilde{p}$ in $\tilde{T}$ are such
that $i \mapsto \xi_i$ embeds $(\omega; <_{\pi(\tilde{p})[n]})$ into $(\kappa; <)$, where

\[\pi(\langle a_0, \xi_0 \rangle, a_1, \langle a_2, \xi_1 \rangle, a_3, \ldots) = \langle a_0, a_1, a_2, a_3, \ldots \rangle\]

and where $n$ is the greatest number such that $2n \leq \ell(h(p)) + 1$. If a terminal
position $\tilde{p}$ in $\tilde{T}$ is such that $\pi(\tilde{p})$ is taboo in $\tilde{T}$, then $\tilde{p}$ is taboo for the same
player in $\tilde{T}$. If $\tilde{p}$ is terminal in $\tilde{T}$ and $\pi(\tilde{p})$ is terminal and normal in $T$, then
$\tilde{p}$ is taboo for $I$ in $\tilde{T}$ if $\pi(\tilde{p}) \not\in B$ and $\tilde{p}$ is normal in $\tilde{T}$ if $\pi(\tilde{p}) \in B$. If $\tilde{p}$ is
terminal in $\tilde{T}$ but $\pi(\tilde{p})$ is not terminal in $T$ then $\tilde{p}$ is taboo for $I$ in $T$. (Such
a $\tilde{p}$ must have some even length $2i$. It occurs when I cannot play $\xi_i$ so as to
obey the order restriction.)

Clearly $|\tilde{T}| \leq \kappa$.

If $\tilde{x}$ is an infinite play in $\tilde{T}$, then $i \mapsto \xi_i$ embeds $(\omega; <_{\pi(\tilde{x})})$ into $(\kappa; <)$,
so $\pi(\tilde{x}) \in B$. If $\tilde{x}$ is a finite normal play in $\tilde{T}$, then also $\pi(x) \in B$. Thus
$\pi^{-1}(B) = [\tilde{T}]$.

To define $\phi$ and $\Psi$, suppose first that $\tilde{\sigma} \in S_1(\tilde{T})$. For positions $\tilde{p}$ consistent
with $\tilde{\sigma}$, let $(\phi(\tilde{\sigma}))(\pi(\tilde{p}))$ be the first component of $\tilde{\sigma}(\tilde{p})$. For other positions
$\tilde{p} \in T$, define $\phi(\tilde{\sigma})$ arbitrarily, subject to the constraints of clause (iii) in
the definition of $\phi: \tilde{T} \rightarrow T$. (In the proof for $k > 0$, the constraints of clause
(iii) in the definition of a $k$-covering must be met also.) We define $\Psi(\tilde{\sigma}, x)$
for every play $x$ consistent with $\phi(\tilde{\sigma})$. Fix such an $x$. There is a unique play
$\hat{x}$ such that $\hat{x}$ is consistent with $\tilde{\sigma}$ and $\pi(\hat{x}) \subseteq x$. Let $\Psi(\tilde{\sigma}, x) = \hat{x}$.
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Now suppose that $\tilde{\tau} \in \mathcal{S}_{\Pi}(\tilde{T})$. Let $n \in \omega$, let $p = \langle a_0, a_1, \ldots, a_{2n} \rangle$ be a position in $T$, and let $v \in [\kappa]^{n+1}$. As in the proof of Theorem 4.1.6 there is a unique 

$$\tilde{q}(p, v) = \langle \langle a_0, \xi_0 \rangle, a_1, \ldots, \langle a_{2n}, \xi_n \rangle \rangle$$

such that $\pi(\tilde{q}(p, v)) = p$ and $i \mapsto \xi_i$ is order-preserving from $(n+1; <_p)$ to $(v; <)$. Let 

$$(\phi(\tilde{\tau}))(p) = a \iff \{ v \in [\kappa]^{n+1} | \tilde{\tau}(\tilde{q}(p, v)) = a \} \in \mathcal{U}^{n+1}.$$ 

(Here, as in the proof of Theorem 4.1.6, we are using the fact that $|T| < \kappa$.) Let 

$$Z_p = \{ v \in [\kappa]^{n+1} | (\phi(\tilde{\tau}))(p) = \tilde{\tau}(\tilde{q}(p, v)) \}.$$ 

$Z_p$ belongs to $\mathcal{U}^{n+1}$. Let $X_p \subseteq \kappa$ be such that $X_p \in \mathcal{U}$ and $[X_p]^{n+1} \subseteq Z_p$. Let 

$$X = \bigcap \{ X_p | p \in T \land \ell h(p) \text{ odd} \}.$$ 

We have that $X \in \mathcal{U}$ and that, for every $n$ and every $p \in T$ of length $2n+1$, 

$$(\forall v \in [X]^{n+1}) \tilde{\tau}(\tilde{q}(p, v)) = (\phi(\tilde{\tau}))(p).$$ 

We define $\Psi(\tilde{\tau}, x)$ for every play $x$ consistent with $\phi(\tilde{\tau})$ such that either $x \in B$ or else $x$ is finite. Fix such an $x$. If $x \in B$ and $x$ is infinite, let $i \mapsto \xi_i$ embed $(\omega; <_x)$ into $(X; <)$. If $x$ is finite, let $i \mapsto \xi_i$ embed $([\ell h(x) <_x]$ into $(X; <)$. Let $\Psi(\tilde{\tau}, x)$ be the play $\tilde{x}$ with these values of the $\xi_i$ and with $\pi(\tilde{x}) = x$.

We leave to the reader the easy verification that our functions $\pi$, $\phi$, and $\Psi$ have the required properties.

For $k > 0$, the main change is that plays in $\tilde{T}$ are of the form 

$$\begin{array}{c|c|c|c|c|c|c|c}
I & a_0 & \ldots & \langle a_{2j}, \xi_0 \rangle & \langle a_{2j+2}, \xi_2 \rangle & \ldots \\
II & a_1 & \ldots & a_{2j+1} & \ldots \\
\end{array}$$

where $j$ is large enough that $2j \geq k$. Other changes are the obvious ones. For example, if $p = \langle a_0, \ldots, a_{2j+2n} \rangle$, then $\tilde{q}(p, v)$ is defined for $v \in [\kappa]^{n+1}$; if $\ell h(p) \leq 2j$ then $\tilde{q}(p, v) = p$. 

Lemmas 4.2.1 and 4.2.2 give a different proof of Theorem 4.1.6. But the importance of these lemmas is that they provide ingredients for proving in Chapter 5 the determinacy of wider classes of games. For the sharpest results in Chapter 5, we need the following refinement of Lemma 4.2.2.
Lemma 4.2.3. Let $T$ be a game tree with taboos. Let $B \subseteq [T]$ with $B \in \Pi_1^1$. Let $k \in \omega$ and $m \in \omega$. Suppose that $\kappa$ is a measurable cardinal such that $|T| \leq \kappa$ and such that

$$(\forall p \in T)(\ell h(p) > m \rightarrow |T_p| < \kappa).$$

(i) There is a $(B, B)$-$k$-semicovering $(\tilde{T}, \pi, \phi, \Psi)$ of $T$ such that $|\tilde{T}| \leq \kappa$ and such that

$$(\forall \tilde{p} \in \tilde{T})(\ell h(\tilde{p}) > \max\{k, m\} + 1 \rightarrow |\tilde{T}_{\tilde{p}}| < \kappa).$$

(ii) There is a $([T] \setminus B, B)$-$k$-semicovering $(\tilde{T}, \pi, \phi, \Psi)$ of $T$ such that $|\tilde{T}| \leq \kappa$ and such that

$$(\forall \tilde{p} \in \tilde{T})(\ell h(\tilde{p}) > \max\{k, m\} + 1 \rightarrow |\tilde{T}_{\tilde{p}}| < \kappa).$$

Proof. The proof is like that of Lemma 4.2.2. We indicate only the changes.

Define $\tilde{T}$ as in the proof of Lemma 4.2.2, but with $\max\{k, m\} + 1$ as the $k$ of that proof, i.e. with moves $\langle a_{2(j+i)}, \xi_i \rangle$ for the least $j$ with $2^j \geq \max\{k, m\}$.

The fact that $|T| < \kappa$ was used twice in the proof of Lemma 4.2.2. The first time was to guarantee, for each position $p$ in $T$ of odd length, that $\tilde{\tau}(\tilde{q}(p, v))$ took fewer that $\kappa$ values. But in the present situation $\tilde{q}(p, v) = p$ unless $\ell h(p) > 2j \geq m$. If $\ell h(p) > 2j$ then, since the values $\tilde{\tau}(\tilde{q}(p, v))$ belong to the set $T_p$ whose size is less than $\kappa$, we get the desired conclusion. The other use of the fact that $|T| < \kappa$ was to guarantee that the set $X = \bigcap\{X_p \mid p \in T \land \ell h(p) \text{ is odd}\}$ belonged to $U$. Instead of considering $X$, we can define sets

$$X^p = \bigcap\{X_{p'} \mid p' \in T_p \land \ell h(p') \text{ is odd}\}$$

for $p \in T$. If $\ell h(p) > m$, then $X^p \in U$. In defining $\Psi(\tilde{\tau}, x)$, we can replace $X$ by $X^p$ for $p \subseteq x$ and $\ell h(p) = m + 1$.

Let us verify that any $|\tilde{T}_{\tilde{p}}| < \kappa$ for every $\tilde{p} \in \tilde{T}$ such that $\ell h(\tilde{p}) > 2j$. Since $2j \leq \max\{k, m\} + 1$, this will complete the proof. Recall that 0 is maximal in $<_p$ for every $p \in T$. Thus every legal position in $\tilde{T}$ is such that $\xi_i < \xi_0$ for each $i > 0$. Therefore, if $\tilde{p} \in \tilde{T}$ and $\ell h(\tilde{p}) > 2j$, then $|\tilde{T}_{\tilde{p}}|$ is no greater than the maximum of (a) $|T_{\pi(\tilde{p})}|$, (b) the cardinal of the $\xi_0$ given by $\tilde{p}$, and (c) $\aleph_0$. But all these cardinal numbers are smaller than $\kappa$. 

Remark. The replacement described above of $X$ by $X^p$ would work for the original proof of Lemma 4.2.2 also, with any value of $n$. Indeed the use in
question of the hypothesis that $|T| < \kappa$ was unnecessary altogether: Instead of considering $X$ or the $X^p$, we could have defined, for each play $x \in |T|$, the set $X^x = \bigcap \{X_p \mid p \subseteq x \wedge \ell h(p) \text{ is odd}\}$. The set $X^x$ is a countable intersection of sets in $\mathcal{U}$, and so it belongs to $\mathcal{U}$. In defining $\Psi(\tilde{\tau}, x)$, we could then have used $X^x$ in place of $X$ or an $X^p$. We did not do this because of a later application of the method (Lemma 5.2.12), where we will not be able to use the $X^x$.

**Exercise 4.2.1.** Let $T$ be any game tree with taboos. Show that there is a semicovering of $T$ that unravels every subset of $|T|$.

*Hint.* Let $\tilde{T} = \{\emptyset\}$.

**Exercise 4.2.2.** Prove that the semicovering of $T$ constructed in the proof of Lemma 4.2.2 need not extend to a covering of $T$. Indeed prove that there need not be any covering of $T$ extending the $\tilde{T}$ and $\pi$ constructed in that proof. *Hint.* Let $T = \langle \omega^2, \emptyset, \emptyset \rangle$, let $A = \{x \mid (\exists n \in \omega) x(2n) = 1\}$, and let $k = 0$. Let $\tilde{\tau}$ be the strategy for II in $\tilde{T}$ such that

$$\tilde{\tau}(\tilde{p}) = 1 \iff \tilde{p}^{-1}(1) \text{ is terminal in } \tilde{T}.$$

### 4.3 Homogeneous Trees

In this section we present still another way to organize the proof of Theorem 4.1.6. No use of the ideas and results of this section will be made until Chapter 8, but they will be the basis of all our determinacy proofs from that point on.

Let $X$ and $Y$ be arbitrary sets. If $B \subseteq X \times Y$, then

$$pB = \{x \in X \mid (\exists y \in Y) (x, y) \in B\}.$$ 

Thus $pB$ is the projection of $B$ onto the first coordinate.

If $E$ is a topological space and $\kappa$ is a cardinal number, a subset $A$ of $E$ is $\kappa$-Souslin if there is a closed $C \subseteq [T] \times \omega^\kappa$ such that $A = pB$.

We will mainly be interested in $\kappa$-Souslin subsets of $[T]$, where $T$ has no terminal positions. As we have remarked on page 182, this involves no loss of generality as far as determinacy results are concerned.

**Lemma 4.3.1.** Let $T$ be a game tree and let $\kappa$ be an infinite cardinal number.
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(a) If \( \lambda < \kappa \) and \( A \subseteq [T] \) is \( \lambda \)-Souslin, then \( A \) is \( \kappa \)-Souslin.

(b) Every \( \Sigma_1^1 \) subset of \([T]\) is \( \kappa \)-Souslin.

(c) The class of \( \kappa \)-Souslin subsets of \([T]\) is closed under unions of size \( \leq \kappa \).

(d) The class of \( \kappa \)-Souslin subsets of \([T]\) is closed under countable intersections.

(e) Both the class of \( \kappa \)-Souslin subsets of \([T]\) and the class of co-\( \kappa \)-Souslin subsets of \([T]\) (the class of complements of \( \kappa \)-Souslin subsets of \([T]\)) are closed under open-separated unions.

**Proof.** (a) is obvious, since if \( \lambda < \kappa \) then \( \omega^\lambda \) is a closed subset of \( \omega^\kappa \).

It is immediate from the definitions that

\[
A \text{ is } \aleph_0\text{-Souslin } \iff A \in \Sigma_1^1.
\]

From this (b) follows with the help of (a).

The proofs of (c) and (d) are a trivial modification of the proof of the proposition (b)(i) occurring in the proof of Theorem 2.2.3, and the proof of (e) is a trivial modification that of (b)(ii) of the the proof of Theorem 2.2.3. We leave them to the reader. (The reader who has skipped §2.2 may skip (e).) \(\square\)

To present and study an alternative characterization of \( \kappa \)-Souslin sets, we need to make a few definitions:

If \( T \) is a game tree, then the field of \( T \), field \( (T) \), is

\[
\{ p(i) \mid p \in T \land i < \ell h(p) \}.
\]

If \( X \) is a set, then a tree on \( X \) is a game tree \( T \) such that field \( (T) \subseteq X \).

If \( p \) and \( q \) are finite sequences of the same length , then we let

\[
\langle p, q \rangle = \langle \langle p(n), q(n) \rangle \mid n < \ell h(p) \rangle.
\]

If \( T \) is a game tree and \( p \) is a finite sequence, then let

\[
T[p] = \{ q \mid \langle p, q \rangle \in T \}.
\]

If \( T \) is a game tree and if \( x \) is an infinite sequence, then let

\[
T(x) = \bigcup_{n \in \omega} T[x \upharpoonright n].
\]

Note that \( T(x) \) is always a game tree if it is nonempty.
Lemma 4.3.2. Let $T$ be a game tree, let $A \subseteq [T]$, and let $\kappa$ be a cardinal number. Then $A$ is $\kappa$-Souslin if and only if there is a tree $U$ on field $(T) \times \kappa$ such that

$$A = \{x \in [T] \mid [U(x)] \neq \emptyset\}.$$ 

Proof. If $C$ witnesses that $A$ is $\kappa$-Souslin, then let

$$U = \{(p, s) \mid (\exists x \supseteq p)(\exists g \supseteq s) (x, g) \in C\}.$$ 

If $U$ is as in the statement of the lemma, then let

$$C = \{(x, g) \mid x \in [T] \land (\forall n \in \omega) \{x \upharpoonright n, g \upharpoonright n\} \in U\}.$$ 

\[\square\] 

Remark. Recall that a game tree is wellfounded if $\supseteq \upharpoonright T$ is wellfounded on $T$, i.e. if every nonempty subset $X$ of $T$ has an element $p$ with no proper extensions in $X$. The Axiom of Choice gives that $T$ is wellfounded if and only if there are no infinite plays in $T$. Thus the last line of the statement of Lemma 4.3.2 can be reformulated as

$$A = \{x \in [T] \mid U(x) \text{ is not wellfounded}\}.$$ 

It will be useful to have a third way to express this relation between $A$ and $U$. For any tree $T$, any set $Y$, and any tree $U$ on field $(T) \times Y$, let us say that the $T$-projection of $U$ is $\{x \in [T] \mid U(x) \neq \emptyset\}$. Thus the last line of the statement of Lemma 4.3.2 says that $A$ is the $T$-projection of $U$.

We have already mentioned the fact that $\Sigma^1_1$ is identical with the class of all $\aleph_0$-Souslin sets. We now show, as is essentially proved in Shoenfield [1961], that all $\Pi^1_1$ sets are $\aleph_1$-Souslin.

Lemma 4.3.3. If $T$ is a game tree and if $A \subseteq [T]$ with $A \in \Pi^1_1$, then $A$ is $\aleph_1$-Souslin, and so $A$ is $\kappa$-Souslin for every uncountable $\kappa$.

Proof. Let $T$ be a game tree and let $A \subseteq [T]$ with $A \in \Pi^1_1$. Though the last clause of the statement of the lemma follows from the preceding one, we will directly verify the last clause. Let $\kappa$ be an uncountable cardinal number.

The set $A \setminus [T]$ is open and so is $\kappa$-Souslin by Lemma 4.3.1. By Lemma 4.1.1, $A \cap [T] \in \Pi^1_1$. By Lemma 4.3.1 again, it suffices to prove that $A \cap [T]$ is $\kappa$-Souslin. Without loss of generality, let us then assume that $A \subseteq [T]$. 

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Let the functions \( p \mapsto \langle p \rangle \) and \( x \mapsto \langle x \rangle \) be as given by Lemma 4.1.4.

We could get a tree \( U \) witnessing that \( A \) is \( \kappa \)-Souslin directly from the tree \( T^* \) occurring in the proof of Lemma 4.1.6 by replacing the moves \( a_i \) of player II in \( T^* \) by \( \langle a_i, 0 \rangle \). Instead we prefer to use a slightly different tree. Let

\[
U = \{ \langle p, s \rangle \mid p \in T \land s \text{ embeds } (\ell h(p); <_p) \text{ into } (\kappa; <) \}.
\]

Let \( x \in [T] \). If \( g \in [T(x)] \), then \( g \) embeds \((\omega; <_x)\) into \((\kappa; <)\), and so \( x \in A \). If \( x \in A \), then there is a \( g \) embedding the wellordering \((\omega; <_x)\) into \((\kappa; <)\), and any such \( g \) belongs to \([T(x)]\). \[ \square \]

Remark. Not every \( \aleph_1 \)-Souslin set belongs to \( \Pi_1^1 \). See Exercises 4.3.2 and 4.3.3.

A set is Souslin if it is \( \kappa \)-Souslin for some \( \kappa \). Sometimes the word “Souslin” is used in a more restricted sense, synonymous with “\( \aleph_0 \)-Souslin.” We will mostly talk of Souslin sets in the context of the concepts that we now define.

Suppose \( U \) and \( V \) are countably complete ultrafilters on sets \( A \) and \( B \) respectively. Suppose that \( \chi : B \to A \) is such that

\[
(\forall X \in U) \{ b \in B \mid \chi(b) \in X \} \in V.
\]

Then we say that \( V \) projects to \( U \) by \( \chi \). In this situation, we can define

\[
i_{U,V,\chi} : \prod_U (V; \in) \prec \prod_V (V; \in)
\]

by

\[
i_{U,V,\chi}([f]_U) = [\chi^*(f)]_V,
\]

where

\[
(\chi^*(f))(b) = f(\chi(b)).
\]

We omit the routine verification that \( i_{U,V,\chi} \) is well-defined and is an elementary embedding.

Let \( T \) be a game tree, let \( Y \) be a nonempty set, and let \( U \) be a tree on field \((T) \times Y\). We say that \( U \) is homogeneous for \( T \) if there is a system

\[
\langle U_p \mid p \in T \rangle
\]

satisfying the following conditions:
(1) Each $U_p$ is a countably complete ultrafilter on $U[p]$.

(2) The $U_p$ are compatible: For all $p \subseteq q \in T$, $U_q$ projects to $U_p$ by $\chi_{q,p}$, where $\chi_{q,p} : U[q] \to U[p]$ is given by $\chi_{q,p}(s) = s \upharpoonright \ell(h(p))$.

(3) Let $x \in [T]$ and let $\langle Z_n \mid n \in \omega \rangle$ be such that each $Z_n$ belongs to $U_{x|n}$. Then
\[ [U(x)] \neq \emptyset \rightarrow (\exists f : \omega \to Y)(\forall n \in \omega) f \mid n \in Z_n. \]

Remarks:

(a) Implicit in clause (1) is the requirement that $U[p]$ be nonempty for each $p \in T$. There is a variant notion of homogeneity that does not make this requirement. (See Exercise 4.3.5.)

(b) Each $U[p]$ is a subset of $\ell(h(p)) Y$, and so each $U_p$ induces—and is essentially the same as—an ultrafilter on $\ell(h(p)) Y$.

(c) The “$\rightarrow$” in the last line of condition (3) can be replaced by a “$\leftrightarrow$,” since any $f$ satisfying the right hand side must belong to $[U(x)]$.

There is an equivalent of condition (3) that will be of use to us later: Suppose that (1) and (2) are satisfied. For $p \in T$, let $\pi_p = \pi_{U_p} : \prod U_p(V; \in ) \cong (\Ult(V; U_p); \in)$. For $p \subseteq q \in T$, let
\[ i_{p,q} = \pi_q \circ i_{U_p,U_q,\chi_{q,p}} \circ \pi_p^{-1}. \]

For each $x \in [T]$, let
\[ (M_x; \langle i_{x|n} \mid n \in \omega \rangle) \]
be the direct limit of the directed system of elementary embeddings
\[ (\langle \Ult(V; U_{x|n}) \mid n \in \omega \rangle; \langle i_{x|m,x|n} \mid m \leq n \in \omega \rangle). \]

(3') $(\forall x \in [T])([U(x)] \neq \emptyset \rightarrow M_x$ is wellfounded).

**Lemma 4.3.4.** If (1) and (2) hold of $T$ and $\langle U_p \mid p \in T \rangle$ and if $x \in [T]$, then $x$ witnesses the falsity of (3) if and only if $x$ witnesses the falsity of (3'). Thus a tree $U$ on field $(T) \times Y$ is homogeneous for $T$ if and only if there is a system $\langle U_p \mid p \in T \rangle$ satisfying (1), (2), and (3').

**Proof.** Let $U$ be a tree on field $(T) \times Y$ and let $\langle U_p \mid p \in T \rangle$ satisfy (1) and (2).
4.3. HOMOGENEOUS TREES

Suppose first that \( x \) and \( \langle Z_n \mid n \in \omega \rangle \) witness the failure of (3). Let

\[
S = \{ s \in U(x) \mid (\forall n \leq \ell h(s)) s \upharpoonright n \in Z_n \}.
\]

\( S \) is a game subtree of \( U(x) \) with no infinite plays. Thus \( S \) is wellfounded. By the elementarity of \( i^x_0 \),

\[
M_x \models i^x_0(S) \text{ is wellfounded.}
\]

But let

\[
s_n = i^x_{x \upharpoonright n}(\pi_{x \upharpoonright n}(\text{id}_{U[x \upharpoonright n]} \upharpoonright U_{x \upharpoonright n})).
\]

It is easy to see that each \( s_n \in i^x_0(S) \) and that, for \( m \leq n \in \omega \), \( s_m \subseteq s_n \). Thus \( \bigcup_{n \in \omega} s_n \) belongs to \( [i^x_0(S)] \), and so \( i^x_0(S) \) is not really wellfounded. Thus \( \| i^x_0(S) \| \) as computed in \( M_x \) is an “ordinal” of \( M_x \) that is not wellordered by the membership relation \( i^x_0(\in) \) of \( M_x \). (See page 25 for the definition of \( \| S \| \).) This implies that \( M_x \) is not wellfounded.

Now suppose that \( x \) witnesses that (3’ fails. Let \( \langle z_n \mid n \in \omega \rangle \) be an infinite descending sequence with respect to \( i^x_0(\in) \). For each \( n \in \omega \), let \( m_n \) and \( a_n \in \text{Ult}(V; U_{x \upharpoonright m_n}) \) be such that \( z_n = i^x_{x \upharpoonright m_n}(a_n) \). Without loss of generality, we may assume that

\[
(\forall n' \in \omega)(\forall n \in \omega)(n' < n \rightarrow m_{n'} < m_n).
\]

Let \( g_n \in U[x \upharpoonright m_n]V \) be such that

\[
\pi_{x \upharpoonright m_n}(\text{g}_n \upharpoonright U_{x \upharpoonright m_n}) = a_n.
\]

For each \( n \in \omega \), let

\[
Z_{m_{n+1}} = \{ s \in U[x \upharpoonright m_{n+1}] \mid g_{n+1}(s) \in g_n(s \upharpoonright m_n) \}.
\]

For each \( m \in \omega \) such that \( m \) is not of the form \( m_{n+1} \), let \( Z_m = U[x \upharpoonright m] \).

For each \( m \in \omega \), we have that \( Z_m \in U_{x \upharpoonright m} \). But if \( f : \omega \rightarrow Y \) is such that \( (\forall m \in \omega) f \upharpoonright m \in Z_m \), then \( \langle f(m_n) \mid n \in \omega \rangle \) is an infinite descending sequence with respect to \( \in \). Thus no such \( f \) exists, and we have a counterexample to (3).

\[\square\]

Remark. By Lemma 4.3.4 and our earlier remark about (3), the “\( \rightarrow \)” in condition (3’) can be replaced by “\( \leftrightarrow \).”
For $T$ a game tree, $Y$ a set, and $\kappa$ a cardinal number, a tree $U$ on field $(T) \times Y$ is $\kappa$-homogeneous for $T$ if there is a system $\langle U_p \mid p \in T \rangle$ witnessing that $U$ is homogeneous for $T$ and having the further property that each $U_p$ is $\kappa$-complete.

Let $T$ be a game tree and let $A \subseteq [T]$. $A$ is homogeneously Souslin if there is a tree $U$ on field $(T) \times Y$ for some $Y$ such that $U$ is homogeneous for $T$ and $A$ is the $T$-projection of $U$. For cardinal numbers $\kappa$, $A$ is $\kappa$-homogeneously Souslin if it is the $T$-projection of a $\kappa$-homogeneous tree.

Remark. Note that the last of the definitions just given says nothing about the size of the $\kappa$-homogeneous tree. A set can be $\kappa$-homogeneously Souslin without being $\kappa$-Souslin.

Theorem 4.3.5. For any game tree $T$, all $|T|^+$-homogeneously Souslin games in $T$ are determined.

Proof. Without loss of generality, we may restrict ourselves to game trees $T$ without terminal positions. Let $T$ be such a tree and let $A \subseteq [T]$ be $|T|^+$-homogeneously Souslin. Let $U$ and $Y$ be such that $U$ is $|T|^+$-homogeneous for $T$, and $A = \{x \in [T] \mid [U(x)] \neq \emptyset\}$. Let $\langle U_p \mid p \in T \rangle$ witness that $U$ is $|T|^+$ homogeneous for $T$. Let $T^*$ be the game tree plays in which are as follows:

\[ I \langle a_0, b_0 \rangle \langle a_2, b_1 \rangle \langle a_4, b_2 \rangle \ldots \]

\[ II a_1 a_3 \ldots \]

Each $\langle a_i \mid i < n \rangle$ must belong to $T$ and each $b_i$ must belong to $Y$.

Define $\pi : T^* \to T$ and the induced $\pi : [T^*] \to [T]$ as in the proof of Theorem 4.1.6.

Let $A^*$ be the set of plays in $T^*$ such that each $\langle \langle a_i, b_i \rangle \mid i < n \rangle$ belongs to $U$. The game $G(A^*; T^*)$ is closed and so is determined.

Suppose first that $\sigma^*$ is a winning strategy I for $G(A^*; T^*)$. Let $\sigma$ be a strategy for I in $T$ gotten as in the proof of Theorem 4.1.6. If $x$ is a play consistent with $\sigma$, then there is an $x^*$ consistent with $\sigma^*$ such that $\pi(x^*) = x$. Since $x^* \in A^*$, $x^*$ gives an element $\langle \langle a_i, b_i \rangle \mid i \in \omega \rangle$ of $[U]$ with $x = \langle a_i \mid i \in \omega \rangle$. Thus $\langle b_i \mid i \in \omega \rangle \in [U(x)]$, and so $x \in A$.

Suppose now that $\tau^*$ is a winning strategy for II for $G(A^*; T^*)$. For each $p = \langle a_i \mid i \leq 2n \rangle \in T$ and each $s \in n^{+1}Y$, let

\[ q^*(p, s) = \langle \langle a_0, s(0) \rangle, a_1, \ldots, \langle a_{2n}, s(n) \rangle \rangle. \]
Each \( q^*(p, s) \) is a position in \( T^* \) and is such that \( \pi(q^*(p, s)) = p \). Define a strategy \( \tau \) for \( II \) in \( T \) setting
\[
\tau(p) = a \iff \{ s \in U[p \upharpoonright n + 1] \mid \tau^*(q^*((p, s))) = a \} \in U_{p\mid n+1},
\]
for \( p \in T \) with \( \ell h(p) = 2n + 1 \). Since \( U_{p\mid n+1} \) is \( |T| \)-complete, \( \tau \) is well-defined. To see that \( \tau \) is a winning strategy for \( II \) for \( G(A; T) \), let \( x \) be a play consistent with \( \tau \). Assume for a contradiction that \( x \in A \), i.e. that \( \left[ U(x) \right] \neq \emptyset \). For each \( n \in \omega \) let
\[
Z_{n+1} = \{ s \in U[x \upharpoonright n + 1] \mid \tau^*(q^*(x \upharpoonright 2n + 1, s)) = x(2n + 1),
\]
and let \( Z_0 = \{ \emptyset \} \). For each \( n \in \omega \), \( Z_n \subseteq U_{x\mid n} \). Hence clause (3) in the definition of homogeneous trees gives us an \( f : \omega \to Y \) such that \( f \upharpoonright n \in Z_n \) for every \( n \in \omega \). By the definition of \( \tau \), this means that
\[
x^* = \langle \langle x(0), f(0) \rangle, x(1), \langle x(2), f(1) \rangle, x(3), \ldots \rangle
\]
is a play in \( T^* \) consistent with \( \tau^* \). Since \( x^* \in A^* \), we have our contradiction. \( \square \)

**Theorem 4.3.6.** If \( T \) is a game tree and \( \kappa \) is a measurable cardinal greater than \( |T| \), then every \( \Pi^1_1 \) subset of \( [T] \) is \( \kappa \)-homogeneously Souslin and is witnessed to be \( \kappa \)-homogeneously Souslin by a tree on field \( (T) \times \kappa \).

**Proof.** Let \( T \) be a game tree, let \( \kappa > |T| \) be a measurable cardinal, let \( \mathcal{U} \) be a uniform normal ultrafilter on \( \kappa \), and let \( A \subseteq [T] \) with \( A \in \Pi^1_1 \). Let \( p \mapsto <_p \) and \( x \mapsto <_x \) be as given by Lemma 4.1.4. Let \( U \) be defined as in the proof of Lemma 4.3.3:
\[
U = \{ \langle p, s \rangle \mid p \in T \land s \text{ embeds } (\ell h(p); <_p) \text{ into } (\kappa; <) \}.
\]

Let \( p \in T \). For each \( v \in [\kappa]^{|\ell h(p)|} \), there is a unique bijection \( s_p^v : \ell h(p) \to v \) such that \( \langle p, s_p^v \rangle \in U \). Define an ultrafilter \( U_p \) on \( U[p] \) by
\[
X \in U_p \iff \{ v \in [\kappa]^{|\ell h(p)|} \mid s_p^v \in X \} \in \mathcal{U}^{[\ell h(p)]}.
\]

We know from the proof of Lemma 4.3.3 that \( A \) is the \( T \)-projection of \( U \). The system \( \langle U_p \mid p \in T \rangle \) obviously has property (1) in the definition of homogeneity, and it is easy to check that it has property (2). For (3), let
$x \in [T]$ and let $\langle Z_n \mid n \in \omega \rangle$ be such that each $Z_n \in U_{x|n}$. Fix for the moment $n \in \omega$. Let

$$\bar{Z}_n = \{ v \in [\kappa]^n \mid s_v^{x|n} \in Z_n \}.$$  

By the definition of $U_{x|n}$, we have that $\bar{Z}_n \in V[n]$. By the definition of $V[n]$, let $X_n \in V$ be such that $[X_n]^n \subseteq Z_n$. Now let $X = \bigcap_{n \in \omega} X_n$. Thus $X \in V$ and, for all $n$, $[X] \subseteq Z_n$. Assume that $[U(x)] \neq \emptyset$. Then $x \in A$, and so $<_x$ is a wellordering of $\omega$. Let $f$ embed $(\omega;<_x)$ into $(X;<_\kappa)$. To see that $f$ is as required by (3), let $n \in \omega$. We have that $f \upharpoonright n \in U[x \upharpoonright n]$ and range $(f \upharpoonright n) \in \bar{Z}_n$. But then $f \upharpoonright n \in Z_n$. □

**Remark.** The name “homogeneous tree” may seem not to be descriptive of the concept: It is not clear that homogeneous trees need be homogeneous in any standard sense. Historically, the paradigm example of a homogeneous tree was essentially the tree $U$ of the proof just given. The tree $U$ is homogeneous in the straightforward sense that

$$(\forall X \subseteq \kappa)(|X| = \kappa \rightarrow U \upharpoonright X \cong U),$$

where $U \upharpoonright X = U \cap \{ \{p, s\} \mid \text{range}(s) \subseteq X \}$. A related property is that membership in $U[p]$ of $s \in \ell \kappa[p]$ depends only on the order type of the sequence $s$. It is these homogeneity properties that made possible the verification of (3). A.S. Kechris and Martin began applying “homogeneous” to trees like $U$ and trees whose fields consist of transfinite sequences and which have similar homogeneity properties. Such trees arose in work on the Axiom of Determinacy by Kenneth Kunen and later by Martin. Finally Kechris and Martin independently abstracted from the particular class of examples and began using “homogeneous tree” in the current sense (Kechris in [Kechris, 1981], and Martin in lectures). The idea is that a tree must have some kind of homogeneity if (3) in the definition of homogeneous trees is to be satisfied. The definition leaves the nature of this homogeneity unspecified.

**Exercise 4.3.1.** Let $A \subseteq [T]$. Show that

$$(\exists n \in \omega) A \text{ is } n\text{-Souslin} \iff A \text{ is } 1\text{-Souslin} \iff A \text{ is closed}.$$

**Exercise 4.3.2.** A subset $X$ of $[T]$ belongs to $\Sigma^1_2$ if there is a subset $B$ of $[T] \times \omega$ such that $B \in \Pi^1_1$ and $A = pB$. Prove that every $A \in \Sigma^1_2$ is $\aleph_1$-Souslin.
4.3. **HOMOGENEOUS TREES**

**Exercise 4.3.3.** It is relatively consistent with the ZFC axioms that in countable trees $\Sigma^1_2$ is identical with the class of $\aleph_1$-Souslin sets. The hypothesis $\text{MA}_{\aleph_1} + (\omega_1)^L = \omega_1$, for example, implies that this is the case. (See [Martin and Solovay, 1970].)

(a) Show, on the other hand, that the continuum hypothesis implies that every subset of $[T]$ is $\aleph_1$-Souslin if $T$ is countable.

(b) Deduce that it is relatively consistent with the ZFC axioms that not every $\aleph_1$-Souslin subset of $\omega^\omega$ belongs to $\Sigma^1_2$.

(c) Show that the determinacy of all $\Pi^1_1$ games in countable trees implies that not every $\aleph_1$-Souslin subset of $\omega^\omega$ belongs to $\Sigma^1_2$.

*Hint.* First prove that for any game tree $T$ that any subset $A$ of $[T]$ is $[A]$-Souslin. This gives (a). For (b), use (a) and the fact that the class of all $\Sigma^1_2$ subsets of $\omega^\omega$ has size $2^{\aleph_0}$. For (c) prove that the determinacy of all $\Pi^1_1$ games in $\omega^\omega$ implies that every uncountable $\Sigma^1_2$ subset of $\omega^\omega$ has a perfect subset. To do this, let $B \in \Pi^1_1$ witness that $A \subseteq \omega^2$ belongs to $\Sigma^1_2$. Use the game of Exercise 4.1.1 modified so that I’s moves have extra components belonging to $\omega$. I wins the modified game if and only if the extra components form a $y$ such that $\langle \bigcup_{i \in \omega} s_i, y \rangle \in B$. (This trick, called *unfolding*, is due to Robert Solovay, though this is not his application of it.) To complete (c) construct, by a diagonalization, an uncountable subset of $\omega^\omega$ without a perfect subset, then from this get a set of size $\aleph_1$ with no perfect subset.

**Exercise 4.3.4.** Show that it is consistent with the ZFC axioms that the only homogeneously Souslin subset of $[T]$ is $[T]$ itself. (But see Exercise 4.3.5.)

**Exercise 4.3.5.** Redefine the concept of a tree’s being homogeneous for $T$ as follows: Replace $\langle U_p \mid p \in T \rangle$ by $\langle U_p \mid p \in T' \rangle$, where $T'$ is allowed to be an arbitrary subtree of $T$. Require that if $x \in [T]$ and $|U(x)| \neq \emptyset$ then $x \upharpoonright n \in T'$ for every $n \in \omega$. Replace “$T$” by “$T'$” in clause (2) of the original definition.

Prove in ZFC that every closed set of $[T]$ is homogeneously Souslin in this modified sense. Prove that if a measurable cardinal exists then the same sets are homogeneously Souslin under the original and the modified definitions.

The modified definition is perhaps more natural, and it has other virtues. We do not adopt it as our official definition simply because it would make our notation more cumbersome.
4.4 Sharps and $\Pi_1^1$ Determinacy

We have already seen in Exercise 4.1.1 that the determinacy of all $\Pi_1^1$ games, even in countable trees, is not provable from the ZFC axioms alone. On the other hand, Theorem 4.1.6 shows that it is provable if we adjoin the hypothesis that there is a measurable cardinal. That hypothesis is, however, stronger than necessary. In this section we show that the determinacy of all $\Pi_1^1$ games in a tree $T$ follows from the hypothesis that every subset of $|T|$ has a sharp. A theorem of Harrington shows that this hypothesis is optimal for countable $T$ in that it follows from the determinacy of all $\Pi_1^1$ games in $^{<\omega}\omega$. (See Exercise 4.4.1.) For $T = ^{<\omega}\omega$, this equivalence is lightface. (See Theorem 4.4.3 and Exercise 4.4.1.)

We first prove a well-known lemma about the existence of definable strategies for open and closed games.

If $T$ is a game tree, then let us say that a subset $D$ of $T$ generates an open subset $A$ of $\lceil T \rceil$ if $A = \{x \in \lceil T \rceil : (\exists d \in D) d \subseteq x\}$.

**Lemma 4.4.1.** Let $T$ be a game tree and let $A \subseteq \lceil T \rceil$. Let $D \subseteq T$ be such that $D$ generates $A$ or $D$ generates $\neg A$. Let $M$ be any transitive class model of ZFC such that $T \in M$ and $D \in M$. Let $<$ be any wellordering of field $(T)$ that belongs to $M$. Then there is a strategy $\sigma \in M$ that is definable in $M$ from $T$, $D$, and $<$ and is a winning strategy for $G(A; T)$. Moreover, $M \models \ "\sigma$ is a winning strategy for $G(\{x \in \lceil T \rceil : (\exists d \in D) d \subseteq x\}; T)."

**Proof.** We may assume that $D$ generates $A$.

We now give what is essentially the construction of Exercise 1.2.4. For each ordinal number $\alpha$, we define $P_\alpha$, a set of positions of even length in $T$. The definition proceeds by transfinite induction on $\alpha$. Let $p \in P_0$ if and only if $(\exists d \in D) d \subseteq p$. For $\alpha > 0$, let $p \in P_\alpha$ if and only if $p \in P_0$ or there is a Move $q$ at $p$ such that either (i) $q \in D \cap \lceil T \rceil$ or (ii) $q \notin \lceil T \rceil$ and, for every Move $r$ at $q$, $r \in \bigcup_{\beta < \alpha} P_\beta$.

If we turn this inductive definition into an explicit definition in the standard way, then it is absolute for $M$: For $\alpha \in \text{Ord} \cap M$, the $P_\alpha$ defined in $M$ is the same as that defined in $V$. 
4.4. SHARPS AND $\Pi_1^1$ DETERMINACY

It is clear that
$$\beta < \alpha \rightarrow P_\beta \subseteq P_\alpha.$$ Since each $P_\alpha \subseteq T$, Comprehension in $M$ gives that $\bigcup_{\alpha \in \text{Ord} \cap M} P_\alpha \in M$. By $\Sigma_1$ Replacement in $M$, there must then be an ordinal $\gamma \in M$ such that $P_\gamma = P_{\gamma+1}$. From this it follows that $(\forall \alpha \geq \gamma) P_\alpha = P_\gamma$. Let $P_\infty = P_\gamma$.

Suppose first that $\emptyset \in P_\infty$. Define a strategy $\sigma$ for I as follows: If $p \in P_\infty \setminus P_0$, let $a$ be the $\prec$-least element of the field of $T$ such that either (i) $p \langle a \rangle \in D \cap \lceil T \rceil$ or (ii), for every Move $r$ at $p \langle a \rangle$,
$$r \in P_\infty \land \mu \beta (r \in P_\beta) < \mu \beta (p \in P_\beta).$$
If $p$ does not belong to $P_\infty \setminus P_0$, then let $\sigma(p)$ be the $\prec$-least element of field $(T)$. Evidently $\sigma$ satisfies the definability condition. It is easy to show by induction that every position consistent with $\sigma$ belongs to $P_\infty$. To see that $\sigma$ is a winning strategy for $G(A; T)$, let $x$ be a play consistent with $\sigma$. For $n \in \omega$ and $2n \leq \ell h(x)$, let
$$\beta_n = \mu \beta (x \upharpoonright 2n \in P_\beta).$$
For each such $n$, it follows from the definitions that one of the following holds:

(a) $\beta_n = 0$ and so $(\exists m \leq 2n) x \upharpoonright m \in D$;
(b) $\ell h(x) = 2n + 1$ and $x \in D$;
(c) $\ell h(x) \geq 2n + 2$ and $\beta_{n+1} < \beta_n$.

Since (c) cannot hold for every $n \in \omega$, there is an $n$ for which (a) or (b) holds. Thus $x \in A$. Notice that the argument shows that $(\exists d \subseteq x) d \in D$. Hence in $M$ the strategy $\sigma$ is winning in the game $G(\{ x \in \lceil T \rceil \mid (\exists d \in D) d \subseteq x \}; T)$.

Now suppose that $\emptyset \notin P_\infty$. If $p \in T \setminus P_\infty$ and if $\ell h(p)$ is even, then for every Move $q$ at $p$ either $q \in \lceil T \rceil \setminus A$ or else there is an $a \in \text{field}(T)$ such that $q \langle a \rangle \in T \setminus P_\infty$. Define a strategy $\sigma$ for II by letting $\sigma(q)$ be the $\prec$-least $a$ such that $q \langle a \rangle \in T \setminus P_\infty$ if such an $a$ exists and 0 otherwise. It is easy to check that $\sigma$ has the required properties. \qed

Remark. The proof does not really require that $M$ is a model of full ZFC.

Theorem 4.4.2. ([Martin, 1970]) Let $\lambda$ be an infinite cardinal number. Assume that
$$(\forall a \subseteq \lambda) a^\# \text{ exists.}$$
Then, for every game tree $T$ such that $|T| \leq \lambda$, all $\Pi_1^1$ games in $T$ are determined.
**Proof.** Let $T$ be a game tree with $|T| \leq \lambda$. Without loss of generality, assume that field $(T) \subseteq \lambda$ and that $T$ has no terminal positions. Let $A \subseteq [T] = [T]$ with $A \in \Pi^1_1$. Let $p \mapsto <_p$ and $x \mapsto <_x$ be as given by Lemma 4.1.4.

Let $T^*$ be defined exactly as in the proof of Theorem 4.1.6, but with $\kappa = \lambda^+$. Let $A^* \subseteq [T^*]$ be defined as in the proof of Theorem 4.1.6. As in that proof, $A^*$ is closed and so $G(A; T^*)$ is determined.

The proof that if $I$ has a winning strategy for $G(A^*; T^*)$ then $I$ also has a winning strategy for $G(A; T)$ is exactly like the corresponding part of the proof of Theorem 4.1.6.

Suppose that $G(A^*; T^*)$ is a win for II.

Let $g : <\omega \lambda \times \omega \times \omega \rightarrow \lambda$ be one-one and such that $g \in L$. Let

$$a = \{g((p, m, n)) \mid p \in T \land m <_p n\}.$$ 

Since

$$T = \{p \in <\omega \lambda \mid g(p, 1, 0) \in a\},$$

we have that $T \in L[a]$ and that $T$ is definable from $a$ in $L[a]$. Since $T^*$ is definable from $T$ and $\lambda^+$ in any transitive class model of ZFC to which both $T$ and $\lambda^+$ belong, it follows that $T^*$ is definable from $a$ and $\lambda^+$ in $L[a]$. Let $D^*$ be the set of all $p^* \in T^*$ such that, for some $n$ with $2n < \ell h(p^*)$, the function $i \mapsto \xi_i$ given by $p^*$ does not embed $(n + 1; <_{(\pi(p^*))|n+1})$ into $(\lambda^+; <)$. We also have that $D^*$ is definable from $a$ and $\lambda^+$ in $L[a]$. Let $\prec^*$ be the restriction to field $(T^*)$ of the wellordering of $\text{Ord} \cup (\text{Ord} \times \text{Ord})$ which is the natural ordering of $\text{Ord}$ followed by the lexicographic ordering of $\text{Ord} \times \text{Ord}$. The relation $\prec^*$ is definable from $a$ and $\lambda^+$ in $L[a]$.

Let $\tau^*$ be the $\sigma$ given by Lemma 4.4.1 with $T^*$ for $T$, $A^*$ for $A$, $L[a]$ for $M$, $D^*$ for $D$, and $\prec^*$ for $\prec$. Since $G(A^*; T^*)$ is a win for II, $\tau^*$ is a strategy for II. Thus $\tau^*$ is a winning strategy for II for $G(A^*; T^*)$, and $\tau^*$ is definable from $a$ and $\lambda^+$ in $L[a]$.

Define the positions $q^*(p, v)$ as in the proof of Theorem 4.1.6. It is easy to see that the function $q^*$ is definable from $a$ and $\lambda^+$ in $L[a]$.

By the hypothesis of the theorem, $a^\#$ exists. Let $C^a$ be the Silver class of indiscernibles for $L[a], a$. Let $\alpha \mapsto c^a_\alpha$ be the order-preserving bijection between $\text{Ord}$ and $C^a$. It follows from (iii) of Lemma 3.4.18 that $c^a_{\lambda^+} = \lambda^+$.

Define a strategy $\tau$ for II in $T$ as follows. For $n \in \omega$ and $p \in T$ with $\ell h(p) = 2n + 1$, let

$$\tau(p) = \tau^*(q^*(p, \{c^a_0, \ldots, c^a_n\})).$$
4.4. SHARPS AND $\Pi_1^1$ DETERMINACY

By indiscernibility and the fact that range ($\tau^*$) $\subseteq \lambda$, we have that

$$(\forall v \in [C^\omega \cap \lambda^+]^{n+1}) \tau(p) = \tau^*(q^*(p, v)).$$

To show that $\tau$ is a winning strategy for $G(A; T)$, let $x$ be a play consistent with $\tau$. Assume for a contradiction that $x \in A$. Then $<_x$ is a wellordering of $\omega$. Let $i \mapsto \xi_i$ embed $(\omega; <_x)$ into $(C^\omega \cap \lambda^+; <)$. Let $x^*$ be the play in $T^*$ with these values of the $\xi_i$ and with $\pi(x^*) = x$. As in the proof of Theorem 4.1.6, one can show that $x^*$ is consistent with $\tau^*$, contradicting the assumption that $x \in A$. \hfill $\Box$

Here is the lightface version of Theorem 4.4.2.

**Theorem 4.4.3.** ([Martin, 1970]) If $0^\#$ exists then all $\Pi_1^1$ games in $<^\omega\omega$ are determined.

**Proof.** Let $A \subseteq ^\omega\omega$ with $A \in \Pi_1^1$. Let $p \mapsto <_p$ and $x \mapsto <_x$ be as given by Lemma 4.1.5. Proceed as in the proof of Theorem 4.4.2, with $T = <^\omega\omega$ and $\lambda = \omega$. The $a$ we get is definable in $L$. Thus $L[a] = L$ and the Silver indiscernibles for $L$ are the Silver indiscernibles for $L[a], a$. Using the existence of $0^\#$, we can then proceed as in the proof of Lemma 4.4.2. \hfill $\Box$

**Exercise 4.4.1.** Show that the determinacy of all $\Pi_1^1$ games in $<^\omega\omega$ implies that $0^\#$ exists. From this, the main result of [Harrington, 1978], and from Theorem 4.4.3, it follows that $\Pi_1^1$ determinacy is equivalent with the existence of $0^\#$. 

**Hint.** First show that the existence of $0^\#$ follows from the existence of an $a \in ^\omega\omega$ such that every $a$-admissible ordinal is a cardinal in $L$. (This result is due to Jack Silver, but the proof we now sketch is due to J.B. Paris.)

Let $a \in ^\omega\omega$ be such that every $a$-admissible is a cardinal in $L$. Work in $L[a]$. Let $X < L_{\omega_1}[a]$ with $|X| = \aleph_1$ and $^\omega X \subseteq X$. Let $\pi : X \cong L_\alpha[a]$. Note that $\alpha$ is $a$-admissible.

Let $j = \pi^{-1} : L_\alpha[a] < L_{\omega_3}[a]$. Let $\gamma = \mathrm{crit}(j)$. Let

$$U = \{Y \in L_\alpha[a] \mid Y \subseteq \gamma \land \gamma \in j(Y)\}.$$

Show that $U$ is a uniform normal $L$-ultrafilter on $\gamma$: i.e., that $U$ is a filter on $\gamma$, that every subset of $\gamma$ in $L$ belongs to $U$ or else its complement does, that for all $f : \gamma \to \mathcal{P}(\gamma)$ with $f \in L$ the set $\{\beta < \gamma \mid f(\beta) \in U\}$ belongs to $L$, and that $U$ is uniform and normal in the obvious senses.
Prove that Rowbottom’s result, Lemma 3.1.8, holds for $U$ in the following sense. If $n \in \omega$ and $Z \in L$ is a subset of $[\gamma]^n$, then there is a $Y \in U \cap L$ such that either $[Y]^n \subseteq Z$ or $[Y]^n \cap Z = \emptyset$. Use this fact and the countable closure of $X$ to get a set of indiscernibles for $L$ of size $\gamma$.

**Remark.** The notion of an $L$-ultrafilter is from [Kunen, 1968]. There Kunen proves that the existence of $0^\#$ follows from the existence of an elementary embedding $j : L \prec L$. Kunen’s proof begins by using $j$ to define an $L$-ultrafilter $U$. But it then proceeds by forming iterated ultrapowers $\text{Ult}_\alpha(L;U)$, showing that they are all wellfounded, and showing that $\{i_{U,\beta}(\gamma) \mid \beta \in \text{Ord}\}$ is a closed unbounded class of indiscernibles for $L$. Kunen’s method also works here, and it could replace the argument suggested in the preceding paragraph.

Now consider the following game $G$ in $<\omega_\omega$. For each play of $G$ let I’s part of the play code a relation $R$ in $\omega$ and let II’s part code a relation $E$ in $\omega$. If $R$ is not a wellordering of $\omega$, then I loses. If $R$ is a wellordering of $\omega$, let $\beta$ be its order type. Then II wins if and only if $(\omega; E)$ is a model of Extensionality and there is a

$$g : L_\beta \to \omega$$

that embeds $(L_\beta; \in)$ into $(\omega; E)$ as an initial segment, i.e. such that

(a) $(\forall u \in L_\beta)(\forall v \in L_\beta)(u \in v \leftrightarrow g(u) E g(v))$;

(b) $(\forall u \in L_\beta)(\forall m E g(u))(\exists v \in L_\beta) m = g(v)$.

Note that $g$ is unique if it exists.

Show that $G$ is $\Pi^1_1$. Assume that $\sigma$ is a winning strategy for I for $G$. Show that there is a countable ordinal $\gamma$ such that the $\beta$ given by any play consistent with $\sigma$ is smaller than $\gamma$. Use this fact to get a contradiction.

Assume $\Pi^1_1$ determinacy, getting that $G$ is a win for II. Let $\tau$ be a winning strategy for II for $G$. Let $a \in <\omega$ code $\tau$. To show that $0^\#$ exists, it is enough to prove that every $a$-admissible ordinal is a cardinal in $L$. By absoluteness under the collapse of cardinals, it is enough to prove that every countable $a$-admissible ordinal is a cardinal in $L$.

Suppose that $\gamma < \beta < \omega_1$, that $b$ is a subset of $L_\gamma$ belonging to $L_\beta$, and that $z$ is a play consistent with $\tau$ whose associated $R$ is a wellordering of $\omega$ of order type $\beta$. Show that $b \in L_{\gamma^+}[z]$. To do this, first let $g$ witness that $z$ is a win for II and prove that $g \upharpoonright L_\gamma \in L_{\gamma^+}$. 


4.4. SHARPS AND $\pi$ DETERMINACY

For each ordinal $\alpha$, let $(Q(\alpha); \leq_{\alpha})$ be the following partial ordering: The members of $Q(\alpha)$ are those pairs $(t, h)$ such that

(i) $t$ is a finite tree on $\omega$;
(ii) $h : t \rightarrow \omega \alpha \cup \{\infty\}$;
(iii) $h(\emptyset) = \infty$;
(iv) $(\forall r \in t)(\forall s \in t)((r \subset s \land h(r) \neq \infty) \rightarrow h(s) < h(r))$.

Let

$$\langle t, h \rangle \leq_{\alpha} \langle t', h' \rangle \iff (t' \subseteq t \land h \upharpoonright t' = h').$$

Show that if $G$ is sufficiently $Q(\alpha)$-generic and $T$ and $H$ are respectively the union of all the first components of elements of $G$ and the union of all the second components of elements of $G$, then

(1) $T$ is a tree on $\omega$, (2) $H$ is a surjection from $T$ onto $\omega \alpha \cup \{\infty\}$, and (3) $(\forall s \in T) H(s) = \|s\|^T$. (Here $\|s\|^T$ is $\|T_s\|$ if $T_s$ is wellfounded and is $\infty$ otherwise.)

If $p = \langle t, h \rangle \in Q(\alpha)$ and $\xi < \alpha$, define $p(\xi) \in Q(\xi)$ by $p(\xi) = \langle t, h' \rangle$, where

$$h'(s) = \begin{cases} h(s) & \text{if } h(s) < \omega \xi; \\ \infty & \text{if } h(s) \geq \omega \xi \end{cases}$$

and where we consider $\infty > \beta$ for every ordinal $\beta$.

Let $p \in Q(\alpha)$, $p' \in Q(\alpha')$, and $\xi \leq \min\{\alpha, \alpha'\}$. Suppose that $p(\xi + 1) = p'(\xi + 1)$. Prove that

$$(\forall q \leq_{\alpha} p)(\exists q' \leq_{\alpha'} p') q(\xi) = q'(\xi).$$

Define a class $S$, the class of ranked sentences, and an ordinal rank of each element of $S$ as follows:

(a) If $s \in T$, then $s \in T$ is a ranked sentence of rank 1.
(b) If $S \subseteq S$ then $\bigwedge S \in S$ and

$$\text{rank}(\bigwedge S) = \sup\{\text{rank}(\varphi) + 1 \mid \varphi \in S\}.$$

(c) If $\varphi \in S$, then $\neg \varphi \in S$ and $\text{rank}(\neg \varphi) = \text{rank}(\varphi) + 1$.

For any tree $T$ on $\omega$, each member of $S$ has an obvious interpretation.

Define a forcing relation $\models_{\alpha}$ between elements $p$ of $Q(\alpha)$ and sentences $\varphi \in S$ inductively as follows:
(a) \( p \models_{\alpha} s \in T \) if and only if \( p = \langle t, h \rangle \) and
\[
s \in t \lor (\exists r \subseteq s)(\ell h(r) + 1 = \ell h(s) \land h(r) \neq 0).
\]
(b) \( p \models_{\alpha} \bigwedge S \) if and only if \( (\forall \varphi \in S) p \models_{\alpha} \varphi \).
(c) \( p \models_{\alpha} \neg \varphi \) if and only if \( (\forall q \leq_{\alpha} p) q \not\models_{\alpha} \varphi \).

Prove that if \( \xi \leq \alpha, \xi \leq \alpha' \), \( p \in \mathbb{Q}_\alpha, p' \in \mathbb{Q}_{\alpha'} \), and \( \varphi \) is a sentence of rank \( \xi \), then
\[
(\dagger) \quad p(\xi) = p'(\xi) \rightarrow (p \models_{\alpha} \varphi \leftrightarrow p' \models_{\alpha'} \varphi).
\]
Proceed by induction on \( \xi \), using (\ast). (This result is from [Steel, 1976], where the partial orderings \( \mathbb{Q}(\alpha) \) are introduced.)

Let \( \alpha < \omega_1 \) be \( a \)-admissible. Assume for a contradiction that \( \alpha \) is not a cardinal in \( L \). Then there are ordinals \( \gamma < \alpha \) and \( \beta < \omega_1 \) and there is a set \( b \in L_\beta \) such that \( b \subseteq \gamma \) and \( b \) codes a wellordering of \( \gamma \) of order type \( \alpha \). Let \( G \) be \( \mathbb{Q}(\beta + 1) \)-generic over \( L_{\omega_{\beta + \omega}}[a] \). Let \( (T, H) \) be given by \( G \). There is an \( s \in T \) such that \( \|s\|^T = \beta \). Hence there is an \( x \in ^\omega \omega \) such that \( x \) is recursive in \( T \) and \( x \) codes a wellordering of \( \omega \) of order type \( \beta \). Let \( z \) be the play of \( G \) consistent with \( x \) in which I plays \( x \). Then \( b \in L_{\gamma + \omega}[z] \) and so \( b \in L_{\gamma + \omega}[a, T] \).

Prove that, for some \( n \in \omega \), there is in \( L_{\gamma + \omega}[a] \) a function that associates with each \( \delta < \gamma \) a ranked sentence which we call \( \delta \in b \) such that
\[
(1) \quad \text{rank}(\delta \in b) < \omega(\gamma + n);
\]
\[
(2) \quad \delta \in b \text{ is true for } T \text{ if and only if } \delta \in b.
\]

To get the sentence \( \delta \in b \), let \( n > 0 \) be such that \( b \in L_{\gamma + n}[a, T] \). There is a formula \( \psi(v) \) with parameters from \( L_{\gamma + n-1}[a] \) that defines \( b \) over \( L_{\gamma + n-1}[a, T] \).

Show that for \( \delta < \gamma \) there is a ranked formula of rank \( < \omega(\gamma + n) \) that is true for any \( T' \) if and only if \( L_{\gamma + n-1}[a, T'] \models \psi[\delta] \). Consider the ranked sentence \( \varphi \):
\[
\bigwedge (\{ \delta \in b \mid \delta \in \gamma \cap b \} \cup \{ -\delta \in b \mid \delta \in \gamma \setminus b \}).
\]

The sentence \( \varphi \) belongs to \( L_{\beta + \omega}[a] \) and has some rank \( \xi < \omega(\gamma + \omega) \). Since \( \varphi \) is true for \( T \), there is some \( p \in G \) such that \( p \models_{\beta + 1} \varphi \). By (\dagger) it follows that \( p(\xi) \models_{\xi} \varphi \). Hence \( p(\xi) \models_{\xi} \delta \in b \) if \( \delta \in b \) and \( p(\xi) \models_{\xi} \neg \delta \in b \) if \( \delta \notin b \). Since \( \{ \delta \in b \mid \delta \in \gamma \} \) belongs to \( L_{\gamma + \omega}[a] \), it follows that \( b \in L_{\gamma + \omega}[a] \). But \( b \) codes a wellordering of \( \gamma \) of order type \( \alpha \), and so this contradicts the \( a \)-admissibility of \( \alpha \).
Chapter 6

Woodin Cardinals

The main goal of this chapter is to introduce Woodin cardinals and prove the consequences of their existence that will be used in the determinacy proofs of Chapter 8. Along the way we give a general survey of large cardinal properties stronger than measurability. The chapter can be read by anyone who has read the first three sections of Chapter 3.

Woodin cardinals and certain other large cardinals cannot be characterized in terms of individual ultrafilters but only in terms of systems of ultrafilters. These systems are called *extenders*, and we will introduce and study them in §1. Extenders will play a central role in Chapters 7 and 8. In §2 we introduce a variety of strong large cardinal axioms and we relate them to one another and to ultrafilters and extenders. We also prove Kunen’s results on the limits of large cardinal axioms. Woodin cardinals are introduced in §2, but they are not singled out for special attention. Section 3 is devoted to some of the basic theory of Woodin cardinals. It ends with a technical result that will be an important tool in Chapter 8.

It is possible to proceed directly from §6.1 to Chapter 7. In a sense this is a more logical order than the order of the book. The concepts and theorems of Chapter 7 do not depend on the material in §6.2–3, and the technical result of §6.3 mentioned above will be used only to construct iteration trees, which are the subject matter of Chapter 7. We chose the actual order only because it put what seemed less technical material first.
6.1 Extenders

Suppose that $j : V \prec M$ with $M$ transitive and $\text{crit}(j) = \kappa$. In the proof of the $(c) \Rightarrow (a)$ part of Theorem 3.2.12, it is shown that if

$$U = \{ X \subseteq \kappa \mid \kappa \in j(X) \},$$

then $U$ is a $\kappa$-complete non-principal ultrafilter on $\kappa$ (and the proof of Lemma 3.2.13 shows further that $U$ is normal). The following lemma gives a general version of this construction of an ultrafilter from such an embedding $j$.

**Lemma 6.1.1.** Let $j : V \prec M$ with $M$ transitive and $\text{crit}(j) = \kappa$. Let $y \in M$. Let $A$ be any set such that $y \in j(A)$.

$$U = \{ X \subseteq A \mid y \in j(X) \}.$$

Then

(i) $U$ is a $\kappa$-complete ultrafilter on $A$;

(ii) $U$ is principal if and only if $y \in \text{range}(j)$.

**Proof.** (i). The proof is very much like that of the $(c) \Rightarrow (a)$ part of Theorem 3.2.12. Since $y \in j(A)$, the definition of $U$ gives that $A \in U$. By the elementarity of $j$, we have that $j(\emptyset) = \emptyset$ and so that $y \notin j(\emptyset)$. Thus $U$ satisfies clause $(a)$ in the definition of a filter. The elementarity of $j$ also gives that $j(X \cap Y) = j(X) \cap j(Y)$, that $X \subseteq Y \Rightarrow j(X) \subseteq j(Y)$, and that $j(A \setminus X) = j(A) \setminus j(X)$; therefore $U$ satisfies clauses $(b)$, $(c)$, and $(d)$ in the definition of an ultrafilter. To verify the $\kappa$-completeness of $U$, let $\delta < \kappa$ and let $X = \langle X_\gamma \mid \gamma < \delta \rangle$ be a sequence of elements of $U$. The elementarity of $j$ and the fact that $\delta < \text{crit}(j)$ yield that

$$j(\bigcap_{\gamma<\delta} X_\gamma) = \bigcap_{\gamma<\delta}(j(X))_\gamma = \bigcap_{\gamma<\delta} j(X_\gamma).$$

But $y \in \bigcap_{\gamma<\delta} j(X_\gamma)$, so $\bigcap_{\gamma<\delta} X_\gamma \in U$.

(ii). The ultrafilter $U$ is principal if and only if there is an $a \in A$ such that $\{a\} \in U$. But $\{a\} \in U$ if and only if $y \in j(\{a\}) = \{j(a)\}$, i.e. if and only if $y = j(a)$. \qed
Many large cardinal properties are like measurability in that they can be formulated in two basic ways: in terms of ultrafilters and in terms of elementary embeddings. In some cases an elementary embedding corresponds to a single ultrafilter, as is the case for measurability, but in other important cases an elementary embedding corresponds to whole system of ultrafilters. Such systems of ultrafilters were first studied in [Mitchell, 1979], and a refinement of Mitchell’s concept was formulated by Dodd and Jensen. (See [Dodd, 1982].) This refinement is the notion of an extender, to which we now turn. All the results of this section were known to Dodd and Jensen and, in a different form, to Mitchell.

First we introduce a standard item of notation, related to the notation \([z]^\gamma\). For sets \(z\) and cardinals \(\gamma\), define

\[
[z]^<\gamma = \{x \subseteq z \mid |x| < \gamma\}.
\]

Let \(j : V \prec M\) with \(M\) transitive and crit \((j) = \kappa\). Let \(\lambda\) be an ordinal number with \(\kappa < \lambda \leq j(\kappa)\). The \((\kappa, \lambda)\)-extender derived from \(j\) is the system

\[
\langle E_a \mid a \in [\lambda]^<\omega\rangle,
\]

where the \(E_a\) are defined by

\[
E_a = \{X \subseteq [\kappa]^{|a|} \mid a \in j(X)\}.
\]

To state the next lemma, we state a useful convention and a few definitions. First the convention: If \(n \in \omega\) and \(z \in [\text{Ord}]^n\) then we write \(z_i\) for the \(i\)th member of \(z\) in order of magnitude, that is, \(z = \{z_1, \ldots, z_n\}\) with \(z_1 < \cdots < z_n\). To give the definitions, let us fix \(n \in \omega\), \(b \in [\text{Ord}]^n\), and \(a \subseteq b\). Let \(a = \{b_{i_1}, \ldots, b_{i_k}\}\), with \(i_1 < \cdots < i_k\). For \(z \in [\text{Ord}]^n\) set

\[
z_{a,b} = \{z_{i_1}, \ldots, z_{i_k}\}.
\]

For \(\alpha \in \text{Ord}\) and \(X \subseteq [\alpha]^k\), define \(X_{a,b}^{\alpha} \subseteq [\alpha]^n\) by

\[
X_{a,b}^{\alpha} = \{z \mid z_{a,b} \in X\}.
\]

Similarly, for \(\alpha \in \text{Ord}\) and \(f : [\alpha]^k \to V\), define \(f_{a,b}^{\alpha} : [\alpha]^n \to V\) by

\[
f_{a,b}^{\alpha}(z) = f(z_{a,b}).
\]
Before proceeding to the lemma, let us introduce one more piece of notation. For any function \( f \) and any set \( x \subseteq \text{domain}(f) \) let

\[
f''x = \text{range}(f \upharpoonright x).
\]

We have earlier in the book written “\( f(x) \)” for \( f''x \), but from now on we will reserve “\( f(x) \)” for the value of \( f \) on the argument \( x \).

**Lemma 6.1.2.** Let \( j : V \prec M \) with \( M \) transitive and \( \text{crit}(j) = \kappa \). Let \( \kappa < \lambda \leq j(\kappa) \) and let \( E = \{ E_a \mid a \in [\lambda]^{<\omega} \} \) be the \((\kappa, \lambda)\)-extender derived from \( j \). Then \( E \) has the following properties:

1. For each \( a \in [\lambda]^{<\omega} \), \( E_a \) is a \( \kappa \)-complete ultrafilter on \( [\kappa]^{\omega} \), and \( E_a \) is principal if and only if \( a \subseteq \kappa \).
2. (Compatibility) If \( a \subseteq b \in [\lambda]^{<\omega} \) and \( X \in E_a \), then \( X^{a,b}_\kappa \in E_b \).
3. (Normality) Let \( a \in [\lambda]^{<\omega} \). Let \( f : [\kappa]^{\omega} \to \kappa \) and \( i \leq |a| \) be such that

\[
\{ z \mid f(z) < z_i \} \in E_a.
\]

Then there is a \( \beta < a_i \) such that

\[
\{ z \in [\kappa]^{\omega} \mid f(z) = z_k \} \in E_{a_i \cup \{ \beta \}},
\]

where \( \beta = (a \cup \{ \beta \})_k \).
4. (Countable Completeness) Let \( \{ a_i \mid i \in \omega \} \) be such that each \( a_i \in [\lambda]^{<\omega} \). Let \( X_i \in E_{a_i} \) for each \( i \in \omega \). Then there is an order preserving \( h : \bigcup_{i \in \omega} a_i \to \kappa \) such that \( h''a_i \in X_i \) for all \( i \in \omega \).

**Proof.** The first assertion of (1) follows directly from part (i) of Lemma 6.1.1. Since if \( a \subseteq j(\kappa) \) but \( a \not\subseteq \kappa \) then \( a \not\in \text{range}(j) \), the second assertion of (1) follows from part (ii) of Lemma 6.1.1.

For (2), let \( n \in \omega \), let \( a \subseteq b \in [\lambda]^n \), and let \( X \in E_a \). By definition, we have that \( a \in j(X) \). Now \( a = b_{a,b} \), so \( b_{a,b} \in j(X) \). But this just means that \( b \in (j(X))^{a,b}_{j(\kappa)} = j(X^{a,b}_\kappa) \) and so that \( X^{a,b}_\kappa \in E_b \).

Let \( a, i, \) and \( f \) be as in the hypothesis of (3). By the definition of \( E_a \), we get that \( a \in \{ z \in [j(\kappa)]^{\omega} \mid (j(f))(z) < z_i \} \). Hence \( (j(f))(a) < a_i \). Let \( \beta = (j(f))(a) \). Let \( k \) be such that \( (a \cup \{ \beta \})_k = \beta \). Then

\[
(j(f))(a) = \beta = (a \cup \{ \beta \})_k.
\]
By the definition of $E_{a \cup \{\beta\}}$, we have that

$$\{z \in [\kappa]^{\omega \setminus \{\beta\}} \mid f(z_{a \cup \{\beta\}}) = z_k\} \in E_{a \cup \{\beta\}}.$$ 

Let $a = \langle a_i \mid i \in \omega \rangle$ and $X = \langle X_i \mid i \in \omega \rangle$ be as in the hypotheses of (4). Let $b = \bigcup_{i \in \omega} a_i$. If $b$ is finite, then $(j \upharpoonright b)^{-1} : j(b) \to b$ witnesses that

$$M \models (\exists h)(h : j(b) \to j(\kappa) \land h \text{ is order preserving} \land (\forall i \in \omega) h''(j(a))_i \in (j(X))_i).$$

The desired conclusion follows by the elementarity of $j$. If $b$ is infinite, then we cannot assume that $j \upharpoonright b \in M$, and so we do not know that its inverse belongs to $M$. Instead we let

$$U = \{s \mid (\exists k \in \omega)(s : \bigcup_{i \leq k} a_i \to \kappa \land s \text{ is order preserving} \land (\forall i \leq k) s''a_i \in X_i)\}.$$ 

If $s$ and $t$ belong to $U$ define

$$s \prec t \iff s \supseteq t.$$ 

The inverse of $j \upharpoonright b$ witnesses that $j(\prec)$ is not wellfounded in $V$. The absoluteness of wellfoundedness implies that $j(\prec)$ is not wellfounded in $M$. The elementarity of $j$ then implies that $\prec$ is not wellfounded. If $\langle s_i \mid i \in \omega \rangle$ is an infinite descending sequence in $\prec$, then $\bigcup_{i \in \omega} s_i$ is our desired $h$. \hfill $\square$

If $\kappa$ is an uncountable cardinal number and $\lambda > \kappa$ is an ordinal number, then a $(\kappa, \lambda)$-extender is a system $\langle E_a \mid a \in [\lambda]^{<\omega} \rangle$ that satisfies clauses (1)–(4) of Lemma 6.1.2. An extender is anything that is a $(\kappa, \lambda)$-extender for some pair $\langle \kappa, \lambda \rangle$.

Remarks.

(1) There can be a $(\kappa, \lambda)$-extender only if $\kappa$ is a measurable cardinal. A $(\kappa, \kappa + 1)$-extender is essentially a uniform normal ultrafilter on $\kappa$. See Exercise 6.1.1.

(2) There is no real reason for the demand that $\lambda \leq j(\kappa)$ in order for the $(\kappa, \lambda)$-extender derived from $j$ to be defined. Removing this requirement would give us ultrafilters $E_a$ that are not all on $[\kappa]^{<\omega}$. See Exercise 6.1.2.
Let $E = \langle E_a \mid a \in [\lambda]^{<\omega} \rangle$ be a $(\kappa, \lambda)$-extender. We define

$$\prod_E (V; \in),$$

which we call the *ultrapower* of $(V; \in)$ with respect to $E$, even though it is not literally an ultrapower.

Let $D_E = \{ \langle a, f \rangle \mid a \in [\lambda]^{<\omega} \land f : [\kappa]|a| \to V \}$. If $\langle a, f \rangle$ and $\langle b, g \rangle$ are elements of $D_E$, define

$$\langle a, f \rangle \sim_E \langle b, g \rangle \iff \{ z \in [\kappa]|a\cup b| \mid f(z_{a\cup b}) = g(z_{b\cup a}) \} \in E_{a\cup b}.$$

It is easily verified that $\sim_E$ is an equivalence relation on the class $D_E$. For $\langle a, f \rangle \in D_E$, let $[\langle a, f \rangle]_E$ be the set of all elements of minimal rank belonging to the equivalence class of $\langle a, f \rangle$. (We will omit the subscript "$_E$" when there is no danger of confusion.) The universe of our ultrapower $\prod_E (V; \in)$ is the class of all the $[\langle a, f \rangle]_E$ for $\langle a, f \rangle \in D_E$. The relation, which we write $\in_E$, is given by

$$[\langle a, f \rangle]_E \in_E [\langle b, g \rangle]_E \iff \{ z \in [\kappa]|a\cup b| \mid f(z_{a\cup b}) \in g(z_{b\cup a}) \} \in E_{a\cup b}.$$

**Remark.** An alternative way of defining $\prod_E (V; \in)$ is as a direct limit of the ordinary ultrapowers $\prod_{E_a} (V; \in)$. See Exercise 6.1.3.

**Theorem 6.1.3.** Let $E = \langle E_a \mid a \in [\lambda]^{<\omega} \rangle$ be a $(\kappa, \lambda)$-extender. Let $\varphi(v_1, \ldots, v_n)$ be any formula of the language of set theory. Let $\langle a_1, f_1 \rangle, \ldots, \langle a_n, f_n \rangle$ be elements of $D_E$. Let $b = \bigcup_{1 \leq i \leq n} a_i$. Then

$$\prod_E (V; \in) \models \varphi[a_1, f_1, \ldots, f_n] \iff \{ z \in [\kappa]|b| \mid (V; \in) \models \varphi[f_1(z_{a_1, b}), \ldots, f_n(z_{a_n, b})] \} \in E_b.$$

**Proof.** The proof is similar to those of Theorem 3.2.1 and Theorem 3.2.5, and we omit it.

As with ordinary ultrapowers, we get a canonical elementary embedding which we call $i'_E$ of $(V; \in)$ into $\prod_E (V; E)$, where $i'_E$ is defined by, e.g.,

$$i'_E(x) = [\emptyset, c_x]_E.$$

Here $\emptyset$ could be replaced by any other $a \in [\lambda]^{<\omega}$ without affecting the definition.
Lemma 6.1.4. If \( E \) is an extender then \( \prod_E(V; \in) \) is wellfounded.

**Proof.** Let \( E = \langle E_a \mid a \in [\lambda]^{<\omega} \rangle \) be an extender and suppose for a contradiction that

\[
\cdots \in_E [a_2, f_2] \in_E [a_1, f_1] \in_E [a_0, f_0].
\]

Replacing each \( a_i \) by \( \bigcup_{i \leq j} a_j \) and each \( f_i \) by \( f_i^{a_{i-1}, a_i} \), we may assume that \( a_0 \subseteq a_1 \subseteq \ldots \). Let \( X_0 = [\kappa]^{|a_0|} \) and for each \( i \) let

\[
X_{i+1} = \{ z \in [\kappa]^{|a_{i+1}|} \mid f_{i+1}(z) \in f_i(z_{a_i, a_{i+1}}) \}.
\]

For each \( i \in \omega \), we have that \( X_i \in E_{a_i} \). By countable completeness (property (4) of extenders), let \( h : \bigcup_{i \in \omega} a_i \to \kappa \) be order preserving and such that \( h''a_i \in X_i \) for all \( i \). We get the contradiction that

\[
(\forall i \in \omega) f_{i+1}(h''a_{i+1}) \in f_i(h''a_i).
\]

The wellfoundedness of \( \prod_E(V; \in) \) is actually equivalent with the countable completeness of \( E \):

Lemma 6.1.5. If \( E = \langle E_a \mid a \in [\lambda]^{<\omega} \rangle \) has properties (1)–(3) of \((\kappa, \lambda)\)-extenders, then \( E \) is countably complete (i.e., \( E \) is an extender) if and only if \( \prod_E(V; \in) \) is wellfounded.

**Proof.** The “only if” part of the corollary is Lemma 6.1.4.

For the “if” part, assume that \( \langle a_i \mid i \in \omega \rangle \) and \( \langle X_i \mid i \in \omega \rangle \) are a counterexample to the countable completeness of \( E \). Replacing, if necessary, each \( a_k \) by \( \bigcup_{i \leq k} a_i \) and \( X_k \) by \( (X_k)^{a_{i-1}, a_i} \), we may assume that \( a_i \subseteq a_k \) for all \( i \leq k \in \omega \). Next replacing, if necessary, each \( X_k \) by \( \bigcap_{i \leq k} (X_k)^{a_{i-1}, a_i} \), we may assume that

\[
(\forall z \in X_k)(\forall i \leq k) z_{a_i, a_k} \in X_i.
\]

As in the proof of Lemma 6.1.2, let

\[
U = \{ s \mid (\exists k \in \omega) (s : a_k \to \kappa \land s \text{ is order preserving} \land s''a_k \in X_k) \}
\]

and define, for elements \( s \) and \( t \) of \( U \),

\[
s < t \iff s \supseteq t.
\]
CHAPTER 6. WOODIN CARDINALS

To say that $\langle a_i \mid i \in \omega \rangle$ and $\langle X_i \mid i \in \omega \rangle$ are a counterexample to the countable completeness of $E$ is just to say that $\prec$ is wellfounded. Define $\parallel \parallel \prec : U \to \text{Ord}$ by induction on $\prec$ as follows.

$$\parallel s \parallel ^E = \sup \{ \parallel t \parallel ^E + 1 \mid t \prec s \}.$$  

For each $k \in \omega$ and each $z \in X_k$, there is a unique $s_z \in U$ such that $z = s_z''^a_k$. For $k \in \omega$, define $f_k : X_k \to \text{Ord}$ by

$$f_k(z) = \parallel s_z \parallel ^E.$$  

For each $k$ and each $z \in X_{k+1}$,

$$f_{k+1}(z) = \parallel s_z \parallel ^E > \parallel s_z \restriction a_k \parallel ^E = \parallel s_{z_{a_k,a_{k+1}}} \parallel ^E = f_k(z_{a_k,a_{k+1}}).$$

Hence the $[a_k,f_k]$ witness that $\prod E(V; \in) \not\text{ is wellfounded.}$ □

Lemma 6.1.6. If $E$ is an extender then $\prod E(V; \in)$ is set-like.

We omit the proof, which is similar to that of Lemma 3.2.9.

If $E$ is an extender then Lemmas 6.1.4, 6.1.6, and 3.2.8 give us a unique $\pi_E : \prod E(V; \in) \cong (\text{Ult}(V; E); \in)$, with Ult$(V; E)$ transitive. Let $i_E : V \prec \text{Ult}(V; E)$ be given by $i_E = \pi_E \circ i_E$. (Note that we continue the convention whereby we may write, for example, “$V$” instead of “$(V; \in)$.”)

Lemma 6.1.7. Let $E$ be an extender. Then $i_E$ is the identity on $V_\kappa$, and $\kappa = \text{crit}(i_E)$.

Proof. The proof is like that of Lemma 3.2.10.

We first show that $i_E$ is the identity on $\kappa$. To do this we prove by induction that $i_E(\alpha) = \alpha$ for all $\alpha < \kappa$. Suppose then that $\alpha < \kappa$ and that $i_E(\beta) = \beta$ for all $\beta < \alpha$. For each $\beta \in \alpha$, $i_E(\beta) \in i_E(\alpha)$, by the elementarity of $i_E$. Suppose that $\pi([a,f]) \in i_E(\alpha)$, where $\pi = \pi_E : \prod E(V; \in) \cong (\text{Ult}(V; E); \in)$. 

Then \( \{ z \in \kappa^{|a|} \mid f(z) \in \alpha \} \in E_a \). But then the \( \kappa \)-completeness of \( E_a \) implies that there is a \( \beta \prec \alpha \) such that \( \{ z \in \kappa^{|a|} \mid f(z) = \beta \} \in E_a \). This means that

\[
\pi([a, f]) = \pi([\emptyset, c_\beta]) = i_E(\beta) = \beta.
\]

This completes the inductive proof that \( i_E \) is the identity on \( \kappa \). From this fact it follows exactly as in the proof of Lemma 3.2.10 that \( i_E \) is the identity on \( V_\kappa \).

To see that \( i_E(\kappa) \prec \kappa \), consider \( \{ \{ \kappa \}, f \} \), where \( f = \bigcup \{ \kappa \}^1 \). Note that \( \bigcup \{ \alpha \} = \alpha \) for each \( \{ \alpha \} \in \kappa \). Since \( E_{\{\kappa\}} \) is non-principal and \( \kappa \)-complete by part (1) of Lemma 6.1.2, we have that

\[
(\forall \beta \prec \kappa) \{ \{ \alpha \} \mid \beta \prec \alpha \} \in E_{\{\kappa\}}.
\]

Thus \( \pi([\{ \kappa \}, f]) \prec \beta \) for all \( \beta \prec \kappa \); hence \( \pi([\{ \kappa \}, f]) \prec \kappa \). (It follows from Lemma 6.1.8 below that \( \pi([\{ \kappa \}, f]) = \kappa \).) But \( f : \kappa^1 \rightarrow \kappa \), so \( \pi([\{ \kappa \}, f]) \prec \pi([\emptyset, c_\kappa]) = i_E(\kappa) \).

If \( \mathcal{U} \) is a uniform normal ultrafilter on \( \kappa \), then (Exercise 3.2.2) \( \pi_\mathcal{U}([id]_\mathcal{U}) = \kappa \). The next lemma is the version of this fact for extenders, and it is proved using the normality of extenders (property (3)).

**Lemma 6.1.8.** Let \( E = \langle E_a \mid a \in [\lambda]^{<\omega} \rangle \) be a \( (\kappa, \lambda) \)-extender.

(i) For each \( a \in [\lambda]^{<\omega} \) and each \( i \), \( 1 \leq i \leq |a| \), \( \pi_E([a, z \mapsto z_i]) = a_i \).

(ii) For each \( a \in [\lambda]^{<\omega} \), \( \pi_E([a, id]) = a \).

**Proof.** Let \( \pi = \pi_E \). We prove (i) by induction on the ordinal \( a_i \), simultaneously for all \( a \in [\lambda]^{<\omega} \). Assume then that, for all \( b \in [\lambda]^{<\omega} \) and all \( k \), \( 1 \leq k \leq |b| \), if \( b_k \prec a_i \), then \( \pi([b, z \mapsto z_k]) = b_k \).

Let \( \beta \prec a_i \). Let \( \beta = (a \cup \{ \beta \})_k \). Using our induction hypothesis, we get that \( \beta = \pi([a \cup \{ \beta \}, z \mapsto z_k]) \). But \( z_k \prec (z_{a \cup \{ \beta \}})_i \) for all \( z \in [\kappa]^{|a \cup \{ \beta \}|} \), and so Theorem 6.1.3 implies that \( \pi([a \cup \{ \beta \}, z \mapsto z_k]) < \pi([a, z \mapsto z_i]) \), and so that \( \beta < \pi([a, z \mapsto z_i]) \).

Assume now that \( \pi([c, f]) \) is an ordinal smaller than \( \pi([a, z \mapsto z_i]) \). We may assume that range \( (f) \subseteq \text{Ord} \). Let \( b = a \cup c \). We have by Theorem 6.1.3 that

\[
\{ z \in [\kappa]^{\cdot} \mid f(z_{a,b}) < (z_{a,b})_i \} \in E_b.
\]
By normality this gives us a $\beta < a_i$ such that
\[
\{ z \in [\kappa]| f(z_{c,b\cup\{\beta\}}) = z_k \} \in E_{b\cup\{\beta\}},
\]
where $\beta = (b \cup \{\beta\})_k$. Thus $\pi([c, f]) = \pi([b \cup \{\beta\}, z \mapsto z_k])$. Our induction hypothesis then gives us that $\pi([c, f]) = (b \cup \{\beta\})_k = \beta$.

We have shown that the ordinals smaller than $\pi([a, z \mapsto z_i])$ are precisely the ordinals smaller than $a_i$; hence $\pi([a, z \mapsto z_i]) = a_i$.

It is easy to see that (i) implies (ii). $\square$

**Corollary 6.1.9.** If $E$ is a $(\kappa, \lambda)$-extender, then $\lambda \leq i_E(\kappa)$, and $E$ is the $(\kappa, \lambda)$-extender derived from $i_E$.

**Proof.** Let $\alpha < \lambda$. By Lemma 6.1.8, $\alpha = \pi_E([\{\alpha\}, z \mapsto z_1])$. Since $\{z \in [\kappa]| z_1 < \kappa \} \in E_{\alpha}$, Theorem 6.1.3 gives that $\pi_E([\{\alpha\}, z \mapsto z_1]) < \pi_E([\emptyset, \kappa]) = i_E(\kappa)$.

Let $a \in [\lambda]^\omega$ and let $X \in [\kappa]^{|a|}$. By Lemma 6.1.8, $a \in i_E(X)$ if and only if $\pi_E([a, id]) \in E_{\emptyset, \kappa}$. By Theorem 6.1.3, this holds if and only if $\{z \in [\kappa]| z \in X\} \in E_{\alpha}$, i.e., if and only if $X \in E_{\alpha}$. $\square$

Now let us return to the topic with which we began this section. Let $j : V \prec M$ with $M$ transitive and $\text{crit}(j) = \kappa$. Let $\lambda \leq j(\kappa)$. Let $\langle E_a | a \in [\lambda]^\omega \rangle$ be the $(\kappa, \lambda)$-extender derived from $j$. The next two lemmas will show that $i_E : V \prec \text{Ult}(V; E)$ is an approximation of $j : V \prec M$. Define $k : \text{Ult}(V; E) \rightarrow M$ by
\[
k(\pi_E([a, f])_E) = (j(f))(a).
\]
The function $k$ is well-defined, since
\[
[a, f] = [b, g] \rightarrow \\
\{z \in [\kappa]| z_1 < \kappa \} | f(z_{a,b\cup\{\beta\}}) = g(z_{b,a\cup\{\beta\}}) \} \in E_{a\cup\{\beta\}} \rightarrow \\
(j(f))(a) = (j(g))(b).
\]

**Lemma 6.1.10.** Let $j$, $M$, $\lambda$, $E$, and $k$ be as in the preceding paragraph. Then
\[
(a) \ k : \text{Ult}(V; E) \prec M; \\
(b) \ k \circ i_E = j;
\]
(c) $k \upharpoonright \lambda$ is the identity.

**Proof.** The proof of (a) is similar to the proof that $k$ is well-defined. Let $\varphi(v_1, \ldots, v_n)$ be any formula of the language of set theory. Let $\langle a_1, f_1 \rangle, \ldots, \langle a_n, f_n \rangle$ be elements of $D_E$. Let $b = \bigcup_{1 \leq i \leq n} a_i$. Then

$$\prod_E (V; \in) \models \varphi[[a_1, f_1], \ldots, [a_n, f_n]] \Leftrightarrow \text{(by Theorem 6.1.3)}$$

$$\{ z \in [\kappa]^{|k|} \mid (V; \in) \models \varphi[f_1(z_{a_1, b}), \ldots, f_n(z_{a_n, b})] \} \in E_b \Leftrightarrow$$

$$(M; \in) \models \varphi[(j(f_1))(a_1), \ldots, (j(f_n))(a_n)] \Leftrightarrow$$

$$(M; \in) \models \varphi[k(\pi_E([a_1, f_1]), \ldots, k(\pi_E([a_n, f_n]))].$$

To see that $k \circ i_E = j$, observe that

$$k(i_E(x)) = k(\pi_E([\emptyset, c_x])) = c_{j(x)}(\emptyset) = j(x).$$

We finish the proof of the lemma by showing that $k \upharpoonright [\lambda]^{<\omega}$ is the identity. This is clearly equivalent with (c). Let $a \in [\lambda]^{<\omega}$. Lemma 6.1.8 implies that $k(a) = k(\pi_E([a, \text{id}])) = (j(\text{id}))(a) = a$. \hfill \Box

**Lemma 6.1.11.** Let $j$, $M$, $\lambda$, $E$, and $k$ be as in the paragraph preceding Lemma 6.1.10. Let $\eta < \lambda$ be such that

$$|V_{\eta}^{M}|^M \leq \lambda.$$ 

Then $V_{\eta}^{\text{Ult}(V;E)} = V_{\eta}^M$ and $k \upharpoonright V_{\eta}^{\text{Ult}(V;E)}$ is the identity.

**Proof.** Let $\gamma = |V_{\eta}^{\text{Ult}(V;E)}|^{\text{Ult}(V;E)}$. Since

$$\gamma \leq k(\gamma) = |V_{\eta}^M|^M \leq \lambda,$$

we must have $k(\gamma) = \gamma$.

Let $\langle X_\beta \mid \beta < \gamma \rangle$ be an enumeration of all elements of $V_{\eta}^{\text{Ult}(V;E)}$. Then $\langle k(X_\beta) \mid \beta < \gamma \rangle$ is an enumeration of all elements of $V_{\eta}^M$. By the elementarity of $k$, this means that

$$k \upharpoonright V_{\eta}^{\text{Ult}(V;E)} : (V_{\eta}^{\text{Ult}(V;E)}; \in) \cong (V_{\eta}^M; \in).$$

But an isomorphism between transitive sets must be the identity. \hfill \Box

For the case $j = i_E$, the elementary embedding $k$ of the preceding lemmas is the identity:
Lemma 6.1.12. For any extender $E$ and any $[a,f]$, 

$$(i_E(f))(a) = \pi_E([a,f]).$$

Proof. For $E$, $a$, and $f$, as in the statement of the lemma,

$$(i_E(f))(a) = (\pi_E([\emptyset,cf]))(\pi_E([a,id]))$$

$$= \pi_E([a,f]),$$

where the last equality is by Theorem 6.1.3, with “$v_1$ is a function and $v_3 = v_1(v_2)$” as the formula $\varphi$, with $\emptyset$ as $a_1$ and $a$ as $a_2$ and $a_3$, with $cf$ as $f_1$, with $id$ as $f_2$, and with $f$ as $f_3$. □

Exercise 6.1.1. (a) Let $\langle E_a | a \subseteq [\kappa+1]^{<\omega} \rangle$ be a $(\kappa,\kappa+1)$-extender. Show that

$$\{X \subseteq \kappa | (\exists Y \in E_\kappa)(\forall \alpha < \kappa)(\alpha \in X \leftrightarrow \{\alpha\} \in Y)\}$$

is a uniform normal ultrafilter on $\kappa$.

(b) Prove that a cardinal $\kappa$ is measurable if and only if there exists a $(\kappa,\kappa+1)$-extender.

Exercise 6.1.2. Let us generalize the notion of extender as follows. Let $j : V \prec M$ with $M$ transitive and $\text{crit}(j) = \kappa$. Let $\lambda$ be any ordinal number such that $\kappa < \lambda$. For $a \in [\lambda]^{<\omega}$, let $\gamma_a$ be the least ordinal $\gamma \geq \kappa$ such that $a \in [j(\gamma)]^{<\omega}$. The $(\kappa,\lambda)$-extender derived from $j$ is the system

$$\langle E_a | a \in [\lambda]^{<\omega} \rangle,$$

where the $E_a$ are defined by

$$E_a = \{X \subseteq [\gamma_a]^{[a]} | a \in j(X)\}.$$ 

Remark. The requirement that $\gamma_a \geq \kappa$ has no purpose other than to make ordinary extenders be extenders in the generalized sense.

Prove that Lemma 6.1.2 becomes true for derived extenders in this generalized sense when (1)–(4) are replaced by the following clauses (1′)–(4′).

(1′) For each $a \in [\lambda]^{<\omega}$, $E_a$ is a $\kappa$-complete ultrafilter on $[\gamma_a]^{[a]}$, and $E_a$ is principal if and only if $a \in \text{range} (j)$.

(2′) (Compatibility) If $a \subseteq b \in [\lambda]^{<\omega}$ and $X \in E_a$, then $X_{\gamma_b}^a \in E_b$. 

(3′) (Normality) Let $a \in [\lambda]^{<\omega}$. Let $f : [\gamma_a]^{|a|} \to \gamma_a$ and $i \leq |a|$ be such that
\[
\{ z \mid f(z) < z_i \} \in E_a.
\]
Then there is a $\beta < a_i$ such that
\[
\{ z \in [\gamma_a]^{a \cup \{ \beta \}} \mid f(z_{a,a \cup \{ \beta \}}) = z_k \} \in E_{a \cup \{ \beta \}},
\]
where $\beta = (a \cup \{ \beta \})_k$.

(4′) (Countable Completeness) Let $\langle a_i \mid i \in \omega \rangle$ be such that each $a_i \in [\lambda]^{<\omega}$. Let $X_i \in E_{a_i}$ for each $i \in \omega$. Then there is an order preserving $h : \bigcup_{i \in \omega} a_i \to \bigcup_{i \in \omega} \gamma_{a_i}$ such that $h'' a_i \in X_i$ for all $i \in \omega$.

A $(\kappa, \lambda)$-extender in the generalized sense is defined using (1′)–(4′) and (if one wants) the condition that no bounded subset of $\gamma_a$ belongs to $E_a$ unless $\gamma_a = \kappa$.

Exercise 6.1.3. Let $E = \langle E_a \mid a \in [\lambda]^{<\omega} \rangle$ be a $(\kappa, \lambda)$-extender.
(a) For $a \subseteq b \in [\lambda]^{<\omega}$, show that
\[
 f \mapsto f^{a,b}
\]
induces an elementary embedding
\[
i_{E_a,E_b} : \text{Ult}(V; E_a) \prec \text{Ult}(V; E_b).
\]
(b) Prove that
\[
(\langle \text{Ult}(V; E_a) \mid a \in [\lambda]^{<\omega} \rangle, \langle i_{E_a,E_b} \mid a \subseteq b \in [\lambda]^{<\omega} \rangle)
\]
is a directed system of elementary embeddings.
(c) Let
\[
(\hat{\mathcal{M}}, \langle i_{E_a} \mid a \in [\lambda]^{<\omega} \rangle)
\]
be the direct limit of the directed system of (b). Prove that there is a (unique) $\pi : \mathcal{M} \cong (\text{Ult}(V; E); \in)$ and that $i_E = \pi \circ i_{E_a} \circ i_{E_a}$. 

6.2 Large Large Cardinals

By Lemma 3.2.11 and Theorem 3.2.12, the measurability of a cardinal $\kappa$ is equivalent with each of the following:

(a) There are a transitive class $M$ and an embedding $j : V \prec M$ such that $\text{crit}(j) = \kappa$ and such that $V_{\kappa+1} \subseteq M$.

(b) There are a transitive class $M$ and an embedding $j : V \prec M$ such that $\text{crit}(j) = \kappa$ and such that $^{*}M \subseteq M$.

These two equivalents of measurability lead to two different ways to generalize the notion of a measurable cardinal. The one corresponding to (a) was considered in [Gaifman, 1974], but became prominent only through work of [Mitchell, 1979] and of Anthony Dodd and Ronald Jensen. (See [Dodd, 1982] and [Dodd, ].) The one corresponding to (b) is was pursued earlier, by William Reinhardt and Robert Solovay. (See [Solovay et al., 1978].)

If $\kappa$ is a cardinal number and $\eta$ is an ordinal number greater than $\kappa$, then $\kappa$ is $\eta$-strong if there are a transitive class $M$ and an embedding $j : V \prec M$ such that $\text{crit}(j) = \kappa$, $\eta < j(\kappa)$, and $V_{\eta} \subseteq M$. A cardinal $\kappa$ is strong if $\kappa$ is $\eta$-strong for every ordinal $\eta > \kappa$.

If $\kappa$ and $\lambda \geq \kappa$ are cardinal numbers, then $\kappa$ is $\lambda$-supercompact if there are a transitive class $M$ and an embedding $j : V \prec M$ such that $\text{crit}(j) = \kappa$, $\lambda < j(\kappa)$, and $^{*}M \subseteq M$. A cardinal $\kappa$ is supercompact if $\kappa$ is $\lambda$-supercompact for every cardinal $\lambda \geq \kappa$.

The condition $\eta < j(\kappa)$ can be dropped from the definition of $\eta$-strong without changing the concept, and the condition $\lambda < j(\kappa)$ can similarly be dropped from the definition of $\lambda$-supercompact. This will be proved later (Theorem 6.2.15).

A cardinal $\kappa$ is measurable if and only if $\kappa$ is $(\kappa + 1)$-strong if and only if $\kappa$ is $\kappa$-supercompact. It is clear that if $\kappa$ is $2^\kappa$-supercompact then $\kappa$ is $(\kappa + 2)$-strong. But the converse fails: If $\kappa$ is $2^\kappa$-supercompact then, as we will see below, $\kappa$ is the $\kappa$th cardinal $\gamma$ such that $\gamma$ is $(\gamma + 2)$-strong—indeed there are $\kappa$ cardinals $\gamma < \kappa$ such that $\gamma$ is $\kappa$-strong.

There is an equivalent definition of $\lambda$-supercompactness that generalizes our basic definition of measurability in terms of ultrafilters. To state this definition, we need to make some preliminary definitions.

For cardinals $\kappa$ and $\lambda$,

$$P_\kappa(\lambda) = \{x \subseteq \lambda \mid |x| < \kappa\}.$$
In other notation, $\mathcal{P}_\kappa(\lambda) = [\lambda]^{<\kappa}$. An ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(\lambda)$ (i) is fine if
\[ \{ x \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in x \} \in \mathcal{U} \]
for each $\alpha < \lambda$ and (ii) is normal if, for every $f : \mathcal{P}_\kappa(\lambda) \to \lambda$, if
\[ \{ x \in \mathcal{P}_\kappa(\lambda) \mid f(x) \in x \} \in \mathcal{U} \]
then there is an $\alpha < \lambda$ such that
\[ \{ x \in \mathcal{P}_\kappa(\lambda) \mid f(x) = \alpha \} \in \mathcal{U}. \]

Note that an ultrafilter $\mathcal{V}$ on an infinite cardinal $\kappa$ generates an ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(\kappa)$: $X \in \mathcal{U} \iff X \cap \kappa \in \mathcal{V}$ ($\iff X \cap \text{Ord} \in \mathcal{V}$). The ultrafilter $\mathcal{U}$ is $\kappa$-complete if and only if $\mathcal{V}$ is $\kappa$-complete; $\mathcal{U}$ is fine if and only if $\mathcal{V}$ is uniform; $\mathcal{U}$ is normal if and only if $\mathcal{V}$ is normal.

**Theorem 6.2.1.** (Reinhardt, Solovay; see [Solovay et al., 1978]) If $\kappa$ and $\lambda \geq \kappa$ are cardinals, then the following are equivalent:

1. $\kappa$ is $\lambda$-supercompact.
2. There is a $\kappa$-complete fine normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$.

**Proof.** First suppose that $j : V \prec M$ witnesses that $\kappa$ is $\lambda$-supercompact. Let
\[ \mathcal{U} = \{ X \subseteq \mathcal{P}_\kappa(\lambda) \mid j''\lambda \in j(X) \}. \]
(Recall that $j''\lambda = \text{range}(j \upharpoonright \lambda)$.) This definition is legitimate, for $|j''\lambda| = \lambda$ and $j''\lambda \in M$. Since $\lambda < j(\kappa)$, we have that $j''\lambda \in j(\mathcal{P}_\kappa(\lambda))$. Thus Lemma 6.1.1 implies that $\mathcal{U}$ is a $\kappa$-complete ultrafilter on $\mathcal{P}_\kappa(\lambda)$.

To see that $\mathcal{U}$ is fine, let $\alpha < \lambda$. Since $j(\alpha) \in j''\lambda$, we have that
\[ j''\lambda \in j(\{ x \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in x \}). \]

Hence $\{ x \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in x \} \in \mathcal{U}$.

To verify the normality of $\mathcal{U}$, let $f : \mathcal{P}_\kappa(\lambda) \to \lambda$ be such that
\[ \{ x \in \mathcal{P}_\kappa(\lambda) \mid f(x) \in x \} \in \mathcal{U}. \]

By the definition of $\mathcal{U}$, we have that
\[ (j(f))(j''\lambda) \in j''\lambda. \]
But this means that there is an $\alpha < \lambda$ such that

$$(j(f))(j''\lambda) = j(\alpha).$$

By the definition of $\mathcal{U}$,

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid f(x) = \alpha\} \in \mathcal{U}.$$ 

Now suppose that $\mathcal{U}$ is a $\kappa$-complete fine normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$. Since $i_\mathcal{U} : V \prec \text{Ult}(V; \mathcal{U})$, it is enough to show

(i) $\text{Ult}(V; \mathcal{U}) \subseteq \text{Ult}(V; \mathcal{U})$;
(ii) $\lambda < i_\mathcal{U}(\kappa)$;
(iii) $\text{crit}(i_\mathcal{U}) = \kappa$.

In order to prove (i) we will first show that $i_\mathcal{U}''\lambda \in \text{Ult}(V; \mathcal{U})$.

Let $\pi = \pi_\mathcal{U} : \prod(V; \in) \cong (\text{Ult}(V; \mathcal{U}); \in)$. We show that $i_\mathcal{U}''\lambda = \pi([\text{id}])$. If $\alpha < \lambda$ then, by fineness of $\mathcal{U}$,

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in x\} \in \mathcal{U};$$

hence

$$[c_\alpha] \in_\mathcal{U} [\text{id}]$$

and so

$$i_\mathcal{U}(\alpha) \in \pi([\text{id}]).$$

Thus $i_\mathcal{U}''\lambda \subseteq \pi([\text{id}])$. To establish the reverse inclusion, let

$$[f] \in_\mathcal{U} [\text{id}].$$

Then

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid f(x) \in x\} \in \mathcal{U}.$$ 

By the normality of $\mathcal{U}$, there is an $\alpha < \lambda$ such that

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid f(x) = \alpha\} \in \mathcal{U}.$$ 

But then $[f] = [c_\alpha]$ and so

$$\pi([f]) = i_\mathcal{U}(\alpha).$$
This shows that \( i_U''\lambda \supseteq \pi([\id]) \) and so that \( i_U''\lambda = \pi([\id]) \).

Now suppose that \( h : \lambda \to \Ult(V;\mathcal{U}) \). For each \( \alpha < \lambda \), let \( h(\alpha) = \pi([g_\alpha]) \).

Define \( \tilde{g} : \mathcal{P}_\kappa(\lambda) \to ^\lambda V \) by
\[
(\tilde{g}(x))(\alpha) = g_\alpha(x).
\]
The function \( \pi([\tilde{g}]) \) has domain \( i_U(\lambda) \). For each \( \alpha < \lambda \),
\[
(\pi([\tilde{g}]))(i_U(\alpha)) = \pi([g_\alpha]) = h(\alpha).
\]
Thus
\[
h = \pi([\tilde{g}]) \circ (i_U \upharpoonright \lambda) \in \Ult(V;\mathcal{U}).
\]

For (ii), note that
\[
\lambda = \text{the order type of } i_U''\lambda
\]
\[
= \text{the order type of } \pi([\id])
\]
\[
< \pi([c_\kappa])
\]
\[
= i_U(\kappa).
\]

Now by Lemma 3.2.10, \( \text{crit } (i_U) \) is the completeness of \( \mathcal{U} \), which is \( \geq \kappa \). If \( \text{crit } (i_U) > \kappa \), then we have the contradiction that \( \lambda \geq \kappa = i_U(\kappa) > \lambda \). Thus (iii) is proved. \( \square \)

There is no analogue of Theorem 6.2.1 that characterizes \( \eta \)-strength in terms of the existence of an ultrafilter. To get an analogue of Theorem 6.2.1 we need to use extenders instead of ultrafilters.

**Lemma 6.2.2.** Let \( \kappa \) be an \( \eta \)-strong cardinal. Then there is an extender \( E \) such that \( i_E \) witnesses that \( \kappa \) is \( \eta \)-strong.

**Proof.** Let \( j : V \prec M \) witness that \( \kappa \) is \( \eta \)-strong. Thus \( V_\eta^M = V_\eta \). Let \( \lambda = [V_\eta]^M \). Let \( E \) be the \( (\kappa,\lambda) \)-extender derived from \( j \). By Lemma 6.1.11, \( V_\eta^{\Ult(V;E)} = V_\eta \). Hence \( i_E \) witnesses that \( \kappa \) is \( \eta \)-strong. \( \square \)

For any extender \( E \), let \( \text{strength } (E) \) be the largest ordinal \( \eta \) such that \( V_\eta \subseteq \Ult(V;E) \). The next theorem is a direct consequence of Lemma 6.2.2.

**Theorem 6.2.3.** (Mitchell; Dodd and Jensen) For cardinals \( \kappa \) and \( \eta > \kappa \), the following are equivalent:
(a) \( \kappa \) is \( \eta \)-strong.

(b) There is an extender \( E \) with \( \text{crit}(i_E) = \kappa \), strength \( (E) \geq \eta \), and \( \eta < i_E(\kappa) \).

Remark. In [Martin and Steel, 1989], the word extender is used for a wider class than that of \((\kappa, \lambda)\)-extenders. The extenders of that paper do not necessarily have the form \( \langle E_a \mid a \in [\lambda]^{<\omega} \rangle \). They can have the more general form \( \langle E_a \mid a \in [Y]^{<\omega} \rangle \), where \( Y \) is required only to be a transitive set. If \( j : V \prec M \) and \( Y \subseteq V^M_{j(\kappa)} \) is transitive, then we can get such an extender by setting \( E_a = \{ X \subseteq [Y]^{\alpha} \mid a \in j(X) \} \). \( Y \) is called the support of \( E \). A cardinal \( \kappa \) is \( \eta \)-strong if and only if there is an extender \( E \) in the sense of [Martin and Steel, 1989] such that \( \text{crit}(i_E) = \kappa \) and the support of \( E \) contains \( V_\eta \).

Let us begin to show that supercompactness is a much stronger property than strength. To do this we introduce three classes of large cardinals that lie between strong cardinals and supercompact cardinals. Among these will be Woodin cardinals, the cardinals we will use in determinacy proofs.

A cardinal \( \kappa \) is called superstrong if there is an elementary embedding \( j : V \prec M \) such that \( \text{crit}(j) = \kappa \) and \( V^M_{j(\kappa)} \subseteq M \). Superstrong cardinals, like strong cardinals, can be characterized in terms of extenders:

**Theorem 6.2.4.** ([Dodd, 1982]) For cardinals \( \kappa \) the following are equivalent:

(a) \( \kappa \) is superstrong.

(b) There is a \( \lambda > \kappa \) and a \((\kappa, \lambda)\)-extender \( E \) such that \( \text{strength}(E) \geq \lambda = i_E(\kappa) \).

**Proof.** If \( E \) witnesses that (b) holds, then clearly \( i_E \) witnesses that \( \kappa \) is superstrong.

Suppose that \( j : V \prec M \) witnesses that \( \kappa \) is superstrong. Let \( E \) be the \((\kappa, j(\kappa))\)-extender derived from \( j \). Let \( k : \text{Ult}(V; E) \prec M \) be defined as on page 328. By Lemma 6.1.10, \( k \upharpoonright j(\kappa) \) is the identity.

We now apply Lemma 6.1.11 with \( \lambda = j(\kappa) \). Since \( j(\kappa) \) is a strong limit cardinal in \( M \), the hypotheses of Lemma 6.1.11 hold for every \( \eta < j(\kappa) \). The lemma thus yields for every \( \eta < j(\kappa) \) that \( V^M_\eta = V^\text{Ult}(V;E)_\eta \). It follows that \( V^M_{j(\kappa)} = V^\text{Ult}(V;E)_{j(\kappa)} \).
To finish the proof, we need only show that \( i_E(\kappa) = j(\kappa) \). Part (b) of Lemma 6.1.10 gives that \( k(i_E(\kappa)) = j(\kappa) \). Since \( i_E(\kappa) \leq k(i_E(\kappa)) \) and since \( k(\alpha) = \alpha \) for all \( \alpha < j(\kappa) \), this implies that \( i_E(\kappa) = j(\kappa) \). \( \square \)

The next lemma will be used in proving that supercompactness is essentially a stronger property than superstrength. But it—and variants of it—will also be useful on other occasions.

To state the lemma, we need to introduce the analogue for extenders of the \( \text{Ult}(\mathcal{M}; \mathcal{U}) \) of Chapter 3. Suppose that \( \mathcal{M} \) is a transitive class model of ZFC and that \( E \) is a \( (\kappa, \lambda) \)-extender in \( \mathcal{M} \), i.e. that \( E \in \mathcal{M} \) and \( \mathcal{M} \models "E \text{ is a } (\kappa, \lambda)\text{-extender}". Then we can form what is in \( \mathcal{M} \) the ultrapower of \( \mathcal{M} \) with respect to \( E \). The universe of this ultrapower consists of equivalence classes (modified \( \text{à la} \) Scott) of pairs \( \langle a, f \rangle \), where \( a \in [\lambda]^{<\omega} \) and \( f \in \mathcal{M} \) is such that \( f : [\kappa]^{|a|} \to \mathcal{M} \). Let us denote the class of all such pairs by \( \mathcal{D}_E^M \), and let us denote the equivalence class of \( \langle a, f \rangle \) by \( [\langle a, f \rangle]_E \). The relation of the ultrapower, which we call \( \in_E^M \), is given by

\[
[a, f]^M_E \in [b, g]^M_E \iff \{ z \in [\kappa]^{|a\cup b|} \mid f(z_{a\cup b}) \in g(z_{b\cup a}) \} \in E_{a\cup b}.
\]

The ultrapower we will denote by

\[
\prod_E^M (\mathcal{M}; \in).
\]

By ZFC in \( \mathcal{M} \), this ultrapower is well-founded and set-like, and so we have a unique \( \pi_E^M : \prod_E^M (\mathcal{M}; \in) \prec (\text{Ult}(\mathcal{M}; E); \in) \). We also have the canonical elementary embedding \( i_E^M : \mathcal{M} \prec \text{Ult}(\mathcal{M}; E) \).

**Lemma 6.2.5.** Let \( \mathcal{M} \) be a transitive class model of ZFC. Let \( E \) be a \( (\kappa, \lambda) \)-extender such that \( E \in \mathcal{M} \). (This implies in particular that \( V_{\kappa+1}^M = V_{\kappa+1} \).) Let \( \zeta \geq \kappa \) be such that \( V_{\zeta+1}^M = V_{\zeta+1} \). Then

(i) \( \mathcal{M} \models "E \text{ is an extender}"; \\
(ii) \( (\forall \alpha \leq \zeta^+) i_E^M(\alpha) = i_E(\alpha) \); in particular, \( i_E^M(\kappa) = i_E(\kappa) \); \\
(iii) \( V_{i_E(\zeta)+1}^{\text{Ult}(\mathcal{M}; E)} = V_{i_E(\zeta)+1}^{\text{Ult}(\mathcal{M}; E)} \); hence \( V_{i_E(\kappa)+1}^{\text{Ult}(\mathcal{M}; E)} = V_{i_E(\kappa)+1}^{\text{Ult}(\mathcal{M}; E)} \).
Proof. That clauses (1)–(4) in the definition of an extender hold for $E$ in $M$ follows from the facts that $[\lambda]^{<\omega} \subseteq M$ and $V_{\kappa+1} \subseteq M$. Clause (4) can be proved either directly, using the absoluteness of wellfoundedness of trees, or indirectly, using Lemma 6.1.5.

(ii) and (iii) follow from the fact that, for $n \in \omega$, $V$ and $M$ have exactly the same functions $f : [\kappa]^n \to \zeta^+$ and $g : [\kappa]^n \to V_{\zeta+1}$. (Such functions $f$ can be coded by a wellordering $R$ of $\zeta$ of order type $\sup (\text{range}(f))$ and a $\tilde{g} : [\kappa]^n \to \zeta$. The pair $\langle R, \tilde{g} \rangle$ can be coded by a $g : [\kappa]^n \to V_{\zeta+1}$. Such a $g$ can in turn be coded by an element of $V_{\zeta+1} = V_{\zeta+1}^M$.) $\square$

The following lemma is possibly due to Dodd.

**Theorem 6.2.6.** Let $\kappa$ be $2^\kappa$-supercompact. Then there is a uniform normal ultrafilter $U$ on $\kappa$ such that

$$\{ \alpha < \kappa \mid \alpha \text{ is superstrong} \} \in U.$$  

**Proof.** Let $j : V \prec M$ witness that $\kappa$ is $2^\kappa$-supercompact. Let $E$ be the $(\kappa, j(\kappa))$-extender derived from $j$.

For each $a \in [j(\kappa)]^{<\omega}$,

$$E_a = \{ X \subseteq [\kappa]^a \mid a \in j(X) \}.$$  

Now $j \restriction \bigcup_{n \in \omega} \mathcal{P}([\kappa]^n)$ is a subset of $M$ of size $2^\kappa$ and is therefore a member of $M$. It follows that $E \in M$.

As in the proof of Theorem 6.2.4, we get that $i_E(\kappa) = j(\kappa)$ and that $V_{j(\kappa)}^M = V_{j(\kappa)}^{\text{Ult}(V;E)}$. Lemma 6.2.5 gives that $E$ is an extender in $M$, that $i_E(\kappa) = i_M(\kappa)$, and that $V_{i_E(\kappa)}^{\text{Ult}(M;E)} = V_{i_M(\kappa)}^{\text{Ult}(V;E)}$. Putting these facts together, we get that $V_{i_E(\kappa)}^{\text{Ult}(M;E)} = V_{i_M(\kappa)}^{\text{Ult}(V;E)}$. But this means that $M \models \kappa \text{ is superstrong.}$

Let $U = \{ X \subseteq \kappa \mid \kappa \in j(X) \}$. By Lemma 3.2.13 we know that $U$ is a uniform normal ultrafilter on $\kappa$. For $X = \{ \alpha < \kappa \mid \alpha \text{ is superstrong} \}$, we have shown that $\kappa \in j(X)$; hence $X \in U$. $\square$

**Remark.** One thing the theorem does not show is that if $\kappa$ is $2^\kappa$-supercompact then $\kappa$ is superstrong. Assuming that the existence of supercompact cardinals is consistent with ZFC, one can show that it is also consistent with ZFC.
that there is a supercompact cardinal that is not superstrong. (See exercise 6.2.1.) One the other hand, it is trivial that every supercompact cardinal is strong.

For some time, nothing interesting was known between strong and superstrong cardinals. Then Saharon Shelah, in weakening the hypothesis of results of [Foreman et al., 1988] and of related theorems (see [Shelah and Woodin, 1990]), discovered a significant intermediate class of large cardinals.

For any cardinal \( \kappa \) and any \( f: \kappa \to \kappa \), let us say that \( \kappa \) is *Shelah for* \( f \) if there is a \( j: V \prec M \) such that \( M \) is transitive, \( \text{crit}(j) = \kappa \), and \( V_{j(f)(\kappa)} \subseteq M \). A cardinal \( \kappa \) is *Shelah* if, for every \( f: \kappa \to \kappa \), \( \kappa \) is Shelah for \( f \).

A routine argument shows that supercompactness is a stronger property than that of being a Shelah cardinal:

**Theorem 6.2.7.** Let \( \kappa \) be superstrong. Then \( \kappa \) is Shelah and there is a uniform normal ultrafilter \( \mathcal{U} \) on \( \kappa \) such that

\[ \{ \alpha < \kappa \mid \alpha \text{ is Shelah} \} \in \mathcal{U}. \]

**Proof.** Let \( j \) witness that \( \kappa \) is superstrong.

To prove that \( \kappa \) is Shelah, let \( f: \kappa \to \kappa \). Since \( (j(f))(\kappa) < j(\kappa) \), we have that

\[ V_{j(f)(\kappa)} \subseteq V_j(\kappa) \subseteq M. \]

For the second assertion of the theorem, we proceed as in the proof of Theorem 6.2.6. We show that \( M \models \text{"} \kappa \text{ is Shelah}. \)" Just as in the proof of Theorem 6.2.6, this suffices. Let \( f: \kappa \to \kappa \) and set

\[ \lambda = \max \{ \kappa + 1, (j(f))(\kappa) + 1, |V_{j(f)(\kappa)}| \}. \]

(Note that \( |V_{j(f)(\kappa)}| = |V_{j(f)(\kappa)}^M| \).) Let \( E \) be the \( (\kappa, \lambda) \)-extender derived from \( j \). Let \( k: \text{Ult}(V; E) \prec M \) be the canonical embedding, i.e. let \( k \) be defined as on page 328. By Lemma 6.1.10, \( k \rest \lambda \) is the identity. This implies that \( k((j(f))(\kappa)) = (j(f))(\kappa) \). But

\[ k((i_E(f))(\kappa)) = (k(i_E(f)))(k(\kappa)) = (j(f))(k(\kappa)) = (j(f))(\kappa), \]

and so \( (i_E(f))(\kappa) = (j(f))(\kappa) \). By Lemma 6.1.11, \( V^\text{Ult}(V; E)_{j(f)(\kappa)} = V^M_{j(f)(\kappa)} \). By Lemma 6.2.5, we have that \( E \) is an extender in \( M \), that \( i_E^M(f) = i_E(f) \) (and
so these functions agree on the argument \( \kappa \), and that \( V_{\check{U}}^{\text{Ult}(M,E)} = V_{\check{U}}^{\text{Ult}(V;E)} \).

Combining these facts we get that
\[
(i_E^M(f))(\kappa) = (j(f))(\kappa); \\
V_{\check{U}}^{\text{Ult}(M,E)}(i_E^M(f)(\kappa)) = V_{\check{U}}^{\text{Ult}(V;E)}(i_E^M(f)(\kappa)).
\]

Thus \( i_E^M \) witnesses in \( M \) that \( \kappa \) is Shelah for \( f \).

Hugh Woodin discovered a weakening of the concept of Shelah cardinals that has turned out to be extremely important. For any cardinal \( \kappa \) and any \( f : \kappa \to \kappa \), \( \kappa \) is Woodin for \( f \) if there are \( \delta < \kappa \) and \( j : V \prec M \) such that \( M \) is transitive, \( \delta \) is closed under \( f \), \( \text{crit}(j) = \delta \), and \( V_{\check{U}}(\delta)(\delta) \subseteq M \). A cardinal \( \kappa \) is Woodin if, for every function \( f : \kappa \to \kappa \), \( \kappa \) is Woodin for \( f \).

The next two theorems, both known to Woodin, show how Woodin cardinals sit within the large cardinal hierarchy.

**Theorem 6.2.8.** Let \( \kappa \) be Shelah. Then \( \kappa \) is Woodin and there is a uniform normal ultrafilter \( \mathcal{U} \) on \( \kappa \) such that
\[
\{ \alpha < \kappa \mid \alpha \text{ is Woodin} \} \in \mathcal{U}.
\]

**Proof.** Let \( f : \kappa \to \kappa \). Define \( g : \kappa \to \kappa \) by
\[
g(\alpha) = \max \{ \alpha + 2, f(\alpha) + 1, |V(\alpha)| \} + 1.
\]

Let \( j : V \prec M \) witness that \( \kappa \) is Shelah for \( g \). Let
\[
\lambda = \max \{ \kappa + 1, (j(\kappa))(\kappa) + 1, |V(j(\kappa))| \}.
\]

Let \( E \) be the \((\kappa, \lambda)\)-extender derived from \( j \). Since \( E : [\lambda]^{\omega} \to V_{\kappa+2} \), it is easy to see that \( E \) can be coded by an element of \( V_{\max\{\lambda, \kappa+2\}+1} \subseteq V_{j(\kappa))}(\kappa) \). Thus \( E \) belongs to \( M \). Using Lemmas 6.1.10, 6.1.11, and 6.2.5 as in the preceding two proofs, we get that \( E \) is an extender in \( M \), that \( (i_E^M(f))(\kappa) = (j(f))(\kappa) \), and that \( V_{\check{U}}^{\text{Ult}(M,E)}(i_E^M(f)(\kappa)) = V_{\check{U}}^{\text{Ult}(M,E)}(i_E^M(f)(\kappa)) \). Since \( (j(f)) \upharpoonright \kappa = f \), we have that \( \kappa \) is closed under \( j(f) \) and that \( (i_E^M(j(f))(\kappa) = (i_E^M(f))(\kappa) \). The latter of these facts gives, since \( V_{\check{U}}^{\text{Ult}(M;E)}(i_E^M(j(f)))(\kappa) \subseteq \text{Ult}(M;E) \), that \( V_{\check{U}}^{\text{Ult}(M;E)}(i_E^M(j(f)))(\kappa) \subseteq \text{Ult}(M;E) \). Thus \( \kappa \) and \( i_E^M \) witness in \( M \) that \( j(\kappa) \) is Woodin for \( j(f) \). By the elementarity of \( j \), we get that in \( V \) there is an extender \( F \) such that \( \text{crit}(i_E) \) and \( i_E \) witness that \( \kappa \) is Woodin for \( f \). Since \( f \) was arbitrary, we have shown that \( \kappa \) is Woodin.
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Now $E \in V_{j(\kappa)}^M$, so the elementarity of $j$ gives the stronger fact that there is an extender $F \in V_\kappa$ such that $\text{crit}(i_F)$ and $i_F$ witness that $\kappa$ is Woodin for $f$. But such an $F$ belongs to $M$. Moreover Lemma 6.2.5 implies that $i_F^M(f) = i_F(f)$ and $V^\text{Ult}(M;F)_\kappa = V^\text{Ult}(V;F)_\kappa$. Hence $(i_F^M)$ and $(i_F^M)$ witness in $M$ that $\kappa$ is Woodin for $f$. The second assertion of the Theorem follows as in the two preceding proofs.

Woodinness is different from the other large cardinal properties we have studied in this chapter in that it is not characterized in terms of elementary embeddings whose critical point is the cardinal itself. Indeed a Woodin cardinal need not be measurable. (See Exercise 6.3.2.) Nevertheless we have the following result, which shows that Woodinness is a stronger property than strength.

**Theorem 6.2.9.** Let $\kappa$ be Woodin. Then

1. $\kappa$ is inaccessible;
2. The set of cardinals $\delta < \kappa$ such that $(\forall \eta) (\delta < \eta < \kappa \rightarrow \delta \text{ is } \eta\text{-strong})$

is unbounded in $\kappa$.

**Proof.** (1) To show that $\kappa$ is regular, suppose that $\gamma < \kappa$ and that $f : \gamma \rightarrow \kappa$. Set

\[
g(0) = \gamma; \\
g(1 + \alpha) = f(\alpha) \text{ for } \alpha < \gamma; \\
g(\alpha) = 0 \text{ for } \gamma \leq \alpha.
\]

Since $\kappa$ is Woodin, there must in particular be a non-zero ordinal $\beta < \kappa$ that is closed under $g$. But any such $\beta$ must be larger than every element of the range of $f$.

To show that $\kappa$ is a strong limit cardinal, let $\gamma < \kappa$ be a cardinal number. Let $f : \kappa \rightarrow \kappa$ be such that $f(0) = \gamma$. Let $\delta < \kappa$ and $j : V \prec M$ witness that $\kappa$ is Woodin for $f$. Then $\delta > \gamma$ and $\delta$ is measurable. Hence $\delta > 2^\gamma$.

(2) Assume for a contradiction that there is a $\beta < \kappa$ such that

$(\forall \delta)(\beta \leq \delta < \kappa \rightarrow (\exists \eta)(\delta < \eta < \kappa \land \delta \text{ is not } \eta\text{-strong}))$. 

Without loss of generality we may take $\beta$ to be a limit ordinal. For ordinals $\alpha$ such that $\beta \leq \alpha < \kappa$, let $\eta(\alpha)$ be the least $\eta > \alpha$ such that $\alpha$ is not $\eta$-strong. Let

$$g(\alpha) = \begin{cases} \beta & \text{if } \alpha < \beta; \\ \max\{\eta(\alpha) + 1, |V_{\eta(\alpha)}|\} + 1 & \text{if } \beta \leq \alpha < \kappa. \end{cases}$$

Note that $g(\alpha) \geq \alpha + 2$ for all $\alpha$. Since $\kappa$ is inaccessible by (1), we have that $g : \kappa \to \kappa$. Let $\delta$ and $j : V \prec M$ witness that $\kappa$ is Woodin for $g$. Clearly $\delta > \beta$. Let

$$\lambda = \max\{(j(\eta))(\delta) + 1, |V_{(j(\eta))(\delta)}|\}.$$

Let $E$ be the $(\delta, \lambda)$-extender derived from $j$. Arguing just as in the proof of Theorem 6.2.8, we get that $E \in M$ and so, by Lemma 6.2.5, that $E$ is an extender in $M$. By Lemmas 6.2.5 and 6.1.11,

$$V_{(j(\eta))(\delta)}^{\Ult(M;E)} = V_{(j(\eta))(\delta)}^{\Ult(V;E)} = V_{(j(\eta))(\delta)}^{M}.$$

Thus $i_{E}^{M}$ witnesses in $M$ that $\delta$ is $(j(\eta))(\delta)$-strong. This contradicts the elementarity of $j$. $\square$

Remark. Theorem 6.2.9 implies that if $\kappa$ is Woodin then $V_{\kappa} \models \text{ZFC + "There is a proper class of strong cardinals."}$ Theorem 6.3.1 will shed more light on the relation between strong cardinals and Woodin cardinals.

We will develop the theory of Woodin cardinals in the next section. In the rest of this section, we will briefly discuss some very strong large cardinal properties and in doing so prove that, for example, the condition $\lambda < j(\kappa)$ in the definition of $\lambda$-supercompactness is unnecessary.

The following definitions and theorem appear in [Solovay et al., 1978]. For $n \in \omega$, a cardinal $\kappa$ is said to be $n$-huge if there is a $j : V \prec M$ such that $M$ is transitive, $\text{crit}(j) = \kappa$, and

$$\kappa_{n}M \subseteq M,$$

where $\kappa_{n} = j_{0,n}(\kappa)$. Being 0-huge is the same as being measurable. Cardinals that are 1-huge are simply called huge. Huge cardinals were introduced in the early 1970’s by Kenneth Kunen.

As with $\lambda$-supercompactness, $n$-hugeness can be characterized in terms of ultrafilters. If $A$ is any set, an ultrafilter $\mathcal{U}$ on $\mathcal{P}(A)$ is fine if

$$(\forall a \in A)\{x \subseteq A \mid a \in x\} \in \mathcal{U}.$$
and is normal if, for all \( f : \mathcal{P}(A) \to A \), if

\[
\{ x \subseteq A \mid f(x) \in x \} \in \mathcal{U}
\]

then \( f \) is constant on a set in \( \mathcal{U} \).

If \( x \) is a set of ordinals, let \( \text{ot}(x) \) be the order type of \( x \).

**Theorem 6.2.10.** ([Solovay et al., 1978]) If \( n \in \omega \) and \( \kappa \) is an infinite cardinal, then the following are equivalent:

1. \( \kappa \) is \( n \)-huge.
2. There are cardinals \( \kappa = \lambda_0 < \cdots < \lambda_n = \lambda \) and there is a \( \kappa \)-complete fine normal ultrafilter \( \mathcal{U} \) on \( \mathcal{P}(\lambda) \) such that

\[
(\forall i < n) \{ x \subseteq \lambda \mid \text{ot}(x \cap \lambda_{i+1}) = \lambda_i \} \in \mathcal{U}.
\]

**Proof.** Let \( j : V \prec M \) witness that \( \kappa \) is \( n \)-huge. Let \( \kappa_i = j_{0,i}(\kappa) \) for \( i \leq n+1 \). Let

\[
\mathcal{U} = \{ X \subseteq \mathcal{P}(\kappa_n) \mid j''\kappa_n \in j(X) \}.
\]

Evidently \( \mathcal{U} \) is an ultrafilter on \( \mathcal{P}(\kappa_n) \). The proofs that \( \mathcal{U} \) is \( \kappa \)-complete, fine, and normal are exactly like the corresponding parts of the proof of Theorem 6.2.1. Fix \( i < n \). We have that

\[
j''\kappa_n \cap j(\kappa_{i+1}) = j''\kappa_n \cap \kappa_{i+2} = j''\kappa_{i+1};
\]

\[
\text{ot}(j''\kappa_{i+1}) = \kappa_{i+1} = j(\kappa_i).
\]

Thus

\[
j''\kappa_n \in j(\{ x \subseteq \kappa_n \mid \text{ot}(x \cap \kappa_{i+1}) = \kappa_i \}).
\]

But this means that

\[
\{ x \subseteq \kappa_n \mid \text{ot}(x \cap \kappa_{i+1}) = \kappa_i \} \in \mathcal{U}.
\]

Thus we can set \( \lambda_i = \kappa_i \) for each \( i < n \) and satisfy all the clauses of condition (2).

Now suppose that \( \kappa = \lambda_0 < \ldots < \lambda_n = \lambda \) and \( \mathcal{U} \) satisfy (2). We will show that \( i_\mathcal{U} : V \prec \text{Ult}(V; \mathcal{U}) \) witnesses that \( \kappa \) is \( n \)-huge. By Lemma 3.2.10, \( \text{crit}(i_\mathcal{U}) \geq \kappa \). For \( \gamma \leq \lambda \) let \( \text{id}_\gamma : \mathcal{P}(\lambda) \to V \) be given by

\[
\text{id}_\gamma(x) = x \cap \gamma.
\]
By an argument like the one in the corresponding part of the proof of Theorem 6.2.1, we can show that
\[(\forall i < n) i''u \gamma = \pi_u([\text{id}])\].

One consequence of this is that \(i''u \in \text{Ult}(V; U)\); by another argument like one in the proof of Theorem 6.2.1, this implies that \(\lambda(\text{Ult}(V; U)) \subseteq \text{Ult}(V; U)\). Another consequence is that \(i_u(\lambda_i) = \lambda_{i+1}\) for all \(i < n\). This follows by Theorem 3.2.5, the elementarity of \(\pi_u\), the hypothesis that \(\{x \subseteq \lambda \mid \text{ot}(x \cap \lambda_{i+1}) = \lambda_i\} \in U\), and the fact that \(\text{ot}(i''_u \lambda_{i+1}) = \lambda_{i+1}\). Since, in particular, \(\lambda_i = \lambda_1\), the proof that \(\text{crit}(i_u) = \kappa\) is now complete. Moreover we have that \(\lambda_i = (i_u)_{0,i}(\kappa)\), so the proof of the theorem is complete. \(\square\)

The property \(n\)-hugeness is related to supercompactness rather than to strength. The large cardinal property that bears an analogous relation to strength can be defined by replacing the condition \(\kappa^\kappa M \subseteq M\) in the definition of \(n\)-hugeness by the weaker condition \(V^{\kappa^\kappa} \subseteq M\). This property, which has no standard name, is weaker than \(n\)-hugeness. On the other hand, it implies \((n-1)\)-hugeness when \(n > 0\). (See Exercise 6.2.4.)

The following observation in [Kunen, 1978] shows that hugeness is a more powerful large cardinal property than supercompactness.

**Theorem 6.2.11.** Let \(\kappa\) be huge. Then there is a uniform normal ultrafilter \(U\) on \(\kappa\) such that \(\{\alpha < \kappa \mid (\forall \beta < \kappa) \alpha \text{ is } \beta\text{-supercompact}\} \in U\).

**Proof.** Let \(j : V < M\) witness that \(\kappa\) is huge. Let \(\lambda < j(\kappa)\). Then \(j\) witnesses that \(\kappa\) is \(\lambda\)-supercompact. By Theorem 6.2.1, let \(V\) be a \(\kappa\)-complete fine normal ultrafilter on \(P_\kappa(\lambda)\). It is clear that \(V \in M\) and that \(M \models \text{"}V\text{ is a } \kappa\text{-complete fine normal ultrafilter on } P_\kappa(\lambda)\text{."}\) Hence \(\kappa\) is \(\lambda\)-supercompact in \(M\). The conclusion of the theorem follows as in the proof of Theorem 6.2.6. \(\square\)

**Remark.** The proof of the theorem would go through unchanged if we weakened the \(\kappa_1 M \subseteq M\) part of the hugeness hypothesis to \(V_{\kappa_1} \subseteq M\). A number of other large cardinal properties have been studied that lie between hugeness and supercompactness. See [Solovay et al., 1978].

The notion of \(n\)-huge cardinals cries out for generalization to the transfinite. One could define \(\kappa\) to be \(\alpha\text{-huge}\), for \(\alpha\) an arbitrary ordinal, if there...
is a \( j : V \prec M \) such that \( M \) is transitive, \( \text{crit}(j) = \kappa \), and \( \kappa^\omega M \subseteq M \), where \( \kappa_\alpha = j_\alpha(\kappa) \). Unfortunately [Kunen, 1971] shows that even \( \omega \)-huge cardinals in this sense do not exist. Kunen’s proof uses the following result of [Erdős and Hajnal, 1966]:

**Theorem 6.2.12.** Let \( \lambda \) be an infinite cardinal. There is a function \( f : [\lambda]^{\omega} \to \lambda \) such that

\[
(\forall X \subseteq \lambda)(|X| = \lambda \rightarrow f''(X) = \lambda).
\]

([\( X \) is the set of all countably infinite subsets of \( X \).])

**Proof.** The proof we give is from [Galvin and Prikry, 1976]. Let \( E \) be the set of all elements \( x \) of \([\lambda]^{\omega}\) such that \( \text{ot}(x) = \omega \). For \( x \) and \( y \) belonging to \( E \), say that \( x \sim y \) if the symmetric difference of \( x \) and \( y \) is finite. For \( x \in E \), let \( g([x]) \subseteq x \) for each \( x \in E \). Let \( h : E \to \lambda \) be given by

\[
h(x) = \begin{cases} 
\text{the greatest element of } g([x]) \setminus x \text{ if } g([x]) \not\subseteq x; \\
0 \text{ otherwise.}
\end{cases}
\]

We will show that there is an \( A \subseteq \lambda \) such that \( |A| = \lambda \) and such that

\[
(\forall X \subseteq A)(|X| = \lambda \rightarrow h''([X]^{\omega} \cap E) \supseteq A).
\]

Given such an \( A \), one can easily construct an \( f \) with the required properties.

Suppose that no such \( A \) exists. We construct a strictly increasing sequence \( \langle \alpha_i \mid i \in \omega \rangle \) of elements of \( \lambda \) and a sequence \( \langle B_i \mid i \in \omega \rangle \) of subsets of \( \lambda \) of cardinality \( \lambda \). Let \( B_0 = \lambda \). Given \( B_i \), let \( \alpha_i \in B_i \) and \( B_{i+1} \subseteq B_i \setminus \alpha_i + 1 \) be such that \( |B_{i+1}| = \lambda \) and

\[
\alpha_i \notin h''([B_{i+1}]^{\omega} \cap E).
\]

The existence of such a pair follows easily from the nonexistence of \( A \). Now let \( x = \{\alpha_i \mid i \in \omega \} \). Let \( \alpha_n \) be the least element of \( x \) that is larger than every element of the symmetric difference of \( g([x]) \) and \( x \). Let \( y = \{\alpha_i \mid i > n\} \). Now \( h(y) = \alpha_n \), since \( \alpha_n \) is the greatest element of \( g([x]) \setminus y \). But this is a contradiction, for \( y \subseteq B_{n+1} \).

\( \square \)

Now we are ready to prove Kunen’s theorem.
**Theorem 6.2.13.** ([Kunen, 1971]) Let \( j : V \prec M \) with \( M \) transitive. Let \( \kappa = \text{crit}(j) \). Let \( \lambda = j_0, \omega(\kappa) \). (\( \lambda \) can also be characterized as the least fixed point of \( j \) that is greater than \( \kappa \).) Then \( j'' \lambda \notin M \).

**Proof.** Let \( f : [\lambda]^{\omega} \rightarrow \lambda \) be given by Theorem 6.2.12. By the elementarity of \( j \), if \( X \subseteq \lambda \) belongs to \( M \) and if \( |X| = \lambda \), then \( j(f''[X]^{\omega}) = \lambda \). We will prove that \( j'' \lambda \notin M \) by showing that \( j(f'')[j'' \lambda]^{\omega} \neq \lambda \).

Since \( \omega < \kappa = \text{crit}(j) \), it is easy to see that
\[
(\forall x \in [\lambda]^{\omega}) j(x) = \{j(\alpha) \mid \alpha \in x\}.
\]
In particular this means that every \( y \in [j'' \lambda]^{\omega} \) belongs to the range of \( j \upharpoonright [\lambda]^{\omega} \).

If \( y \in [j'' \lambda]^{\omega} \) and \( y = j(x) \), then \( (j(f))(y) = (j(f))(j(x)) = j(f(x)) \). This shows that
\[
j(f'')[j'' \lambda]^{\omega} \subseteq j'' \lambda \neq \lambda.
\]
(That \( j'' \lambda \neq \lambda \) follows from the fact that \( \kappa \in \lambda \setminus j'' \lambda \).) □

For other proofs of Theorem 6.2.13, see §23 of [Kanamori, 1994].

Kunen’s theorem and its proof give some more negative results. For \( j : V \prec M \) or \( j : V_\eta \prec M \) with \( M \) transitive, let us for the moment denote by \( \lambda \) the first fixed point of \( j \) greater than \( \text{crit}(j) \), if it exists.

**Theorem 6.2.14.** (1) If \( j : V \prec M \), \( j \) is not the identity, and \( M \) is transitive, then (a) \( V_{\lambda+1} \notin M \) and (b) \( \omega(V_\lambda) \subseteq M \). (2) There is no \( j : V_{\lambda+2} \prec V_{\lambda+2} \).

**Proof.** (1)(a) follows immediately from Theorem 6.2.13. (1)(b) follows from the fact that, since \( \text{cf}(\lambda) = \omega \), every subset of \( \lambda \) is the union of countably many elements of \( V_\lambda \). (Of course, (1)(b) implies (1)(a).) To verify (2), note that the \( f \) of the proof of Theorem 6.2.13 belongs to \( V_{\lambda+2} \). □

No inconsistency has been derived from any of the following (where we continue to use “\( \lambda \)” as above, so that the embeddings are implicitly asserted to be non-trivial):

(a) There is a \( j : V_\lambda \prec V_\lambda \).
(b) There is a \( j : V \prec M \) with \( M \) transitive and \( V_\lambda \subseteq M \).
(c) There is a \( j : V_{\lambda+1} \prec V_{\lambda+1} \).
(d) There is a \( j : L(V_{\lambda+1}) \prec L(V_{\lambda+1}) \).
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These assertions are listed in order of (strictly) increasing strength. [Martin, 1980] proved the determinacy of all Π^1_2 games from a hypothesis intermediate between (a) and (b). Hugh Woodin (unpublished) subsequently proved AD^{L(R)} and much more from (d). The determinacy proofs we will give in Chapters 8 and 9 will have, of course, much weaker hypotheses.

No inconsistency with ZFC minus Choice is known for the existence of an elementary embedding of the whole universe into itself.

Kunen’s results, as he noted, make it possible to simplify the definitions of η-strong and λ-supercompact in the way mentioned earlier:

Theorem 6.2.15. Let κ be a cardinal number.

(1) For cardinal numbers λ ≥ κ, κ is λ-supercompact if and only if there is a j : V ≺ M such that M is transitive, crit (j) = κ, and λ^M ⊆ M.

(2) For ordinal numbers η > κ, κ is η-strong if and only if there is a j : V ≺ M such that M is transitive, crit (j) = κ, and V_η ⊆ M.

Proof. (1) Let λ ≥ κ. Clearly we need only prove the “if” part. Let j : V ≺ M be as in the statement of (1). For ordinals α, let κ_α = j_0,α(κ) and let M_α = j_0,α(V). (See §3.3.) By Theorem 6.2.13, λ < κ_ω. Let n be the least number such that λ < κ_n. Since j_{i, i+1} = j_{0, i}(j) for each i, the elementarity of j_{0, i} implies that (M_i ∩ λ^M_{i+1}) ⊆ M_{i+1} and so that (M_i ∩ λ^M_{i+1}) ⊆ M_{i+1}.

By induction we then get that λ^M_n ⊆ M_n. Thus j_0, n : V ≺ M_n witnesses that κ is λ-supercompact.

(2) Let η > κ. As for (1) we need only prove the “if” part. Let j : V ≺ M be as in the statement of (2). Define κ_α and M_α, α ∈ Ord, as in the proof of (1). By Theorem 6.2.14, we know that η ≤ κ_ω. An argument as in the proof of (1) shows that V_η ⊆ M_i for each i ∈ ω. Since η ≤ κ_ω, this implies that V_η ⊆ M_ω. Since κ_ω = crit (j_{ω, ω+1}), we finally get that V_η ⊆ M_{ω+1}. Thus j_0, ω+1 : V ≺ M_{ω+1} witnesses that κ is η-strong. (If η < κ_ω then j_0, n will also work for any n such that η < κ_n.)

Exercise 6.2.1. (a) Show that if κ is superstrong then there are measurable cardinals larger than κ.

(b) Let κ be any cardinal number. Show that either κ is not superstrong or else there is an inaccessible δ > κ such that V_δ |= “κ is not superstrong.”

(c) Show that if κ is supercompact and δ > κ is inaccessible then V_δ |= “κ is supercompact.”

(d) Prove that if ZFC + “There is a supercompact cardinal” is consistent then so is ZFC + “There is a supercompact cardinal that is not superstrong.”
Hint. For (a) first show that $\kappa$ is measurable in $M$, where $j : V \prec M$ witnesses that $\kappa$ is superstrong. Use this to show that $\kappa$, and hence, $j(\kappa)$, is a limit of measurable cardinals. For (c) use Theorem 6.2.1.

**Exercise 6.2.2.** Let $\kappa$ be a regular cardinal. Assume that the set of Woodin cardinals smaller than $\kappa$ is stationary in $\kappa$. (See Exercise 3.2.7 for the definition of stationary.) Prove that $\kappa$ is Woodin.

**Exercise 6.2.3.** Prove that every Woodin cardinal is Mahlo. (See Exercise 3.2.7.)

**Exercise 6.2.4.** Let $n \in \omega$ and let $\kappa$ be a cardinal number. Assume that there is a $j : V \prec M$ such that $M$ is transitive, $\text{crit}(j) = \kappa$, and $V_{j(\kappa)} \subseteq M$. Prove that $\kappa$ is $n$-huge.

Hint. The proof of the $(1) \Rightarrow (2)$ part of Theorem 6.2.10 goes through under our present hypotheses.

### 6.3 Equivalents of Woodinness

The main aim of this section is to prove a property of Woodin cardinals (actually an equivalent of Woodinness) that will be the basis for our constructions in the determinacy proofs of Chapter 8 and to prove the technical consequence of this property that is actually used in the constructions.

We begin by giving an equivalent of Woodinness that is very useful in applications and that throws into clear relief the relation between Woodin cardinals and strong cardinals.

If $A$ is any class, $\kappa$ is a cardinal, and $\eta > \kappa$ is an ordinal, then $\kappa$ is $\eta$-strong in $A$ if there is a $j : V \prec M$ such that

(i) $j$ witnesses that $\kappa$ is $\eta$-strong;

(ii) $j(A) \cap V_{\eta} = A \cap V_{\eta}$.

The following fact was surely first noticed by Woodin.

**Theorem 6.3.1.** Let $\kappa$ be any infinite cardinal number. The following are equivalent:

(1) $\kappa$ is Woodin.
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(2) \((\forall A \subseteq V_\kappa)(\exists \delta < \kappa)(\forall \eta)(\delta < \eta < \kappa \rightarrow \delta \text{ is } \eta\text{-strong in } A)\).

**Proof.** First suppose that \(\kappa\) satisfies (2). Let \(f : \kappa \rightarrow \kappa\). By (2) let \(\delta < \kappa\) be such that \(\delta\) is \(\eta\)-strong in \(f\) (i.e. in graph \((f)\)) for all \(\eta, \delta < \eta < \kappa\). For \(\beta \leq \delta\), let

\[ \eta_\beta = \max \{ \beta, f(\beta) \} + 3 \]

and let \(j_\beta : V \prec M_\beta\) witness that \(\delta\) is \(\eta_\beta\)-strong in \(f\). For any \(\beta \leq \delta\), we have that \(\langle \beta, f(\beta) \rangle \in f \cap V_{\eta_\beta}\) and so that \(\langle \beta, f(\beta) \rangle \in j_\beta(f) \cap V_{\eta_\beta}\). But this means that

\[(\forall \beta \leq \delta)(j_\beta(f))(\beta) = f(\beta) < \eta_\beta.\]

Taking \(\beta = \delta\), we deduce that \((j_\delta(f))(\delta) < \eta_\delta\). Hence

\[ V_{(j_\delta(f))(\delta)} \subseteq V_{\eta_\delta} \subseteq M_{\eta_\delta}. \]

To show that \(\delta\) and \(j_\delta\) witness that \(\kappa\) is Woodin for \(f\), we need only prove that \(\delta\) is closed under \(f\). For this, assume that \(\beta < \delta\). Then

\[ f(\beta) = (j_\delta(f))(\beta) = (j_\delta(f))(j_\delta(\beta)) = j_\delta(f(\beta)). \]

But this implies that \(f(\beta) < \delta\), for if \(f(\beta) \geq \delta\) then \(j_\delta(f(\beta)) \geq j_\delta(\delta) > \eta_\delta > f(\beta)\).

Now we turn to the proof that (1) implies (2). This will be similar to the proof of part (2) of Theorem 6.2.9, with one ingredient missing (the ordinal \(\beta\)) and another ingredient added. Suppose that \(\kappa\) is Woodin. Let \(A \subseteq V_\kappa\). Assume that (2) fails for \(A\). For \(\alpha < \kappa\) let \(\eta(\alpha)\) be the least \(\eta > \alpha\) such that \(\alpha\) is not \(\eta\)-strong in \(A\). For \(\alpha < \kappa\) let

\[ g(\alpha) = \max \{ \eta(\alpha) + 1, |V_{\eta(\alpha)}| \} + 1. \]

As with the analogous function in the proof of Theorem 6.2.9, we have that \(g(\alpha) \geq \alpha + 2\) for all \(\alpha\) and that \(g : \kappa \rightarrow \kappa\). Let \(\delta < \kappa\) and \(j : V \prec M\) witness that \(\kappa\) is Woodin for \(g\). Let

\[ \lambda = \max \{ (j(\eta))(\delta) + 1, |V_{(j(\eta))(\delta)}| \}. \]

Let \(E\) be the \((\delta, \lambda)\)-extender derived from \(j\). As in the proof of Theorem 6.2.9, we get that \(E\) is an extender in \(M\) and that

\[ V_{(j(\eta))(\delta)}^{\text{Ult}(M; E)} = V_{(j(\eta))(\delta)}^{\text{Ult}(V; E)} = V_{(j(\eta))(\delta)}^M. \]
To derive the contradiction that $i_E$ witnesses in $M$ that $\delta$ is $(j(\eta))(\delta)$-strong in $j(A)$, we need only show that

$$i^M_E(j(A)) \cap V^M_{(\eta)(\delta)} = j(A) \cap V^M_{(\eta)(\delta)}.$$ 

Let $k : \text{Ult}(V;E) < M$ be defined as on page 328. Since $(j(\eta))(\delta) < \lambda$, Lemma 6.1.10 implies that $(i_E(\eta))(\delta) = (j(\eta))(\delta)$. (This is proved using $k$ just as we showed, in the proof of Theorem 6.2.7, that $(i_E(f))(\kappa) = (j(f))(\kappa)$.) It follows that $(j(\eta))(\delta) < i_E(\delta) = (by \ Lemma \ 6.2.5) i^M_E(\delta)$. Since $A \cap V_\delta = j(A) \cap V_\delta$, we have that $i^M_E(A) \cap V^M_{(\eta)(\delta)} = i^M_E(j(A)) \cap V^M_{(\eta)(\delta)}$.

What we must prove is thus that

$$i^M_E(A) \cap V^M_{(\eta)(\delta)} = j(A) \cap V^M_{(\eta)(\delta)}.$$ 

By Lemma 6.1.10, $k \circ i_E = j$. By Lemma 6.1.11, $k \upharpoonright V^{\text{Ult}(V;E)}_{(\eta)(\delta)}$ is the identity. If $x \in V^M_{(\eta)(\delta)}$ then $x \in V^{\text{Ult}(V;E)}_{(\eta)(\delta)}$ and

$$x \in i_E(A) \iff k(x) \in k(i_E(A)) \iff x \in j(A).$$

We are thus finally reduced to showing that

$$i^M_E(A) \cap V^M_{(\eta)(\delta)} = i^M_E(A) \cap V^M_{(\eta)(\delta)}.$$ 

But this follows from Lemma 6.2.5. \qed

The $(1) \Rightarrow (2)$ half of Theorem 6.3.1 isn’t the full “$A$-strong” analogue of part $(2)$ of Theorem 6.2.9. The latter says that a certain set is unbounded in $\kappa$, while the former says only that the analogous set is non-empty. The full analogue is nevertheless true. The next theorem records this fact.

**Theorem 6.3.2.** Let $\kappa$ be any infinite cardinal number. The following are equivalent:

1. $\kappa$ is Woodin.
2. For all $A \subseteq V_\kappa$ the set of cardinals $\delta < \kappa$ such that

   $$(\forall \eta)(\delta < \eta < \kappa \rightarrow \delta \text{ is } \eta\text{-strong in } A)$$

   is unbounded in $\kappa$. 


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Proof. That (2) implies (1) follows from Theorem 6.3.1. That (1) implies (2) can be demonstrated by a routine combination of the proofs of the corresponding half of Theorem 6.3.1 and part (2) of Theorem 6.2.9. We leave this to the reader. \square

Remark. Theorem 6.3.2 remains true if “unbounded” is replaced by “stationary” in its statement. See Exercise 6.3.1. The hint for that exercise also indicates a way to prove (2) of Theorem 6.3.2 directly from (2) of Theorem 6.3.1.

From now through Theorem 6.3.8, we will be showing that Woodinness of \( \kappa \) is witnessed by embeddings coming from extenders in \( V_\kappa \), extenders whose ultrapowers may be taken to have certain closure properties. These results are pretty routine, but they should be attributed to Woodin if to anyone.

We begin with the following fact, which gives another useful strengthening of (2) of Theorem 6.3.1.

**Theorem 6.3.3.** Let \( \kappa \) be a strong limit cardinal. Let \( A \subseteq \kappa \). Let \( \delta < \eta < \kappa \) be such that \( \delta \) is \( \eta \)-strong in \( A \). Then there is an extender \( E \in V_\kappa \) such that \( i_E \) witnesses that \( \delta \) is \( \eta \)-strong in \( A \).

Proof. Let \( j : V \prec M \) witness that \( \delta \) is \( \eta \)-strong in \( A \). Let \( E \) be the \((\delta, |V_{\eta+1}|)\)-extender derived from \( j \). Lemma 6.1.11 implies that \( V^{\text{Ult}(V;E)}_{\eta} = V_{\eta}^M \) and gives the first and third equalities of the following chain.

\[
\begin{align*}
 i_E(A) \cap V_\eta &= k(i_E(A) \cap V_\eta) \\
 &= k(i_E(A)) \cap k(V_\eta) \\
 &= k(i_E(A)) \cap V_\eta \\
 &= j(A) \cap V_\eta \\
 &= A \cap V_\eta.
\end{align*}
\]

Here \( k \) is as usual. \square

Remark. The proof of the \((1) \Rightarrow (2)\) part of Theorem 6.3.1 could have been simplified very slightly if we had, in analogy with the proof of Theorem 6.3.3, defined \( \lambda \) as \( |V_{(j(\eta))(\delta+1)}| \). We chose instead to keep a closer correspondence with the proof of Theorem 6.2.9.

**Theorem 6.3.4.** Let \( \kappa \) be Woodin and let \( f : \kappa \rightarrow \kappa \). There is an extender \( E \in V_\kappa \) such that \( \text{crit}(i_E) \) and \( i_E \) witness that \( \kappa \) is Woodin for \( f \).
**Proof.** As in the first half (the \((2) \Rightarrow (1)\) half) of the proof of Theorem 6.3.1, let \(A = f\). Also let \(\delta, \eta, \) and \(j : V \prec M\) be as in the first half of the proof of Theorem 6.3.1. By Theorem 6.3.3, let \(E\) be an extender in \(V_\kappa\) such that \(i_E\) witnesses that \(\delta\) is \(\eta\)-strong in \(A\). The first half of the proof of Theorem 6.3.1 shows that \(\text{crit}(i_E)\) and \(i_E\) witness that \(\kappa\) is Woodin for \(f\).

\(\square\)

**Corollary 6.3.5.** Let \(\kappa\) be Woodin and let \(M\) be a transitive class model of ZFC such that \(V_\kappa \subseteq M\). Then \(M \models \text{"\(\kappa\) is Woodin."}\)

**Proof.** The corollary follows easily from Theorem 6.3.4 or from Theorems 6.3.1 and 6.3.3.

\(\square\)

In Chapter 9 we will need to know that we can demand, of \(\delta\) and \(j : V \prec M\) witnessing Woodinness of a cardinal \(\kappa\), that \(<\delta M \subseteq M\). In fact, we can demand even that \(\delta M \subseteq M\). In order to prove this, we need the following lemma.

**Lemma 6.3.6.** Let \(E\) be a \((\delta, \lambda)\)-extender such that \(\delta \lambda \subseteq \text{Ult}(V; E)\). Then \(\delta \text{Ult}(V; E) \subseteq \text{Ult}(V; E)\).

**Proof.** Let \(\langle a_\beta \mid \beta < \delta \rangle\) be elements of \(M\). For \(\beta < \delta\) let \(\pi_E([a_\beta, f_\beta]) = x_\beta\). The hypothesis that \(\delta \lambda \subseteq \text{Ult}(V; E)\) implies that \(\delta ([\lambda]^{<\omega}) \subseteq \text{Ult}(V; E)\). Hence \(\langle a_\beta \mid \beta < \delta \rangle \in \text{Ult}(V; E)\). Define \(g : <\delta([\delta]^{<\omega}) \to V\) whose values are functions as follows. For \(h \in <\delta([\delta]^{<\omega})\), set

(i) \(\text{domain}(g(h)) = \text{domain}(h)\);

(ii) \((g(h))(\beta) = f_\beta(h(\beta))\) for all \(\beta \in \text{domain}(h)\).

The function \(\langle a_\beta \mid \beta < \delta \rangle\) belongs to domain \((i_E(g))\). Thus

\[
\text{domain}((i_E(g))(\langle a_\beta \mid \beta < \delta \rangle)) = \delta
\]

and, for all \(\beta < \delta\), Lemma 6.1.12 gives that

\[
((i_E(g))(\langle a_\beta \mid \beta < \delta \rangle))(\beta) = (i_E(f_\beta))(a_\beta) = \pi_E([a_\beta, f_\beta]) = x_\beta.
\]

Thus \((i_E(g))(\langle a_\beta \mid \beta < \delta \rangle) = \langle x_\beta \mid \beta < \delta \rangle\).

\(\square\)
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Theorem 6.3.7. Let \( \kappa \) be inaccessible. Let \( \delta < \kappa \) be such that, for all \( \eta \) such that \( \delta < \eta < \kappa \), \( \delta \) is \( \eta \)-strong in \( A \). Then, for all \( \eta \) such that \( \delta < \eta < \kappa \), there is an extender \( E \in V_\kappa \) such that \( i_E \) witnesses that \( \delta \) is \( \eta \)-strong in \( A \) and such that \( \delta \) \( \Ult(V; E) \subseteq \Ult(V; E) \).

Proof. Let \( \delta < \eta < \kappa \). Let \( \lambda \) be a strong limit cardinal of cofinality \( > \delta \) such that \( \eta \leq \lambda < \kappa \). By Theorem 6.3.3, let \( \hat{E} \in V_\kappa \) be an extender such that \( i_{\hat{E}} \) witnesses that \( \delta \) is \( \lambda \)-strong in \( A \). Let \( E \) be the \((\delta, \lambda)\)-extender derived from \( i_{\hat{E}} \). Lemma 6.1.11 implies that \( V_\lambda \subseteq \Ult(V; E) \) and implies that \( i_E \) witnesses that \( \delta \) is \( \lambda \)-strong, and therefore \( \eta \)-strong, in \( A \). (See the proof of Lemma 6.3.3.) Since \( \text{cf}(\lambda) > \delta \), we have that \( \delta \lambda \subseteq V_\lambda \subseteq \Ult(V; E) \). By Lemma 6.3.6, we get that \( \delta \) \( \Ult(V; E) \subseteq \Ult(V; E) \). \( \Box \)

Theorem 6.3.8. Let \( \kappa \) be a Woodin cardinal and let \( f : \kappa \rightarrow \kappa \). There is an extender \( E \in V_\kappa \) such that \( \delta = \text{crit}(i_E) \) and \( E \) witness that \( \kappa \) is Woodin for \( f \) and such that \( \delta \) \( \Ult(V; E) \subseteq \Ult(V; E) \).

Proof. The proof is just like that of Theorem 6.3.4, except that we apply Theorem 6.3.7 instead of Theorem 6.3.3. \( \Box \)

We now turn to an equivalent of Woodinness that is rather technical but that will play a central role in the constructions of Chapter 8. The definitions and results that follow are, with minor changes, from [Martin and Steel, 1988] and [Martin and Steel, 1989].

For ordinals \( \alpha, \beta, \) and \( \gamma \geq \alpha \), let \( L_{\gamma, \beta}^\alpha \) be the result of adding to the language of set theory a constant \( c_a \) for each element \( a \) of \( V_\alpha \) and, if \( \beta > 0 \), a constant \( d \). Let \( V_{\gamma, \beta}^\alpha \) be the expansion of the model \((V_{\gamma+\beta}; \in)\) gotten by interpreting each \( c_a \) by \( a \) and, if \( \beta > 0 \), \( d \) by \( \gamma \). For \( z \in V_{\gamma+\beta}^{\lessdot \omega} \), let \( \text{tp}_{\gamma, \beta}^\alpha(z) \) be the type realized by \( z \) in \( V_{\gamma, \beta}^\alpha \), i.e. let

\[
\text{tp}_{\gamma, \beta}^\alpha(z) = \{ \varphi(v_1, \ldots, v_{\text{lh}(z)}) \in L_{\gamma, \beta}^\alpha \mid V_{\gamma, \beta}^\alpha \models \varphi[z] \}.
\]

We want to think of the objects \( \text{tp}_{\gamma, \beta}^\alpha(z) \) as sets, so let us choose a way of so representing them. Let the symbols of the language of set theory be the odd natural numbers in some reasonable order. Let the constant \( d \) be 0. For sets \( a \) let \( c_a \) be \( a \) itself unless \( a \in \omega \); for \( a \in \omega \) let \( c_a \) be \( 2a + 2 \). It would be natural to take formulas simply to be finite sequences of symbols. We do not do so, for we would like to make the formulas of \( L_{\gamma, \beta}^\alpha \) be members of \( V_\alpha \) for each infinite \( \alpha \), but \( \lessdot \omega(V_\alpha) \subseteq V_\alpha \) only for limit \( \alpha \). Instead we choose
an injection \( f : \langle \omega \rangle V \rightarrow V \) with the property that \( f''(\langle \omega \rangle (V_\alpha)) \subseteq V_\alpha \) for all infinite \( \alpha \), and we let the formula corresponding to a sequence \( s \) of symbols be \( f(s) \). To be explicit, let \( f \upharpoonright \langle \omega \rangle(V_\omega) \) be the identity and, for \( \alpha \geq \omega \) and \( s = \langle s_n \mid n < \ell h(s) \rangle \in \langle \omega \rangle(V_{\alpha+1}) \setminus \langle \omega \rangle(V_\alpha) \), let

\[
f(s) = \{ f(\langle n, y \rangle) \mid n < \ell h(s) \land y \in s_n \}.
\]

It is easy to check that this \( f \) is one-one and that the rank of \( f(s) \) is the maximum of the rank \((s_n)\), \( n < \ell h(s) \), for all \( s \) of infinite rank. Thus we have, for all ordinals \( \gamma \) and \( \beta \) and all \( z \in \langle \omega \rangle(V_{\gamma+\beta}) \),

\[
\begin{align*}
(\forall \alpha)(\omega \leq \alpha \leq \gamma & \rightarrow \text{tp}_{\gamma,\beta}^\alpha(z) \subseteq V_\alpha); \\
(\forall \alpha)(\forall \alpha^*)(\omega \leq \alpha \leq \alpha^* \leq \gamma & \rightarrow \text{tp}_{\gamma,\beta}^\alpha(z) \cap V_\alpha = \text{tp}_{\gamma,\beta}^{\alpha^*}(z)).
\end{align*}
\]

For ordinals \( \beta \), limit ordinals \( \gamma \), cardinals \( \delta < \gamma \), and elements \( z \) of \( \langle \omega \rangle(V_{\gamma+\beta}) \), we say that \( \delta \) is \( \beta \)-reflecting in \( z \) relative to \( \gamma \) if

\[
(\forall \eta)(\delta < \eta < \gamma \rightarrow \delta \text{ is } \eta \text{-strong in } \text{tp}_{\gamma,\beta}^\gamma(z)).
\]

**Theorem 6.3.9.** Let \( \kappa \) be a cardinal. The following are equivalent.

1. \( \kappa \) is Woodin.
2. For all ordinals \( \beta \) and for all \( z \in \langle \omega \rangle(V_{\kappa+\beta}) \), the set of all \( \delta < \kappa \) such that \( \delta \) is \( \beta \)-reflecting in \( z \) relative to \( \kappa \) is unbounded in \( \kappa \).
3. For all \( z \in \langle \omega \rangle(V_{\kappa+1}) \), there is a \( \delta < \kappa \) such that \( \delta \) is 1-reflecting in \( z \) relative to \( \kappa \).

**Proof.** (1) implies (2) by Theorem 6.3.2. (2) trivially implies (3). Thus we need only show that (3) implies (1). Assume then that \( \kappa \) satisfies (3). Let \( f : \kappa \rightarrow \kappa \). Let \( \delta \) be 1-reflecting in \( \langle f \rangle \) relative to \( \kappa \). Let \( \alpha < \kappa \) be such that

\[
\alpha > \max\{\delta, \sup\{f(\xi) \mid \xi \leq \delta\}\}.
\]

Let \( j : V \prec M \) witness that \( \delta \) is \( \alpha \)-strong in \( \text{tp}_{\kappa,1}^\alpha(\langle f \rangle) \). Thus

\[
\begin{align*}
\text{tp}_{\kappa,1}^\alpha(\langle j(f) \rangle) &= \text{tp}_{\kappa,1}^\alpha(\langle f \rangle) \cap V_\alpha^M \\
&= j(\text{tp}_{\kappa,1}(\langle f \rangle)) \cap V_\alpha \\
&= \text{tp}_{\kappa,1}(\langle f \rangle) \cap V_\alpha \\
&= \text{tp}_{\kappa,1}(\langle f \rangle).
\end{align*}
\]
6.3. EQUIVALENTS OF WOODINNESS

Let $\xi \leq \delta$ and let $\gamma = f(\xi)$. Then both $\xi$ and $f(\xi)$ are smaller than $\alpha$ and so belong to $V_\alpha$. The fact that $\gamma = f(\xi)$ is thus expressed by a member of $\text{tp}_ {\kappa,1}(\langle f \rangle)$. Hence the same member of $\text{tp}_ {\kappa,1}(\langle f(\xi) \rangle)$ expresses the fact that $(j(f))(\xi) = \gamma$. It follows that

$$(j(f))(\xi) = f(\xi) < \alpha.$$ 

For $\xi < \delta$ this gives us that $(j(f))(\xi) < \alpha < j(\delta)$ and so that $f(\xi) < \delta$. Thus $\delta$ is closed under $f$. For $\xi = \delta$ we get that $V_{(j(f))(\delta)} \subseteq V_\alpha \subseteq M$. Therefore $j$ witnesses that $\kappa$ is Woodin for $f$. □

Theorem 6.3.10. Let $\kappa$ be a strong limit cardinal, let $\delta < \kappa$ be a cardinal, let $\beta$ be an ordinal, and let $z \in ^{<\omega}(V_{\kappa+\beta})$. Then $\delta$ is $\beta$-reflecting in $z$ relative to $\kappa$ if and only if for all $\alpha$ such that $\delta < \alpha < \kappa$ there is an extender $E \in V_\kappa$ such that

(a) $\text{crit}(i_E) = \delta$;
(b) $\text{strength}(E) \geq \alpha$ (i.e. $V_\alpha \subseteq \text{Ult}(V;E)$);
(c) $\alpha < i_E(\delta)$;
(d) $\text{tp}_ {\kappa,\beta}(z) = (\text{tp}_ {\kappa,i_E(\beta)})^{\text{Ult}(V;E)}(i_E(z))$.

Proof. The theorem follows easily from Theorem 6.3.3. □

If $\kappa$ is inaccessible and $E$ is an extender belonging to $V_\kappa$, then $i_E(\kappa) = \kappa$. Thus we have

Corollary 6.3.11. Theorem 6.3.10 remains true if “$\kappa$ is a strong limit cardinal” is replaced by “$\kappa$ is inaccessible” and clause (d) is replaced by

(e) $i_E(\kappa) = \kappa$ ∧ $\text{tp}_ {\kappa,\beta}(z) = (\text{tp}_ {\kappa,i_E(\beta)})^{\text{Ult}(V;E)}(i_E(z))$.

Lemma 6.3.12. Let $n \in \omega$. There is a formula $\text{TYPE}_n(v_1, \ldots, v_{n+4})$ of the language of set theory such that, for all $\alpha$, $\beta$, $\gamma$, $\alpha'$, and $\beta'$ with

$$\omega \leq \alpha' < \alpha \leq \gamma \land \beta' < \beta,$$

for all $z \in ^n(V_{\gamma+\beta})$, and for all $a \in V_\alpha$,

$$a = \text{tp}_ {\gamma',\beta'}(z)$$
$$\leftrightarrow V_{\gamma+\beta} \models \text{TYPE}_n[z^\sim(\beta',a,\alpha',\gamma)]$$
$$\leftrightarrow \text{TYPE}_n(v_1, \ldots, v_{n+1}, c_a, c_{\alpha'}, d) \in \text{tp}_ {\gamma,\beta}(z^\sim(\beta')).$$

Thus $\text{tp}_ {\gamma,\beta}(z)$ is identified by a single element of $\text{tp}_ {\gamma,\beta}(z^\sim(\beta'))$. 
The proof is routine, and we omit it.

**Lemma 6.3.13.** Let \( n \in \omega \). There is a formula \( \text{REFL}_n(v_1, \ldots, v_{n+3}) \) of the language of set theory such that, for all \( \kappa, \delta, \beta, \beta' \), and \( z \) such that \( \kappa \) is a strong limit cardinal, \( \delta < \kappa \), \( \beta' < \beta \), and \( z \in \langle V_{\kappa+\beta'} \rangle \),

\[
\delta \text{ is } \beta'-\text{reflecting in } z \text{ relative to } \kappa \leftrightarrow V_{\kappa+\beta} \models \text{REFL}_n[z^{-}\langle \beta', \delta, \kappa \rangle] \leftrightarrow \text{REFL}_n(v_1, \ldots, v_{n+1}, c_{\delta}, d) \in \text{tp}_{\kappa, \beta}(z^{-}\langle \beta' \rangle).
\]

**Proof.** The construction of \( \text{REFL}_n \) is a straightforward application of Theorem 6.3.10, except perhaps for the matter of clause (d) from that theorem. We have to say, for a \((\delta, \lambda)\)-extender \( E \in V_\kappa \),

\[
\text{tp}_{\kappa, \beta}^\alpha(z) = (\text{tp}_{i_\kappa(\kappa), i_E(\beta')}(V; E)(i_E(z)))^\text{Ult}(V; E)(i_E(z))
\]

by a formula of \( L_{\kappa, \beta}^{\delta+1} \), using the parameters \( \beta', z, \kappa, \alpha, E, \delta, \) and \( \lambda \). What our formula must say can be rephrased as

For all \( a \in <\omega(V_\alpha) \) and for every formula \( \varphi(v_1, \ldots, v_{\ell(h(z)+h(\alpha)+1)}) \) of the language of set theory,

\[
V_{\kappa+\beta} \models \varphi[z^{-}a^{-}\langle \kappa \rangle] \leftrightarrow V_{i_\kappa(\kappa)+i_E(\beta')}^\text{Ult}(V; E) \models \varphi[i_E(z)^{-}a^{-}\langle i_E(\kappa) \rangle].
\]

(For uniformity of notation, we are dealing only with the case \( \beta' > 0 \).) Here the only problem is with the second part of the biconditional, which can be rephrased as

There are \( b \in [\lambda]^{<\omega} \) and \( f : [\delta]^{h(b)} \to V_\beta \) such that \( a = \pi_E([b, f]_E) \) and

\[
\{ x \in [\delta]^{h(b)} \mid V_{\kappa+\beta} \models \varphi[z^{-}f(x)^{-}\langle \kappa \rangle] \} \in E_b.
\]

This is easily expressible by a formula of \( L_{\kappa, \beta}^{\delta+1} \). \( \square \)

The property of being \( \beta \)-reflecting in \( z \) is preserved by "decreasing" \( z \) but not by decreasing \( \beta \). Suppose that \( \delta \) is \( \beta \)-reflecting in \( z \) relative to \( \gamma \). If \( z' \) is a subsequence of the finite sequence \( z \), then it clearly follows that \( \delta \) is \( \beta \)-reflecting in \( z' \) relative to \( \gamma \). On the other hand, it is not necessarily true that if \( \beta' < \beta \) and \( z \in <\omega(V_{\gamma+\beta'}) \) then \( \delta \) is \( \beta' \)-reflecting in \( z \) relative to \( \gamma \). (For \( \gamma \) Woodin, one can get a counterexample with \( \beta = \kappa \) and \( z = \emptyset \),...
using the fact that no $\delta$ can be $\delta$-reflecting in $\emptyset$ relative to $\gamma$.) If, however, $\delta$ is also $\beta$-reflecting in $z^{\prec}\langle \beta' \rangle$ relative to $\gamma$, then it does follow easily that it is $\beta'$-reflecting in $z$. In particular, this is the case if $\beta'$ is definable in $V_{\gamma+\beta}$ from elements of $V_{\gamma} \cup \{ \gamma \}$, for then $\text{tp}^\alpha_{\gamma,\beta}(z^{\prec}\langle \beta' \rangle)$ is, for all sufficiently large $\alpha < \gamma$, determined by $\text{tp}^\alpha_{\gamma,\beta}(z)$. The following theorem gives an additional consequence of the assumption that $\delta$ is $\beta$-reflecting in $z^{\prec}\langle \beta' \rangle$ relative to $\gamma$, when $\gamma$ is a strong limit cardinal.

**Theorem 6.3.14.** Let $\kappa$ be a strong limit cardinal, let $\beta$ and $\beta' < \beta$ be ordinals, let $z \in <^\omega(V_{\kappa+\beta})$, and let $\delta < \kappa$ be $\beta$-reflecting in $z^{\prec}\langle \beta' \rangle$ relative to $\kappa$. Then the set of $\delta' < \kappa$ such that $\delta'$ is $\beta'$-reflecting in $z$ relative to $\kappa$ is unbounded in $\kappa$.

**Proof.** As we remarked above, the hypothesis of the theorem implies that $\delta$ is $\beta'$-reflecting in $z$ relative to $\kappa$. Let $\delta < \alpha < \kappa$. Let $j : V \prec M$ witness that $\delta$ is $\alpha+1$-strong in $\text{tp}^\kappa_{\kappa,\beta}(z^{\prec}\langle \beta' \rangle)$. Then $j(\delta) > \alpha$ and

\[ M \models "j(\delta) is j(\beta')-reflecting in j(z) relative to j(\kappa)." \]

Thus by Lemma 6.3.13,

\[ V^M_{j(\kappa)+j(\beta)} \models \text{REFL}_{\text{th}(z)}[j(z)^{\prec}\langle j(\beta'), j(\delta), j(\kappa) \rangle], \]

From this and the fact that $j(\kappa) > j(\delta) > \alpha$ it follows directly that $V^M_{j(\kappa)+j(\beta)}$ satisfies

\[ (\exists v_{\text{th}(z)+2})(d > v_{\text{th}(z)+2} > v_{\text{th}(z)+4} \land \text{REFL}_{\text{th}(z)}[j(z)^{\prec}\langle j(\beta'), j(\delta), j(\kappa) \rangle]), \]

where $\alpha$ is assigned to $v_{\text{th}(z)+4}$. Hence

\[ (\exists v_{\text{th}(z)+2})(d > v_{\text{th}(z)+2} > c_\alpha \land \text{REFL}_{\text{th}(z)}(v_1, \ldots, v_{\text{th}(z)+2}, d)) \]

belongs to $(\text{tp}^{\alpha+1}_{j(\kappa),j(\beta)}(z^{\prec}\langle \beta' \rangle))$. But then this formula also belongs to $\text{tp}^{\alpha+1}_{\kappa,\beta}(z^{\prec}\langle \beta' \rangle)$, and so there is a $\delta'$ such that $\kappa > \delta' > \alpha$ and $\delta'$ is $\beta'$-reflecting in $z$ relative to $\kappa$. □

We will not make any direct use of Theorem 6.3.14. In our constructions $\kappa$ will be Woodin, and therefore (2) of Theorem 6.3.9 will give us anything we could get from Theorem 6.3.14.
CHAPTER 6. WOODIN CARDINALS

The remainder of this section will be devoted to establishing a technical lemma that will be directly used in Chapter 8.

If \( X \) and \( Y \) are classes and \( \alpha \) is an ordinal, let us say that \( X \) and \( Y \) agree through \( \alpha \) if \( X \cap V_\alpha = Y \cap V_\alpha \).

Suppose that \( M \) and \( N \) are transitive class models of ZFC and that \( M \) and \( N \) agree through \( \kappa + 1 \). Suppose that \( E \in M \) is a \((\kappa, \lambda)\)-extender in \( M \) for some \( \lambda \). Then we can define an ultrapower

\[
\prod^N_E (N; \in)
\]

of \( N \) with respect to \( E \). The universe of this ultrapower is

\[
\{ [a, f]^N_E \mid \langle a, f \rangle \in D^N_E \},
\]

where

\[
D^N_E = \{ \langle a, f \rangle \mid a \in \lambda^{<\omega} \land f \in N \land f : [\kappa]^a \to N \}.
\]

The relation of \( \prod^N_E (N; \in) \), which we call \( \in^N_E \), is given by

\[
[a, f]^N_E \in^N_E [b, g]^N_E \leftrightarrow \{ z \in [\kappa]^{a \cup b} \mid f(z_{a, a \cup b}) \in g(z_{b, a \cup b}) \} \in E_{a \cup b}.
\]

It is easy to check that Lo"{s}' Theorem generalizes to such ultrapowers:

**Theorem 6.3.15.** Let \( M, N, E, \kappa, \) and \( \lambda \) be as in the preceding two paragraphs. Let \( \varphi(v_1, \ldots, v_n) \) be any formula of the language of set theory. Let \( \langle a_1, f_1 \rangle, \ldots, \langle a_n, f_n \rangle \) be elements of \( D^N_E \). Let \( b = \bigcup_{1 \leq i \leq n} a_i \). Then

\[
\prod^N_E (N; \in) \models \varphi[f_1, f_1]_{E}^N, \ldots, [a_n, f_n]_{E}^N \leftrightarrow \{ z \in [\kappa]^{[a \cup b]} \mid (N; \in) \models \varphi[f_1(z_{a_1, b}), \ldots, f_n(z_{a_n, b})] \} \in E_{a \cup b}.
\]

Thus we get a canonical \((i')^N_E : (N; \in) \prec \prod^N_E (N; \in)\).

It is also easy to prove that Lemma 3.2.9 generalizes to these ultrapowers:

**Lemma 6.3.16.** Let \( M, N, \) and \( E \) be as above. Then \( \prod^N_E (N; \in) \) is set-like.

Unfortunately wellfoundedness does not in general hold. In the next section we will prove wellfoundedness for important special cases. If \( \prod^N_E (N; \in) \) is wellfounded, let us denote by \( \text{Ult}(N; E) \) the unique transitive class \( N' \) such that \( \prod^N_E (N; \in) \cong (N'; \in) \) and let us denote by \( \pi^N_E \) the unique isomorphism. In this case we get as usual the canonical

\[
i^N_E : N \prec \text{Ult}(N; E),
\]

given by \( i^N_E = \pi^N_E \circ (i')^N_E \).
Lemma 6.3.17. Let $M$ be a transitive class model of ZFC. Let $E$ be a $(\kappa, \lambda)$-extender in $M$. Let $\zeta \geq \kappa$ be an ordinal of $M$. Let $N$ be a transitive class model of ZFC such that $M$ and $N$ agree through $\zeta + 1$. Assume that $\prod_E(N; \in)$ is wellfounded. Then

(a) If $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^a \to (\zeta^+)^M \cup V_{\zeta+1}^M$, then

$$\pi_E^M([a, f]) = \pi_E^N([a, f]).$$

(b) $(\forall \alpha)(\alpha \leq (\zeta^+)^M \to i^M_E(\alpha) = i^N_E(\alpha))$; in particular, $i^M_E(\kappa) = i^N_E(\kappa)$;

(c) Ult$(M; E)$ and Ult$(N; E)$ agree through $i^M_E(\zeta)+1$; in particular, they agree through $i^M_E(\kappa) + 1$.

Proof. For $n \in \omega$, the models $M$ and $N$ have exactly the same functions $f : [\kappa]^n \to (\zeta^+)^M \cup V_{\zeta+1}^M (= (\zeta^+)^N \cup V_{\zeta+1}^N)$. (See the proof of Lemma 6.2.5.) This implies (a), from which (b) and (c) follow. □

Remark. Parts (b) and (c) of Lemma 6.3.17 are analogous to parts (ii) and (iii) of Lemma 6.2.5. The analogue of (a) is true in the case of the earlier lemma; we simply didn’t bother to state it as part of the lemma.

The technical lemma that follows is called the “One-Step Lemma” in [Martin and Steel, 1989], and we will use the same name for it here. It will be used in Chapter 8. Readers may want to skip it and return to it when its use is imminent. For those who do not skip it, the remarks after its proof may be of some help in understanding it.

Lemma 6.3.18 (One-Step-Lemma) Let $M$ and $N$ be transitive class models of ZFC. Let $\kappa \in M \cap N$ be inaccessible in $V$ and Woodin in $M$. Let $\delta$ and $\eta$ be ordinals such that $\delta \leq \eta < \kappa$. Let $\beta$ and $\xi < \beta$ be ordinals of $M$. Let $\beta'$ be an ordinal of $N$. Let $x$ and $y$ belong to $<\omega(V_{\kappa+\beta})$ and let $x'$ belong to $<\omega(V_{\kappa+\beta'})$ with $\ell h(x') = \ell h(x)$. Let $\chi(v)$ be a formula of the language of set theory. Suppose that

1. $M$ and $N$ agree through $\delta + 1$;
2. $(tp^\delta_{\kappa, \beta})^M(x) = (tp^\delta_{\kappa, \beta'})^N(x')$;
3. $\delta$ is $\beta$-reflecting in $x$ relative to $\kappa$ in $M$;
4. $V_{\kappa+\beta}^M \models \chi[\xi]$.
Then there are a $\lambda < \kappa$ and an $E$ such that $E$ is a $(\delta, \lambda)$-extender in $M$ and such that either (a) $\prod^N_N(N; \in) \in M$, or (b) there are $\delta^*, \xi^*$, and $y^*$ such that $\eta < \delta^* < i^N_N(\delta) < \kappa$, $\xi^* < i^N_N(\beta')$, $i^N_N(x')$ and $y^*$ both belong to $\langle \omega(V^\text{	ext{Ult}}(N; E)), \rangle$, and

1. $\text{Ult}(N; E)$ and $M$ agree through $\delta^* + 1$;
2. $\text{(tp}_{\kappa, \xi}^\delta)^{\text{Ult}(N; E)}(i^N_N(x') \sim y^*) = (\text{tp}_{\kappa, \xi}^\delta)^M(x \sim y)$;
3. $\delta^*$ is $\xi^*$-reflecting in $i^N_N(x') \sim y^*$ relative to $\kappa$ in $\text{Ult}(N; E)$;
4. $V^\text{Ult}(N; E)_{\kappa + 1} \models \chi[\xi^*]$.

Furthermore, let $\alpha$ be any ordinal of $\text{Ult}(N; E)$ and let $z$ be any element of $\langle \omega(V^\text{Ult}(N; E)), \rangle$ such that

$$(\text{tp}_{\kappa, \alpha}^\delta)^{\text{Ult}(N; E)}(z) = (\text{tp}_{\kappa, \alpha}^\delta)^{\text{Ult}(N; E)}(i^N_N(x')).$$ 

Then there are $\hat{\xi}$ and $\hat{y}$ such that $\hat{\xi} < \alpha$, $\hat{y} \in \langle \omega(V^\text{Ult}(N; E)), \rangle$, and

1. $\hat{\xi}$ and $\hat{y}$ are such that $\hat{\xi} \prec \alpha$, $\hat{y} \in \langle \omega(V^\text{Ult}(N; E)), \rangle$, and
2. $\text{(tp}_{\kappa, \xi}^\delta)^{\text{Ult}(N; E)}(z \sim \hat{y}) = (\text{tp}_{\kappa, \xi}^\delta)^M(x \sim y)$;
3. $\delta^*$ is $\hat{\xi}^*$-reflecting in $z \sim \hat{y}$ relative to $\kappa$ in $\text{Ult}(N; E)$;
4. $V^\text{Ult}(N; E)_{\kappa + 1} \models \chi[\hat{\xi}]$.

**Proof.** By Theorem 6.3.9, let $\delta^*$ be such that $\eta < \delta^* < \kappa$ and $\delta^*$ is $\xi$-reflecting in $x \sim y$ in $M$. By (3) and Corollary 6.3.11, let $\lambda < \kappa$ and $E$ be such that $E$ is a $(\delta, \lambda)$-extender in $M$, strength$^M(E) \geq \delta^* + 1$, $i^M_E(\kappa) = \kappa$, and

$$(\text{tp}_{\kappa, \alpha}^\delta)^{\text{Ult}(M; E)}(i^M_E(x)) = (\text{tp}_{\kappa, \alpha}^\delta)^M(x).$$

Assume that $\prod^N_N(N; \in)$ is wellfounded, since otherwise there is nothing to prove. Note that the inaccessibility of $\kappa$ in $V$ guarantees that $i^N_E(\kappa) = \kappa$.

$\text{Ult}(M; E)$ and $M$ agree through $\delta^* + 1$. By hypothesis (1) and part (c) of Lemma 6.3.17, $\text{Ult}(N; E)$ and $\text{Ult}(M; E)$ also agree through $\delta^* + 1$. Thus we have (1*).

Before choosing $\xi^*$ and $y^*$, let us prove that

$$(\text{tp}_{\kappa, \alpha}^\delta)^{\text{Ult}(N; E)}(i^N_N(x')) = (\text{tp}_{\kappa, \alpha}^\delta)^M(x).$$
For this it is enough to show that
\[(tp_{\kappa,\pi_E}(\delta^+))^{\text{ult}(\text{Ult}(\text{Ult}(M_\lambda,M,F))}(i^N_E(x')) = (tp_{\kappa,\pi_E^N(\beta^+)}(\delta^+))^{\text{ult}(\text{Ult}(M_\lambda,F))}(i^N_E(x)).\]

Let \(\varphi(v_1, \ldots, v_{\ell(x)+n+1})\) be a formula of the language of set theory and let \(b = \langle b_1, \ldots, b_n \rangle \in n(V^{\text{ult}(\text{Ult}(M,F)))\). For \(1 \leq j \leq n\) let \(\langle a_j, f_j \rangle\) be such that \(b_j = \pi_E^M([a_j, f_j]_E^M)\). Letting \(a = \bigcup_{1 \leq j \leq n} a_j\) and replacing each \(f_j\) by \(f_j^a\), we may assume that each \(a_j = a\) and that each \(b_j = \pi_E^M([a, f_j]_E^M)\). Since \(\delta^* < \text{strength}^M(E) \leq \lambda < i^M_E(\delta)\), we may assume that each \(f_j : [\delta]^{\alpha} \to V_\delta\). By (a) of Lemma 6.3.17, \(b_j = \pi_E^N([a, f_j]_E^N)\) for \(1 \leq j \leq n\). By Theorem 6.3.15 and hypothesis (2),

\[
\varphi(v_1, \ldots, v_{\ell(x)}, c_1, \ldots, c_m, d) \in (tp_{\kappa,\pi_E}(\delta^+))^{\text{ult}(\text{Ult}(\text{Ult}(M,F))}(i^N_E(x')) \quad \leftrightarrow \quad \{ z \in [k]^{\alpha} \mid \varphi(v_1, \ldots, v_{\ell(x)}, c_{f_1(z)}, \ldots, c_{f_m(z)}, d) \in (tp_{\kappa,\pi_E}(\delta^+))^{\text{ult}(\text{Ult}(\text{Ult}(M,F))}(i^N_E(x')) \in E_a \\
\leftrightarrow \quad \{ z \in [k]^{\alpha} \mid \varphi(v_1, \ldots, v_{\ell(x)}, c_{f_1(z)}, \ldots, c_{f_m(z)}, d) \in (tp_{\kappa,\pi_E}(\delta^+))^{\text{ult}(\text{Ult}(\text{Ult}(M,F))}(i^N_E(x')) \in E_a \\
\leftrightarrow \quad \varphi(v_1, \ldots, v_{\ell(x)}, c_1, \ldots, c_m, d) \in (tp_{\kappa,\pi_E}(\delta^+))^{\text{ult}(\text{Ult}(\text{Ult}(M,F))}(i^N_E(x')).
\]

Next we turn to the choice of \(\xi^*\) and \(y^*\). Let

\[A = (tp_{\kappa,\xi^*}(\delta^+))^{\text{ult}(\text{Ult}(\text{Ult}(M,F))}(x^\sim y).
\]

Let \(\psi(v_1, \ldots, v_{\ell(x)+\ell(y)+1})\) be the formula of \((L_{\kappa,\beta}^{\delta^+})^M\) given as follows:

\[
v_{\ell(x)+\ell(y)+1} \in \text{Ord} \\
\wedge \text{TYPE}_{\ell(x)+\ell(y)}(v_1, \ldots, v_{\ell(x)+\ell(y)+1}, c_A, c_{\delta^*}, d) \\
\wedge \text{REFL}_{\ell(x)+\ell(y)}(v_1, \ldots, v_{\ell(x)+\ell(y)+1}, c_{\delta^*}, d) \\
\wedge \chi(v_{\ell(x)+\ell(y)+1}).
\]

The finite sequence \(y\) and the ordinal \(\xi\) witness that

\[(\exists v_{\ell(x)+1}) \cdots (\exists v_{\ell(x)+\ell(y)+1}) \psi(v_1, \ldots, v_{\ell(x)+\ell(y)+1})\]

belongs to \((tp_{\kappa,\beta}^{\delta^+})^M(x)\). Thus this formula also belongs to

\[(tp_{\kappa,\pi_E}(\beta^+))^{\text{ult}(\text{Ult}(\text{Ult}(M,F))}(i^N_E(x')).
\]

If we let \(y^*\) and \(\xi^*\) witness this, then (2\*), (3\*), and (4\*) hold.

For the second part of the conclusion of the lemma, let \(\alpha\) and \(z\) be as in the hypotheses. Then the formula above also belongs to \((tp_{\kappa,\alpha}^{\delta^+})^{\text{ult}(\text{Ult}(\text{Ult}(M,F))}(z)\). Let \(\tilde{y}\) and \(\tilde{\xi}\) witness this. \(\square\)
Remarks:

(a) The lemma does not assert that $\prod^N_E (N; \in)$ is wellfounded; indeed the lemma is vacuously satisfied by an $E$ such that this ultrapower is illfounded. In our applications, we will be able to prove wellfoundedness. Since a long-enough initial part of $\prod^N_E (N; \in)$ is always wellfounded, we could have formulated the lemma so that it would have had real content independently of full wellfoundedness. In [Martin and Steel, 1989] the problem of wellfoundedness is handled in a different way, by an assumption that $M$ and $N$ are countably closed.

(b) The hypothesis that $\kappa$ is inaccessible in $V$ was included for its notationally simplifying consequence that $i_E^N (\kappa) = \kappa$.

(c) It will be crucial in our applications of the lemma to know that $\kappa$ is Woodin in Ult$(N; E)$, for we will want to apply the lemma iteratively. In fact, it follows from the hypotheses of the Lemma that $\kappa$ is Woodin in $N$. (See Exercise 6.3.5.) This implies that it is Woodin in Ult$(N; E)$. In our applications $\kappa$ will be a Woodin cardinal in $V$, and there will be an elementary embedding of $V$ into $N$ that fixes $\kappa$. Hence we will know immediately (without Exercise 6.3.5) that $\kappa$ is Woodin in $N$ and so in Ult$(N; E)$.

(d) The ordinal $\delta^*$ is larger than the given $\delta$. In fact, the arbitrariness of $\eta < \kappa$ means that it can be made as large as one wants, subject to being smaller than $\kappa$. On the other hand, the ordinal $\xi$ is smaller than the given $\beta$. This will give us problems, though not insurmountable ones, in generating an infinite sequence of applications of the lemma. Of course $\xi^*$ need not be smaller that $\beta$, but there is no obvious way to make use of this.

(e) The pair of clauses (4) and (4*) will be needed for technical reasons in the applications. The lemma could be strengthened by allowing the formula $\chi$ in (4) to be any element of $(\operatorname{tp}^N_{\beta, \delta})^M (x \dashv y)$ and demanding in (4*) that $\chi \in (\operatorname{tp}^N_{\xi, i_E^N (\beta)})^{\operatorname{Ult}(N; E)} (i_E^N (x') \dashv y^*)$.

(f) The second part of the lemma was not used in [Martin and Steel, 1989]. Using it will allow us to avoid a good deal of work that was done in that paper. (Nevertheless, we will do the work, in §8.3.)

**Exercise 6.3.1.** Let $\kappa$ be Woodin. Prove that for all $A \subseteq V_\kappa$ the set of cardinals $\delta < \kappa$ such that

$$(\forall \eta) (\delta < \eta < \kappa \rightarrow \delta \text{ is } \eta\text{-strong in } A)$$
is stationary in $\kappa$. (See Exercise 3.2.7 for the definition of “stationary.”)

**Hint.** One way to proceed is to modify the proof of the $(1) \Rightarrow (2)$ half of Theorem 6.3.1. Another way is to argue directly from $(2)$ of Theorem 6.3.1, using the following fact: If $j : V \prec M$, $\text{crit}(j) = \delta$, and $C \cap \delta$ is bounded in $\delta$, then $j(C) \cap j(\delta) \subseteq \delta$.

**Exercise 6.3.2.** Use Corollary 6.3.5 to show that if there is a Woodin cardinal then the least Woodin cardinal is not measurable.

**Exercise 6.3.3.** Prove that a cardinal $\kappa$ is Woodin if and only if for all $A \subseteq V_\kappa$ there is a $\delta < \kappa$ such that, for every $\eta$ with $\delta < \eta < \kappa$,

$$(\exists j : V \prec M) (M \text{ is transitive } \land \text{crit}(j) = \delta \land j(A) \cap V_\eta = A \cap V_\eta).$$

Note that the displayed statement would say that $\delta$ is $\eta$-strong in $A$ if we added the condition that $V_\eta \subseteq M$.

**Exercise 6.3.4.** Suppose that $\kappa$ is a strong limit cardinal and that $\delta < \kappa$ is 0-reflecting in $\emptyset$ relative to $\kappa$. Show that $V_\delta \prec V_\kappa$.

**Hint.** If $V_\kappa \models (\exists v) \varphi(v)$, then there is an $\alpha < \kappa$ and an $x \in V_\alpha$ such that $V_\kappa \models \varphi[x]$. Get $j : V \prec M$ from the hypothesis about $\delta$. Note that $x \in V_M^M$.

**Exercise 6.3.5.** Assume the hypotheses of Lemma 6.3.18. Prove that $\kappa$ is Woodin in $N$. (The only hypotheses actually needed are that $\kappa$ is Woodin in $M$, that $\beta > 0$, and the consequence of $(2)$ that

$$(\text{tp}^0_{\kappa, \beta})^M(\emptyset) = (\text{tp}^0_{\kappa, \beta})^N(\emptyset).$$

**Exercise 6.3.6.** Call a $(\delta, \lambda)$-extender $E$ strong if $\delta + 1 < \text{strength}(E) = \lambda < i_E(\delta)$. Let $\kappa$ be Woodin. Say that a set $E$ of extenders strongly witnesses that $\kappa$ is Woodin if

(i) $E \subseteq V_\kappa$;

(ii) each $E \in E$ is strong;

(iii) for every $A \subseteq V_\kappa$ and for every $\eta < \kappa$, there are a $\delta < \kappa$ and an $E \in E$ such that $i_E$ witnesses that $\delta$ is $\eta$-strong in $A$.

Prove that there is an $E$ strongly witnessing that $\kappa$ is Woodin. (Of course, the set of all strong extenders that belong to $V_\kappa$ works if any $E$ does.)
Exercise 6.3.7. The following construction and the results of this and the next exercise are due to Woodin.

Let \( \neg, \lor \), and \( a_n, n \in \omega \), be distinct sets, say natural numbers. The class of \( \infty \)-Borel codes is the smallest class satisfying the following conditions.

(a) For each \( n \in \omega \), \( a_n \) is an \( \infty \)-Borel code.
(b) If \( c \) is an \( \infty \)-Borel code, then \( \langle \neg, c \rangle \) is an \( \infty \)-Borel code.
(c) If \( \beta \) is an ordinal and \( c_\alpha, \alpha < \beta \), are \( \infty \)-Borel codes, then \( \langle \lor, \langle c_\alpha | \alpha < \beta \rangle \rangle \) is an \( \infty \)-Borel code.

We write \( \neg c \) for \( \langle \neg, c \rangle \), and we write \( \lor \langle c_\alpha | \alpha < \beta \rangle \) for \( \langle \lor, \langle c_\alpha | \alpha < \beta \rangle \rangle \).

We associate with each \( \infty \)-Borel code \( c \) a subset \( B_c \) of \( \omega^2 \) inductively as follows:

\[
B_{a_n} = \{ x \in \omega^2 | x(n) = 1 \};
\]
\[
B_{\neg c} = \omega^2 \setminus B_c;
\]
\[
B_{\lor \langle c_\alpha | \alpha < \beta \rangle} = \bigcup_{\alpha < \beta} B_{c_\alpha}.
\]

Let \( C \) be the class of all \( \infty \)-Borel codes. If \( I \subseteq C \times C \) and \( x \in \omega^2 \), then \( I \) is \( x \)-consistent if

\[
(\forall c \in C)(\forall c' \in C)(\langle c, c' \rangle \in I \to (x \in B_c \leftrightarrow x \in B_{c'})).
\]

If \( B \) is a complete Boolean algebra and if \( \tau : \{ a_n | n \in \omega \} \to B \) (as in the example above with \( B = \mathcal{P}(\omega^2) \)). Say that \( \tau : \{ a_n | n \in \omega \} \to B \) respects \( I \) if whenever \( \langle c, c' \rangle \in I \) then \( \tau^*(c) = \tau^*(c') \). (Thus \( a_n \mapsto \{ x | x(n) = 1 \} \) respects \( I \) if and only if \( I \) is \( x \)-consistent for every \( x \in \omega^2 \).

Let \( I \subseteq C \times C \). For \( \infty \)-Borel codes \( c \) and \( c' \), define \( c \sim_I c' \) to hold if, for every complete Boolean algebra \( B \) and every \( \tau : \{ a_n | n \in \omega \} \to B \), if \( \tau \) respects \( I \) then \( \tau^*(c) = \tau^*(c') \). (One can also define \( \sim_I \) by transfinite induction, using the laws of complete Boolean algebras.) It is evident that \( \sim_I \) is an equivalence relation. For \( c \in C \), let \( [c]_I \) be the equivalence class of \( c \) with respect to \( I \), fixed up à la Scott to make it a set. Let \( C(I) \) be the class of all \( [c]_I \). If \( C(I) \) has more than one element (as it will if \( I \) is \( x \)-consistent for some \( x \in \omega^2 \)), then \( C(I) \) is a complete Boolean (class) algebra under the obvious complement and join operations. If \( I \) is \( x \)-consistent for every \( x \in \omega^2 \), then \( [c]_I \mapsto B_c \) is a complete homomorphism of \( C(I) \) onto \( \mathcal{P}(\omega^2) \).
For any set $E$ of extenders, we define a set $I_E$ of pairs of elements of $C$. A pair belongs to $I_E$ only if this is required by the following. Let $E \in \mathcal{E}$. Let $E$ be a $(\delta, \lambda)$-extender. Let $c_\alpha$, $\alpha < \delta$, be $\infty$-Borel codes belonging to $V_\delta$.

Let $i_E(\langle c_\alpha \mid \alpha < \delta \rangle) = \langle \hat{c}_\alpha \mid \alpha < i_E(\delta) \rangle$.

(Note that $\hat{c}_\alpha = c_\alpha$ for $\alpha < \delta$.) All the $\hat{c}_\alpha$ are $\infty$-Borel codes. If $\hat{c}_\delta \in V_\lambda$, then

$$\langle \bigvee \langle c_\alpha \mid \alpha < \delta \rangle, \bigvee \langle \hat{c}_\alpha \mid \alpha \leq \delta \rangle \rangle \in I_E.$$ 

(a) Prove that $I_E$ is $x$-consistent for every $x \in \omega^2$.

Suppose $\kappa$ is Woodin. Let $E$ be a collection of extenders strongly witnessing that $\kappa$ is Woodin. (See exercise 6.3.6.)

(b) Prove that $C(I_E)$ is a set of size $\kappa$ and, as a Boolean algebra, has the $\kappa$-chain condition.

Hint. Let $C_\kappa = C \cap V_\kappa$. Let $C_\kappa(I_E)$ be the corresponding Boolean subalgebra of $C(I_E)$. It suffices to prove that $C_\kappa(I_E)$ has the $\kappa$-chain condition. To prove this, use the fact that $E$ strongly witnesses that $\kappa$ is Woodin.

Exercise 6.3.8. Let $E$ be a set of extenders. Let $M$ be a transitive class model of ZFC with $E \in V_{\text{Ord}^M}$. Note that $C^M = C \cap M$. For $E \in \mathcal{E}$ with $E = \langle E_a \mid a \in [\lambda]^{<\omega} \rangle$, let $E \upharpoonright M = \langle E_a \cap M \mid a \in [\lambda]^{<\omega} \rangle$. Let $\mathcal{E} \upharpoonright M = \{ E \upharpoonright M \mid E \in \mathcal{E} \}$. Suppose that $\mathcal{E} \upharpoonright M \in M$ and that $\mathcal{E} \upharpoonright M$ strongly witnesses in $M$ that $\kappa$ is Woodin. In $M$ we have the algebra $(C^M(I^M_E \upharpoonright M))^M$.

Let $P_{E \upharpoonright M}$ be the partially ordered set $(C^M(I^M_E \upharpoonright M))^M \setminus \{0\}$. Let $x \in \omega^2$.

Attempt to define $G_x \subseteq P_{E \upharpoonright M}$ by

$$[\mathbf{c}^M_{E \upharpoonright M}] \in G_x \leftrightarrow x \in B_x,$$

where $B_x$ is as in Exercise 6.3.7. Prove that $G_x$ is well-defined and is $P_{E \upharpoonright M}$-generic over $M$ and that $M[G_x] = M[x]$.

Hint. Use an absoluteness argument to show that

$$(\sim_{I^M_E \upharpoonright M})^M = \sim_{I^M_E \upharpoonright M} \upharpoonright C^M.$$ 

(This is perhaps easier using the inductive definition of $\sim_I$.) It follows that $(C^M(I^M_E \upharpoonright M))^M$ is an $M$-complete subalgebra of $C(I^M_E \upharpoonright M)$. 
Next observe that
$$\mathcal{I}^M_{\mathcal{E}|M} = \mathcal{I}_\mathcal{E} \cap M.$$ This implies, by part (a) of Exercise 6.3.7, that $\mathcal{I}^M_{\mathcal{E}|M}$ is $y$-consistent in $V$ for every $y \in \omega_2$. In particular, $\mathcal{I}^M_{\mathcal{E}|M}$ is $x$-consistent in $V$. Thus, in $V$, a complete homomorphism $\sigma : C(\mathcal{I}^M_{\mathcal{E}|M}) \to \{0, 1\}$ is given by
$$\sigma([c]_{\sim_{\mathcal{I}^M_{\mathcal{E}|M}}}) = 1 \iff x \in B_c.$$ The restriction of $\sigma$ to $(C^M(\mathcal{I}^M_{\mathcal{E}|M}))^M$ is thus an $M$-complete homomorphism. The preimage of $\{1\}$ is just $G_x$. 
Chapter 7

Iteration Trees

In this chapter, which can be read immediately after §6.1 if one is willing to refer back to Chapter 6 for one or two definitions, we introduce and prove some basic results about the main technical tool of the determinacy proofs of Chapter 8. This material comes from Martin–Steel [1988], [Martin and Steel, 1989], and [Martin and Steel, 1994]. Our treatment will follow that of [Martin and Steel, 1994], which is a bit more general than that of the other two papers.

Iteration trees are a generalization of iterated ultrapowers, which we studied in §3.3. They are more general in three ways: (1) The individual ultrapowers are with respect to extenders and not just ultrafilters. (2) The individual ultrapowers are not all with respect to images of the same ultrafilter or extender. (3) The iteration is not linear, but has a tree structure, and the individual ultrapowers are of models at one node of the tree but with respect to extenders in models at possibly different nodes.

In fact we have already introduced an even wider generalization of type (1). In §3.3 we defined transfinite iterations of an arbitrary elementary embedding $j : V \prec M$.

Before considering iterations with all three properties, we will consider, in §1, those with properties (1) and (2). Iterations with property (2) were first used in [Kunen, 1970], and [Mitchell, 1979] introduced iterations which essentially also had property (1)

In §2 we introduce iteration trees. In §3 we study finite iteration trees, and in §4 we study those of length $\omega$. The definitions and results in the text of §2–§4 are almost all from [Martin and Steel, 1994], though—in order to avoid continually citing the paper—we will mostly omit explicitly citation.
7.1 Internal Iterations

Most of the business of this chapter will be proving that the direct limit models of various iterations are wellfounded. Even when the initial model of the iteration is \( V \), we will have to deal in the proofs with iterations whose models are sets that may not satisfy full ZFC. To handle this, we introduce a concept from [Martin and Steel, 1994] that is general enough to cover all the models that will arise in our proofs.

First we need two preliminary definitions.

The Lévy hierarchy of formulas defined on page 19 can be defined for any language extending the language of set theory. The clauses in the definition are exactly the same as those on page 19, but now “atomic formula” means atomic formula of the extended language. Let \( \mathcal{L}(P) \) be the result of adding to the language of set theory a one-place function symbol \( P \). A class model \((M; E)\) satisfies \( \Sigma_1(P) \) Replacement if \( M \) satisfies the Power Set Axiom and the expansion of \((M; \in)\) in which \( P \) is interpreted as the power set operation satisfies Replacement for \( \Sigma_1 \) formulas of \( \mathcal{L}(P) \).

If \((M; E)\) is a class model and \( u \in M \), then \((M; E)\) satisfies Replacement for the domain \( u \) if, for every formula \( \varphi(u, x, y, z_1, \ldots, z_n) \) of the language of set theory,

\[
(M; E) \models (\forall z_1) \cdots (\forall z_n) (((\forall x \in u)(\exists! y) \varphi) \rightarrow (\exists v)(\forall x \in u)(\exists y \in v) \varphi).
\]

Replacement for domain \( u \) implies that the range of any class function whose domain is \( u \) is a set, and so that the class function is a set function.

Now we turn to the concept from [Martin and Steel, 1994] that was mentioned above. A class model \( \mathcal{M} = (M; \in, \delta) \) is a premouse if

(a) \( M \) is transitive;
(b) \( \delta \) is an ordinal belonging to \( M \);
(c) \( (M; \in) \models \text{ZC + } \Sigma_1(P) \) Replacement + Replacement for the domain \( V_\delta \).

For premice \( \mathcal{M} = (M; \in, \delta) \) we write \( \delta = \delta^\mathcal{M} \).

It is easy to show that if \((M; \in)\) satisfies \( \text{ZC + } \Sigma_1(P) \) Replacement, then

\[
(M; \in) \models (\forall \alpha \in \text{Ord}) V_\alpha \text{ exists.}
\]

Thus clauses (a), (b), and the first two parts of clause (c) imply that \( M \) satisfies “\( V_\delta \) exists,” and so that the third part of clause (c) makes sense.
Remark. The name “premouse” may seem an odd one. Premice and mice were introduced in [Dodd and Jensen, 1981] and generalizations have been defined by various authors. In all these versions, including ours, a premouse is required to satisfy some fragment of ZFC. In most versions—though not in ours, i.e., not in that of [Martin and Steel, 1994]—it is required also that a premouse have some specific structure, such as being $L[E]$ for $E$ a “coherent” sequence of extenders. In all versions, what makes a premouse a mouse is iterability, the existence of wellfounded limit models of any appropriate iteration whose initial model is the premouse. We will not define or discuss mice, but the main results of this chapter can be thought of as establishing mouse-like properties for certain classes of premice.

We have already noted that every premouse $\mathcal{M}$ satisfies the sentence “$(\forall \alpha \in \text{Ord}) V_\alpha$ exists.” It is also easy to show that, for premice $\mathcal{M}$,

$$\mathcal{M} \models (\forall x)(\exists \alpha \in \text{Ord}) x \in V_\alpha.$$ 

If $\mathcal{M} = (M; \in, \delta)$ is a premouse and $E$ is an extender in $M$, i.e. $E \in M$ and $\mathcal{M} \models “E$ is an extender,” then we can define $\prod^\mathcal{M}_E$ just as we defined, on page 337. $\prod^\mathcal{M}_E (M; \in)$ for class models $M$ of ZFC. Our notation for such ultrapowers will be like that for the earlier ones. Since there seems no reason for preferring, e.g., one of the notations $[a, f]^M_E$ and $[a, f]^M_E$ over the other, we will in this and other cases indiscriminately use both notations. Loś’ Theorem generalizes to these ultrapowers, except that we must restrict $E$ to be a $(\kappa, \lambda)$-extender with $\kappa \leq \delta$:

**Theorem 7.1.1.** Let $\mathcal{M} = (M; \in, \delta)$ be a premouse. Let $\kappa$ and $\lambda > \kappa$ be ordinals of $M$ with $\kappa \leq \delta$. Let $E$ be an $(\kappa, \lambda)$-extender in $\mathcal{M}$. Let $\varphi(v_1, \ldots, v_n)$ be any formula of the language of set theory. Let $\langle a_1, f_1 \rangle, \ldots, \langle a_n, f_n \rangle$ be elements of $\mathcal{D}_E$. Let $b = \bigcup_{1 \leq i \leq n} a_i$. Then

$$\prod^\mathcal{M}_E \mathcal{M} \models \varphi[[a_1, f_1]^M_E, \ldots, [a_n, f_n]^M_E] \leftrightarrow \{ z \in [\kappa]^{|b|} | \mathcal{M} \models \varphi[f_1(z_{a_1, b}), \ldots, f_n(z_{a_n, b})] \} \in E_b.$$ 

**Proof.** The proof is by an induction on $\varphi$ as usual. We sketch the case $\varphi$ is $(\exists v_0)\psi$ to indicate how the axioms that hold in premice are used. In that
case we have, suppressing some subscripts and superscripts,

\[
\prod^M_E \mathcal{M} \models \varphi[[a_1, f_1], \ldots, [a_n, f_n]]
\]

\[
\leftrightarrow (\exists a_0 \in [\lambda]^{<\omega}) (\exists f_0 \in [\kappa]^{\alpha_0}, M \cap M)
\]

\[
(\prod^M_E \mathcal{M} \models \psi[[a_0, f_0], \ldots, [a_n, f_n]])
\]

\[
\leftrightarrow (\exists a_0 \in [\kappa]^{<\omega}) (\exists f_0 \in [\kappa]^{\alpha_0}, M \cap M)
\]

\[
(\{z \in [\kappa]^{\alpha_0, \beta_0} \mid \mathcal{M} \models \psi[f_0(z_{a_0, a_0, \beta_0}), \ldots, f_n(z_{a_n, a_0, \beta_0})] \in E_{a_0, \beta_0}\})
\]

\[
\leftrightarrow \{z \in [\kappa]^{\beta_0} \mid (\exists x \in M) \mathcal{M} \models \psi[x, f_1(z_{a_1, \beta_0}), \ldots, f_n(z_{a_n, \beta_0})] \in E_b\}
\]

\[
\leftrightarrow \{z \in [\kappa]^{\beta_0} \mid \mathcal{M} \models \varphi[f_1(z_{a_1, \beta_0}), \ldots, f_n(z_{a_n, \beta_0})] \in E_b\}
\]

To show that the fourth line implies the third, one argues as follows: Because Replacement for the domain \(V_\delta\) holds in \(\mathcal{M}\), the fourth line implies that there is an ordinal \(\alpha \in M\) such that

\[
\{z \in [\kappa]^{\beta_0} \mid (\exists x \in V_\alpha^M) \mathcal{M} \models \psi[x, f_1(z_{a_1, \beta_0}), \ldots, f_n(z_{a_n, \beta_0})] \in E_b\}
\]

Since this set belongs to \(M\) as well as to \(E_b\), Choice in \(M\) yields the third line.

\(\square\)

Theorem 7.1.1 gives us the canonical \((i')^M_E : \mathcal{M} \prec \prod^M_E \mathcal{M}\). The usual proofs give the following two results.

**Lemma 7.1.2.** Let \(\mathcal{M}\) be a premouse, let \(\kappa \leq \delta^\mathcal{M}\), and let \(E\) be a \((\kappa, \lambda)\)-extender in \(\mathcal{M}\) for some \(\lambda\). Then \(\mathcal{M} \models \text{“}\prod^M_E \mathcal{M}\) is set-like,” and so \(\prod^M_E \mathcal{M}\) is set-like.

**Lemma 7.1.3.** Let \(\mathcal{M}\) be a premouse, let \(\kappa \leq \delta^\mathcal{M}\), and let \(E\) be a \((\kappa, \lambda)\)-extender in \(\mathcal{M}\) for some \(\lambda\). Then \(\mathcal{M} \models \text{“}\prod^M_E \mathcal{M}\) is wellfounded,” and so \(\prod^M_E \mathcal{M}\) is wellfounded.

Thus, if \(\mathcal{M} = (M; \in, \delta)\), we get a unique

\[
\pi^M_E : \prod^M_E \mathcal{M} \cong \text{Ult}(\mathcal{M}; E) = (\text{Ult}(M; E); \in, \delta^{\text{Ult}(M; E)})
\]

with \(\text{Ult}(M; E)\) transitive, and we define

\[
i^M_E = \pi^M_E \circ (i')^M_E : \mathcal{M} \prec \text{Ult}(\mathcal{M}; E).
\]
Note that all of the classes $\prod_E M, (i^E)^M, \pi_E^M, \text{Ult}(M; E), \text{Ult}(M; E)$, and $i^E_M$ are classes in $M$. Note also that $\text{Ult}(M; E)$ is a premouse.

We now begin our study of iterations that have the first two of the properties mentioned in the introduction to this Chapter. We will define such iterations for transitive class models of ZFC and then for premice.

First we need some more terminology for talking about direct limits. Suppose that $M_d$ is a transitive class for each $d \in D$ and that

$$\langle \langle \langle M_d; \in \rangle | d \in D \rangle, \langle j_{d,d'} | d \in D \land d' \in D \land d R d' \rangle \rangle$$

is a directed system of elementary embeddings. Let $(\tilde{M}, \langle \tilde{j}_d | d \in D \rangle)$ be the direct limit of this directed system. We say that $\tilde{M}$ is the direct limit model of the directed system. If $\tilde{M}$ is wellfounded and set-like, let $\pi : \tilde{M} \cong (N; \in)$, with $N$ transitive. We say that

$$\langle \langle N; \in \rangle, \langle \pi \circ \tilde{j}_d | d \in D \rangle \rangle$$

is the canonical limit of the directed system and that $(N; \in)$ is the canonical limit model of the directed system. If either wellfoundedness or set-likeness fails, then there is no canonical limit and no canonical limit model. Note that if there is an $R$-maximal element $d$ of $D$, then the direct limit model is isomorphic to $M_d$, the canonical limit is $(M_d, \langle j_{d,d'} | d' \in D \rangle)$, and the canonical limit model is $M_d$.

Similarly define the direct limit model, the canonical limit, and the canonical limit model when the individual models of the directed system have additional structure, e.g., when they are premice $(M_d; \in, \delta_d)$.

If $M$ is a transitive class model of ZFC and $\theta$ is a non-zero ordinal number, an internal iteration of $M$ of length $\theta$ is a sequence $\langle E_\alpha | \alpha + 1 < \theta \rangle$ such that there are transitive classes $M_\alpha, \alpha < \theta$, and embeddings $j_{\alpha,\beta}, \alpha \leq \beta < \theta$, satisfying

(a) $M_0 = M$;
(b) each $E_\alpha$ is an extender in $M_\alpha$;
(c) for $\alpha \leq \beta \leq \gamma < \theta$, $j_{\alpha,\gamma} = j_{\beta,\gamma} \circ j_{\alpha,\beta}$;
(d) for each $\alpha$ such that $\alpha + 1 < \theta$, $M_{\alpha+1} = \text{Ult}(M_\alpha; E_\alpha)$ and $j_{\alpha,\alpha+1} = i^{M_\alpha}_{E_\alpha}$;
(e) for each limit $\lambda < \theta$, $(M_\lambda, \langle j_{\alpha,\lambda} | \alpha < \lambda \rangle)$ is the canonical limit of

$$\langle \langle M_\alpha | \alpha < \lambda \rangle, \langle j_{\alpha,\beta} | \alpha \leq \beta < \lambda \rangle \rangle.$$
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Note that the $M_\alpha$ and the $j_{\alpha,\beta}$ are uniquely determined by $M$ and the $E_\alpha$.

Remark. Our notion of internal iteration is in one sense less general and in another sense more general than the name suggests. A broader notion would permit the $E_\alpha$ to be extenders in the general sense of Exercise 6.1.2; an even broader notion would replace the $i_{E_\alpha}^{M_\alpha}$ by embeddings $j_\alpha : M_\alpha \prec M_{\alpha+1}$, requiring only that each $j_\alpha$ be a class in $M_\alpha$. A narrower notion would require the iteration $\langle E_\alpha \mid \alpha + 1 < \theta \rangle$ to belong to $M$.

If $\mathcal{M}$ is a premouse and $\theta$ is a non-zero ordinal number, an internal iteration of $\mathcal{M}$ of length $\theta$ is a sequence $\langle E_\alpha \mid \alpha < \theta \rangle$ such that there are premice $\mathcal{M}_\alpha$, $\alpha < \theta$, and embeddings $j_{\alpha,\beta}$, $\alpha \leq \beta < \theta$, satisfying

(a) $\mathcal{M}_0 = \mathcal{M}$;
(b) each $E_\alpha$ is an extender in $\mathcal{M}_\alpha$ with $E_\alpha \in V^{\mathcal{M}_\alpha}_{\delta^{\mathcal{M}_\alpha}}$;
(c) for $\alpha \leq \beta \leq \gamma < \theta$, $j_{\alpha,\gamma} = j_{\beta,\gamma} \circ j_{\alpha,\beta}$;
(d) for each $\alpha$ such that $\alpha + 1 < \theta$, $\mathcal{M}_{\alpha+1} = \text{Ult}(\mathcal{M}_\alpha; E_\alpha)$ and $j_{\alpha,\alpha+1} = i_{E_\alpha}^{\mathcal{M}_\alpha}$;
(e) for each limit $\lambda < \theta$, $(\mathcal{M}_\lambda, \langle j_{\alpha,\lambda} \mid \alpha < \lambda \rangle)$ is the canonical limit of $(\langle \mathcal{M}_\alpha \mid \alpha < \lambda \rangle, \langle j_{\alpha,\beta} \mid \alpha \leq \beta < \lambda \rangle)$.

Remark. The condition that $E_\alpha \in V^{\mathcal{M}_\alpha}_{\delta^{\mathcal{M}_\alpha}}$ is stronger than necessary. One could simply require that $E_\alpha$ be a $(\kappa, \lambda)$-extender in $\mathcal{M}_\alpha$ for some $\kappa$ and $\lambda$ with $\kappa \leq \delta^{\mathcal{M}_\alpha}$.

We would like to think of transitive class models $M$ of ZFC as giving “premice” $(M; \in, \text{Ord} \cap M)$, so that, for example, we can think of the internal iterations of such $M$ as special cases of internal iterations of premice. Let us therefore say that $\mathcal{M}$ is a premouse* if either of the following holds:

(i) $\mathcal{M}$ is a premouse;
(ii) $\mathcal{M} = (M; \in)$ for some transitive class model $M$ of ZFC.

If $\mathcal{M} = (M; \in)$ is a premouse*, then by $\delta^\mathcal{M}$ we mean $\text{Ord} \cap M$ (which is not a genuine ordinal number if $M$ is a proper class).

[Mitchell, 1974] proved wellfoundedness results for internal iterations in the special case that all the $E_\alpha$ are given by normal ultrafilters on their critical points. His methods extend to the case of general extenders. We will now present a different and simpler approach to the same theorems. This
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The approach is attributed to R. Jensen in Dodd [1982]. The results we prove with it in this section are essentially from Dodd [19??]. The first lemma is the key to the whole method.

**Lemma 7.1.4.** Let \( \mathcal{M} = (M; \in, \delta^M) \) be a countable premouse\(^*\). Suppose that, for some ordinals \( \eta \) and \( \delta \leq \eta \), there is a \( \tau : \mathcal{M} \prec (V_\eta; \in, \delta) \).

Let \( E \) be a \((\kappa, \lambda)\)-extender in \( \mathcal{M} \) with \( \kappa \leq \delta^M \). Then there is a \( \sigma : \text{Ult}(\mathcal{M}; E) \prec (V_\eta; \in, \delta) \) such that \( \sigma \circ i_E^M = \tau \).

**Proof.** By the elementarity of \( \tau \), we have that \( \tau(E) \) is a \((\tau(\kappa), \tau(\lambda))\)-extender in the premouse\(^*\) \( (V_\eta; \in, \delta) \) and so in \( V \). For each \( a \in [\lambda]^{<\omega} \), let

\[
X_a = \bigcap_{Y \in E_a} \tau(Y).
\]

Since \( M \) is countable, each \( X_a \) is a countable intersection of elements of \( (\tau(E))_{\tau(a)} \); hence each

\[
X_a \in (\tau(E))_{\tau(a)}.
\]

By the countability of \([\lambda]^{<\omega}\) and the countable completeness of \( \tau(E) \) (clause (4) of Lemma 6.1.2), there is an order preserving \( h : \tau''[\lambda]^{<\omega} \to \tau(\kappa) \) such that

\[
(\forall a \in [\lambda]^{<\omega}) h''\tau(a) \in X_a.
\]

We define \( \sigma \) by setting

\[
\sigma(\pi_E^M([a, f]^M_E)) = (\tau(f))(h''\tau(a)).
\]

To see that \( \sigma \) is well-defined, suppose that \([a, f]^M_E = [b, g]^M_E\). Then

\[
Y = \{ z \in [\kappa]^{a \cup b} \mid f(z_{a,a \cup b}) = g(z_{b,a \cup b}) \} \in E_{a \cup b}.
\]

By the definition of \( X_{a \cup b} \),

\[
h''\tau(a \cup b) \in X_{a \cup b} \subseteq \tau(Y).
\]
Thus

\[(\tau(f))(h''\tau(a \cup b))_{\tau(a),\tau(a \cup b)} = (\tau(g))(h''\tau(a \cup b))_{\tau(b),\tau(a \cup b)}.\]

Since, e.g., \((h''\tau(a \cup b))_{\tau(a),\tau(a \cup b)} = h''\tau(a)\), we get that

\[(\tau(f))(h''\tau(a)) = (\tau(g))(h''\tau(b)),\]

i.e., that

\[\sigma(\pi^\mathcal{M}_E([a, f]^\mathcal{M}]) = \sigma(\pi^\mathcal{M}_E([b, g]^\mathcal{M})).\]

The proof that \(\sigma\) is elementary is similar to the proof that it is well-defined, and we omit it.

Finally we must prove commutativity. Let \(x \in M\).

\[\sigma(i^\mathcal{M}_E(x)) = \sigma(\pi^\mathcal{M}_E([\emptyset, c_x])) = c_{\tau(x)}(h''\tau(\emptyset)) = \tau(x),\]

as required. \(\square\)

Remark. Note that the proof gives directly an elementary embedding of \(\prod^\mathcal{M}_E \mathcal{M}\) into \((V_{\eta}, \in, \delta)\). Since \((V_{\eta}; \in, \delta)\) is wellfounded it follows that \(\prod^\mathcal{M}_E \mathcal{M}\) is wellfounded. Thus the proof gives a different way of showing e.g. that ultrapowers of \(V\) with respect to extenders are wellfounded.

The next lemma extends Lemma 7.1.4 to countable internal iterations.

**Lemma 7.1.5.** Let \(\mathcal{M} = (M; \in, \delta^\mathcal{M})\) be a countable premouse\(^*\). Suppose that, for some ordinals \(\eta\) and \(\delta \leq \eta\), there is a \(\tau : \mathcal{M} \prec (V_{\eta}; \in, \delta)\). Let \(\theta > 0\) be a countable ordinal and let \(\langle \mathcal{M}_\alpha \mid \alpha < \theta \rangle\) and \(\langle j_{\alpha, \beta} \mid \alpha \leq \beta < \theta \rangle\) witness that \(\langle E_\alpha \mid \alpha + 1 < \theta \rangle\) is an internal iteration of \(\mathcal{M}\). Let \(\langle \mathcal{M}_\beta, (j_{\alpha, \beta} \mid \alpha < \theta) \rangle\) be the direct limit of \(\langle \mathcal{M}_\alpha \mid \alpha < \theta \rangle\). Then there is a \(\tau^* : \tilde{\mathcal{M}}_\theta \prec (V_{\eta}; \in, \delta)\)

such that \(\tau^* \circ j_{0, \theta} = \tau\).

**Proof.** Since \(\mathcal{M}\) is countable and \(\theta\) is countable, it follows by induction that all the \(M_\alpha, \alpha < \theta\), are countable.
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By induction on $\alpha < \theta$, we define embeddings

$$\tau_\alpha : \mathcal{M}_\alpha \prec (V_\eta; \in, \delta).$$

When we define $\tau_\alpha$, we will make sure that

$$(\forall \beta < \alpha) \tau_\alpha \circ j_{\beta, \alpha} = \tau_\beta.$$ 

Let $\tau_0 = \tau$.

Next consider the case that $\alpha = \gamma + 1$ for some $\gamma$. We apply Lemma 7.1.4 with $\mathcal{M}_\gamma$ as the $\mathcal{M}$ of that lemma, with $\tau_\gamma$ as the $\tau$, and with $E_\gamma$ as the $E$. Let $\tau_\alpha$ be the $\sigma$ given by this application of Lemma 7.1.4. It is easy to see that our induction hypotheses for $\alpha$ are satisfied.

Now consider the case that $\alpha < \theta$ is a limit ordinal. Let $x \in M_\alpha$, where each $M_\beta = (M_\beta, \in; \delta^{M_\beta})$. Then $x = j_{\beta, \alpha}(y)$ for some $\beta < \alpha$ and some $y \in M_\beta$. Set

$$\tau_\alpha(x) = \tau_\beta(y).$$

It is easy to see that $\tau_\alpha$ is well-defined and that $\tau_\alpha \circ j_{\beta, \alpha} = \tau_\beta$ for all $\beta < \alpha$.

The definition of $\tau^*$ is similar to that of limit $\tau_\alpha$. Let $x \in \tilde{M}_\theta$, where $\tilde{M}_\theta = (\tilde{M}_\theta; \tilde{\in}_\theta, \tilde{\delta}_\theta)$. Then $x = \tilde{j}_{\alpha, \theta}(y)$ for some $\alpha < \theta$ and some $y \in M_\alpha$. Set

$$\tau^*(x) = \tau_\alpha(y).$$

It is easy to see that $\tau^*$ has the required properties. \qed

**Corollary 7.1.6.** Assume all the hypotheses of Lemma 7.1.5. Then the direct limit model $\mathcal{M}_\theta$ is wellfounded.

**Proof.** If $\tau^*$ is given by the lemma, then $\tau^*$ embeds $\tilde{M}_\theta$ into a wellfounded structure. \qed

**Remark.** The corollary is trivial in the case $\theta$ is a successor ordinal, but the lemma is not.

**Theorem 7.1.7.** Every internal iteration of $V$ has a wellfounded direct limit model.
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**Proof.** Let $\mathcal{J}$ be an internal iteration of $V$. Assume for a contradiction that the direct limit model of $\mathcal{J}$ is not wellfounded. Let $\delta$ be such that $\mathcal{J} \in V_\delta$ and let $\eta > \delta$ be such that $(V_\eta; \in, \delta)$ is a premouse satisfying “the direct limit model of $\mathcal{J}$ is not wellfounded.” Let $(X; \in, \delta)$ be a countable elementary submodel of $(V_\eta; \in, \delta)$ such that $\mathcal{J} \in X$. Let $\pi : X \cong M$ with $M$ transitive. Let $\mathcal{I} = \pi(\mathcal{J})$. Then $(M; \in, \pi(\delta))$ is a countable premouse and

$$\pi^{-1} : (M; \in, \pi(\delta)) \prec (V_\eta; \in, \delta).$$

Moreover

$$(M; \in, \pi(\delta)) \models \text{“the direct limit model of $\mathcal{I}$ is not wellfounded.”}$$

By the absoluteness of wellfoundedness, it follows that the direct limit model of $\mathcal{I}$ is not wellfounded. But this contradicts Corollary 7.1.6.

If $M$ is a model of ZFC, then applying Theorem 7.1.7 in $M$ gives that, if $\mathcal{I}$ is an internal iteration of $M$ and $\mathcal{I} \in M$, then the direct limit model of $\mathcal{I}$ is wellfounded. Exercise 7.1.1 eliminates the assumption that $\mathcal{I} \in M$ in the case that $\text{Ord} \cap M$ has uncountable cofinality.

**Exercise 7.1.1.** Let $M$ be transitive class model of ZFC such that $\text{Ord} \cap M$ is not an ordinal number of cofinality $\omega$. Prove that every internal iteration of $M$ has a wellfounded direct limit model.

**Hint.** Suppose that $\mathcal{J}$ is a counterexample.

First assume that $M$ is a proper class. Deduce that there is a premouse $(V^M_\eta; \in, \delta) \in M$ such that $\mathcal{J}$ is also an internal iteration of $(V^M_\eta; \in, \delta)$ with illfounded direct limit model. Next use the Löwenheim–Skolem Theorem to show that there are a premouse $\bar{M}$ with countable universe, an embedding $\tau : \bar{M} \prec (V^M_\eta; \in, \delta)$, and a countable iteration $\mathcal{I}$ of $\bar{M}$ whose direct limit model is not wellfounded. Now use an absoluteness argument to show that Corollary 7.1.6 fails in $M$.

Now assume that $M$ is a set. Use the Löwenheim–Skolem Theorem as above to get a countable $\bar{M}$, an embedding $\tau : \bar{M} \prec M$, and a countable iteration $\mathcal{I}$ of $\bar{M}$ with illfounded direct limit model. Use the hypothesis about $\text{cf}((\text{Ord} \cap M)$ to prove the existence of an ordinal $\eta$ of $M$ such that $\text{range}(\tau) \subseteq V^M_\eta \prec M$. Now contradict Corollary 7.1.6 in $M$ as in the first case.
7.2 General Iteration Trees

The main applications of iterations and iteration trees are in the study of canonical inner models for large cardinal axioms. One defines the property of being a “canonical” model, proves that canonical models exist and satisfy the large cardinal axiom, and one proves (the “Comparison Lemma”) that canonical models are indeed canonical by proving that any two of them can be elementarily embedded into a third. The elementary embeddings in question are some $j_{0,\alpha}$ of an iteration or an iteration tree. This technique was used in a primitive form in [Kunen, 1970], where the large cardinal axiom is (mainly) the existence of a measurable cardinal, and the iterations were the iterated ultrapowers we discussed in Chapter 3. [Mitchell, 1974] introduced the general Comparison Lemma method, in the context of axioms asserting the existence of measurable cardinals with a rich array of normal ultrafilters. [Mitchell, 1979] extended the method beyond the range of normal ultrafilters. [Dodd, ] and [Baldwin, 1986] developed it far enough to get canonical inner models for strong cardinals and more, employing internal iterations in our general sense. At, or a little before, the level of Woodin cardinals, internal iterations are no longer adequate. [Steel,?] used primitive iteration trees in studying inner model theory at about this level. Iteration trees proper were introduced in [Martin and Steel, 1988] and [Martin and Steel, 1989], and they were used as we will use them in Chapter 8, to prove determinacy results. In [Martin and Steel, 1994] their theory was further developed and they were applied to get inner models for Woodin cardinals.

Remark. The historical sketch just given omitted a major part of inner model theory: fine structure and core models. This omitted subject was invented by Ronald Jensen. Dodd, Mitchell, and Steel have, after Jensen, probably made the most important contributions to it. See the introduction to [Martin and Steel, 1994] for a longer historical sketch that does not omit fine structure and core models.

The basic step in generating an iteration tree is, like the basic step in generating an internal iteration, the ultrapower of a given model with respect to an extender. In the case of internal iterations, the extender belongs to the given model and is an extender in it. In the case of iteration trees, the extender may be an extender in a different model and may not belong to the given model at all. Already in the last chapter (page 358), we discussed this
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kind of ultrapower for the case of transitive class models of ZFC. We now must extend the discussion to the case of premice.

Let us say that premice $M = (M; \in, \delta^M)$ and $N = (N; \in, \delta^N)$ agree through $\alpha$ if $\alpha \leq \min\{\delta^M, \delta^N\}$ and $M$ and $N$ agree through $\alpha$. More generally, let us say that premice $^*M$ and $N$ agree through $\alpha$ if $\alpha \leq \min\{\delta^M, \delta^N\}$ and the universes (first components) of $M$ and $N$ agree through $\alpha$. (The hybrid concept of a premouse $^*$ was defined on page 372.)

Suppose that $\kappa$ is an ordinal and that $M$ and $N$ are premice $^*$ agreeing through $\kappa + 1$. Suppose that $E$ is a $(\kappa, \lambda)$-extender in $M$, for some $\lambda$. Then we can define $\prod^N_E N$ just as we defined, on page 358, $\prod^N_E (N; \in)$ for $N$ a transitive class model of ZFC and $E$ a $(\kappa, \lambda)$-extender in another such model $M$ agreeing with $N$ through $\kappa + 1$. Our notation for such ultrapowers will be the obvious combination of that for the $\prod^N_E (N; \in)$ and that for the $\prod^M_E M$ of the preceding section.

The proof of the following theorem is just like that of the special case $M = N$ (Theorems 7.1.1 and 6.1.3).

**Theorem 7.2.1.** Let $M$ and $N$ be premice $^*$. Let $\kappa$ and $\lambda > \kappa$ be ordinals of $M$ with $\kappa < \delta^N$. Assume that $M$ and $N$ agree through $\kappa + 1$. Let $E$ be an $(\kappa, \lambda)$-extender in $M$. Let $\varphi(v_1, \ldots, v_n)$ be any formula of the language of set theory. Let $\langle a_1, f_1 \rangle, \ldots, \langle a_n, f_n \rangle$ be elements of $D^N_E$. Let $b = \bigcup_{1 \leq i \leq n} a_i$. Then

$$\prod^N_E N \models \varphi([a_1, f_1]^N_E, \ldots, [a_n, f_n]^N_E) \leftrightarrow \{z \in [\kappa]^{|b|} \mid N \models \varphi(f_1(z_{a_1, b}), \ldots, f_n(z_{a_n, b})) \} \in E_b.$$ 

Theorem 7.2.1 gives us the canonical $(i')^N_E : N \prec \prod^N_E N$. The next lemma is proved as were earlier analogous lemmas.

**Lemma 7.2.2.** Let $M$, $N$, $\kappa$, $\lambda$, and $E$ be as in the statement of Theorem 7.2.1. Then $\prod^N_E N$ is set-like.

$\prod^N_E N$ may not be wellfounded. If it is, and if $N$ is a premouse $(N; \in, \delta)$, then we get a unique

$$\pi^N_E : \prod^N_E N \cong \text{Ult}(N; E) = (\text{Ult}(N; E); \in, \delta^{\text{Ult}(N; E)})$$
with \( \text{Ult}(N; E) \) transitive. If it is wellfounded and if \( \mathcal{N} = (N; \in) \), then we get a unique

\[
\pi_{E}^{\mathcal{N}} = \pi_{E}^{\mathcal{N}} : \prod_{E}^{\mathcal{N}} \cong \text{Ult}(\mathcal{N}; E) = (\text{Ult}(N; E); \in)
\]

with \( \text{Ult}(N; E) \) transitive. In either case we define

\[
i_{E}^{\mathcal{N}} = \pi_{E}^{\mathcal{N}} \circ (i')^{\mathcal{N}} : N \prec \text{Ult}(\mathcal{N}; E).
\]

When \( \mathcal{N} = (N; \in) \), we may of course also write \( i_{E}^{\mathcal{N}} \) for \( i_{E}^{\mathcal{N}} \). Note that \( \text{Ult}(\mathcal{N}; E) \) is a premouse* and is a premouse if and only if \( \mathcal{N} \) is a premouse.

The following analogue of Lemma 6.3.17 is proved just as was that lemma.

**Lemma 7.2.3.** Let \( \mathcal{M} \) be a premouse*. Let \( E \) be a \((\kappa, \lambda)\)-extender in \( \mathcal{M} \). Let \( \zeta \geq \kappa \) be an ordinal of \( \mathcal{M} \). Let \( \mathcal{N} \) be a premouse* such that \( \kappa \leq \delta^{\mathcal{N}} \) and such that \( \mathcal{M} \) and \( \mathcal{N} \) agree through \( \zeta + 1 \). Assume that \( \prod_{E}^{\mathcal{N}} \mathcal{N} \) is wellfounded. Then

(a) If \( a \in [\lambda]^{<\omega} \) and \( f : [\kappa]^{a} \rightarrow (\zeta^{+})^{\mathcal{M}} \cup V_{\zeta+1}^{\mathcal{M}} \), then

\[
\pi_{E}^{\mathcal{M}}([a, f]^{\mathcal{M}}) = \pi_{E}^{\mathcal{N}}([a, f]^{\mathcal{N}}); \]

(b) \((\forall \alpha)(\alpha \leq (\zeta^{+})^{\mathcal{M}} \rightarrow i_{E}^{\mathcal{M}}(\alpha) = i_{E}^{\mathcal{N}}(\alpha)); \) in particular, \( i_{E}^{\mathcal{M}}(\kappa) = i_{E}^{\mathcal{N}}(\kappa); \)

(c) \( \text{Ult}(\mathcal{M}; E) \) and \( \text{Ult}(\mathcal{N}; E) \) agree through \( i_{E}^{\mathcal{M}}(\zeta) + 1 \); in particular, they agree through \( i_{E}^{\mathcal{M}}(\kappa) + 1 \).

If there is a premouse* \( \mathcal{M} \) such that \( E \) is an \((\kappa, \lambda)\)-extender in \( \mathcal{M} \), set \( \text{crit}(E) = \kappa \). Thus, for any premouse* \( \mathcal{N} \) such that \( i_{E}^{\mathcal{N}} \) exists, we have

\[
\text{crit}(E) = \text{crit}(i_{E}^{\mathcal{M}}) = \text{crit}(i_{E}^{\mathcal{N}}).
\]

The next lemma is a consequence of Lemma 7.2.3 describing the amount of agreement between \( \mathcal{M} \) and \( \text{Ult}(\mathcal{N}; E) \) for \( E \) an extender in \( \mathcal{M} \).

**Lemma 7.2.4.** Assume all the hypotheses of Lemma 7.2.3. Let \( \rho = \text{strength}^{\mathcal{M}}(E) \). Then

(1) \( \text{Ult}(\mathcal{N}; E) \) and \( \mathcal{M} \) agree through \( \rho; \)
By the definition of strength^M(E) and the fact that Ult(M; E) is a class in M,
\[ V_{\rho+1}^{\text{Ult}(\mathcal{M}; E)} \subseteq V_{\rho+1}^\mathcal{M} \]  
By (c) of Lemma 7.2.3 and the fact that \( \rho \leq i_E^\mathcal{M}(\kappa) \),
\[ V_{\rho+1}^{\text{Ult}(\mathcal{N}; E)} = V_{\rho+1}^{\text{Ult}(\mathcal{M}; E)}. \]
The lemma follows. \(\square\)

Let \( \theta \) be a non-zero ordinal number. A partial ordering \( T\) of \( \theta \) is a tree ordering of \( \theta \) if

(i) for all \( \beta < \theta \), the set of all \( \alpha \) such that \( \alpha T \beta \) is wellordered by \( T \);
(ii) \( T \) respects the natural order: if \( \alpha T \beta \) then \( \alpha < \beta \);
(iii) \( 0 \) is the \( T \)-least element of \( \theta \): if \( 0 < \alpha < \theta \) then \( 0T\alpha \);
(iv) for all \( \alpha < \theta \), \( \alpha \) is a successor ordinal if and only if \( \alpha \) is a \( T \)-successor, i.e., if and only if \( \alpha \) has an immediate predecessor with respect to \( T \);
(v) for all limit ordinals \( \lambda < \theta \), the set of all \( \alpha \) such that \( \alpha T \lambda \) is an unbounded subset of \( \lambda \) (with respect to \(<\)).

For successor ordinals \( \alpha < \theta \), we define \( \alpha^-T \) to be the immediate predecessor of \( \alpha \) with respect to \( T \), which exists by clause (iv). When there is no ambiguity, we write \( \alpha^- \) for \( \alpha^-T \).

To avoid giving two definitions of “iteration tree,” we make use of the concept of premice*. An iteration tree is a triple
\[ \mathcal{T} = (\mathcal{M}, T, \langle E_\alpha | \alpha + 1 < \theta \rangle), \]
such that there are premice* \( \mathcal{M}_\alpha \), \( \alpha < \theta \), and embeddings \( j_{\alpha,\beta}, \alpha T \beta < \theta \), satisfying

(a) \( T \) is a tree ordering of \( \theta \);
(b) \( \mathcal{M}_0 = \mathcal{M} \);
(c) each \( E_\alpha \) is an extender in \( \mathcal{M}_\alpha \) with \( E_\alpha \in V_{\delta^\mathcal{M}_\alpha} \);
(d) for \( \alpha T \beta T \gamma < \theta \), \( j_{\alpha,\gamma} = j_{\beta,\gamma} \circ j_{\alpha,\beta} \);
(e) for each \( \alpha \) such that \( \alpha + 1 < \theta \),

(i) \( M_\alpha \) and \( M_{(\alpha+1) \tau} \) agree through \( \text{crit}(E_\alpha) + 1 \);
(ii) \( M_{\alpha+1} = \text{Ult}(M_{(\alpha+1) \tau}; E_\alpha) \);
(iii) \( j_{(\alpha+1) \tau, \alpha+1} = i_{E_\alpha}^M \).

(f) for each limit \( \lambda < \theta \), \((M_\lambda, \langle j_{\alpha, \lambda} \mid \alpha T \lambda \rangle)\) is the canonical limit of \((\langle M_\alpha \mid \alpha T \lambda \rangle, \langle j_{\alpha, \beta} \mid \alpha T \beta T \lambda \lor \alpha = \beta T \lambda \rangle)\), where \( j_{\alpha, \alpha} \) is the identity embedding of \( M_\alpha \) into itself for each \( \alpha < \theta \).

Note that (c) and (e)(i) imply that \( \text{crit}(E_\alpha) < \delta^{M_{(\alpha+1) \tau}} \). Note also that \( \mathcal{T} \) uniquely determines the \( M_\alpha \) and the \( j_{\alpha, \beta} \). By \( T^\mathcal{T}, E^T_\alpha \), and \( j^T_{\alpha, \beta} \) we mean respectively the tree ordering, the \( \alpha \)th extender, and the \( \alpha \)th embedding of \( \mathcal{T} \). Our notation for the premice* of \( \mathcal{T} \) will be in terms of the first component of \( \mathcal{T} \): If the first component is \( N \), then \( N^T_\alpha \) is the \( \alpha \)th premouse* of \( \mathcal{T} \).

If \( \mathcal{T} = (M, T, \langle E_\alpha \mid \alpha + 1 < \theta \rangle) \) is an iteration tree, then \( \mathcal{T} \) is an iteration tree on \( M \), and \( \theta \) is the length of \( \mathcal{T} \). We write \( \ell(\mathcal{T}) \) for the length of \( \mathcal{T} \). If \( M \) is a transitive class model of ZFC and \( \mathcal{T} \) is an iteration tree on \( \langle M; \in \rangle \), then we will also say that \( \mathcal{T} \) is an iteration tree on \( M \).

If \( I \) is an internal iteration of \( M \), then \((M, <, I)\) is an iteration tree on \( M \). Thus internal iterations are essentially iteration trees whose tree orderings are linear and so, by property (ii) of tree orderings, are the natural orderings of their lengths.

It would accord better with the concept of internal iterations if we defined iteration trees on \( M \) to be of the form \((T, \langle E_\alpha \mid \alpha + 1 < \theta \rangle)\). This would have the additional virtue of making iteration trees sets, where the actual definition makes them proper classes if \( M \) is a proper class. Indeed it was for just this reason that we did not make \( M \) a component of internal iterations on \( M \). Unfortunately, our notation would become too cumbersome if we were to do likewise for iteration trees on \( M \). We will often need notation such as \( j^T_{\alpha, \beta} \), and we do not want to write instead \( j^T_{\alpha, \beta}^M \) or something more complicated when we have a more complex name than \( M \).

The amount of agreement between two of the models of an iteration tree is related to the strength of the extenders of the tree. If \( \mathcal{T} \) is an iteration tree on \( M \) and \( \alpha < \beta \leq \ell(\mathcal{T}) \), then set

\[
\rho^\mathcal{T}(\alpha, \beta) = \min \{ \text{strength}^{M^{\mathcal{T}}_\gamma}(E^T_\gamma) \mid \alpha \leq \gamma < \beta \}.
\]
Lemma 7.2.5. Let $T$ be an iteration tree on $M$ and let $\alpha < \beta < \ell h(T)$. Then

1. $M^\alpha_T$ and $M^\beta_T$ agree through $\rho^T(\alpha, \beta)$;
2. $V^M_{\rho^T(\alpha, \beta)+1} \subseteq V^M_{\rho^T(\alpha, \beta)+1}$.

Proof. We suppress the superscript $T$, and we suppress the subscript $T$. Fix $\alpha < \ell h(T)$. We proceed by induction on $\beta$ for $\alpha < \beta < \ell h(T)$.

First suppose that $\beta = \gamma + 1$ for some $\gamma \geq \alpha$. Let $\delta = \beta^-$. Now $M_\delta = \text{Ult}(M_\delta; E_\gamma)$, and by part (1) of Lemma 7.2.4 $\text{Ult}(M_\delta; E_\gamma)$ agrees with $M_\gamma$ through $\text{strength}^M_{\gamma}(E_\gamma)$. If $\alpha = \gamma$, then $\rho(\alpha, \beta) = \text{strength}^M_{\gamma}(E_\gamma)$. If $\alpha < \gamma$, then $\rho(\alpha, \beta) = \text{min}\{\text{strength}^M_{\gamma}(E_\gamma), \rho(\alpha, \gamma)\}$, and induction gives us that $M_\alpha$ and $M_\gamma$ agree through $\rho(\alpha, \gamma)$. In either case, we have (1). If $\alpha = \gamma$ or if $\text{strength}^M_{\gamma}(E_\gamma) \leq \rho(\alpha, \gamma)$, then part (2) of Lemma 7.2.4 and (if $\alpha < \gamma$) induction give that

$$V_{\text{strength}^M_{\gamma}(E_\gamma)+1} \subseteq V_{\text{strength}^M_{\gamma}(E_\gamma)+1} \subseteq V_{\text{strength}^M_{\gamma}(E_\gamma)+1}.$$ 

If $\alpha < \gamma$ and $\rho(\alpha, \gamma) \leq \text{strength}^M_{\gamma}(E_\gamma)$, then part (2) of Lemma 7.2.4 and induction give that

$$V_{\rho(\alpha, \gamma)+1} \subseteq V_{\rho(\alpha, \gamma)+1} \subseteq V_{\rho(\alpha, \gamma)+1}.$$ 

In either case (2) follows.

Now suppose that $\beta$ is a limit ordinal. We first show that there are only finitely many $\gamma$ such that

$$(\gamma + 1) T \beta \land \text{crit}(E_\gamma) \leq \rho(\alpha, \beta).$$

Assume that $\gamma_0 < \gamma_1 < \cdots$ witness that this fails. For each $i \in \omega$, $\text{crit}(E_{\gamma_i}) \leq \rho(\alpha, \beta)$. Hence we have for each $i$ that

$$j(\gamma_{i+1}^-)(\gamma_{i+1}^-) - (\text{crit}(E_{\gamma_i})) \geq j(\gamma_{i+1}) - \gamma_{i+1}(\text{crit}(E_{\gamma_i}))$$

$$= i^{M_{\gamma_{i+1}}^-}_{E_{\gamma_i}}(\text{crit}(E_{\gamma_i}))$$

$$= i^{M_{\gamma_{i+1}}}_{E_{\gamma_i}}(\text{crit}(E_{\gamma_i}))$$

$$\geq \text{strength}^M_{\gamma_{i+1}}(E_{\gamma_i})$$

$$\geq \rho(\alpha, \beta).$$
Since $j_{(\gamma_i+1)^-} \cdot (j_{(\gamma_i+1)^-} \cdot (\text{crit}(E_{\gamma_i})) > \text{crit}(E_{\gamma_i})$, it follows that

$$j_{(\gamma_i+1)^-} \cdot (\rho(\alpha, \beta)) > \rho(\alpha, \beta).$$

This gives us the contradiction that $\langle j_{(\gamma_i+1)^-} \cdot (\rho(\alpha, \beta)) \mid i \in \omega \rangle$ is an infinite descending sequence of ordinals. Next we observe that, since $\{\delta \mid \delta T \beta\}$ is unbounded in $\beta$, there is a $\delta$ such that $\alpha < \delta T \beta$ and

$$(\forall \gamma)((\delta < \gamma \land (\gamma + 1) T \beta) \to \text{crit}(i_{E_{\gamma}}) > \rho(\alpha, \beta)).$$

Thus $\text{crit}(j_{\delta, \beta}) = \rho(\alpha, \beta)$, and so $M_\beta$ and $M_\delta$ agree through $\rho(\alpha, \beta) + 1 = \rho(\alpha, \delta) + 1$. Hence (1) and (2) for $\alpha$ and $\beta$ follow from (1) and (2) for $\alpha$ and $\delta$.

**Corollary 7.2.6.** Let $T = (M, T, \langle E_\alpha \mid \alpha + 1 < \ellh(T) \rangle)$ be an iteration tree and let $\alpha + 1 < \ellh(T)$. Then

(a) $\text{crit}(E_\alpha) + 1 \leq \rho^T((\alpha + 1)^-, \alpha)$;

(b) $((\forall \beta)((\alpha + 1)^- T \leq \beta < \alpha \to \text{crit}(E_\alpha) + 1 \leq \text{strength}^M_\beta(E_\beta))$.

**Proof.** We omit the superscript $T$ and the subscript $T$.

By clause (e) in the definition of an iteration tree, $M_\alpha$ and $M_{(\alpha+1)^-}$ agree through $\text{crit}(E_\alpha) + 1$. But part (2) of Lemma 7.2.5 implies that they do not agree through $\rho((\alpha + 1)^-, \alpha) + 1$. Thus $\text{crit}(E_\alpha) + 1 \leq \rho((\alpha + 1)^-, \alpha)$.

Assume that $(\alpha + 1)^- \leq \beta < \alpha$. By the definition of the $\rho$ function, $\rho((\alpha + 1)^-, \alpha) \leq \text{strength}^M_\beta(E_\beta)$. Thus (b) for $\beta$ follows from (a).

The main constructions of Chapter 8 will be constructions of iteration trees. There are three ingredients needed to construct an iteration tree on $M$:

1. the extenders $E_\alpha$;
2. wellfoundedness at successor ordinals $\alpha + 1$, i.e., wellfoundedness of $\prod_{E_\alpha} M_{(\alpha+1)^-}$;
3. wellfoundedness at limit ordinals $\lambda$, i.e., the existence of a $T$-chain that is unbounded in $\lambda$ such that the corresponding direct limit model is wellfounded.
For (1) we will use the One-Step Lemma, Lemma 6.3.18. The only successor ordinals that will concern us are the finite ordinals, and in the next section we will prove that, for \( \alpha \) finite, (2) holds very generally, e.g. it holds if \( \mathcal{M} \) is a model of ZFC. The only limit ordinal that will concern us is \( \omega \), and in §4 we will prove that (3) holds for \( \lambda = \omega \) under conditions that will be satisfied by our constructions in Chapter 8.

7.3 Finite Trees

In our wellfoundedness proofs, we will be given an iteration tree \( \mathcal{T} \) on a premouse \( \mathcal{M} \), a tree for which wellfoundedness fails, and we will also be given an elementary embedding \( \tau : \mathcal{M} \prec Q \subseteq \mathcal{N} = (V; \in, \delta) \). We will construct an iteration tree \( \mathcal{U} \) on \( \mathcal{N} \) and embeddings \( \tau_\alpha : \mathcal{M}_\alpha^\mathcal{T} \prec Q_\alpha \subseteq \mathcal{N}_\alpha^\mathcal{U} \). In the first proof, we will make sure that each \( \tau_\alpha \) belongs to the universe of \( \mathcal{N}_\alpha^\mathcal{U} \) and use this fact to derive a contradiction. In the second proof, we will get our contradiction from some further models and embeddings that are constructed at the same time as the \( \mathcal{N}_\alpha^\mathcal{U} \) and the \( \tau_\alpha \). To keep our inductive constructions going, we will in both cases need a certain amount of agreement among the \( \tau_\alpha \). The next definition gives the appropriate notion of “agreement.”

Suppose that \( \tau : \mathcal{M} \prec \mathcal{N} \) and \( \tau' : \mathcal{M}' \prec \mathcal{N}' \), where \( \mathcal{M} \), \( \mathcal{M}' \), \( \mathcal{N} \), and \( \mathcal{N}' \) are premice. For ordinals \( \eta \), say that \( \tau \) and \( \tau' \) agree through \( \eta \) if

(a) \( \mathcal{M} \) and \( \mathcal{M}' \) agree through \( \eta \);  
(b) \( \tau(V_\eta^\mathcal{M}) = \tau'(V_\eta^\mathcal{M}') \);  
(c) \( \tau \mid V_\eta^\mathcal{M} = \tau' \mid V_\eta^\mathcal{M}' \).

Note that (b) can be restated as follows: \( \tau(\eta) = \tau'(\eta) \) and \( \mathcal{N} \) and \( \mathcal{N}' \) agree through \( \tau(\eta) \). Note also that (b) follows from (c) if \( \eta \) is a successor ordinal.

The next lemma will be one of the main tools in our construction of the embeddings \( \tau_\alpha \). A slight variant of it is called the Shift Lemma in [Martin and Steel, 1994].

**Lemma 7.3.1.** Let \( \tau : \mathcal{M} \prec \mathcal{N} \) and \( \tau' : \mathcal{M}' \prec \mathcal{N}' \), with \( \mathcal{M} \), \( \mathcal{M}' \), \( \mathcal{N} \), and \( \mathcal{N}' \) premice. Suppose that \( \tau \) and \( \tau' \) agree through \( \kappa + 1 \). Suppose that \( E \) is an extender in \( \mathcal{M} \) with \( \text{crit}(E) = \kappa \). Suppose that \( \prod_{\tau(E)}^{\mathcal{N}'} \mathcal{N}' \) is wellfounded. Then \( \prod_{E}^{\mathcal{M}'} \mathcal{M}' \) is wellfounded. Moreover, if \( \sigma : \text{Ult}(\mathcal{M}'; E) \to \text{Ult}(\mathcal{N}'; \tau(E)) \)
is given by
\[ \sigma(\pi^M_E([a, f]^M_E)) = \pi^{N'}_{\tau(E)}([\tau(a), \tau'(f)]^{N'}_{\tau(E)}), \]
then \( \sigma \) is well-defined and elementary, and \( \sigma \) and \( \tau \) agree through strength \( \mathcal{M}(E) \).

Furthermore, the following diagram commutes:
\[
\begin{array}{ccc}
\text{Ult}(\mathcal{M}; E) & \xrightarrow{\sigma} & \text{Ult}(N'; \tau(E)) \\
\uparrow i^M_E & & \uparrow i^{N'}_E \\
\mathcal{M}' & \xrightarrow{\tau'} & N'
\end{array}
\]

**Proof.** For \([a, f]^M_E \in \prod^M_E \mathcal{M}'\), set
\[
\tilde{\sigma}([a, f]^M_E) = [\tau(a), \tau'(f)]^{N'}_{\tau(E)}.
\]
We will show that \( \tilde{\sigma} \) is well-defined and that
\[
\tilde{\sigma} : \prod^M_E \mathcal{M}' \prec \prod^{N'}_{\tau(E)} N'.
\]
Since \( \prod^{N'}_{\tau(E)} N' \) is wellfounded by hypothesis, this will show that \( \prod^M_E \mathcal{M}' \) is wellfounded. It will also show that \( \sigma \) is welldefined and elementary.

To show that \( \tilde{\sigma} \) is well-defined, suppose that \([a, f]^M_E = [b, g]^M_E\). Then
\[
X = \{ z \in [\kappa]^{|a|} | f(z_{a,a,b}) = g(z_{b,a,b}) \} \in E_{a,b}.
\]
Now \( X \in V_{\kappa+1} \), so our hypotheses about agreement imply that \( X \in V^M_{\kappa+1} \) and that \( \tau(X) = \tau'(X) \). The elementarity of \( \tau \) gives that \( \tau(X) \in (\tau(E))_{\tau(a,b)} \).

Using these facts and the elementarity of \( \tau' \), we get that
\[
\{ z \in [\tau'(\kappa)]^{|a|} | (\tau'(f))(z_{a,a,b}) = (\tau'(g))(z_{b,a,b}) \} = \tau'(X) = \tau(X) \in (\tau(E))_{\tau(a,b)}.
\]
Hence \([\tau(a), \tau'(f)]^{N'}_{\tau(E)} = [\tau(b), \tau'(g)]^{N'}_{\tau(E)}\).

We omit the proof that \( \tilde{\sigma} \) is elementary, as it is similar to the proof that \( \tilde{\sigma} \) is well-defined.

By Lemma 7.2.4, \( \text{Ult}(\mathcal{M}; E) \) and \( \mathcal{M} \) agree through strength \( \mathcal{M}(E) \). To show that \( \sigma \) and \( \tau \) agree through strength \( \mathcal{M}(E) \), what we must show is that
\[
\sigma \upharpoonright (V^\text{Ult}(\mathcal{M}; E)_{\text{strength}^\mathcal{M}(E)}) \cup \{ V^\text{Ult}(\mathcal{M}; E)_{\text{strength}^\mathcal{M}(E)} \}) = \tau \upharpoonright (V^\text{Ult}(\mathcal{M}; E)_{\text{strength}^\mathcal{M}(E)}) \cup \{ V^\text{Ult}(\mathcal{M}; E)_{\text{strength}^\mathcal{M}(E)} \}).
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Now strength\(^{\mathcal{M}}(E) \leq i_E(\kappa)\), and Lemma 7.2.3 gives that \(i_E(\kappa) = i_E(\kappa)\) and that\
\[
V_{i_E(\kappa)+1}^{\mathcal{M}'(E)} = V_{i_E(\kappa)+1}^{\mathcal{M}(E)},
\]
with the latter clearly a subset of the universe of \(\mathcal{M}\). Thus it suffices to prove that\
\[
\sigma \upharpoonright V_{i_E(\kappa)+1}^{\mathcal{M}'(E)} = \tau \upharpoonright V_{i_E(\kappa)+1}^{\mathcal{M}(E)}.
\]
Every element of \(V_{i_E(\kappa)+1}^{\mathcal{M}'(E)}\) is of the form \(\pi_{\mathcal{M}'}([a,f]_{\mathcal{M}'})\), with \(f : [\kappa]^a \to V_{\kappa+1}\). Consider such a \(\pi_{\mathcal{M}'}([a,f]_{\mathcal{M}'})\). By definition,\
\[
\sigma(\pi_{\mathcal{M}'}([a,f]_{\mathcal{M}'})) = \pi_{\mathcal{N}'}([\tau(a),\tau'(f)]_{\tau(E)}^{\mathcal{N}'}).\n\]
Since \(f\) can be coded by an element of \(V_{\kappa+1}\), we get that \(\tau(f) = \tau'(f)\) and so that\
\[
\tau_{\tau(E)}^{\mathcal{N}'}([\tau(a),\tau'(f)]_{\tau(E)}^{\mathcal{N}'}) = \pi_{\mathcal{N}'}([\tau(a),\tau(f)]_{\tau(E)}^{\mathcal{N}'}).\n\]
The agreement of \(\tau\) and \(\tau'\) through \(\kappa + 1\) means, in particular, that \(\mathcal{N}\) and \(\mathcal{N}'\) agree through \(\tau(\kappa) + 1\), and so Lemma 7.2.3 gives that\
\[
\tau_{\tau(E)}^{\mathcal{N}'}([\tau(a),\tau(f)]_{\tau(E)}^{\mathcal{N}'}) = \pi_{\tau(E)}^{\mathcal{N}'}([\tau(a),\tau(f)]_{\tau(E)}^{\mathcal{N}'}).\n\]
By the elementarity of \(\tau\),\
\[
\pi_{\tau(E)}^{\mathcal{N}'}([\tau(a),\tau(f)]_{\tau(E)}^{\mathcal{N}'}) = \tau(\pi_{\mathcal{M}'}([a,f]_{\mathcal{M}'})).\n\]
Since \(\mathcal{M}\) and \(\mathcal{M}'\) agree through \(\kappa + 1\), another application of Lemma 7.2.3 gives that\
\[
\tau(\pi_{\mathcal{M}'}([a,f]_{\mathcal{M}'})) = \tau(\pi_{\mathcal{M}'}([a,f]_{\mathcal{M}'})).\n\]
By this chain of equalities, it follows that\
\[
\sigma(\pi_{\mathcal{M}'}([a,f]_{\mathcal{M}'})) = \tau(\pi_{\mathcal{M}'}([a,f]_{\mathcal{M}'})).\n\]
It remains only to show that the diagram commutes. Let \(x\) belong to the universe of \(\mathcal{M}'\). Then\
\[
\sigma(i_{\mathcal{M}'}(x)) = \sigma(\pi_{\mathcal{M}'}([\emptyset,c_x]_{\mathcal{M}'})) = \pi_{\tau(E)}^{\mathcal{N}'}([\emptyset,c_{\tau'(x)}]_{\tau(E)}^{\mathcal{N}'}) = i_{\tau(E)}^{\mathcal{N}'}(\tau'(x)).\n\]
We are now ready to prove our wellfoundedness results for finite iteration trees.

**Theorem 7.3.2.** Let \( n \in \omega \).

1. Let \( T \) be an iteration tree of length \( n + 1 \) on a premouse \( \mathcal{M} \). Suppose that \( \tau : \mathcal{M} \prec (V_\nu; \in, \delta) \) for some ordinals \( \nu \) and \( \delta \). Suppose that \( n^* < n \) and that \( E \) is an extender in \( \mathcal{M}^T_n \) with \( \text{crit}(E) < \rho^T(n^*, n) \). Then \( \prod_{E}^{\mathcal{M}^T_n} \mathcal{M}^T_n \) is wellfounded.

2. Let \( U \) be an iteration tree of length \( n + 1 \) on \( V \). Suppose that \( n' < n \) and that \( F \) is an extender in \( (V; \in)^U_n \) with \( \text{crit}(F) < \rho^U(n', n) \). Then \( \prod_{F}^{(V; \in)^U_n} (V; \in)^U_n \) is wellfounded.

**Proof.** We first observe that, for each \( n \), (1) implies (2). To see this, suppose that \( U, n', F \) witness that (2) fails for \( n \). Let \( \delta \) be large enough that all the \( E^U_m \) and \( F \) belong to \( V_\delta \), and let \( \nu > \delta \) be such that \( \mathcal{M} = (V_\nu; \in, \delta) \) is a premouse and such that \( (V; \in)^U_n \) is not wellfounded. (This last condition is actually automatic.) Then \( (\mathcal{M}, T^U, (E^U_m \mid m < n)) \), the identity embedding, \( n' \), and \( F \) witness that (1) fails for \( n \).

Assume that the theorem is false. We may assume that \( n \) is the least number for which the theorem is false.

We will also assume that the universe \( M \) of \( \mathcal{M} \) (given by the hypotheses of (1)) is countable. To see that this assumption involves no loss of generality, let \( \eta \) be a limit ordinal such that our given \( T \) belongs to \( V_\eta \). Let \( X \) be a countable elementary submodel of \( V_\eta \) such that \( T \) and \( E \) are members of \( X \). Let \( \pi : X \models M' \) with \( M' \) transitive. Then \( \tau \circ (\pi^{-1} \upharpoonright \pi(M)) \) elementarily embeds \( \pi(M) \) into \( (V_\nu; \in, \delta) \) and, along with \( \pi(T), n^*, \) and \( \pi(E) \), witnesses that (1) fails for \( n \).

We will get a contradiction by building an iteration tree \( U \) on \( \mathcal{N} = (V; \in, \delta) \) of length \( n + 1 \) and proving that for some extender \( F \) in \( \mathcal{N}^U_n \) the ultrapower \( \prod_{E}^{\mathcal{N}^U_n} \mathcal{N}^U_n \) is not wellfounded.

The tree ordering of \( U \) will be \( T \). We will mostly suppress the subscript \( T \) and the superscripts \( T \) and \( U \). In particular, we will write \( E^U_m \) for the extender \( E^T_m \) (but never for \( E^U_m \)).

As we construct \( U \), we will also construct, for \( m \leq n \), embeddings

\[ \tau_m : \mathcal{M}_m \prec (V_{\nu_m}; \in, \delta_m) \]

where in fact \( \nu_m = j_{0,m}^U(\nu) \) and \( \delta_m = j_{0,m}^U(\delta) \). We denote \( V_{\nu_m}^N \) by \( Q_{\nu_m} \) and \( (V_{\nu_m}^N; \in, \delta_m) \) by \( Q_m \). We will make sure that

\[
(\forall m \leq n) \tau_m \in N_m.
\]

This will be the key to obtaining our contradiction.

The iteration tree \( \mathcal{U} \) will be

\[
(\mathcal{N}, T, (\tau_m(E_m) \mid m \leq n)).
\]

Let \( \mathcal{N}_0 = \mathcal{N} = (V; \in, \delta) \). Let \( \delta_0 = \delta, \nu_0 = \nu, \) and \( \tau_0 = \tau \).

Let \( m < n \) and assume that we have defined, for each \( k \leq m, N_k = (N_k; \in, \delta_k) \) and

\[
\tau_k : M_k \prec Q_k = (Q_k; \in, \delta_k) = (V_{\nu_k}^N; \in, \delta_k),
\]

in such a way that \( (\mathcal{N}, T \upharpoonright m+1, (\tau_k(E_k) \mid k < m)) \) is an iteration tree whose premice* are the \( N_k \) and that \( \tau_k \in N_k \) for each \( k \leq m \). Assume also that

(a) for all \( k < n \) such that \( (k+1)^- \leq m \leq k, \tau_{(k+1)^-} \) and \( \tau_m \) agree through \( \text{crit} (E_k) + 1 \);

(b) if \( n^* \leq m \) then \( \tau_{n^*} \) and \( \tau_m \) agree through \( \text{crit} (E) + 1 \).

The elementarity of \( \tau_m \) implies that \( \tau_m(E_m) \) is an extender in the premouse \( Q_m \) and so is an extender in \( N_m \).

If we can show that

\[
(m+1)^- < m \rightarrow \text{crit} (\tau_m(E_m)) < \rho^U((m+1)^-, m),
\]

then we can deduce from the minimality of \( n \) that \( \prod_{\tau_m(E_m)}^{N_{(m+1)^-}} N_{(m+1)^-} \) is well-founded. By Lemma 7.2.5, this is equivalent with showing that \( N_{(m+1)^-} \) and \( N_m \) agree through \( \text{crit} (\tau_m(E_m)) + 1 \). But this is a consequence of the assumption—a special case of (a)—that \( \tau_{(m+1)^-} \) and \( \tau_m \) agree through \( \text{crit} (E_m) + 1 \).

Let

\[
N_{m+1} = \text{Ult}(N_{(m+1)^-}; \tau_m(E_m)).
\]

Since \( E_m \in V_{\delta M_m}^\mathcal{U} \), it follows that \( \tau_m(E_m) \in V_{\delta_m}^\mathcal{U} = V_{\delta_m}^N \). Hence \( (N_0, T \upharpoonright m+2, (\tau_k(E_k) \mid k < m+1)) \) is an iteration tree on \( N_0 \).
Set $ν_{m+1} = i^{N_{(m+1)^-}}_{τ_m(E_m)}(ν_{(m+1)^-})$ and $δ_{m+1} = i^{N_{(m+1)^-}}_{τ_m(E_m)}(δ_{(m+1)^-})$. Now

$$\text{Ult}(Q_{(m+1)^-}; τ_m(E_m)) = (V^{N_{m+1}}; ε, δ_{m+1}).$$

Thus we may apply the Shift Lemma (Lemma 7.3.1) with $M_m$ for $M$, $M_{(m+1)^-}$ for $M'$, $Q_m$ for $N$, $Q_{(m+1)^-}$ for $N'$, $τ_m$ for $τ$, $τ_{(m+1)^-}$ for $τ'$, $\text{crit}(E_m)$ for $κ$, and $E_m$ for $E$. This gives us an embedding

$$σ : M_{m+1} ≺ \text{Ult}(Q_{(m+1)^-}; τ_m(E_m)) = Q_{m+1},$$

such that $σ$ and $τ_m$ agree through $\text{strength}^{M_m}(E_m)$.

Suppose that $k < n$ is such that $(k+1)^- < m+1 ≤ k$; i.e., suppose that $(k+1)^- ≤ m < k < n$. Our induction hypothesis (a) for $m$ guarantees that $τ_{(k+1)^-}$ and $τ_m$ agree through $\text{crit}(E_k) + 1$. By part (b) of Corollary 7.2.6, $\text{crit}(E_k) + 1 ≤ \text{strength}^{M_m}(E_m)$. Thus $τ_{(k+1)^-}$ and $σ$ agree through $\text{crit}(E_k) + 1$.

The argument just given shows that, for all $k < n$,

$$(k+1)^- < m+1 ≤ k → τ_{(k+1)^-} \text{ and } σ \text{ agree through } \text{crit}(E_k) + 1.$$

Hence induction hypothesis (a) would be true for $m+1$ if we were to set $τ_{m+1} = σ$.

If $n^* < m+1$ then the fact that $\text{crit}(E)+1 ≤ ρ^T(n^*, n) ≤ \text{strength}^{M_m}(E_m)$ and our induction hypothesis (b) for $m$ imply that $τ_{n^*}$ and $σ$ agree through $\text{crit}(E) + 1$. It follows easily that induction hypothesis (b) would also be true for $m+1$ if we made $τ_{m+1} = σ$.

Nevertheless, we cannot take $σ$ for $τ_{m+1}$ because $σ$ might not belong to $N_{m+1}$. Let

$$μ(m) = \sup\{\text{crit}(E_k) \mid (k+1)^- ≤ m < k < n\}.$$  

Let

$$μ'(m) = \begin{cases} μ(m) & \text{if } m < n^*; \\ \max\{μ(m); \text{crit}(E)\} & \text{if } n^* ≤ m. \end{cases}$$

If we can find a $\bar{τ} : M_{m+1} ≺ Q_{m+1}$ such that

$$\bar{τ} \upharpoonright (V^{M_{m+1}}; μ'(m+1)) = σ \upharpoonright (V^{M_{m+1}}; μ'(m+1)) \land \bar{τ} ∈ N_{m+1},$$

then, since $\bar{τ}$ and $σ$ will agree through $μ'(m+1) + 1$, we can set $τ_{m+1} = \bar{τ}$, and the induction step of our construction will be complete.
Since \( n \) is finite, \( \mu(m) \) and hence also \( \mu'(m) \) are suprema of finite sets of ordinals, each of which is smaller than \( \text{strength}^M_m(E_m) \). Hence \( \mu'(m) + 1 \leq \text{strength}^M_m(E_m) \). Let \( \chi = \sigma \upharpoonright (V^\text{M}_{\mu(m)+1}) = \tau_m \upharpoonright (V^\text{M}_{\mu'(m)+1}) \). Since \( \tau_m \in N_m \), it follows that \( \chi \in N_m \). All the universes \( M_k \) of the \( \mathcal{M}_k \) are countable transitive sets; since \( \text{crit} (\mathcal{M}_k) \) is a measurable cardinal, they all belong to and are countable in each \( N_k, k \leq m + 1 \). The function \( \chi \) has a subset of \( M_{m+1} \) for its domain, thus \( \chi \) can be coded by a subset of \( V^\text{M}_{\mu'(m)+1} \) that is countable in \( N_m \). Now \( \mu'(m) \) is either 0 or an infinite ordinal. For any infinite ordinal \( \alpha \), a countable subset of \( V_{\alpha+1} \) can be coded by an element of \( V_{\alpha+1} \). Hence \( \chi \) can be coded by an element of \( V^\text{M}_{\mu'(m)+1} \). But \( N_m \) and \( N_{m+1} \) agree through \( \mu'(m) + 1 \), and therefore \( \chi \) is an element of \( N_{m+1} \).

Let \( \langle a_i \mid i \in \omega \rangle \in N_{m+1} \) enumerate \( M_{m+1} \). Let \( U \) be the tree of all \( u \in \omega^{\omega}(Q_{m+1}) \) such that

1. \( u(i) = \chi(a_i) \) for all \( i < \ell \text{h}(u) \) with \( a_i \in \text{domain} (\chi) \);
2. for all formulas \( \varphi(v_1, \ldots, v_k) \) of the language of set theory and for all natural numbers \( i_1, \ldots, i_k \) with each \( i_j < \ell \text{h}(u) \), \( \mathcal{M}_{m+1} \models \varphi[a_{i_1}, \ldots, a_{i_k}] \) if and only if \( Q_{m+1} \models \varphi[u(i_1), \ldots, u(i_k)] \).

The function \( i \mapsto \sigma(a_i) \) belongs to \( [U] \). By absoluteness, there is an \( f \in N_{m+1} \) that belongs to \( [U] \). Then \( a_i \mapsto f(i) \) is our desired \( \bar{\tau} : M_{m+1} \prec Q_{m+1} \) such that \( \bar{\tau} \in N_{m+1} \) and \( \bar{\tau} \) extends \( \chi \).

Since \( E \) is an extender in \( \mathcal{M}_n \), we know that \( \tau_n(E) \) is an extender in \( Q_n \) and so in \( N_n \). To finish the proof, we will derive a contradiction from the fact that \( N_n \models \text{"\( \tau_n(E) \) is countably complete."} \) (By Lemma 6.1.5, this is the same as contradicting the wellfoundedness of \( \prod_{\tau_n(E), N_n, \text{M}_n} \)).

Let \( \kappa = \text{crit} (E) \). By assumption, we have that \( \prod_{\text{M}_n} \text{M}_n \) is not well-founded. Let then

\[
\cdots \in \text{M}_n^* \quad [a_2, f_2]_{E}^{\text{M}_n^*} \in \text{M}_n^* \quad [a_1, f_1]_{E}^{\text{M}_n^*} \in \text{M}_n^* \quad [a_0, f_0]_{E}^{\text{M}_n^*}.
\]

Without loss of generality, we may assume that

\[
(\forall i \in \omega) a_i \subseteq a_{i+1}.
\]

For each \( i \in \omega \), let

\[
X_{i+1} = \{ z \in [\kappa]^{a_{i+1}} \mid f_{i+1}(z) \in f_i(z_{a_i, a_{i+1}}) \}.
\]
and let $X_0 = [\kappa]^{\omega_0}$. By Theorem 7.2.1, $X_{i+1} \in E_{a_{i+1}}$ for all $i \in \omega$; $X_0 \in E_{a_0}$ trivially. Since $\tau_n$ is elementary,

$$\forall i \in \omega \exists \tau_n(X_i) \in (\tau_n(E))_{\tau_n(a_i)}.$$ 

All subsets of the hereditarily countable $M_n$ belong to $N_n$, and therefore both $\langle a_i \mid i \in \omega \rangle$ and $\langle X_i \mid i \in \omega \rangle$ belong to $N_n$. But $\tau_n \in N_n$, and so

$$\langle \tau_n(a_i) \mid i \in \omega \rangle \in N_n \land \langle \tau_n(X_i) \mid i \in \omega \rangle \in N_n.$$ 

Let $b = \bigcup_{i \in \omega} \tau_n(a_i)$. Since $\tau_n(E)$ is countably complete in $N_n$, there is a function $h : b \to \tau_n(\kappa)$ such that

$$\forall i \in \omega \exists \tau_n(a_i) \in \tau_n(X_i).$$ 

Since $\mathcal{M}_n$ and $\mathcal{M}_n^*$ agree through $\rho^T(n^*, n) \geq \kappa + 1$, all the $X_i$ belong to $M_n^*$. By the elementarity of $\tau_n^*$,

$$\forall i \in \omega \forall z \in \tau_n^*(X_i) \exists (\tau_n^*(f_i+1))(z) \in (\tau_n^*(f_i))(z_{a_i,a_{i+1}}).$$ 

But $\tau_n$ and $\tau_n^*$ agree through $\kappa + 1$, so

$$\forall i \in \omega \forall z \in \tau_n(X_i) \exists (\tau_n^*(f_i+1))(z) \in (\tau_n^*(f_i))(z_{a_i,a_{i+1}}).$$

Hence

$$\forall i \in \omega \exists (\tau_n^*(f_i+1))(h''\tau_n(a_{i+1})) \in (\tau_n^*(f_i))(h''\tau_n(a_i)).$$ 

But this contradicts the wellfoundedness of $N_n^*$. \hfill \Box \\

Remark. Theorem 7.3.2 does not cover the case $n^* = n$, but of course $\prod_{E \in \mathcal{M}_n^T} \mathcal{M}_n^T$ is always wellfounded for $E$ an extender in $\mathcal{M}_n^T$ with $\text{crit}(E) \leq \delta^{\mathcal{M}_n^T}$. Part (2) of Theorem 7.3.2 thus guarantees that failure of wellfoundedness will never interfere with our construction of iteration trees on $V$ of length $\omega$.

### 7.4 Trees of length $\omega$

If $\mathcal{T}$ is an iteration tree of length $\theta$, then a branch of $\mathcal{T}$ is a nonempty subset $b$ of $\theta$ such that

1. $b$ has no $<$-greatest element;
(ii) $b$ is linearly ordered by $T^T$;
(iii) if $\beta \in b$ and $\alpha T^T \beta$, then $\alpha \in b$.

Note that being a branch of $T$ depends only on $T^T$. If $b$ is a branch of $T$ then $b$ is a branch of any other iteration tree with the same tree ordering.

If $T$ is an iteration tree on $M$ and $b$ is a branch of $T$, then we denote by

$$(\hat{M}_b^T, \langle j^T_{\alpha,b} | \alpha \in b \rangle)$$

the direct limit of

$$(M_b^T, \langle j^T_{\alpha,b} | \alpha T^T \beta \in b \rangle).$$

We say that a branch $b$ of $T$ is wellfounded with respect to $T$ if the direct limit model $\hat{M}_b^T$ is wellfounded. When there is no ambiguity, we will omit the phrase “with respect to $T$.” If $b$ is wellfounded, then we denote by

$$(M_b^T, \langle j^T_{\alpha,b} | \alpha T^T \beta \in b \rangle)$$

the canonical limit of

$$(M_b^T, \langle j^T_{\alpha,b} | \alpha T^T \beta \in b \rangle).$$

If $T$ is an iteration tree of limit length $\theta$, then a cofinal branch of $T$ is a branch $b$ of $T$ such that $b$ is an unbounded subset of $\theta$. For $\theta = \omega$, all branches are thus cofinal branches.

Two possible problems can arise at limit steps $\theta$ in the construction of an iteration tree: (1) There may be no cofinal branch. (2) There may be cofinal wellfounded branches. By clause (f) in the definition of iteration trees, either (1) or (2) would make it impossible to extend the iteration tree to one of length $\theta + 1$. Since the iteration trees we construct in Chapter 8 will all be of length $\leq \omega$, we will not be worried about this problem per se. Nevertheless, we will have to rule out (1) and (2) for the trees of length $\omega$ that we construct, for it will be crucial for us that each of our trees has a cofinal wellfounded branch. (In fact, problem (1) will not arise: it will be obvious that our trees have cofinal branches.)

It is an open question whether every iteration tree of length $\omega$ on $V$ has a wellfounded branch. The trees we construct in Chapter 8 will, fortunately, have two special properties. We next introduce these two properties, one at a time.

An iteration tree $T$ of length $\omega$ on $M$ is continuously illfounded if there are $\xi_n$, $n \in \omega$, such that each $\xi_n$ is an ordinal of $M_n^T$ and such that, for $m$ and $n \in \omega$,

$$m T n \to j^T_{m,n}(\xi_m) > \xi_n.$$ 

A continuously illfounded tree cannot have wellfounded branches, for if $b$ is a branch then

$$\langle j^T_{n,b}(\xi_n) | n \in \omega \rangle$$
is an infinite descending sequence of ordinals of $\mathcal{M}_{b}^{\uparrow}$.

**Lemma 7.4.1.** Let $T$ be an iteration tree of length $\omega$ on $V$ with no branches. Then $T$ is continuously illfounded.

**Proof.** Let $T$ be the tree ordering of $T$. The absence of branches is equivalent with the wellfoundedness of $T^{*}$, where where $mTn$ if and only if $nTm$. For $n \in \omega$ define, by induction on $T^{*}$,

$$\xi_{m} = \sup\{\xi_{n} + 1 \mid nT^{*}m\} = \sup\{\xi_{n} + 1 \mid mTn\}.$$  

For $mTn$ we have that

$$j_{m,n}^{T}(\xi_{m}) \geq \xi_{m} > \xi_{n}.$$  

Hence the $\xi_{n}$ witness that $T$ is continuously illfounded.  

If $T$ is an iteration tree of length $\omega$ with tree ordering $T$ and if $b$ is a branch of $T$, then say that $T$ is continuously illfounded off $b$ if there are $\xi_{n}, n \in \omega$, such that each $\xi_{n}$ is an ordinal of $\mathcal{M}_{n}^{\uparrow}$ and such that, for all $m$ and $n \in \omega$,

$$mTn \rightarrow \left\{ \begin{array}{ll} j_{m,n}^{T}(\xi_{m}) > \xi_{n} & \text{if } n \notin b; \\ j_{m,n}^{T}(\xi_{m}) = \xi_{n} & \text{if } n \in b. \end{array} \right.$$  

Each iteration tree $T$ we construct in Chapter 8 will be continuously illfounded off some branch $b$ of $T$, and we will want to know that $b$ is wellfounded. Lemma 7.4.3 below shows that this will follow if we know that $T$ is not continuously illfounded.

First we need to prove an equivalent of illfoundedness for such limit models.

**Lemma 7.4.2.** Let $\mathcal{M}_{n}, n \in \omega$, be premice* and let $j_{m,n} : \mathcal{M}_{m} \prec \mathcal{M}_{n}$ for $m \leq n \in \omega$. Assume that whenever $m \leq n \leq p \in \omega$ then $j_{m,p} = j_{n,p} \circ j_{m,n}$. Assume also that $j_{0,n}^{\#	ext{Ord}^{\mathcal{M}_{0}}}$ is unbounded in the ordinals of $\mathcal{M}_{n}$ for each $n \in \omega$. Let $(\mathcal{M}, \langle j_{n} \mid n \in \omega\rangle)$ be the direct limit of $(\mathcal{M}_{n}, \langle j_{m,n} \mid m \leq n \in \omega\rangle)$. The following are equivalent:

(a) $\tilde{\mathcal{M}}$ is not wellfounded.

(b) There are $\zeta_{n}, n \in \omega$, such that each $\zeta_{n}$ is an ordinal of $\mathcal{M}_{n}$ and such that, for all $m$ and $n \in \omega$,

$$m < n \rightarrow j_{m,n}(\zeta_{m}) > \zeta_{n}.$$  

Proof. If $\zeta_n$, $n \in \omega$, witness that (b) holds, then $\langle j_n(\zeta_n) \mid n \in \omega \rangle$ is an infinite descending sequence of ordinals of $\hat{M}$.

To see that (a) implies (b), assume that $\hat{M}$ is illfounded. If $\langle x_i \mid i \in \omega \rangle$ is an infinite descending sequence with respect to the membership relation of $\hat{M}$, then $\langle \text{rank}^{\hat{M}}(x_i) \mid i \in \omega \rangle$ is an infinite descending sequence of ordinals of $\hat{M}$. Since every ordinal of $\hat{M}$ is of the form $\tilde{j}_n(\gamma)$ for some $n \in \omega$ and some ordinal $\gamma$ of $M_n$, we may assume that there are sequences $\langle n_i \mid i \in \omega \rangle$ and $\langle \gamma_i \mid i \in \omega \rangle$ such that

(i) $\forall i \in \omega. \; \gamma_i$ is an ordinal of $M_{n_i}$;
(ii) $\langle \tilde{j}_n(\gamma_i) \mid i \in \omega \rangle$ is an infinite descending sequence of ordinals of $\hat{M}$.

If some number $n$ were $n_i$ for infinitely many $i$, then the corresponding subsequence of the $\gamma_i$ would be an infinite descending sequence of ordinals of $M_n$. Thus we may assume that

$$i < i' \rightarrow n_i < n_{i'}.$$ 

Since $j_{0,n_0} \text{"Ord}^{M_0}$ is unbounded in the ordinals of $M_{n_0}$, we may assume also without loss of generality that $n_0 = 0$. Replacing, if necessary, each $\gamma_i$ by $\omega \gamma_i$, we may assume that each $\gamma_i$ is a limit ordinal. For $i \in \omega$ and $n_i \leq n < n_{i+1}$, set

$$\zeta_n = j_{n_i,n}(\gamma_i) + n_{i+1} - n.$$ 

The $\zeta_n$, $n \in \omega$, witness that (b) holds.

Lemma 7.4.3. Let $T$ be an iteration tree of length $\omega$ and let $b$ be a branch of $T$. Assume that $T$ is continuously illfounded off $b$ and that $b$ is not well-founded. Then $T$ is continuously illfounded.

Proof. Let $\langle \xi_i \mid i \in \omega \rangle$ witness that $T$ is continuously illfounded off $b$. Let $\langle \xi_n \mid n \in \omega \rangle$ be as given by the illfoundedness of $b$ and Lemma 7.4.2. For $i \in \omega$ let

$$\xi^*_i = \begin{cases} 
\xi_i & \text{if } i \notin b; \\
\xi_i + \zeta_i & \text{if } i \in b.
\end{cases}$$

The $\xi^*_i$, $i \in \omega$, witness that $T$ is continuously illfounded.

It is unknown whether there is a continuously illfounded iteration tree of length $\omega$ on $V$. But we can show that there are no such trees that obey a certain technical restriction, which we now describe.
Let $\mathcal{T} = (\mathcal{M}, T, \langle E_\alpha \mid \alpha + 1 < \ellh(\mathcal{T}) \rangle)$ be an iteration tree. For $\beta + 2 < \ellh(\mathcal{T})$, let

$$
\mu^\mathcal{T}(\beta) = \sup \{ \crit(E_\alpha) \mid (\alpha + 1)^\mathcal{T} \leq \beta < \alpha \}.
$$

For $\mathcal{T}$ the tree of Theorem 7.3.2 and for $m + 2 < \ellh(\mathcal{T})$, $\mu^\mathcal{T}(m)$ is the same as the ordinal $\mu(m)$ defined in the proof of Theorem 7.3.2. By part (a) of Corollary 7.2.6, $\mu^\mathcal{T}(\beta) \leq \strength^{\mathcal{M}_\beta}(E_\beta)$. In the proof of Theorem 7.3.2, we used the fact that, for $\mathcal{T}$ finite, $\mu^\mathcal{T}(\beta) < \strength^{\mathcal{M}_\beta}(E_\beta)$. Unfortunately, this may fail for infinite iteration trees.

For $n \in \omega$, we say that an iteration tree $\mathcal{T}$ is a plus $n$ iteration tree if, for every $\beta$ such that $\beta + 2 < \ellh(\mathcal{T})$,

$$
\mu^\mathcal{T}(\beta) + n \leq \strength^{\mathcal{M}_\beta}(E_\beta).
$$

The technical restriction mentioned above is thus being a plus one iteration tree. Some of the results of [Martin and Steel, 1994] require plus two trees.

**Remark.** Iteration trees in the sense of [Martin and Steel, 1988] and [Martin and Steel, 1989] are a special kind of plus one trees. See Exercise 7.4.1.

**Theorem 7.4.4.** Let $\mathcal{M}$ be a premouse and suppose that $\tau : \mathcal{M} \prec (V_\nu; \in, \delta)$ for some ordinals $\nu$ and $\delta$. Then there is no continuously illfounded plus one iteration tree of length $\omega$ on $\mathcal{M}$.

**Proof.** Assume for a contradiction that $\mathcal{T} = (\mathcal{M}, T, \langle E_n \mid n \in \omega \rangle)$ is a plus one iteration tree and that $\langle \xi_n \mid n \in \omega \rangle$ witnesses that $\mathcal{T}$ is continuously illfounded.

As in the proof of Theorem 7.3.2, we may assume that the universe of $M$ is countable.

We will construct an iteration tree $\mathcal{U} = (\mathcal{N}, T, \langle F_m \mid m \in \omega \rangle)$ with $\mathcal{N} = (V; \in, \delta)$. For each $m \in \omega$, we will also construct

(a) an uncountable premouse $\tilde{\mathcal{N}}_m$;
(b) $\psi_m : \tilde{\mathcal{N}}_m \prec P_m = (V_{\tilde{\nu}_{\tilde{m}}}; \in, \tilde{\delta}_m)$, where $\tilde{\delta}_m = j^\mathcal{U}_{0,m}(\delta)$;
(c) $\tilde{\tau}_m : \mathcal{M}_m \prec \tilde{\mathcal{Q}}_m = (V_{\tilde{\nu}_{\tilde{m}}}; \in, \tilde{\delta}_m)$, where $\tilde{\delta}_m = \delta_{\tilde{N}}$ and hence $\psi_m(\tilde{\delta}_m) = \delta_m$, and where $\psi_m(\tilde{\nu}_m) = \nu_m = j^\mathcal{U}_{0,m}(\nu)$. 


We set $\tau_m = \psi_m \circ \bar{\tau}_m$. Thus we will have

$$\tau_m : M_m \prec Q_m = (V_{\nu_m}^{\mathcal{P}_m}, \in, \delta_m) = (V_{\nu_m}^{\mathcal{N}_m}, \in, \delta_m).$$

The extender $F_m$ will be $\tau(E_m)$.

For each of our premites whose name is a subscripted calligraphic letter, we denote the universe of the premouse by the corresponding roman letter. We will suppress the subscript $T$ and—as we already have done—suppress the superscripts $T$ and $U$ except where there is ambiguity.

For all $m \in \omega$, the following conditions will be satisfied:

(i) for all $k \in \omega$, if $(k + 1)^- \leq m \leq k$ then $\bar{\tau}_{(k+1)^-}$ and $\bar{\tau}_m$ agree through $\text{crit}(E_k) + 1$;

(ii) for all $k \leq m$, $\psi_k$ and $\psi_m$ agree through

$$\min \{\text{strength}^N(\bar{\tau}_i(E_i)) \mid k \leq i < m\}$$

(iii) $\bar{\tau}_m \in \bar{N}_m$;

(iv) $\{\alpha \mid \bar{\nu}_m < \alpha < \text{Ord}^N \land (V_\alpha^{\mathcal{N}_m}, \in, \bar{\delta}_m) \text{ is a premouse}\}$ has order type at least $\bar{\tau}_m(\xi_m)$.

(v) $\bar{N}_{m+1} \in \bar{N}_m$.

Because condition (v) contradicts the Axiom of Foundation, our construction will give the desired reductio ad absurdum.

The independent objects we must define are $\bar{N}_m$, $\eta_m$, $\psi_m$, and $\bar{\tau}_m$.

Let $\eta_0$ be such that $(V_{\eta_0}; \in, \delta)$ is a premouse and $\{\alpha \mid \nu < \alpha < \eta_0 \land (V_\alpha^{\mathcal{N}_m}, \in, \bar{\delta}_m) \text{ is a premouse}\}$ has order type $\tau(\xi_0)$. Let $\bar{N}_0 = P_0 (= V_{\eta_0})$. Let $\psi_0$ be the identity. Let $\bar{\tau}_0 = \tau$.

Let $m \in \omega$ and suppose we have defined $\bar{N}_k$, $\eta_k$, $\psi_k$, and $\bar{\tau}_k$ for all $k \leq m$. Suppose that $(\mathcal{N}, T \upharpoonright m + 1, \langle F_k \mid k < m \rangle)$ is an iteration tree. Suppose that, for all $m' \leq m$, (a)–(c) hold with "$m$" replacing "$m$." Suppose that (i)–(iv) hold and that, for all $m' < m$, (v) holds with "$m'$" replacing "$m$.”

The elementarity of $\tau_m$ gives that $F_m = \tau_m(E_m)$ is an extender in $Q_m$ and so in $\mathcal{N}_m$.

We first show that conditions (i) and (ii) imply that $\tau_{(m+1)^-}$ and $\tau_m$ agree through $\text{crit}(E_m) + 1$. By condition (i), we will have shown this if we prove that conditions (i) and (ii) imply that $\psi_{(m+1)^-}$ and $\psi_m$ agree through
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crit $(\bar{\tau}_t(E_m)) + 1$. Corollary 7.2.6 gives that crit $(E_m) + 1 \leq \rho^T (m+1)^-, m$, i.e., that crit $(E_m) + 1 \leq \text{strength}^{M_i}(E_i)$ for all $i$ such that $(m+1)^- \leq i < m$. Since condition (i) gives that $\bar{\tau}_t(\text{crit} (E_m)) = \bar{\tau}_t(\text{crit} (E_m))$ for all such $i$, the elementarity of the $\bar{\tau}_t$ implies that crit $(\bar{\tau}_t(E_m)) + 1 \leq \text{strength}^{Q_i}(\bar{\tau}_t(E_i)) = \text{strength}^{N_i}(\bar{\tau}_t(E_i))$ for all such $i$. The desired conclusion follows from condition (ii).

The agreement of $\tau_{(m+1)^-}$ and $\tau_m$ implies that $Q_{(m+1)^-}$ and $Q_m$ agree through crit $(\tau_m(E_m)) + 1$, and so $N_{(m+1)^-} \text{ and } N_m$ agree through crit $(\tau_m(E_m)) + 1$. Thus either $(m+1)^- = m$ or else crit $(\tau_m(E_m)) + 1 < \rho^T ((m+1)^-, m)$. Thus we can apply part (2) of Lemma 7.3.2 to deduce that $\prod_{\bar{\tau}_t(E_m)}^{\bar{\tau}_m(E_m)^+} (\bar{\tau}_t(E_m)^+)$ is wellfounded. Since $F_m = \tau_m(E_m)$ belongs to $V_{\delta_m}^\mathcal{N}$, it follows that $(\mathcal{N}, T \upharpoonright m + 2, \langle F_k \mid k < m + 1 \rangle)$ is an iteration tree.

By the elementarity of $\bar{\tau}_m$, we get that $\bar{\tau}_m(E_m)$ is an extender in $\bar{Q}_m$ and so in $\bar{\mathcal{N}}_m$ and that $\bar{\tau}_m(E_m)$ belongs to $V_{\delta_m}^{\bar{\mathcal{N}}_m}$. Moreover $\psi_m(\bar{\tau}_m(E_m)) = F_m$. Thus we can apply the Shift Lemma (Lemma 7.3.1) with $\bar{\mathcal{N}}_m$ for $\mathcal{M}$, $\bar{\mathcal{N}}_{(m+1)^-}$ for $\mathcal{M}'$, $\mathcal{P}_m$ for $\mathcal{N}$, $\mathcal{P}_{(m+1)^-}$ for $\mathcal{N}'$, $\psi_m$ for $\tau$, $\psi_{(m+1)^-}$ for $\tau'$, crit $(\bar{\tau}_m(E_m))$ for $\kappa$, and $F_m$ for $E$. This gives us that $\prod_{\mathcal{F}_m}^{\mathcal{N}_{(m+1)^-}} (\mathcal{N}_{(m+1)^-})$ is wellfounded, and it gives us an embedding

$$\hat{\sigma} : \text{Ult}(\mathcal{N}_{(m+1)^-}; \bar{\tau}_m(E_m)) \prec \text{Ult}(\mathcal{P}_{(m+1)^-}; F_m) = \mathcal{P}_{m+1},$$

such that $\hat{\sigma}$ and $\psi_m$ agree through strength$^{\mathcal{N}_m}(\bar{\tau}_m(E_m))$.

Let us next see that, for all $k \leq m$, $\psi_k$ and $\hat{\sigma}$ agree through

$$\min \{\text{strength}^{\mathcal{N}_m}(\bar{F}_i) \mid k \leq i < m + 1\},$$

where each $\bar{F}_i = \bar{\tau}_t(E_i)$. We know that $\psi_m$ and $\hat{\sigma}$ agree through strength$^{\mathcal{N}_m}(\bar{\tau}_m(E_m))$. This gives us the case $k = m$ and, together with condition (ii), gives the case $k < m$ also.

Another application of the Shift Lemma gives us an embedding

$$\hat{\sigma} : \mathcal{M}_{m+1} \prec \text{Ult}(\mathcal{Q}_{(m+1)^-}; \bar{\tau}_m(E_m)),$$

such that $\hat{\sigma}$ and $\bar{\tau}_m$ agree through strength$^{\mathcal{M}_m}(E_m)$ and such that $\hat{\sigma} \circ \mathcal{I}_{(m+1)^-}^{\mathcal{M}_m} = \mathcal{I}_{\mathcal{Q}_{(m+1)^-}}^{\mathcal{M}_m} \circ \bar{\tau}_{(m+1)^-}$. (We are finally going to make use of the commutativity clause of the Shift Lemma.)
By an argument exactly like that in the analogous step of the proof of Theorem 7.3.2, we get that, for all $k \in \omega$,

$$(k + 1)^- \leq m + 1 \leq k \rightarrow \tilde{\tau}(k+1)^- \text{ and } \tilde{\sigma} \text{ agree through } \text{crit}(E_k) + 1.$$ 

We next show that there is a $\tilde{\tau} \in \text{Ult}(\tilde{N}_{(m+1)^-}; \tilde{F}_m)$ such that $\tilde{\tau} : M_{m+1} \prec \text{Ult}(\tilde{Q}_{(m+1)^-}; \tilde{F}_m)$, such that $\tilde{\tau}$ and $\tilde{\sigma}$ agree through $\mu^\tau(m) + 1$, and such that $\tilde{\tau}(\xi_{m+1}) = \tilde{\sigma}(\xi_{m+1})$. The argument is like that in the proof of Theorem 7.3.2 of the existence of the embedding there called $\tilde{\tau}$. We will mention only the points of difference. The fact that all the $M_k$ belong to, and are countable in, all the $\tilde{N}_k'$ follows from the uncountability of the $\tilde{N}_k$. In the earlier proof, the finiteness of the length of the iteration tree gave us that $\mu^\tau(m) + 1 \leq \text{strength}^{M_{m+1}}(E_m)$ and so that $\mu^\tau(m) + 1 \leq \text{strength}^{M_m}(E_m)$. Here the fact that $\mathcal{T}$ is a plus one tree gives us directly that $\mu^\tau(m) + 1 \leq \text{strength}^{M_m}(E_m)$. To take care of the extra condition that $\tilde{\tau}(\xi_{m+1}) = \tilde{\sigma}(\xi_{m+1})$, we simply add to our new version of requirement (i) in the definition of the tree $U$ the clause “and $u(i) = \tilde{\sigma}(a_i)$ for the $i$ such that $a_i = \xi_{m+1}$.”

The only thing preventing us from setting $\tilde{N}_{m+1} = \text{Ult}(\tilde{N}_{(m+1)^-}; \tilde{F}_m)$, $\eta_{m+1} = \tilde{\iota}^{(m+1)^-}_{E_m} ; (\eta_m)$, $\tilde{\tau}_{m+1} = \tilde{\tau}$, and $\psi_{m+1} = \tilde{\sigma}$ is condition (v).

By hypothesis, $\xi_{m+1} < \tilde{\iota}^{(m+1)^-}_{E_m} ; (\xi_{(m+1)^-})$. Thus

$$\tilde{N}_{(m+1)^-}^{\tilde{\iota}_{E_m}^{(m+1)^-} ; (\xi_{(m+1)^-})} \ (\tilde{\tau}_{(m+1)^-} ; (\xi_{(m+1)^-})) = \tilde{\iota}^{\tilde{\iota}^{(m+1)^-}_{E_m} ; (\xi_{(m+1)^-})} \ (\tilde{\tau}_{(m+1)^-} ; (\xi_{(m+1)^-})) = \tilde{\sigma}(\tilde{\iota}^{\tilde{\iota}^{(m+1)^-}_{E_m} ; (\xi_{(m+1)^-})} \ (\tilde{\tau}_{(m+1)^-} ; (\xi_{(m+1)^-}) \ (\xi_{m+1}^+)) \ (\xi_{(m+1)^-})) = \tilde{\tau}(\xi_{m+1}).$$

Let $\tilde{\delta} = \tilde{\iota}^{\tilde{\iota}^{(m+1)^-}_{E_m} ; (\xi_{(m+1)^-})}$ is an ordinal of $\text{Ult}(\tilde{N}_{(m+1)^-}; \tilde{F}_m)$ that are larger than $\tilde{\iota}^{\tilde{\iota}^{(m+1)^-}_{E_m} ; (\xi_{(m+1)^-})}$ and are such that $(\tilde{V}_\alpha^{\text{Ult}(\tilde{N}_{(m+1)^-}; \tilde{F}_m)} ; \xi, \tilde{\delta})$ is a premouse. Let $\tilde{\eta}$ be the $\tilde{\tau}(\xi_{m+1})$st such $\alpha$. Let $\eta_{m+1} = \tilde{\sigma}(\tilde{\eta})$.

Let $\tilde{N} = \text{Ult}(\tilde{N}_{(m+1)^-}; \tilde{F}_m)$ and let $\tilde{\mathcal{N}} = \text{Ult}(\tilde{N}_{(m+1)^-}; \tilde{F}_m)$. Applying the Löwenheim-Skolem Theorem in $\tilde{\mathcal{N}}$, we get an $X \subseteq V_{\tilde{\eta}}^{\tilde{\mathcal{N}}}$ such that

1. $(X; \xi, \tilde{\delta}) < (V_{\tilde{\eta}}^{\tilde{\mathcal{N}}}; \xi, \tilde{\delta})$;
2. $V_{\text{strength}^{M_{m+1}}(E_m)} \cup \{\tilde{\tau}, \tilde{\iota}_{E_m}^{(m+1)^-} ; (\tilde{\nu}_m)\} \subseteq X$;
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(3) in $\mathcal{N}$, the cardinal of $X$ is the same as that of $V^\mathcal{N}\text{strength}^{\mathcal{N}_m}(F_m)^{+1}$.

Since $(X; \in, \bar{\delta})$ is an elementary submodel of a premouse, the structure to which it is isomorphic via Lemma 3.2.4 is a premouse. Let then

$$\pi : (X; \in, \bar{\delta}) \cong (\bar{\mathcal{N}}_{m+1}; \bar{\in}, \bar{\delta}_{m+1}) = \mathcal{N}_{m+1}.$$ 

Define $\psi_{m+1}$ and $\bar{\tau}_{m+1}$ by

$$\psi_{m+1} = \bar{\sigma} \circ \pi^{-1} : \bar{\mathcal{N}}_{m+1} \prec \mathcal{P}_{m+1};$$

$$\bar{\tau}_{m+1} = \pi(\bar{\tau}) : \mathcal{M}_{m+1} \prec (\mathcal{V}^\mathcal{N}_{m+1}; \pi(i_{\mathcal{F}_m}^{\mathcal{N}_m}((\bar{\nu}_m)^-)) ; \in, \bar{\delta}_{m+1}) = \bar{\mathcal{Q}}_{m+1}.$$ 

Note that $\bar{\tau}_{m+1} = \pi \circ \bar{\tau}$, since, for $x \in M_{m+1}$, we have that $\bar{\tau}_{m+1}(x) = (\pi(\bar{\tau}))(x) = \pi(\tau(x)).$

Since $V^\mathcal{N}\text{strength}^{\mathcal{N}_m}(\mathcal{F}_m) \subseteq X$, we have that $\pi \upharpoonright V^\mathcal{N}\text{strength}^{\mathcal{N}_m}(\mathcal{F}_m)$ is the identity and that $\pi(V^\mathcal{N}\text{strength}^{\mathcal{N}_m}(\mathcal{F}_m)) = V^\mathcal{N}\text{strength}^{\mathcal{N}_m}(\mathcal{F}_m)$. This implies that $\bar{\sigma}$ and $\psi_{m+1}$ agree through strength $\mathcal{N}_m(\mathcal{F}_m)$, and it also implies that $\bar{\tau}$ and $\bar{\tau}_{m+1}$ agree through strength $\mathcal{M}_m(\mathcal{E}_m)$.

The agreement of $\bar{\sigma}$ and $\psi_{m+1}$ through strength $\mathcal{N}_m(\mathcal{F}_m)$ and the agreement of $\psi_m$ and $\bar{\sigma}$ through this same ordinal imply that $\psi_m$ and $\psi_{m+1}$ agree through strength $\mathcal{N}_m(\mathcal{F}_m)$. Together with condition (ii), this gives condition (ii) with “$m+1$” replacing $m$.

Let $k \in \omega$ be such that $(k+1)^- \leq m+1 \leq k$. Since $\bar{\tau}$ and $\bar{\tau}_{m+1}$ agree through strength $\mathcal{M}_m(\mathcal{E}_m)$, they agree through $\text{crit}(\mathcal{E}_k)+1$. But $\bar{\tau}$ and $\bar{\sigma}$ agree through $\mu^T(m)+1$, so it follows that $\bar{\tau}_{m+1}$ and $\bar{\sigma}$ agree through $\text{crit}(\mathcal{E}_k)+1$. Because $\bar{\sigma}$ and $\bar{\tau}_{(k+1)^-}$ also agree through $\text{crit}(\mathcal{E}_k)+1$, we finally deduce that $\bar{\tau}_{(k+1)^-}$ and $\bar{\tau}_{m+1}$ agree through $\text{crit}(\mathcal{E}_k)+1$. Thus we have verified condition (i) with “$m+1$” replacing “$m$.”

Condition (iii) with “$m+1$” replacing “$m$” follows from the fact that $\bar{\tau} \in X$.

By the definition of $\bar{\eta}$, there are $\bar{\tau}(\xi_{m+1})$ ordinals $\alpha$ of $\mathcal{N}$ that are larger than $i_{\mathcal{F}_m}^{\mathcal{N}_m}((\bar{\nu}_m)^-)$ such that $(V^\mathcal{N}_\alpha; \in, \bar{\delta})$ is a premouse. It follows that there are $\bar{\tau}_{m+1}(\xi_{m+1})$ ordinals of $\bar{\mathcal{N}}$ that are larger than $\bar{\nu}_{m+1}$ such that $(V^\mathcal{N}_{\alpha,m+1}; \in, \bar{\delta}_{m+1})$ is a premouse. Thus condition (iv) holds with “$m+1$” replacing “$m$.”

By property (3) of $X$, the cardinal in $\mathcal{N}$ of $\mathcal{N}_{m+1}$ is the same as that of $V^\mathcal{N}\text{strength}^{\mathcal{N}_m}(\mathcal{F}_m)^{+1}$. Hence $\mathcal{N}_{m+1}$ can be coded as an element of $V^\mathcal{N}\text{strength}^{\mathcal{N}_m}(\mathcal{F}_m)^{+1}$. 

But Lemma 7.2.3 implies that \( \text{Ult}(\bar{N}_{(m+1)^-}; \bar{F}_m) \) (i.e., \( \bar{N} \)) and \( \text{Ult}(\bar{N}_m; \bar{F}_m) \) agree through \( \psi^\bar{N}_m(\text{crit}(\bar{F}_m)) + 1 \), which is at least as large as strength \( \bar{N}_m(\bar{F}_m) + 1 \). Hence

\[ \bar{N}_{m+1} \in \text{Ult}(\bar{N}_m; \bar{F}_m) \subseteq \bar{N}_m. \]

We have verified all the induction hypotheses for \( m + 1 \), and so we have completed our construction and reached our contradiction. \( \square \)

**Theorem 7.4.5.** There is no continuously illfounded plus one iteration tree of length \( \omega \) on \( V \).

**Proof.** By an argument like the proof that part (1) of Theorem 7.3.2 implies part (2) of that theorem, any counterexample to the the present theorem would give rise to a counterexample to Theorem 7.4.4. \( \square \)

**Corollary 7.4.6.** Let \( T \) be a plus one iteration tree of length \( \omega \) on \( V \) and let \( b \) be a branch of \( T \). If \( T \) is continuously illfounded off \( b \) then \( b \) is wellfounded.

**Proof.** The corollary follows directly from Lemma 7.4.3 and Theorem 7.4.5. \( \square \)

Corollary 7.4.6 is the result needed in Chapter 8. Nevertheless, we will now make a few more remarks about further results and questions concerning wellfounded cofinal branches.

Suppose that \( M \) is a premouse and that \( \tau : M \prec (V_\nu; \in, \delta) \). It follows from Lemma 7.4.3 and Theorem 7.4.4 that, if \( T \) is a plus one iteration tree of length \( \omega \) on \( M \) and if \( T \) is continuously illfounded off \( b \), then \( b \) is wellfounded. One can also prove this assertion directly, without going through Theorems 7.4.4 and 7.4.5 (and doing so gives an alternate proof of Corollary 7.4.6). Such a direct proof is like the proof of Theorem 7.4.4, except for two modifications. The first is that condition (v) of that proof \( (\bar{N}_{m+1} \in \bar{N}_m) \) is now restricted to the case \( m + 1 \notin b \). For \( m + 1 \in b \), one just sets

\[ \bar{N}_{m+1} = \text{Ult}(\bar{N}_{(m+1)^-}; \bar{F}_m); \]
\[ \psi_{m+1} = \bar{\sigma}; \]
\[ \bar{\tau}_{m+1} = \bar{\tau}. \]

From the modified condition (v), it follows that

\[ (\forall m)(m + 1 \in b \rightarrow ((m + 1)^- = m \lor \bar{N}_m \in \bar{N}_{(m+1)^-})) \]
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and hence that

$$(\forall m)(m + 1 \in b \rightarrow \tilde{F}_m \in \tilde{N}_{(m+1)^-}).$$

If $\langle m_i \mid i \in \omega \rangle$ is an enumeration of $b$ in order of magnitude, then one has that $\langle \tilde{F}_{m_i} \mid i \in \omega \rangle$ is an internal iteration of $\tilde{N}_0 = (V_{\eta_0}; \in, \delta)$. It is thus a consequence of Theorem 7.1.5 that there is a canonical limit $(\tilde{N}_b, \langle \tilde{j}_{n,b} \mid n \in b \rangle)$. To finish the proof, one constructs an elementary embedding $\tilde{\tau}_b : \tilde{M}_b \prec \tilde{N}_b$. The purpose of the second modification of the proof of Theorem 7.4.4 is to make this possible. One arranges, for all $k \in \omega$ and all $x \in M_k$, that

$$(\exists k' \geq k)(\forall m \geq k') i^{N_{(m+1)^-}}_{E_m} (\tilde{\tau}_{(m+1)^-}(j^{T}_{k,m}(x))) = \tilde{\tau}_{m+1}(i^{M_{(m+1)^-}}_{E_m}(j^{T}_{k,m}(x))).$$

This can be done, since it involves, for each stage $m$ of the construction, making the function $\tilde{\tau}$ agree with $\tilde{\sigma}$ on finitely many additional arguments. One then sets

$$\tilde{\tau}_b(j_{k,b}(x)) = \lim_{n \in \omega \cap b} \tilde{\tau}_n(j_{k,n}(x)).$$

Suppose that $M$ is a premouse and that $\tau : M \prec (V_\nu; \in, \delta)$. Suppose in addition that the universe of $M$ is countable. If $T$ is a plus one iteration tree of length $\omega$ on $M$, then $T$ has a wellfounded branch, and indeed $T$ has a branch $b$ such that there is a $\tau^* : M^T_b \prec (V_\nu; \in, \delta)$ with $\tau^* \circ j^T_{0,b} = \tau$. (See Exercise 7.4.3.)

If $T = (M, T, \langle E_{\alpha} \mid \alpha + 1 < \theta \rangle)$ is an iteration tree and if $\theta' \leq \theta$, then by $T \upharpoonright \theta'$ we mean the iteration tree $(M, T, \langle E_{\alpha} \mid \alpha + 1 < \theta' \rangle)$.

Let $\theta$ be a countable limit ordinal and let $T$ be an iteration tree of length $\theta$ on a premouse $^* M$.

(a) $T$ is continuously illfounded if there is a subset $X$ of $\theta$ of order type $\omega$ and there are $\xi_\alpha$, $\alpha < \theta$, such that

(i) each $\xi_\alpha$ belongs to the universe of $M^T_{\alpha}$;

(ii) $(\forall \alpha)(\forall \beta)((\alpha T^T \beta \land (\exists \gamma \in X) \alpha < \gamma \leq \beta) \rightarrow j^T_{\alpha,\beta}(\xi_\alpha) > \xi_\beta)$;

(iii) $(\forall \alpha)(\forall \beta)((\alpha T^T \beta \land \neg(\exists \gamma \in X) \alpha < \gamma \leq \beta) \rightarrow j^T_{\alpha,\beta}(\xi_\alpha) = \xi_\beta)$.

(b) If $b$ is a cofinal branch of $T$, then $T$ is continuously illfounded off $b$ if there is a subset $X$ of $\theta$ of order type $\omega$ and there are $\xi_\alpha$, $\alpha < \theta$, such that

(i) each $\xi_\alpha$ belongs to the universe of $M^T_{\alpha}$;
(ii) \((\forall \alpha)(\forall \beta)((\alpha \mathcal{T} \beta \wedge \beta \notin b \wedge (\exists \gamma \in X) \alpha < \gamma \leq \beta) \rightarrow j_{\alpha,\beta}^{\mathcal{T}}(\xi_{\alpha}) > \xi_{\beta})\); 
(iii) \((\forall \alpha)(\forall \beta)((\alpha \mathcal{T} \beta \wedge (\beta \in b \vee (\exists \gamma \in X) \alpha < \gamma \leq \beta)) \rightarrow j_{\alpha,\beta}^{\mathcal{T}}(\xi_{\alpha}) = \xi_{\beta})\).

An iteration tree \(\mathcal{T}\) of countable length \(\theta\) is self-justifying if, for all limit \(\theta' < \theta\), \(\mathcal{T}\) is continuously illfounded off the branch \(\{\alpha < \theta' \mid \alpha \mathcal{T} \theta'\}\).

Let us consider the problem of extending our results to trees of length greater than \(\omega\). (1) Can we show that Theorem 7.3.2 remains true if we replace the natural numbers \(n\) and \(n^* < n\) by arbitrary countable ordinals \(\alpha\) and \(\alpha^* < \alpha\), and if we assume that the given iteration tree is a plus one tree and is self-justifying? (2) Can we similarly generalize Theorems 7.4.4 and 7.4.5, i.e., can we prove, for all countable ordinals \(\theta\), that if \(\mathcal{M}\) is a premouse and if \(\tau : \mathcal{M} \prec (V_{\nu}; \in, \delta)\) then there is no continuously illfounded, self-justifying, plus one iteration tree of length \(\theta\) on \(\mathcal{M}\)? For \(\alpha\) of the form \(\omega + n\), the answer to (1) is yes. We first do the construction outlined on page 400; then do a construction like that of the proof of Theorem 7.3.2. The fact that \((\omega + n) - \omega\) is finite implies, at step \(\omega\), that \(\chi\) (the analogue of the \(\chi\) of the earlier proof) belongs to some \(\bar{N}_k\), \(k \in \omega\), that agrees enough with \(\bar{N}\) (the analogue of the \(\bar{N}\) of the proof of Theorem 7.4.4) to give that \(\xi \in \bar{N}\). This allows us to construct the required \(\bar{\tau} \in \bar{N}\), and so to get a \(\bar{\tau}_\omega \in \bar{N}_\omega\). We do, however, encounter an obstacle if we try to get a positive answer to (2) for the case \(\theta = \omega + \omega\). Now step \(\omega\) cannot be carried out, for we cannot show that \(\chi\) belongs to \(\bar{N}\).

In [Martin and Steel, 1994], positive answers to questions (1) and (2) are given, except that “plus one” is replaced by “plus two.” In other words the following is proved:

Let \(\mathcal{T}\) be a self-justifying, plus two iteration tree of countable length \(\theta\) on a premouse \(\mathcal{M}\) that is elementarily embeddable into some \((V_{\nu}; \in, \delta)\). Then

(a) if \(\theta\) is a limit ordinal, then \(\mathcal{T}\) is not continuously illfounded;
(b) if \(\theta = \alpha + 1\), \(\alpha^* < \alpha\), \(E\) is an extender in \(M^\mathcal{T}_\alpha\), and \(\text{crit}(E) < \rho^\mathcal{T}(\alpha^*, \alpha)\), then \(\prod_E M^\mathcal{T}_{\alpha^*}\) is wellfounded.

Instead of merely replacing \(\bar{\sigma}\) by \(\bar{\tau}\), one uses a tree argument to replace the whole construction up to step \(\omega\) by a new one. This argument requires the assumption that \(\mathcal{T}\) is a plus two tree. The necessity of such an argument also means that, for general countable \(\theta\), one needs not just a single
construction of the kind we have been discussing but a transfinite sequence of such constructions.

The **Cofinal Branches Hypothesis** (the CBH) is the assertion that if $\mathcal{T}$ is an iteration tree on $\mathcal{M} = (V; \in)$ then

(a) if $\mathcal{T}$ has limit length, then $\mathcal{T}$ has a wellfounded cofinal branch;

(b) if $\alpha^* < \alpha < \ell h(\mathcal{T})$, if $\mathcal{M}_\alpha^\mathcal{T} \models "E \text{ is an extender},"$ and if $\text{crit}(E) < \rho(\alpha^*, \alpha)$, then $\prod_{E \in \mathcal{M}_\alpha^\mathcal{T}} \mathcal{M}_\alpha^\mathcal{T}$ is wellfounded.

The **Unique Branches Hypothesis** (the UBH) says that every iteration tree on $V$ has at most one wellfounded cofinal branch.

The CBH, if true, would guarantee that illfoundedness never blocks the construction of iteration trees on $V$. For sufficiently closed iteration trees, the UBH implies the CBH. Unfortunately, large cardinal hypotheses in the range of Woodin cardinals imply that both the CBH and the UBH are false. See [Neeman and Steel, 2006].

Provable special cases of the UBH are very useful. Knowing that iteration trees have at most one wellfounded cofinal branch is often important in proving the existence of wellfounded cofinal branches of trees. For example, the theorem of [Martin and Steel, 1994] mentioned on page 402 gives wellfounded cofinal branches only for trees whose restrictions have unique wellfounded cofinal branches. Indeed, the application of this result in [Martin and Steel, 1994] is in a situation where the result of Exercise 7.4.6 gives such uniqueness.

Any failure of the UBH gives an inner model with a Woodin cardinal. (See Exercise 7.4.6.) In [Steel, 2002], Steel gets inner models with more Woodin cardinals from the failure of UBH for non-overlapping iteration trees. An iteration tree $\mathcal{T}$ is non-overlapping if whenever $(\alpha + 1)^T \mathcal{M} (\beta + 1)$ then $\text{crit}(E^T_\beta)$ is greater than the $\lambda$ such that $E^T_\alpha$ is a $(\kappa, \lambda)$-extender in $\mathcal{M}_\alpha^\mathcal{T}$. The trees are used in inner model theory are essentially only non-overlapping trees. Non-overlapping trees are involved also in Exercises 7.4.12 and 7.4.13. [Sargsyan and Trang, 2016] shows that the failure of UBH for *tame* trees yields inner models of strong large cardinal hypotheses.

An important weakening of the CBH is the **Strategic Branches Hypothesis** (the SBH). For each ordinal $\theta$, consider the game in which players $I$ and $II$ attempt to build an iteration tree $\mathcal{T}$ of length $\theta$ on $V$. $I$ must pick the extenders $E^T_\alpha$, satisfying the obvious conditions. At limit ordinals $\gamma$, player $II$ must choose a cofinal branch of $\mathcal{T} \upharpoonright \gamma$. Any failure of wellfoundedness, either
at successor or limit steps, results in a loss for \( II \). If \( T \) is actually built, then \( II \) wins. The SBH says that, for every ordinal \( \theta \), player \( II \) has a winning strategy. Clearly the CBH implies the SBH, since the CBH provides \( II \) with a trivial winning quasistrategy. See §5 of [Martin and Steel, 1994] for more on the SBH.

**Exercise 7.4.1.** In [Martin and Steel, 1988] and [Martin and Steel, 1989] it is required in the definition of an iteration tree \( T \) that there be a nondecreasing sequence \( \langle \rho_\alpha \mid \alpha + 1 < \text{lh}(T) \rangle \) such that, for each \( \alpha \),

\[
\rho_\alpha < \text{strength}^{M_\alpha}(E_\alpha) \wedge \text{crit}(E_\alpha) \leq \rho(\alpha+1)^-.
\]

Prove that every such iteration tree is a plus one tree.

**Exercise 7.4.2.** Let \( M \) be a premouse and suppose that \( \tau : M < (V_\nu; \in, \delta) \). Let \( T \) be an iteration tree of length \( \omega \) on \( M \). Construct an iteration tree \( U \) on \( V \) and a sequence \( \langle \tau_n \mid n \in \omega \rangle \) such that

- (a) \( T^U = T^T \);
- (b) if \( k \in \omega \) and \( T \) is a plus \( k \) tree, then \( U \) is a plus \( k \) tree;
- (c) \( \tau_0 = \tau \);
- (d) for \( n \in \omega \), \( \tau_n : M^T_n < j^U_0,n(V_\nu; \in, \delta) \);
- (e) for \( m T^T n \in \omega \), \( \tau_n \circ j^T_{m,n} = j^U_{m,n} \circ \tau_m \).

**Exercise 7.4.3.** Let \( M \) be a premouse whose universe is countable and suppose that \( \tau : M < (V_\nu; \in, \delta) \). Let \( T \) be a plus one iteration tree of length \( \omega \) on \( M \). Show that there are a branch \( b \) of \( T \) and a \( \tau^* : \tilde{M}_b < (V_\nu; \in, \delta) \) such that

\[
\tau^* \circ j^T_{0,b} = \tau.
\]

**Hint.** Let \( T = T^T \). Form a tree \( W \) whose members are initial segments of attempts to enumerate \( b \) and \( \tau^* \) with the required properties. To define \( W \), let \( \langle y^k_i \mid i \leq k \in \omega \rangle \) be such that each \( y^k_i \) belongs to the universe \( M^T_k \) of \( M^T_k \), such that \( j^T_{k',k}(y^k_{i'}) = y^k_i \) for \( i \leq k' T k \in \omega \), and such that, for any branch \( b \) of \( T \),

\[
\{ j^T_{k,b}(y^k_i) \mid i \leq k \in b \} = \tilde{M}^T_b,
\]

where \( \tilde{M}^T_b \) is the universe of \( \tilde{M}^T_b \). Let \( W \) be the set of all \( \langle k_i, a_i \mid i \leq n \rangle \) such that

1. \( n \in \omega \),
2. \( k_i T k_{i'} \) for \( i < i' \leq n \),
3. \( a_i \in V_\nu \),
4. \( y^{k_i}_{i'} \rightarrow a_i \) is a partial elementary embedding of \( M^T_{k_i} \) into \( (V_\nu; \in, \delta) \),
5. for all \( x \) in
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the universe of $\mathcal{M}$ and for all $i \leq n$, if $j_{0,k}^T(x) = y_i^k$, then $a_i = \tau(x)$. If the desired $b$ and $\tau^*$ do not exist, then $W$ is wellfounded.

Assume that $W$ is wellfounded. Let $U$ and $\langle \tau_n \mid n \in \omega \rangle$ be as given by Exercise 7.4.2. For $k \in \omega$, define $w_k = \langle k, a_i \mid i \leq n \rangle$ as follows. Let $n$ be the number of $T$-predecessors of $k$. Let $k_0 T \cdots T k_n = k$. Let $a_i = \tau_k(y_i^k)$. Show that $w_k \in j_{0,k}(W)$. Now set

$$\xi_k = \|w_k\|_{j_{0,k}(W)}.$$

Show that the $\xi_k$ witness that $U$ is continuously illfounded.

**Exercise 7.4.4.** Assume the result from [Martin and Steel, 1994] stated on page 402, and prove the following (the main wellfoundedness result for iteration trees in that paper).

Let $\mathcal{M}$ be a premouse whose universe is countable and suppose that $\tau : \mathcal{M} \rightarrow (V_\nu, \in, \delta)$. Let $T$ be a plus two iteration tree of countable length $\theta$ on $\mathcal{M}$. Assume that there is no maximal (not properly extendable) non-cofinal branch $b$ of $T$ such that there is a $\tau^* : \tilde{\mathcal{M}}_b \rightarrow (V_\nu, \in, \delta)$ such that $\tau^* \circ j_{0,b}^T = \tau$.

Then

(a) if $\theta$ is a limit ordinal, then there are a cofinal branch $b$ of $T$ and a $\tau^* : \mathcal{M}_b^T \rightarrow (V_\nu, \in, \delta)$ such that $\tau^* \circ j_{0,b}^T = \tau$;

(b) if $\theta = \alpha + 1$, $\alpha^* < \alpha$, $E$ is an extender in $\mathcal{M}_\alpha^T$, and $\crit(E) < \rho(\alpha^*, \alpha)$, then the ultrapower $\prod_{E/\alpha^*} \mathcal{M}_\alpha^T$ is wellfounded and there is a $\tau^* : \text{Ult}(\mathcal{M}_{\alpha^*}^T; E) \rightarrow (V_\nu, \in, \delta)$ such that $\tau^* \circ i_{E/\alpha^*}^T \circ j_{0,\alpha^*}^T = \tau$.

Hint. By taking direct limits at limit ordinals, construct $U$ and $\langle \tau_\gamma \mid \gamma < \theta' \rangle$ having the properties (a)–(e) of Exercise 7.4.2, except that $\omega$ is replaced by $\theta'$, where $\theta'$ is either $\theta$ or the least ordinal at which illfoundedness prevents continuing the construction. Use an argument similar to the one in the hint for Exercise 7.4.3 to show that $U$ is self-justifying. Part (b) of the result on page 402 implies that $\theta'$ is not a successor ordinal < $\theta$. Thus $\theta' = \theta$. Another Exercise 7.4.3 argument then shows that (a) follows from part (a) of the page 402 result. For (b), use part (b) of the page 402 result to establish the wellfoundedness of $\prod_{\tau_{\alpha}(E)} j_{0,\alpha^*}(V)$. This gives a $\tilde{\tau} : \text{Ult}(\mathcal{M}_{\alpha^*}^T; E) \rightarrow \text{Ult}(j_{0,\alpha^*}^T(V_\nu, \in, \delta); \tau_{\alpha}(E))$ such that $\tilde{\tau} \circ i_{E/\alpha^*}^T \circ j_{0,\alpha^*}^T = \tau^*$. 


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\[ i_{\tau_\alpha(E)} \circ j^{\mu}_{0,\alpha^*} \circ \tau = \hat j_{\tau_\alpha(E)}^{\mu} \circ j^{\mu}_{0,\alpha^*} \circ (\tau). \]

Use the absoluteness of illfoundedness of trees to show that there is such a \( \hat \tau \) belonging to \( \text{Ult}(j^{\mu}_{0,\alpha^*}(V); \tau_\alpha(E)) \). The existence of \( \tau^* \) follows from the absoluteness of \( i_{\tau_\alpha(E)} \circ j^{\mu}_{0,\alpha^*} \).

Remark. The proof of this theorem, and the proof of the theorem on which it depends, go through under weaker assumptions than ZFC. For example (and we mention this example only because it will be used in subsequent exercises), if \( \kappa \) is an ordinal number then the theorem holds in any transitive proper class satisfying ZF + DC_{<\kappa} + V = L(V^\kappa). \( \text{(DC}_{<\kappa} \) is the assertion that sequences of dependent choices of arbitrary length \( \beta < \kappa \) can always be made.)

Exercise 7.4.5. For any class model \( \tilde M \) for the language of set theory (or an expansion of that language), we let wford (\( \tilde M \)) be the largest ordinal that is order isomorphic to a not necessarily proper initial segment of the ordinals of \( \tilde M \) if not every ordinal is so isomorphic, and let wford (\( \tilde M \)) = Ord otherwise. (This is the same as wfo(\( A \)), where \( A \) is the ordering of the ordinals of \( \tilde M \). See page 287.)

Assume the result of Exercise 7.4.4, in the version mentioned in the remark above, and prove the following theorem of Hugh Woodin.

Let \( \kappa \) be an ordinal and let \( M \) be a transitive proper class satisfying ZF + DC_{<\kappa} + V = L(V^\kappa). \text{Note that all extenders of } M \text{ belong to } V^M_\kappa. \text{Let } \theta \text{ be an ordinal number and let } T \text{ be a plus two iteration tree of length } \theta \text{ on } M.

(a) If \( \theta \) is a limit ordinal, then for every ordinal \( \lambda \) there is a generic maximal branch of \( T \) (i.e., there is in some forcing extension of \( V \) a maximal branch of \( T \)) such that wford (\( \tilde v_3 \)) \geq \lambda.

(b) If \( \theta = \alpha + 1 \), \( \alpha^* < \alpha \), \( E \) is an extender in \( M^T_{\alpha^*} \), and \( \text{crit } (E) < \rho(\alpha^*, \alpha) \), then either the conclusion of (a) holds or else \( \prod_{\gamma < \kappa} M^T_{\alpha^*} \) is wellfounded.

Hint (for part (a); the proof of (b) is similar). Assume that (a) fails. Let \( \lambda > \max\{\kappa, \text{th}(T)\} \) be arbitrary. Let \( \psi(v_1, v_2, v_3, v_4) \) be a formula saying that \( v_1 \) and \( v_2 > v_1 \) are ordinals, that \( v_3 \) is a premouse with \( \text{Ord}^{v_3} = v_2 \), that \( v_4 \) is a countable iteration tree of countable limit length on \( v_3 \), and that there is no maximal branch \( b \) of \( v_4 \) such that wford (\( (v_3)_b \)) \geq v_1. \text{Let } \gamma \text{ be any ordinal greater than } \lambda \text{ such that } (V^M_\gamma; \in, \kappa) \text{ is a premouse. Let Coll}(\omega, \gamma) \text{ be the usual partial ordering for collapsing } \gamma \text{ to } \omega. \text{(See 537 for the definition.)}
Show that if $G$ is $\text{Coll}(\omega,\gamma)$-generic over $V$, then

$$V[G] \models \psi[\lambda,\gamma, (V^M_\gamma;\in,\kappa), \mathcal{T}(\gamma)],$$

where $\mathcal{T}(\gamma)$ is the iteration tree on $(V^M_\gamma;\in,\kappa)$ with the same tree ordering and extenders as $\mathcal{T}$. Argue by absoluteness that there is a $\mathcal{T}' \in M[G]$ such that

$$M[G] \models \psi[\lambda,\gamma, (V^M_\gamma;\in,\kappa), \mathcal{T}'].$$

Let $\eta > (\gamma^+)^M$ be such that $(V^M_\eta;\in,\kappa)$ is a premouse. Let $X \in M$ be such that $X$ is countable, $\gamma \in X$, and $(X;\in,\kappa) \prec (V^M_\eta;\in,\kappa)$. Let $N$ be transitive with $\pi : X \cong N$. Let $\bar{G} \in M$ be $\text{Coll}(\omega,\pi(\gamma))$-generic over $N$. Then

$$N[\bar{G}] \models \psi[\pi(\lambda),\pi(\gamma), (V^N_\pi(\gamma);\in,\pi(\kappa)), \pi(\mathcal{T}')] .$$

Now use the absoluteness of $\psi$ and Exercise 7.4.4, applied in $M$, to get a contradiction.

**Exercise 7.4.6.** Let $\theta$ be a limit ordinal and let $\mathcal{T} = (\mathcal{M}, T, \langle E_\alpha | \alpha + 1 < \theta \rangle)$ be an iteration tree. Suppose that $b$ and $c$ are distinct wellfounded cofinal branches of $\mathcal{T}$. Let

$$\kappa^* = \sup \{ \rho^T(\alpha, \theta) | \alpha < \theta \}.$$

Assume that $\kappa^*$ is an ordinal both of $\mathcal{M}^T_b$ and of $\mathcal{M}^T_c$. (This is automatically true if $\mathcal{M}$ is a premouse or if it is a proper class.) Let $\eta = \min \{ \text{Ord}^M_b, \text{Ord}^M_c \}$. Show that in $L_\eta(V^M_{\kappa^*})$ ($= L_\eta(V^M_{\kappa^*})$) the ordinal $\kappa^*$ is a Woodin cardinal. This is a result of [Martin and Steel, 1994].

**Hint.** First show that

$$\kappa^* = \sup \{ \text{crit} (j^T_{\alpha, b}) | \alpha \in b \} = \sup \{ \text{crit} (j^T_{\alpha, c}) | \alpha \in c \} .$$

Next define inductively

\[
\begin{align*}
\kappa_0 &= \min \{ \kappa | (\exists \alpha)(\alpha + 1 \in b \setminus c \land \kappa = \text{crit} (E_\alpha)) \}; \\
\alpha_0 &= \max \{ \alpha | \alpha + 1 \in b \land \kappa_0 = \text{crit} (E_\alpha) \}; \\
\nu_n &= \min \{ \nu | (\exists \beta)(\beta + 1 \in c \setminus (\alpha_n + 1) \land \nu = \text{crit} (E_\beta)) \}; \\
\beta_n &= \max \{ \beta | \beta + 1 \in c \land \nu_n = \text{crit} (E_\beta) \}; \\
\kappa_{n+1} &= \min \{ \kappa | (\exists \alpha)(\alpha + 1 \in b \setminus (\beta_n + 1) \land \kappa = \text{crit} (E_\alpha)) \}; \\
\alpha_{n+1} &= \max \{ \alpha | \alpha + 1 \in b \land \kappa_{n+1} = \text{crit} (E_\alpha) \} .
\end{align*}
\]
Clearly all four sequences are strictly increasing, and \( \alpha_n < \beta_n < \alpha_{n+1} \) for all \( n \in \omega \). Show that the \( \alpha_n \) and \( \beta_n \) converge to \( \theta \) and that the \( \kappa_n \) and \( \nu_n \) converge to \( \kappa^* \). Show that, for all \( n \in \omega \),

\[
\nu_n = \text{crit}(E_{\beta_n}) < \text{strength}^{M_{\alpha_n}}(E_{\alpha_n});
\]

\[
\kappa_{n+1} = \text{crit}(E_{\alpha_{n+1}}) < \text{strength}^{M_{\beta_n}}(E_{\beta_n}).
\]

Now fix \( n \in \omega \) and let \( z \) belong belong to the ranges of both \( j^{(\alpha_n+1)}_{\omega} \) and \( j^{(\beta_n+1)}_{\omega} \). Let \( \varphi(v_1, v_2, v_3) \) be a \( \Sigma_0 \) formula of the language of set theory.

Let \( \gamma = \min\{\kappa_n, \nu_n\} \) and let \( \gamma' = \min\{\nu_n, \kappa_{n+1}\} \). Prove that

1. \( (x \in V^{{M_b}}_{\gamma} \land (\exists y \in V^{{M_b}}_{\kappa_n}) \varphi(x, y, z)) \rightarrow (\exists y \in V^{{M_b}}_{\gamma}) \varphi(x, y, z); \)
2. \( (x \in V^{{M_b}}_{\gamma} \land (\exists y \in V^{{M_b}}_{\kappa^*}) \varphi(x, y, z)) \rightarrow (\exists y \in V^{{M_b}}_{\gamma^*}) \varphi(x, y, z). \)

Now let \( f : \kappa^* \rightarrow \kappa^* \) with \( f \in L_\eta(V^{{M_b}}_{\kappa^*}) \). For some \( n \), \( f \) belongs to the ranges of both \( j^{(\alpha_n+1)}_{\omega} \) and \( j^{(\beta_n+1)}_{\omega} \). Assume for definiteness that \( \kappa_n < \nu_n \). Let \( \xi = \min\{\nu_n, \kappa_{n+1}\} \). Observe that \( \kappa_n < \xi \leq \nu_n < \text{strength}^{M_{\alpha_n}}(E_{\alpha_n}) \). By the result just proved, \( \kappa_n \) is closed under \( f \) and \( f(\kappa_n) < \xi \). Let

\[
F = \langle (E_{\alpha_n})_a \mid a \in [\xi]^{<\omega}\rangle.
\]

Prove that in \( L_\eta(V^{{M_b}}_{\kappa^*}) \) the cardinal \( \kappa_n \) and the embedding \( i^L_{\eta}(V^{{M_b}}_{\kappa^*}) \) witness that \( \kappa^* \) is Woodin for \( f \).

Remarks:

(a) If \( \mathcal{M} \) satisfies ZFC, then \( L_\eta(V^{{M_b}}_{\kappa^*}) \) satisfies ZF but may not satisfy the Axiom of Choice. However, Woodin has shown that there is a generic extension of it in which Choice holds and \( \kappa^* \) is still Woodin.

(b) Suppose that \( \hat{\mathcal{M}}^\beta_b \) and \( \hat{\mathcal{M}}^\gamma_b \) are not necessarily wellfounded, but that \( \eta \) is the minimum of wford \( (\hat{\mathcal{M}}^\beta_b) \) and \( \text{wford} (\hat{\mathcal{M}}^\gamma_b) \). (See page 406.) Then \( L_\eta(V^{{M_b}}_{\kappa^*}) \) \( (L_\eta(V^{{M_b}}_{\kappa^*})) \) still makes sense, though it may not satisfy ZF. The argument of the hint shows that \( \kappa^* \) is Woodin in this model.

Exercise 7.4.7. Let \( \kappa \) be an ordinal and let \( M \) be a transitive proper class model of ZF + DC \( \prec \kappa \) + “\( \kappa \) is inaccessible” + \( V = L(\kappa) \). Assume also that there is no \( \kappa' < \kappa \) such that \( L(V^M_{\kappa'}) \models “\kappa' \text{ is Woodin}.” \) Show that no iteration tree on \( M \) has more than one wellfounded cofinal branch.
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**Hint.** Show that the hypotheses about $M$ imply, for any proper class $X$ of ordinals, that the set of all ordinals $< \kappa$ that are definable in $M$ from $\kappa$ and elements of $X$ is unbounded in $\kappa$.

Now assume that $b$ and $c$ are wellfounded cofinal branches of an iteration tree $T$ on $M$. Apply the result of the preceding paragraph to $M_b$ and $M_c$ with

$$X = \{\alpha \in \text{Ord} \mid \alpha = j_{0,b}(\alpha) = j_{0,c}(\alpha)\}.$$  

Now get a contradiction by generalizing to arbitrary formulas the proposition about $\Sigma_0$ formulas in the hint to Exercise 7.4.6.

**Exercise 7.4.8.** This exercise and the next give corollaries of Woodin’s result of Exercise 7.4.5

Let $\kappa$ and $M$ be as in Exercise 7.4.7. For iteration trees $T$ of limit length $\theta$ on $M$, let us make the following definitions. Let $\kappa^*(T) = \sup\{\rho^T(\alpha, \theta) \mid \alpha < \theta\}$. For each $\gamma < \kappa^*(T)$, there is an $\alpha < \theta$ such that $V^{M^T}_\gamma = V^{M^T}_\gamma$ for all $\beta$ and $\beta'$ such that $\alpha \leq \beta \leq \beta' < \theta$. Define $M(T)$ by letting $M(T) = L(V^{M(T)}_{\kappa^*(T)})$, where $V^{M(T)}_{\kappa^*(T)}$ is the limit of the $V^{M^T}_{\kappa^*(T)}$. If $M \models \text{“\mathcal{E} is a set of extenders,”}$ then let $\mathcal{E}(T)$ be the limit of the $j^T_{0,\beta}(\mathcal{E}) \cap V^{M(T)}_{\kappa^*(T)}$.

Let $T$ be a plus two iteration tree on $M$ of limit length. Assume that $M(T) \not\models \text{“}\kappa^*(T) \text{ is Woodin.”}$. Prove that $T$ has a wellfounded cofinal branch (which must be unique, by Exercise 7.4.7).

**Hint.** Assume $T$ has no wellfounded cofinal branch. By Exercise 7.4.6, $T$ has no wellfounded maximal branch. Use Exercise 7.4.5 and remark (b) following Exercise 7.4.6 to get the contradiction that some $\kappa' \leq \kappa^*$ is Woodin in $L(V^{M(T)}_{\kappa^*(T)})$.

**Exercise 7.4.9.** Let $\kappa$ and $M$ be as in Exercises 7.4.7 and 7.4.8. Let $T$ be a plus two iteration tree on $M$ of successor length $\alpha + 1$. Let $\alpha^* < \alpha$, and let $E$ be an extender in $M^T_\alpha$ with $\text{crit} (E) < \rho(\alpha^*, \alpha)$. Show that $\prod_{E}^{M^T_{\alpha^*}} M^T_{\alpha^*}$ is wellfounded.

**Exercise 7.4.10.** This exercise gives an improvement, due to Woodin, of the result of Exercise 7.4.8.

For $M$ as in Exercise 7.4.7 and for positive integers $n$, say that $M$ is $n$-iterable if, for any plus two iteration tree $T$ of limit length on $M$, the
following game $G_T$ is a win for II. Plays of $G_T$ are as follows:

$$
\begin{array}{c}
\text{I} & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-1} \\
\text{II} & \beta_0 & \beta_1 & \cdots & \beta_{n-1}
\end{array}
$$

All $\alpha_i$ and all $\beta_i$ must be ordinal numbers. II wins such a play if and only if there is a generic cofinal branch $b$ of $T$ such that

1. for all $i < n$, both $\alpha_i$ and $\beta_i$ belong to wford $(\mathcal{M}_b^T)$;
2. $(\forall i < n) j^T_{0,b}(\alpha_i) = \beta_i$, where $j^T_{0,b}$ is the obvious partial function.

Note that the assertion that $G_T$ is a win for II is expressed by a formula of the language of set theory in the parameters $T$ (i.e., $T$’s extender sequence) and $V^M$. 

(a) Let $n \in \omega$. Suppose that $M$ (as in Exercise 7.4.7 and) is $(n + 1)$-iterable. Suppose that $T$ is a plus two iteration tree on $M$ of limit length with no wellfounded cofinal branch. Prove that $M$ and $M(T)$ satisfy the same $\Sigma_n$ sentences. ($M(T)$ is defined in Exercise 7.4.8.)

(b) Assume that there is a transitive proper class $M$ that satisfies ZFC + “There is a Woodin cardinal.” Prove that, for every $n \geq 1$, there is an $M$ as in Exercise 7.4.7 such that $M$ is $n$-iterable and $M$ satisfies “$\kappa$ is Woodin.”

*Hint.* To prove (b), first show that there is an $M$ as in Exercise 7.4.7 such that $M$ satisfies “$\kappa$ is Woodin.” Fix such an $M$ and assume for a contradiction that $M$ is not $n$-iterable.

Show that there exist $\langle T_i \mid i \in \omega \rangle$ and $\langle M_i \mid M \in \omega \rangle$ such that

1. $M_0 = M$;
2. for each $i \in \omega$, $T_i$ is a plus two iteration tree of limit length on $M_i$;
3. for each $i \in \omega$, $M_{i+1} = M_i(T_i)$;
4. for each $i \in \omega$, $T_i$ witnesses that $M_i$ is not $n$-iterable.

Say that an ordinal $\gamma$ has property P if, for any $G$ that is $\text{Coll}(\omega, \gamma)$-generic over $V$, there exist in $V[G]$ a transitive set $N$ and an ordinal $\delta \leq \kappa$ such that

(a) $V^M_\delta \in N$ and $N$ is a model of ZC plus, say, $\Sigma_{100}$ Replacement;
(b) $\text{Ord} \cap N = \gamma$;
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(c) $N$ satisfies the formula asserting that there are $\langle T_i \mid i \in \omega \rangle$ and $\langle M_i \mid i \in \omega \rangle$ such that $M_0 = L_\gamma(V_\delta^M)$ and (ii), (iii), and (iv) above hold.

Let $\gamma$ be the least ordinal that has property P. By the absoluteness for $M$ of property P, $M$ satisfies that $\gamma$ is the least ordinal with property P. Let $G$ be $\text{Coll}(\omega, \gamma)$-generic over $V$. Work in $M[G]$.

Let $N \in M[G]$ witness (in $M[G]$) that $\gamma$ has property P. Let $\langle T_i \mid i \in \omega \rangle$ and $\langle M_i \mid i \in \omega \rangle$ be given by (c).

Use Exercises 7.4.8, 7.4.9, and 7.4.5 to show that the tree ordering and extenders of $T_0$ yield a plus two iteration tree $T_0^*$ on $M$ and that $T_0^*$ has a cofinal branch $b$ whose wellfounded part is at least $(\gamma^+)^{M[G]}$. This branch is of course also a branch of $T_0$. To simplify notation, let us identify the transitive part of the model $\mathcal{M}_b^{T_0}$ with the transitive set isomorphic to it. With this identification,

$$M(T_0^*) = L(V_{\kappa^*(T_0^*)}).$$

(See Exercise 7.4.8 for the definition of $\kappa^*(T_0^*)$.)

The model $N$ and the sequences $\langle T_i \mid 1 \leq i \in \omega \rangle$ and $\langle M_i \mid 1 \leq i \in \omega \rangle$ witness that clauses (a)–(c) above hold with $M$ replaced by $\mathcal{M}_b^{T_0}$ and $\delta$ replaced by $\kappa^*(T_0^*)$. Hence $\gamma$ has what we might call property $j_{T_0^*}^\gamma(P)$. By absoluteness, it is true in $\mathcal{M}_b^{T_0}$ that $\gamma$ has property $j_{T_0^*}^\gamma(P)$.

It follows that $j_{T_0^*}^\gamma(\gamma)$ $\leq$ $\gamma$ and so that $j_{T_0^*}^\gamma(\gamma)$ $=$ $\gamma$. Hence $b$ is a well-founded cofinal branch of $T_0$ and $\text{Ord} \cap (M_0)^{T_0} = \gamma$. But this means that $j_{T_0^*}^\gamma$ gives a winning strategy for II for the game $G_{T_0}$ (where, of course, moves are restricted to ordinals < $\gamma$). By absoluteness, we get that $N$ satisfies that $II$ wins $G_{T_0}$, and that is a contradiction.

Exercise 7.4.11. Let $\mathcal{M}$ be a premouse*. Let $T$ be an iteration tree of length $|\delta^M|^+$ on $\mathcal{M}$. Let $b$ be a cofinal branch of $T$. Note that $b$ is closed and unbounded in $|\delta^M|^+$. Prove that there is a stationary subset $X$ of $b$ such that

$$(\forall \alpha \in X)(\forall \beta \in X)(\forall \gamma \in X)(\alpha \leq \beta \leq \gamma \rightarrow j_{a,\beta}^T(\text{crit}(j_{a,\beta}^T)) = \text{crit}(j_{\beta,\gamma}^T)).$$
Exercise 7.4.12. This exercise concerns a theorem of Woodin whose proof uses iteration trees in the way they are used in inner model theory. The exercise is a sequel to Exercises 6.3.7 and 6.3.8.

For $x \in \omega_2$, for $N$ a transitive class model of ZFC, and for $E$ strongly witnessing in $N$ that some $\kappa$ is Woodin, one can try to define, as $G_x$ was defined in Exercise 6.3.8, $G^N_x \subseteq P^N_E$ by

$$[c]^{N_{<\omega_2}} \in G^N_x \iff x \in B_c.$$ 

Suppose that $M$ is a transitive class model of ZFC + “there is a Woodin cardinal” and that every plus two iteration tree on $M$ has a wellfounded cofinal branch. (Woodin has shown that the existence of such a proper class $M$ follows from the existence of a transitive proper class model of ZFC + “there is a Woodin cardinal” plus the hypothesis that every set has a #.) Let $E$ strongly witness in $M$ that $\kappa$ is Woodin.

Let $x \in \omega_2$. Show that there is an iteration tree $T$ of successor length $\theta + 1$ on $M$ such that $G^M_{x,T_0,\theta}(E)$ is well-defined and is $P^{M^T_\theta}_{j^T_{0,\theta}(E)}$-generic over $M^T_\theta$ with $M^T_\theta[G^M_{x,T_0,\theta}(E)] = M^T_\theta[x]$.

**Hint.** By the argument of the hint to Exercise 6.3.8, it is enough to construct an iteration tree of successor length $\theta + 1$ on $M$ (i.e., on $(M; \in)$), such that $T^{M^T_\theta}_{j^T_{0,\theta}(E)}$ is $x$-consistent in $V$. In doing so, one may assume without loss of generality that

$$E = \{E \mid E \in V^M_\kappa \text{ and } M \models \text{“}E \text{ is strong”}\}.$$ 

Construct an iteration tree $T$ on $M$ and a sequence $\langle \rho_\alpha \mid \alpha + 1 < \ellh(T) \rangle$ of ordinals, with the following properties. (We omit the superscript $T$ and the subscript $T$.)

(i) The length of $T$ is $\theta + 1$ for the least $\theta < \kappa^+$ such that $T^{M_\theta}_{j_{0,\theta}(E)}$ is $x$-consistent in $V$, if there is such a $\theta$; otherwise $\ellh(T) = \kappa^+$.

(ii) For each $\alpha < \theta$, $E_\alpha$ belongs to $j_{0,\alpha}(E)$, i.e., is in $M_\alpha$ a strong extender belonging to $V^{M_\alpha}_{j_{0,\alpha}(\kappa)}$.

(iii) For all $\alpha$ and $\beta$ such that $\alpha \leq \beta < \theta$, $\text{strength}^{M_\alpha}(E_\alpha) \leq \text{strength}^{M_\beta}(E_\beta)$.

(iv) For all $\alpha$ and $\beta$ such that $\alpha \leq \beta < \theta$, $\rho_\alpha \leq \rho_\beta$.

(v) For each $\alpha < \theta$, $\rho_\alpha < \text{strength}^{M_\alpha}(E_\alpha)$. 


(vi) For each \( \alpha < \theta \), \( (\alpha+1)^- \) is the least \( \beta \leq \alpha \) such that \( \text{crit} (E_\alpha) \leq \rho_\beta \).

Since the strength of a strong extender must be a limit ordinal, properties (ii) and (v) imply that \( \rho_\alpha + \omega \leq \text{strength}^{M_\alpha} (E_\alpha) \) for each \( \alpha \). Properties (iv) and (vi) and the argument for Exercise 7.4.1 then show that \( T \) will be plus \( n \) for every \( n \).

Suppose that \( T \upharpoonright \alpha + 1 \) and \( \langle \rho_\beta \mid \beta < \alpha \rangle \) have been constructed and that \( \alpha + 1 \) does not meet condition (i) above for being the length \( \theta + 1 \) of \( T \). Then there must exist

1. an \( E \in V^{M_\alpha}_{\rho_\alpha}(\kappa) \) that is a strong \((\delta, \lambda)\)-extender in \( M_\alpha \), for some \( \delta \) and \( \lambda \);
2. a sequence \( \langle c_\gamma \mid \gamma < \delta \rangle \in M_\alpha \) such that each \( c_\gamma \in C^{M_\alpha} \cap V^{M_\alpha}_{\delta} \), such that \( \hat{c} \), the \( \delta \)th term of the sequence \( i^M_E (\langle c_\gamma \mid \gamma < \delta \rangle) \), belongs to \( V^{M_\alpha}_\lambda \), and such that \( x \in B_{\hat{c}} \setminus \bigcup_{\gamma < \delta} B_{c_\gamma} \).

Choose such an \( E \) and \( \langle c_\gamma \mid \gamma < \delta \rangle \) with the least possible \( \lambda \) and, subject to this, with the least possible value of rank (\( \hat{c} \)). Let \( E_\alpha = E \) and let \( \rho_\alpha = \text{rank} (\hat{c}) \). Also let \( \langle c_\alpha^\gamma \mid \gamma < \delta_\alpha \rangle = \langle c_\gamma \mid \gamma < \delta \rangle \).

At limit steps \( \alpha \), get \( M_\alpha \) by choosing any wellfounded cofinal branch.

Show that \( T \) has properties (i)–(vi). The key fact for verifying (iv) is the following: If \( E \) and \( E' \) are strong extenders with \( \text{crit} (E) = \text{crit} (E') \) and \( \text{strength} (E) < \text{strength} (E') \), then \( E' \upharpoonright \text{strength} (E) = \langle E'_a \mid a \in [\text{strength} (E)]^{<\omega} \rangle \) is a strong extender.

Assume that \( \ell \text{h}(T) = \kappa^+ \). Let \( b \) be a wellfounded cofinal branch of \( T \). Prove that there is a stationary set \( X \subseteq b \) such that, for \( (\alpha + 1)^- \leq (\beta + 1)^- \in X \),

\[
\hat{j}_{(\alpha+1)^-, (\beta+1)^-} (\delta_\alpha) = \delta_\beta;
\hat{j}_{(\alpha+1)^-, (\beta+1)^-} (\langle c_\alpha^\gamma \mid \gamma < \delta_\alpha \rangle) = \langle c_\beta^\gamma \mid \gamma < \delta_\beta \rangle.
\]

(The existence of stationary \( X \) satisfying the first equation comes directly from Exercise 7.4.11.) Suppose that \( (\alpha + 1)^- \) and \( (\beta + 1)^- \) are members of \( X \) with \( \alpha < \beta \). Now \( x \in B_{\hat{c}_\alpha} \), and \( \hat{c}_\alpha \) is the \( \delta_\alpha \)th term of the sequence \( i^M_{E_\alpha} (\langle c_\alpha^\gamma \mid \gamma < \delta_\alpha \rangle) \). But \( \hat{c} \) is also the \( \delta_\alpha \)th term of the sequence \( i^M_{E_\alpha} (\langle c_\gamma \mid \gamma < \delta_\alpha \rangle) \). Clause (vi) implies that \( \text{crit} (\hat{j}_{\alpha+1,(\beta+1)^-}) > \rho_\alpha \). By the definition of \( \rho_\alpha \), this means that \( \hat{c}_\alpha \) is also the \( \delta_\alpha \)th element of the sequence \( \hat{j}_{(\alpha+1)^-, (\beta+1)^-} (\langle c_\gamma^\alpha \mid \gamma < \delta_\alpha \rangle) \). This a contradiction.
Exercise 7.4.13. Let $\kappa$ and $M$ be as in Exercises 7.4.7, 7.4.8, and 7.4.9. Suppose also that $\kappa$ is Woodin in $M$. Let $E$ strongly witness in $M$ that $\kappa$ is Woodin. Suppose that $x \in {}^{\omega}2$ is such that $M$ is a class in $L[x]$ and $\kappa < \omega^L_1$.

Prove that there is an iteration tree $T$ on $M$ with the following properties:

1. $\ell h(T) = \omega^{L[x]}$;
2. $T \in L[x]$;
3. $\omega^{L[x]}_1$ is Woodin in $M(T)$;
4. $G^M_{x, T} \in L[x]$ is well-defined and is $P^{M(T), E(T)}$-generic over $M(T)$ with $M(T)[G^M_{x, T}] = M(T)[x] = L[x]$.

(See Exercise 7.4.8 for the definitions of $M(T)$, $E(T)$, and $\kappa^*(T)$.)

This result is due to Woodin. From it, he gets (a) a proof of $\Pi^1_2$ determinacy different from, and using a slightly weaker hypothesis than, the one in Chapter 8 and (b) a proof that the consistency of ZFC + “There is a Woodin cardinal” implies the consistency of ZFC + “$\Pi^1_2$ determinacy.” (See Exercise 8.3.3.) (b) is half of an equiconsistency result.

Hint. Construct an iteration tree $T \in L[x]$ as in the hint for Exercise 7.4.12. We are not assuming that every iteration tree on $M$ has a well-founded cofinal branch, and even if true this might fail in $L[x]$. Therefore we must replace property (i) of construction of Exercise 7.4.12 by

(i) The length of $T$ is as small as possible so the one of the following holds:

(a) $\ell h(T) = \theta + 1$ and $\theta < \omega^{L[x]}_1$ is the least ordinal such that $T^M_{\theta, 0, 0}(E)$ is $x$ consistent in $L[x]$.
(b) Wellfoundedness fails: either $\ell h(T)$ is a limit ordinal and $T$ has no wellfounded cofinal branch belonging to $L[x]$, or $\ell h(T) = \alpha + 1$ and, for the chosen candidate for $E_\alpha$ and the $(\alpha + 1)^-$ given by

(vi), $\prod_{E_\alpha} M^{(\alpha + 1)^-}_\alpha M^{(\alpha + 1)^-}_\alpha$ is not wellfounded.
(c) $\ell h(T) = \omega^{L[x]}_1$.

Since $T$ is, in particular, a plus two tree, Exercises 7.4.8 and 7.4.9 imply that (b) can hold only if $\ell h(T)$ is a limit ordinal and $M(T) \models \text{“} \kappa^*(T) \text{ is Woodin.”}$
If (a) holds, then $j_{0,\theta}(\kappa) < \omega_1^{L[x]}$. Since $P_{j_{0,\theta}(\kappa)}^{M_\theta}$ has the $j_{0,\theta}(\kappa)$ chain condition in $M_\theta$, this gives the contradiction that
\[
\omega_1^{M_\theta[x]} \leq j_{0,\theta}(\kappa) < \omega_1^{L[x]}.
\]
If (c) holds, then the last argument of the hint to Exercise 7.4.12 shows that (b) must hold also. Thus (b) holds.

Show that $I_{\kappa(T)^M}^{M(T)}$ is $x$-consistent in $L[x]$.

If $\kappa^*(T) < \omega_1^{L[x]}$, then we get a contradiction as in the case of (a)’s holding. Thus $\kappa^* = \ell h(T) = \omega_1^{L[x]}$. 

Chapter 8

Projective Games

The projective hierarchy of subsets of a topological space $X$ is defined (by induction on all $X$ simultaneously) as follows:

(a) $A \in \Sigma^1_1$ if and only if there is a closed $C \subseteq X \times \omega^\omega$ such that $A = \{x \in X \mid (\exists y \in \omega^\omega) \langle x, y \rangle \in C\}$.

(b) For all positive integers $n$, $A \in \Pi^1_n$ if and only if $X \setminus A \in \Sigma^1_n$.

(c) $A \in \Sigma^1_{n+1}$ if and only if there is a $B \subseteq X \times \omega^\omega$ such that $B \in \Pi^1_n$ and $A = \{x \in X \mid (\exists y \in \omega^\omega) \langle x, y \rangle \in B\}$.

(d) $A \in \Delta^1_1$ if and only if $A \in \Sigma^1_n$ and $A \in \Pi^1_n$.

The class of projective subsets of $X$ is $\bigcup_n \Sigma^1_n$.

Remark. Instead of defining $\Sigma^1_1$ directly by clause (a), one can start with $n = 0$, letting $\Sigma^1_0$ be the class of all open sets.

In this chapter we will prove, assuming the existence of infinitely many Woodin cardinals greater than $|T|$, the determinacy of all projective games in a game tree $T$. For the determinacy of all $\Pi^1_{n+1}$ games in $T$, we will need $n$ Woodin cardinals greater than $|T|$, plus—say—a measurable cardinal greater than the $n$ Woodin cardinals. These results are from [Martin and Steel, 1988] and [Martin and Steel, 1989].

The proof will proceed via Theorem 4.3.5. By this theorem, the determinacy of all $\Pi^1_{n+1}$ games in $T$ will follow if we can show that every $\Pi^1_{n+1}$ subset of $[T]$ is $|T|^+$-homogeneously Souslin. By Theorem 4.3.6, we already have the special case $n = 0$, provided that there is a measurable cardinal.
greater than $|T|$. What we need is thus a method to propagate homogeneous Souslinness up the projective hierarchy.

In §8.1 we deal with the first half of the problem of propagating homogeneous Souslinness: We give a natural way of transferring homogeneous Souslinness at one level of the projective hierarchy to Souslinness at the next level. We first illustrate the idea by defining an operation

$$\langle U, \langle \mathcal{U}_p \mid p \in U \rangle \rangle \mapsto U^\dagger(\langle \mathcal{U}_p \mid p \in T \rangle).$$

The value $U^\dagger(\langle \mathcal{U}_p \mid p \in T \rangle)$ is defined whenever $\langle \mathcal{U}_p \mid p \in T \rangle$ witnesses that $U$ is homogeneous for $T$, and this value is a tree witnessing that the complement of the $T$-projection of $U$ is Ord-Souslin. We next introduce the notions of weakly homogeneous trees and weakly $\kappa$-homogeneously Souslin sets. We study various equivalents of these notions, some of which will be used in Chapter 9. We show that if $A \subseteq [T] \times \omega$ is $\kappa$-homogeneously Souslin—in the obvious sense—then $pA$ is weakly $\kappa$-homogeneously Souslin. We then define a $U^\dagger$ operation analogous to the $U^\dagger$ operation, but defined on trees $U$ and witnesses to the weak homogeneity of $U$. The ultimate origin of the $U^\dagger$ and $U^\dagger$ operations is [Martin and Solovay, 1969], though the general constructions were discovered a few years later.

In §8.2 we show that, if $\kappa$ is a Woodin cardinal and $U^\dagger$ is the result of applying our operation to a witness that some set is weakly $\kappa^+$-homogeneously Souslin, then (any sufficiently large restriction of) $U^\dagger$ is $(<\kappa)$-homogeneous, i.e., is $\eta$-homogeneous for every $\eta < \kappa$. This theorem will enable us to propagate homogeneous Souslinness up the projective hierarchy and so to prove projective determinacy. It will also be the basis for further determinacy theorems in Chapter 9. To make the ideas of the main construction more comprehensible, we first do an analogous construction for the simpler $U^\dagger$ operation. To motivate our constructions, we aim directly at a weaker property than homogeneous Souslinness: the property of having an embedding normal form. The homogeneity of $U^\dagger$ and $U^\dagger$ falls out of our proofs that the $T$-projections of these trees have embedding normal forms.

The construction given in §8.2 is a modification of that given in [Martin and Steel, 1989]. This modification, based on an idea of Itay Neeman, yields a construction that is slightly more complicated than what could be gotten by a smaller modification of the earlier proof. The slight extra complexity is more than compensated by the new construction’s yielding immediately not just an embedding normal form but also homogeneity. The construction of [Martin and Steel, 1989]
8.1. WEAKLY HOMOGENEOUS TREES

§8.3 is devoted to variations on the proof given in §8.2. First we give a construction that is not too different from the one in [Martin and Steel, 1989]. We then follow [Martin and Steel, 1989] in proving a lemma asserting roughly that, if \( T \subseteq V_\kappa \) is an iteration tree of length \( \omega \) on \( V \) and \( \langle U_p \mid p \in T \rangle \) witnesses that some tree is \( \kappa \)-homogeneous, then the embeddings of the former and the latter act trivially on one another. Armed with this lemma, we show how the construction of §8.3 can yield the results of §8.2. We next prove a theorem of Katrin Windßus stating that an embedding normal form with \( 2^{\aleph_0} \)-closed models directly implies homogeneous Souslinness. Windßus’ theorem provides another way to get the determinacy results of §8.2 from either of our constructions. Finally we mention—and cite references for—machinery due to Neeman for proving the theorem from an optimal hypothesis, machinery that he has used to get a large range of determinacy results.

8.1 Weakly Homogeneous Trees

Let \( T \) be a game tree, let \( Y \) be a set, and let \( U \) be a tree on field \((T) \times Y\). Suppose that \( \langle U_p \mid p \in T \rangle \) witnesses that \( U \) is homogeneous for \( T \). For \( p \in T \), let \( \pi_p = \pi_U : \prod_{U_p} (V; \in) \cong (\Ult(V; U_p); \in) \). For \( p \subseteq q \in T \), let \( i_{p,q} : \Ult(V; U_p) \prec \Ult(V; U_q) \) be defined as on page 200. Similarly define, for \( x \in [T] \), the class model \( M_x \) and the embeddings

\[ i_{x\upharpoonright \alpha} : \Ult(V; U_p) \prec M_x \]

be defined as before.

Define a tree \( U^\uparrow(\langle U_p \mid p \in T \rangle) \) on field \((T) \times \text{Ord}, \text{as follows. If } p \in T \) and \( t \in \text{lh}(p)\text{Ord, then } \{p, t\} \in U^\uparrow(\langle U_p \mid p \in T \rangle) \text{ if and only if } \]

\( (\forall i_1 < \text{lh}(p))(\forall i_2 < \text{lh}(p))(i_1 < i_2 \rightarrow t(i_2) < i_{p|i_1, p|i_2}(t(i_1))) \).

Let \( A \) be the \( T \)-projection of \( U \).

**Theorem 8.1.1.** Let \( T, Y, U, \langle U_p \mid p \in T \rangle, \) and \( A \) be as above. Let \( U^\uparrow = U^\uparrow(\langle U_p \mid p \in T \rangle) \). Then \([T] \setminus A\) is the \( T \)-projection of \( U^\uparrow \). Moreover \([T] \setminus A\) is also the \( T \)-projection of \( U^\uparrow \mid \alpha \) for any ordinal \( \alpha \geq \max\{\omega, (2^{|Y|})^+\} \), where \( U^\uparrow \mid \alpha = U^\uparrow \cap \{\{p, t\} \mid \text{range}(t) \subseteq \alpha\} \).
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**Proof.** Let \( x \in [T] \) be such that \([U^\uparrow(x)] \neq \emptyset \). Let \( f \in [U^\uparrow(x)] \). For each \( n \in \omega \),

\[
i_{x|n,x|n+1}(f(n)) < f(n+1).
\]

Now

\[
i_{x|n}^n(f(n)) = i_{x|n+1}(i_{x|x|n+1}(f(n))) < i_{x|n+1}^n(f(n+1)).
\]

Hence \( \langle i_{x|n}^n(f(n)) \mid n \in \omega \rangle \) is an infinite descending sequence of ordinals of \( \mathcal{M}_x \). Lemma 4.3.4 implies that \( x \notin A \).

Next let \( x \in [T] \setminus A \). Thus \( U(x) \) is a wellfounded tree. Note that, in the case that \( Y \) is finite, it follows by König’s Lemma that \( U(x) \) is finite. This gives us the function

\[
\| U(x) : U(x) \to |U(x)|^+,
\]

defined on page 25. For each \( n \in \omega \), let

\[
f_n : U[x \mid n] \to |U(x)|^+
\]

be given by setting \( f_n(s) = \| s \|^U(x) \) for each \( s \in U[x \mid n] \). For \( n \in \omega \) let \( t_n = \pi_{x|n}(\{f_n\}_{U[x]}) \). Since, for each \( n \), \( f_{n+1}(s) < f_n(s \mid n) \) for every \( s \in U[x \mid n+1] \), it follows that \( t_{n+1} < i_{x|n,x|n+1}(t_n) \) for each \( n \). This shows that \([U^\uparrow(x)] \neq \emptyset \). Moreover, for each \( n \in \omega \),

\[
|t_n| = \{| \{g \mid U[x] \mid \pi_{x|n}(\{g\}_{U[x]}) < \pi_{x|n}(\{f_n\}_{U[x]}) \} \|
\leq \{| \{g : U[x \mid n] \to \text{Ord} \mid (\forall t \in U[x \mid n]) \ g(t) < f_n(t) \} \|
\leq |U(x)|^{\| U(x) \|}.
\]

For \( Y \) infinite, \( |U(x)|^{\| U(x) \|} \leq 2^{Y} \). Hence each \( t_n < \max\{\omega, (2^Y)^+\} \), and so the \( t_n \) witness that \( [(U^\uparrow \uparrow \alpha)(x)] \neq \emptyset \) for all \( \alpha \geq \max\{\omega, (2^Y)^+\} \).

\( \square \)

In order to propagate up the projective hierarchy the property of being homogeneously Souslin, it will be useful to have an operation on homogenous trees that will yield a result like Theorem 8.1.1, but with the complement of \( pA \) replacing the complement of \( A \). For this purpose, and for use in Chapter 9, we now introduce the notion of weak homogeneity.

For trees \( T \) and \( R \), let

\[
T \otimes R = \{ \langle p, r \rangle \mid p \in T \land r \in R \land \ell h(p) = \ell h(r) \}.
\]
Let $T$ be a game tree, let $Y$ be a nonempty set, and let $U$ be a tree on field $(T) \times Y$. We say that $U$ is weakly homogeneous for $T$ if there is a system

$$\langle U_{p,r} \mid \{p, r\} \in T \otimes {}^{<\omega}\omega \rangle$$

satisfying the following conditions:

1. Each $U_{p,r}$ is a countably complete ultrafilter on $U_{p}$. 
2. The $U_{p,r}$ are compatible: For all $\langle \{p, r\} \subseteq \{q, s\} \in T \otimes {}^{<\omega}\omega, U_{q,s}$ projects to $U_{p,r}$ by $\chi_{q,p}$, where $\chi_{q,p} : U[q] \rightarrow U[p]$ is given (as on page 200) by $\chi_{q,p}(t) = t \upharpoonright \ell h(p)$. 
3. Let $x \in [T]$ and $\langle Z_r \mid r \in {}^{<\omega}\omega \rangle$ be such that each $Z_r$ belongs to $U_{x \upharpoonright \ell h(r),r}$. Then $[U(x)] \neq \emptyset \rightarrow (\exists y \in {}^{<\omega}\omega)(\exists f \in {}^{<\omega}Y)(\forall n \in \omega) f \upharpoonright n \in Z_{y \upharpoonright n}$.

As was the case for the corresponding clause in the definition of homogeneous trees, there is an equivalent of condition (3) in terms of wellfoundedness of direct limit models. Suppose that (1) and (2) are satisfied. For $\langle \{p, r\} \subseteq \{q, s\} \in T \otimes {}^{<\omega}\omega \rangle$ let $i_{\langle p,r \rangle,\langle q,s \rangle} = \pi_{q,s} \circ i_{U_{p,r}, U_{q,s}, \chi_{q,p}} \circ \pi_{p,r}^{-1}$. (See page 199 for the definition of $i_{U_{p,r}, U_{q,s}, \chi_{q,p}}$.) For $x \in [T]$ and $y \in {}^{<\omega}\omega$, let

$$M_{x,y} : \langle i_{\langle x \upharpoonright n, y \upharpoonright n \rangle}^\circ n \in \omega \rangle$$

be the direct limit of the directed system of elementary embeddings

$$\langle (\Ult(V; U_{x \upharpoonright n, y \upharpoonright n}) \mid n \in \omega) ; \langle i_{\langle x \upharpoonright m, y \upharpoonright m \rangle}^\circ (x \upharpoonright n, y \upharpoonright n) \mid m \leq n \in \omega \rangle \rangle.$$ 

(3’) $(\forall x \in [T])([U(x)] \neq \emptyset \rightarrow (\exists y \in {}^{<\omega}\omega) M_{x,y} \text{ is wellfounded}).$

Lemma 8.1.2. Let $T$ and $\langle U_{p,r} \mid \{p, r\} \in T \otimes {}^{<\omega}\omega \rangle$ be such that (1) and (2) hold. If $x \in [T]$, then $x$ witnesses the falsity of (3) if and only if $x$ witnesses the falsity of (3’). Thus a tree $U$ on field $(T) \times Y$ is weakly homogeneous for $T$ if and only if there is a system $\langle U_{p,r} \mid \{p, r\} \in T \otimes {}^{<\omega}\omega \rangle$ satisfying (1), (2), and (3’).
Proof. The proof parallels that of Lemma 4.3.4, with an extra wrinkle in the second part.

Suppose first that \( x \) and \( \langle Z_r \mid r \in {\leq^*}_x \rangle \) witness the failure of (3). Let \( y \in \omega \). Let
\[
S_y = \{ s \in U(x) \mid (\forall n \leq \ell h(s)) s \upharpoonright n \in Z_y[n] \}.
\]
Exactly as the tree \( S \) was used in the proof of Lemma 4.3.4 to show that \( M_x \) was not wellfounded, the tree \( S_y \) can be used to show that \( M_{x,y} \) is not wellfounded.

Now suppose that \( x \) witnesses that (3') fails. For each \( y \in \omega \), let \( \langle z_n^y \mid n \in \omega \rangle \) be an infinite descending sequence with respect to \( i^{x,y}_n(\in) \). For each \( y \in \omega \) and each \( n \in \omega \), let \( m_n^y \) and \( a_n^y \in \text{Ult}(V;\mathcal{U}_{x|m_n^y|y/m_n^y}) \) be such that \( z_n^y = i^{x,y}_n(m_n^y,y/m_n^y)(a_n^y) \). Without loss of generality, we may assume that
\[
(\forall y \in \omega)(\forall n' \in \omega)(\forall n \in \omega)(n' < n \rightarrow m_{n'}^y < m_n^y).
\]
Let \( g_n^y \in U[x|m_n^y]V \) be such that
\[
\pi_{x|m_n^y|y/m_n^y}(\langle g_n^y \rangle_{\mathcal{U}_{x|m_n^y|y/m_n^y}}) = a_n^y.
\]
For each \( y \in \omega \) and each \( n \in \omega \), let
\[
Z_{m_n^y} = \{ s \in U[x \upharpoonright m_{n+1}^y] \mid g_{n+1}^y(s \upharpoonright m_n^y) \in g_n^y(s \upharpoonright m_n^y) \}.
\]
For each \( m \in \omega \) such that \( m \) is not of the form \( m_{n+1}^y \), let \( Z_m^y = U[x \upharpoonright m] \). For \( m \in \omega \), we have that \( Z_m^y \in \mathcal{U}_{x|m|y/m} \). For \( r \in {\leq^*}_x \), let
\[
Z_r = \bigcap \{ Z_m^y \mid y \in \omega, r \subseteq y \}.
\]
Since any countably complete ultrafilter is \( 2^{\aleph_0} \)-complete, \( Z_r \) belongs to \( \mathcal{U}_{x|\ell h(r),r} \) for every \( r \in {\leq^*}_x \). To see that \( \langle Z_r \mid r \in {\leq^*}_x \rangle \) witnesses the failure of (3) for \( x \), suppose that \( y \in \omega \) and \( f \in {\leq^*} Y \) are such that \( (\forall m \in \omega) f \upharpoonright m \in Z_m^y \).
Then \( f(m_n^y) \in Z_{m_n^y}^y \) for every \( n \in \omega \), and so we get the contradiction that \( \langle f(m_n^y) \mid n \in \omega \rangle \) is an infinite descending sequence with respect to \( \in \).

Remark. As in the case of homogeneous trees, the “\( \rightarrow \)” in the last line of condition (3) and that in condition (3') can be replaced by “\( \leftrightarrow \).”

For \( T \) a game tree, \( Y \) a set, and \( \kappa \) a cardinal number, a tree \( U \) on field \( (T) \times Y \) is \emph{weakly} \( \kappa \)-homogeneous for \( T \) if there is a system \( \langle U_{p,r} \mid \langle p, r \rangle \in T \otimes \)}
8.1. WEAKLY HOMOGENEOUS TREES

Let $T$ be a game tree. A subset $A$ of $[T]$ is weakly homogeneously Souslin if it is the $T$-projection of a tree weakly homogeneous for $T$; $A$ is weakly $\kappa$-homogeneously Souslin if it is the $T$-projection of a tree weakly $\kappa$-homogeneous for $T$.

We now prove some results giving equivalents of a tree’s being weakly $\kappa$-homogeneous and of a set’s being weakly $\kappa$-homogeneously Souslin. The easy half of the first of these results is directly relevant to our goal of propagating homogeneous Souslinness up the projective hierarchy. The other results, particularly Theorem 8.1.7, will be important in Chapter 9.

It will be convenient to extend our notation $\langle |p,q| \rangle$ to infinite sequences. If $x$ and $y$ are functions with domain $\omega$, let $\langle |x,y| \rangle = \{ \langle |x|_n, y|_n \rangle | n \in \omega \}$.

If $T$ and $T'$ are trees, let us say that a subset $A$ of $[T] \times [T']$ is homogeneously Souslin if $\{ \{x, x'\} | \langle x, x' \rangle \in A \}$ is homogeneously Souslin. Similarly define, for subsets of products, the notions of $\kappa$-homogeneously Souslin, weakly homogeneously Souslin, and weakly $\kappa$-homogeneously Souslin.

Theorem 8.1.3. Let $T$ be a game tree, let $A \subseteq [T]$, and let $\kappa$ be a cardinal number greater than $|T|$. Then the following are equivalent:

(a) $A$ is weakly $\kappa$-homogeneously Souslin.

(b) There is a $B \subseteq [T] \times ^\omega \omega$ such that $B$ is $\kappa$-homogeneously Souslin and $A = pB$

Proof. Suppose first that $B$ is as in (b). Let $\hat{U} \subseteq (T \otimes ^\omega \omega) \otimes ^\omega Y$ be a tree witnessing that $B' = \{ \{x, x'\} | \langle x, x' \rangle \in A \}$ is $\kappa$-homogeneously Souslin. Let $\langle \hat{U}_{|p,r|} | \langle p, r \rangle \in T \otimes ^\omega \omega \rangle$ witness that $\hat{U}$ is $\kappa$-homogeneous for $T \otimes ^\omega \omega$.

Let

$$U = \{ \{p, \{r, s\}\} | \{\{p, r\}, s\} \in \hat{U} \}.$$  

For $x \in [T]$,

$$x \in A \iff (\exists y \in ^\omega \omega) [\hat{U}([x, y])] \neq \emptyset$$

$$\iff (\exists y \in ^\omega \omega)(\exists z \in ^\omega Y) \{\{x, y\}, z\} \in [\hat{U}]$$

$$\iff (\exists y \in ^\omega \omega)(\exists z \in ^\omega Y) \{x, \{y, z\}\} \in [U]$$

$$\iff [U(x)] \neq \emptyset.$$
Fix $p \in T$ and $r \in \omega\omega$. For $Z \subseteq U[p]$, define $\hat{Z} \subseteq U[p, r]$ by

$$\hat{Z} = \{s \in \omega Y \mid \langle r, s \rangle \in Z\}.$$ 

Set

$$U_{p, r} = \{Z \subseteq U[p] \mid \hat{Z} \in \hat{U}_{p, r}\}.$$ 

It is easy to check that $\langle U_{p, r} \mid \langle p, r \rangle \in T \otimes \omega\omega \rangle$ witnesses that $U$ is weakly $\kappa$-homogeneous.

To prove the other half of the lemma, suppose $U$ is a tree on field $(T \times Y)$ witnessing that $A$ is weakly $\kappa$-homogeneously Souslin and suppose that $\langle U_{p, r} \mid \langle p, r \rangle \in T \times \omega\omega \rangle$ witnesses that $U$ is weakly $\kappa$-homogeneous for $T$.

Let $x \in [T]$ and $y \in \omega\omega$. If there is a system $\langle Z_n \mid n \in \omega \rangle$ such that each $Z_n \in U_{x|n, y|n}$ and such that

$$(\forall f \in \omega Y)(\exists n \in \omega) f \upharpoonright n \notin Z_y|n,$$

then choose such a system and set $Z_{x,y}^n = Z_n$ for each $n \in \omega$. Otherwise set $Z_{x,y}^n = U_{x|n, y|n}$ for each $n$.

For $\langle p, r \rangle \in T \otimes \omega\omega$, set

$$Z_{p, r} = \bigcap \{Z_{x,y}^n \mid x \in [T] \land y \in \omega Y \land \langle p, r \rangle \subseteq \langle x, y \rangle\}.$$ 

Since $\kappa > |T|$, we have that $Z_{p, r} \in U_{p, r}$ for every $\langle p, r \rangle \in T \otimes \omega\omega$.

Define a tree $\hat{U}$ on $(T \times Y) \times Y$ by

$$\hat{U} = \{\langle \langle p, r \rangle, s \rangle \mid s \in Z_{p, r}\}.$$ 

For $\langle p, r \rangle \in T \otimes \omega\omega$, set

$$\hat{U}_{p, r} = U_{p, r} \cap \mathcal{P}(\hat{U}[\langle p, r \rangle]).$$

Thus $\hat{U}_{p, r}$ is essentially the same ultrafilter as $U_{p, r}$.

It is evident that the $\hat{U}_{p, r}$ are $\kappa$-complete and that $\langle \hat{U}_{p, r} \mid \langle p, r \rangle \in T \otimes \omega\omega \rangle$ satisfies conditions (1) and (2) for witnessing that $\hat{U}$ is homogeneous for $T \otimes \omega\omega$.

To verify condition (3), let us suppose that $\langle x, y \rangle \in [\hat{U}]$. Let $f \in \hat{U}(\langle x, y \rangle)]$. Let $X_n \mid n \in \omega$ be such that each $X_n$ belongs to $\hat{U}_{x|n, y|n}$. Assume for a contradiction that there is no $g : \omega \to Y$ such that $g \upharpoonright n \in X_n$ for every $n$. By the definition of the $Z_{x,y}^n$, it follows that there is no $g : \omega \to Y$
such that $g \restriction n \in Z^{x,y}_n$ for every $n$. For each $n$, $Z_{x|n,y|n} \subseteq Z^{x,y}_n$. Hence there is no $g : \omega \to Y$ such that $g \restriction n \in Z_{x|n,y|n}$ for every $n$. But

$$(\forall n \in \omega) Z_{x|n,y|n} = \hat{U}[\{x \restriction n, y \restriction n\}].$$

Since $f \restriction n \in \hat{U}[\{x \restriction n, y \restriction n\}]$ for all $n$, we get our contradiction.

Let $B = \{\langle x, y \rangle \mid \langle x, y \rangle \in [U]\}$. It remains to show that $A = pB$.

Suppose first that $x \in pB$. Let $y$ be such that $\langle x, y \rangle \in B$. Let $f$ be such that $f \in [\hat{U}(\langle x, y \rangle)]$. We then have, for all $n \in \omega$, that $f \restriction n \in \hat{U}[\{x \restriction n, y \restriction n\}] = Z_{x|n,y|n} \subseteq Z^{x,y}_n \subseteq U[x \restriction n]$.

Thus $f \in [U(x)]$, and so $x \in A$.

Now suppose that $x \in A$. Let us apply condition (3) to the system $\langle Z_{x|n|r}, r \mid r \in <\omega, \omega\rangle$, which is identical with $\langle \hat{U}[\{x \restriction n, r\} \mid r \in <\omega, \omega\rangle$. This gives us a $y \in \omega, \omega$ and an $f : \omega \to Y$ such that each $f \restriction n$ belongs to $\hat{U}[\{x \restriction n, y \restriction n\}]$. But then $f \in [\hat{U}(\langle x, y \rangle)]$, and therefore $\langle x, y \rangle \in B$.  

$\square$

Note that our proof that (b) implies (a) made no use of the hypothesis that $\kappa > |T|$. Hence we have the following result.

**Corollary 8.1.4.** Let $T$ be a game tree, let $\kappa$ be a cardinal number, and let $B$ be a $\kappa$-homogeneously Souslin subset of $[T] \times \omega$. Then $pB$ is weakly $\kappa$-homogeneously Souslin.

**Remark.** The material between here and the end of the proof of Lemma 8.1.7 will not be used until Chapter 9. The reader may thus prefer to skip this material and return to it only when it is about to be used (in §9.6).

Let $T$ be a game tree, let $Y$ be a set, and let $U$ be a tree on field $(T) \times Y$. A $T$-cover of $U$ by ultrafilters is a set $V$ of of countably complete ultrafilters on $U$ such that, for each member $x$ of the $T$-projection of $U$, there are $V_i$, $i \in \omega$, satifying (i)–(iv) below.

(i) each $V_i$ belongs to $V$;
(ii) each $V_i$ is an ultrafilter on $U[x \restriction i]$;
(iii) for $i < j \in \omega$, $V_j$ projects to $V_i$ by $\chi_{j,i}$, where $\chi_{j,i} : U[x \restriction j] \to U[x \restriction i]$ is given by $\chi_{j,i}(s) = s \restriction i$;
(iv) if \( \langle Z_i \mid i \in \omega \rangle \) is such that each \( Z_i \in V_i \), then there is an \( f : \omega \to Y \) such that \( f \upharpoonright i \in Z_i \) for all \( i \in \omega \).

For cardinal numbers \( \kappa \), a \( T \)-cover \( V \) of \( U \) by ultrafilters is \( \kappa \)-complete if every member of \( V \) is \( \kappa \)-complete. A \( T \)-cover \( V \) of \( U \) by ultrafilters is full if the following holds. Suppose that \( p \in [T] \) and that \( V \in V \) is an ultrafilter on \( U[p] \). Let \( p' \in [T] \) with \( p' \supseteq p \) and \( \ell h(p') = \ell h(p) + 1 \). Then there is a \( W \in V \) such that \( W \) is an ultrafilter on \( U[p'] \) and \( W \) projects to \( V \) by \( \chi_{i+1,i} \), where the \( \chi_{j,i} \) are as in the statement of (iii).

We need the following technical lemma in order to prove Theorem 8.1.7. It would not have been necessary if we had worked with the definition of homogeneity of Exercise 4.3.5 and with the corresponding definition of weak homogeneity. (See Exercise 8.1.1.)

**Lemma 8.1.5.** Let \( \kappa \) be a cardinal number and let \( \rho \) be a measurable cardinal. Let \( T \) be a game tree, let \( Y \) be a set, and let \( U \) be a tree on field \( (T) \times Y \).
Suppose that \( V \) is a \( \kappa \)-complete \( T \)-cover of \( U \) by ultrafilters. Then there exist

1. a set \( Y' \supseteq Y \) such that \( |Y'| \leq \max\{\rho, |Y|\} \);
2. a tree \( U' \supseteq U \) on field \( (T) \times Y' \) such that \( U \) and \( U' \) have the same \( T \)-projection;
3. a set \( V' \supseteq V \) such that \( |V'| \leq \max\{\aleph_0, |V|\} \) and \( V' \) is a full, \( (\min\{\kappa, \rho\}) \)-complete \( T \)-cover of \( U \).

**Proof.** Let \( b_\beta, \beta < \rho \), be distinct from one another and from all elements of \( Y \). Let \( Y' = Y \cup \{b_\beta \mid \beta < \rho\} \).

Let \( U' \) be the set of all \( \langle p, s^{-} \langle b_\beta \mid i < n \rangle \rangle \) such that

(a) \( p \in {}^<\omega \text{field}(T) \);
(b) \( s \in {}^<\omega Y \);
(c) \( n \in \omega \);
(d) \( (\forall i < n) \beta_i < \rho \);
(e) \( \ell h(p) = \ell h(s) + n \);
(f) \( \langle p \upharpoonright \ell h(s), s \rangle \in U \);
(g) \( (\forall i < n)(\forall j < n)(i < j \to \beta_i > \beta_j) \).
8.1. WEAKLY HOMOGENEOUS TREES

Note that $U \subseteq U'$ and that $[U'(x)] = [U'(x)]$ for all $x \in [T]$.

Let $\mathcal{U}$ be a uniform normal ultrafilter on $\rho$. Recall the Robottom ultrafilters $\mathcal{U}^{[n]}$ of §3.1. For $n \in \omega$ and $q \in [\rho]^n$ let $g_q : n \rightarrow q$ be such that $g_q(i) > g_q(j)$ whenever $i < j < n$. For $n \in \omega$, let $\mathcal{U}_n^*$ be the ultrafilter on $n\rho$ defined by

$$Z \in \mathcal{U}_n^* \leftrightarrow (\exists X \in \mathcal{U}^{[n]})(\forall q \in X) g_q \in Z.$$ 

For each $V \in \mathcal{V}$ and each $n \in \omega$, let $\mathcal{W}_{V,n}$ be the iterated product of $V$ and $\mathcal{U}_n^*$, i.e., let

$$X \in \mathcal{W}_{V,n} \leftrightarrow \{s | (\exists Z \in \mathcal{U}_n^*)(\forall t \in Z) s \rightarrow t \in X\} \in \mathcal{V}.$$ 

For each $\langle p, \langle b_\beta | i < n \rangle \rangle \in U'$ and each $V \in \mathcal{V}$ such that $V$ is an ultrafilter on $U[p \upharpoonright m]$ for some $m$, $\mathcal{W}_{V,n}$ is a $(\min\{\kappa, \rho\})$-complete ultrafilter on $U'[p]$. Moreover, for each $V \in \mathcal{V}$ and each $n \in \omega$, $\mathcal{W}_{V,n+1}$ projects to $\mathcal{W}_{V,n}$ by $\chi_{n+1,n}$.

Set

$$\mathcal{V}' = \{\mathcal{W}_{V,n} | V \in \mathcal{V} \wedge n \in \omega\}.$$ 

It is easy to check that $\mathcal{V}'$ has the required properties. □

**Lemma 8.1.6.** Let $T$ be a game tree, let $Y$ be a set, let $U$ be a tree on field $(T) \times Y$, and let $\kappa$ be a cardinal number. Then the following are equivalent:

(a) $U$ is weakly $\kappa$-homogeneous for $T$.

(b) There is a countable, full, $\kappa$-complete $T$-cover of $U$ by ultrafilters.

**Proof.** Suppose first that $\langle \mathcal{U}_{p,r} | \langle p, r \rangle \in T \otimes ^{<\omega}\omega \rangle$ witness that $U$ is weakly $\kappa$-homogeneous. Let

$$\mathcal{V} = \{\mathcal{U}_{p,r} | \langle p, r \rangle \in T \otimes ^{<\omega}\omega \}.$$ 

Suppose that $x \in [T]$ and $[U(x)] \neq \emptyset$. By condition (3'), let $y \in ^{\omega}\omega$ be such that $\mathcal{M}_{x,y}$ is wellfounded, where $\mathcal{M}_{x,y}$ is defined as on page 421. By the proof of Lemma 4.3.4, clauses (i)–(iv) from the definition of a $T$-covering of $U$ by ultrafilters hold if we set $\mathcal{V}_i = \mathcal{U}_{x[i,y][i]}$ for each $i$. Thus $\mathcal{V}$ is a $T$-covering of $U$ by ultrafilters.

Obviously $\mathcal{V}$ is countable, full, and $\kappa$-complete.

Now suppose that $\mathcal{V}$ is a countable, full, $\kappa$-complete $T$-cover of $U$ by ultrafilters.
We construct $\mathcal{U}_{p,r}$, $(p,r) \in T \otimes {\mathcal{V}}$, by induction on $\ell h(p)$. Assume that $U_{p,r}$ has been defined, is an ultrafilter on $U[p]$, and belongs to $\mathcal{V}$. Let $p' \supseteq p$ with $\ell h(p') = \ell h(p) + 1$. Since $\mathcal{V}$ is full, there is at least one $\mathcal{V} \in \mathcal{V}$ such that $\mathcal{V}$ is an ultrafilter on $U'[p']$ and $\mathcal{V}$ projects to $U_{p,r}$ by $\chi_{n+1,n}$. Since $\mathcal{V}$ is countable, we can let $(U_{p',r} - (i) \mid i \in \omega)$ be an enumeration, possibly with repetitions, of all such $\mathcal{V}$.

To verify clause (3) in the definition of weak homogeneity, let $x$ belong to the $T$-projection of $U$ and let $Z_r$, $r \in {\omega}^{\omega}$, be such that each $Z_r \in \mathcal{U}_{x,\ell h(r),r}$. Let $\mathcal{V}_i$, $i \in \omega$, be given by (i)–(iv). By construction, there is a $y \in {\omega}^{\omega}$ such that $\mathcal{V}_i = \mathcal{U}_{x,i,y,i}$. By (iv) we get an $f \in {\omega}^{\omega}$ such that $f \upharpoonright n \in Z_y \upharpoonright i$ for each $i \in \omega$.

**Theorem 8.1.7.** Let $\kappa$ be a cardinal, and assume that there is a measurable cardinal $\geq \kappa$. Let $T$ be a game tree, and let $A \subseteq |T|$. Then the following are equivalent:

(a) $A$ is weakly $\kappa$-homogeneously Souslin.

(b) $A$ is the $T$-projection of a tree $U$ such that the exists a countable, $\kappa$-complete $T$-cover of $U$ by ultrafilters.

**Proof.** That (a) implies (b) follows directly from Lemma 8.1.6.

Let $U$ witness that (b) holds. Applying Lemma 8.1.5 to $U$ with some measurable cardinal $\geq \kappa$ as $\rho$, we get a tree $U'$ whose $T$-projection is (a) and which has a countable, full, $\kappa$-complete cover by ultrafilters. Lemma 8.1.6 gives (a).

We now turn to the $U_{p,r}^+$ construction.

Let $T$ be a game tree, let $Y$ be a set, and let $U$ be a tree on field $(T) \times Y$. Suppose that $(U_{p,r}, (p,r) \in T \otimes {\omega}^{\omega})$ witnesses that $U$ is weakly homogeneous for $T$. Let $(\pi_{p,r}, (p,r) \in T \otimes {\omega}^{\omega})$, $(i_{(p,r),(q,s)}, (p,r) \subseteq (q,s) \subseteq T \otimes {\omega}^{\omega})$, and $(M_{x,y}; (x,y), n \in {\omega})$ be as on page 421. Let $i \mapsto r_i$ be a one-one correspondence between $\omega$ and $\omega^{\omega}$ with the property that

$$\forall i \in \omega)(\forall i' \in \omega)(r_i \subseteq r_{i'} \rightarrow i \leq i').$$

Define a tree $U^+(\mathcal{U}_{p,r}, (p,r) \in T \otimes {\omega}^{\omega})$ on field $(T) \times \text{Ord}$, as follows. If $p \in T$ and $t \in \ell h(p)\text{Ord}$, then $(p,t) \in U^+(\mathcal{U}_{p,r}, (p,r) \in T \otimes {\omega}^{\omega})$ if and only if, for all $i_1$ and $i_2$ less than $\ell h(p)$,

$$r_{i_1} \subseteq r_{i_2} \rightarrow t(i_2) < i_{(p,\ell h(r_{i_1}),r_{i_1}),(p,\ell h(r_{i_2}),r_{i_2})} (t(i_1)).$$

Let $A$ be the $T$-projection of $U$. 

Theorem 8.1.8. Let $T$, $Y$, $U$, $\langle U_p \mid p \in T \rangle$, and $A$ be as above. Let $U^+ = U^+(\{U_{p,r} \mid \{p,r\} \in T \otimes \omega \omega \})$. Then $[T] \setminus A$ is the $T$-projection of $U^+$. Moreover $[T] \setminus A$ is also the $T$-projection of $U^+ \upharpoonright \alpha$ for any ordinal $\alpha \geq \max\{\omega, (2^{|Y|})^+\}$, where $U^+ \upharpoonright \alpha = U^+ \cap \{\{p,t\} \mid \text{range } (t) \subseteq \alpha\}$.

Proof. Let $x \in [T]$ be such that $[U^+(x)] \neq \emptyset$. Let $f \in [U^+(x)]$. Let $y \in \omega \omega$. For each $n \in \omega$, let $i_n$ be the number such that $r_{i_n} = y \upharpoonright n$. For each $n \in \omega$,

$$i_{(x,y/\omega,n)}(f(n)) > f(i_{n+1}).$$

It follows that $(i_{(x,y/\omega,n)}(f(n)) \mid n \in \omega)$ is an infinite descending sequence of ordinals of $M_{x,y}$. Since $M_{x,y}$ is thus illfounded for every $y \in \omega \omega$, Lemma 8.1.2 implies that $x \notin A$.

Next let $x \in [T] \setminus A$. Thus $U(x)$ is a wellfounded tree. As in the proof of Theorem 8.1.1, define, for $n \in \omega$,

$$f_n : U[x \upharpoonright n] \rightarrow |U(x)|^+$$

by setting $f_n(s) = ||s||_U^+(x)$ for each $s \in U[x \upharpoonright n]$. For $i \in \omega$ let

$$t_i = \pi_{x[\text{th}(r_i)r_1]}([f_{\text{th}(r_i)}]_{U_{\text{th}(r_i)r_1}}).$$

To show that $(t_i \mid i \in \omega) \in U^+(x)$, let $i_1$ and $i_2$ be such that $r_{i_1} \subseteq r_{i_2}$. Since $f_{\text{th}(r_{i_2})}(s) < f_{\text{th}(r_{i_1})}(s \upharpoonright \text{th}(r_{i_1}))$ for every $s \in U[x \upharpoonright \text{th}(r_{i_2})]$, it follows that $t_{i_2} < i_{(x,\text{th}(r_{i_1}),r_{i_1}),r_{i_1}}(t_{i_1})$, as required.

The proof that $t_n < \max\{\omega, (2^{|Y|})^+\}$ for each $n$ is just like the corresponding part of the proof of Theorem 8.1.1.

Exercise 8.1.1. Give a modified definition of weakly homogeneous, analogous to the modified definition of homogenous given in Exercise 4.3.5. Prove that, for any game tree $T$, every $\Sigma^1_1$ subset of $[T]$ is weakly homogenously Souslin in the modified sense. Prove that if a measurable cardinal exists then the same sets are weakly homogeneously Souslin under the original and the modified definitions.

8.2 Projective Determinacy

From now until the end of the proof of Theorem 8.2.7, let $\kappa$ be an inaccessible cardinal, let $T \in V_\kappa$ be a game tree, let $Y$ be a set, let $U$ be a tree on field $(T) \times Y$. 
In this section we will prove that if $\kappa$ is Woodin and $\langle U_p, r \mid r \in <^\omega \omega \rangle$ witnesses that $U$ is weakly $\kappa^+$-homogeneous for $T$, then $U^\dagger = U^\dagger(\langle U_p, r \mid \langle p, r \rangle \in T \otimes <^\omega \omega \rangle)$ is $\gamma$-homogeneous for every $\gamma < \kappa$.

For the purpose of motivation, it will be helpful to focus on a more modest goal, the goal of showing that the $T$-projection $A$ of $U^\dagger$ has an embedding normal form, i.e., that there is a system

$$(\langle M_p \mid p \in T \rangle, \langle k_{p_1,p_2} \mid p_1 \subseteq p_2 \in T \rangle)$$

such that

(a) $M_0 = V$ and each $M_p$ is a transitive proper class model of ZFC;
(b) for each $p \in T$, $k_{p_1,p_2} : M_{p_1} \prec M_{p_2}$;
(c) for $p_1 \subseteq p_2 \subseteq p_3 \in T$, $k_{p_1,p_3} = k_{p_2,p_3} \circ k_{p_1,p_2}$;
(d) for each $x \in [T]$, $x \in A$ if and only if the direct limit model of the directed system $((M_x|n \mid n \in \omega), (k_{x|m,x|n} \mid m < n \in \omega))$ is wellfounded.

Note that if $U^\dagger$ is homogeneous for $T$ then $A$ has an embedding normal form. Actually our construction of an embedding normal form for $A$ will give the desired homogeneity of $U^\dagger$.

The construction of the tree for $U^\dagger$ is notationally somewhat complex. To exhibit the main ideas of the construction, we will first do a somewhat simpler construction. We will assume that we have $\langle U_p \mid p \in T \rangle$ witnessing the $\kappa^+$-homogeneity of $U$, and we will show that the $T$-projection of $U^\dagger = U^\dagger(\langle U_p \mid p \in T \rangle)$ has an embedding normal form, indeed that $U^\dagger$ is $\gamma$-homogeneous for every $\gamma < \kappa$.

To get an embedding normal form for the $T$-projection of $U^\dagger$, we will construct, for each $x \in [T]$, a special kind of iteration tree, an alternating chain. An alternating chain is an iteration tree of length $\omega$ whose tree ordering $S$ is the restriction of the tree ordering $C$ of $\omega$ given by

$$m C n \iff (0 = m < n \lor (\exists k \geq 1) m + 2k = n).$$

Thus the models and embeddings of an alternating chain (of length at least 7) begin as follows:
If \( C \) is an alternating chain of length \( \omega \), then \( C \) has exactly two branches:

\[
\text{Even} = \{2n \mid n \in \omega\}; \\
\text{Odd} = \{0\} \cup \{2n + 1 \mid n \in \omega\}.
\]

With each \( x \in [T] \) we will associate an alternating chain \( C_x \) of length \( \omega \) on \( V \). For each \( n \in \omega \), the restrictions \( C_x \upharpoonright 2n + 1 \) will depend only on \( x \upharpoonright n \). To get our embedding normal form for the projection of \( U^\uparrow \), we will use the branches Even of the \( C_x \). We will set \( M_x \upharpoonright n = M_{\omega}^{C_x} \); for \( m \leq n \in \omega \), we will set \( k_{x[m,x]n} = j_{2m,2n}^{C_x} \). Our task will then be to build the \( C_x \) so that, for each \( x \in [T] \),

\[
[U^\uparrow(x)] \neq \emptyset \iff \hat{M}_{\text{Even}}^C \text{ is wellfounded}.
\]

If we are to carry out our plan, then we must find a method of constructing alternating chains, and we must be able to control the wellfoundedness or illfoundedness of the branch Even for these alternating chains. Before giving the full construction we will discuss, one at a time, how we intend to solve these two problems.

Our tool for building alternating chains will be the One-Step Lemma, Lemma 6.3.18. To illustrate the method, we give a result whose proof will show how to build finite alternating chains.

For the tree ordering \( C \) of infinite alternating chains, note that \( (k + 1)^C = k - 1 \), where

\[
m \div n = \begin{cases} 
m - n & \text{if } m \geq n; \\
0 & \text{if } m < n. \end{cases}
\]

Lemma 8.2.1. Assume that \( \kappa \) is Woodin. Let \( n \in \omega \). Then there is an alternating chain \( C \) of length \( n + 1 \) on \( V \) such that \( C \in V_\kappa \).

Proof. If \( n = 0 \), there is nothing to do, so assume that \( n > 0 \). Let \( \gamma < \kappa \) be such that \( T \in V_\gamma \). Let \( \delta_0 > \gamma \) be \((n - 1)\)-reflecting in in \( \emptyset \) relative to \( \kappa \). Let
$k < n$ and assume inductively that we have an alternating chain $C_k$ of length $k + 1$ on $\mathcal{M} = (V; \in)$. We denote the extenders $E^C_m$ by $E_m$, the models $\mathcal{M}^C_m$ by $(M_m; \in)$, and the embeddings $j^C_{m,m'}$ by $j_{m,m'}$. Assume also that there is an ordinal $\delta_k$ such that $\gamma < \delta_k < \kappa$ and such that

(i) $M_k$ and $M_{k-1}$ agree through $\delta_k + 1$;
(ii) $(\text{tp}_{\kappa,n-k-1}^\delta_k(M_k(\emptyset)) = (\text{tp}_{\kappa,n-k-1}^{\delta_k})^M_{k-1}(\emptyset);$ 
(iii) $\delta_k$ is $(n - k - 1)$-reflecting in $\emptyset$ relative to $\kappa$ in $M_k$.

Assume first that $n - k > 1$. Since all the $j_{m,m'}$ fix $\kappa$, $\kappa$ is Woodin in $M_k$. Thus the hypotheses of the One-Step Lemma hold for $\kappa$ with

\[
M = M_k; \\
N = M_{k-1}; \\
\delta = \delta_k; \\
\eta = \delta_k; \\
\beta = n - k - 1; \\
\xi = n - k - 2; \\
\beta' = n - k - 1; \\
x = \emptyset; \\
y = \emptyset; \\
x' = \emptyset; \\
\chi(v) = "\kappa + v is the greatest ordinal." 
\]

Let $\lambda$ and $E$ be given by the One-Step Lemma. If $k = 0$ then the model $\prod_E^{M_{k-1}}(M_{k-1}; \in) = \prod_E^V(V; \in)$ and so is wellfounded. If $k > 0$, then $\prod_E^{M_{k-1}}(M_{k-1}; \in) = \prod_E^{M_{k-1}}(M_{k-1}; \in)$. By part (2) of Lemma 7.2.5 and the (i) above, we have that

$\rho^E(k - 1, k) > \delta_k = \text{crit}(E).$

Thus part (2) of Theorem 7.3.2 implies that $\prod_E^{M_{k-1}}(M_{k-1}; \in)$ is wellfounded. Let then $\delta^*$, $\xi^*$, and $y^*$ be given by the One-Step Lemma. By clause (2*) of the One-Step Lemma, $y^* = \emptyset$. By clause (4*), $\xi^* = n - k - 2$. Extend $C_k$ to an alternating chain $C_{k+1}$ of length $k + 2$ by setting $E_k = E$. Let $\delta_{k+1} = \delta^*$. Our inductive assumptions hold for $k + 1$.

Now assume that $n = k + 1$. Using Lemmas 7.2.5 and 7.3.2 as in the preceding paragraph, we may extend $C_k$ to an alternating chain $C_{k+1}$ of length $k + 1$ by letting $E_k$ witness that $\kappa$ is 0-reflecting in $\emptyset$ relative to $\kappa$ in $M_k$. \qed
Remark. Clause (4∗) of the One-Step Lemma was not really needed for the proof of Lemma 8.2.1. The fact that that $\xi = n - k - 2$ could have been deduced from clause (2∗) instead of from clause (4∗). In the proofs of subsequent lemmas, it will be necessary to use clause (4∗), and we will use it in just the way we used it in the proof of Lemma 8.2.1.

We next show how to use the One-Step Lemma to construct infinite alternating chains. At first this appears impossible, because of that Lemma’s requirement that $\xi < \beta$.

**Lemma 8.2.2.** There exist ordinals $\nu$, $\zeta_0$, $\zeta_1$, and $\rho$ such that

1. $\nu < \zeta_0 < \zeta_1 < \rho$;
2. $\nu$, $\zeta_0$, $\zeta_1$, and $\rho$ are strong limit cardinals of cofinality greater than $\kappa$;
3. $\text{tp}^\nu_{\kappa,0}(\langle \zeta_0 \rangle) = \text{tp}^\nu_{\rho,0}(\langle \zeta_1 \rangle)$;
4. $U \in V_\nu$.

**Proof.** Let $Z$ be the class of all strong limit cardinals of cofinality greater than $\kappa$. Let $\nu$ be the least element of $Z$ such that $U \in V_\nu$. Let $\rho$ be the $|V_{\nu+1}|^{+}$th element of $Z$. There are at most $|V_{\nu+1}|$ distinct values of $\text{tp}^\nu_{\rho,0}(\zeta)$. Hence there must exist $\zeta_0$ and $\zeta_1$ belonging to $Z$ and satisfying (1) and (3).

□

From now until the end of the proof of Theorem 8.2.7, let $\nu$, $\zeta_0$, $\zeta_1$, and $\rho$ be as in the statement of Lemma 8.2.2.

The next lemma gives the key facts about these ordinals.

**Lemma 8.2.3.** (a) If $T \in V_\kappa$ is an iteration tree, then each of the ordinals $\nu$, $\zeta_0$, $\zeta_1$, and $\rho$ is fixed by each of the embeddings $j^T_{\beta,\gamma}$.
(b) If $z \in \langle V_\nu \rangle$ and $\alpha < \kappa$, then $\text{tp}^\alpha_{\kappa,\zeta_0}(z) = \text{tp}^\alpha_{\kappa,\zeta_1}(z)$.
(c) If $z \in \langle V_\nu \rangle$, then $\delta < \kappa$ is $\zeta_0$-reflecting in $z$ relative to $\kappa$ if and only if $\delta$ is $\zeta_1$-reflecting in $z$ relative to $\kappa$.

**Proof.** (a) This follows from property (2) of $\langle \nu, \zeta_0, \zeta_1, \rho \rangle$.
(b) Let $n \in \omega$ and let $z = \langle z_1, \ldots, z_n \rangle \in \langle V_\nu \rangle$. Let $\alpha < \kappa$ and let $a = \text{tp}^\alpha_{\kappa,\zeta_0}$. By Lemma 6.3.12, $\text{TYPE}_n(v_1, \ldots, v_{n+1}, c_a, c_\alpha, d) \in \text{tp}^\kappa_{\alpha,\rho}(z^{-}(\langle \zeta_0 \rangle))$. 


CHAPTER 8. PROJECTIVE GAMES

Since \( \kappa + \rho = \rho \), it follows from the definition of \( \text{tp}_{\beta, \gamma}^\nu \) that

\[
\text{TYPE}_n(c_{z_1}, \ldots, c_{z_n}, v_1, c_\alpha, c_\kappa) \in \text{tp}_{\rho, 0}^\nu(\langle \zeta_0 \rangle).
\]

By property (3) of \( \langle \zeta_0, \zeta_1, \rho \rangle \), we get that

\[
\text{TYPE}_n(c_{z_1}, \ldots, c_{z_n}, v_1, c_\alpha, c_\kappa) \in \text{tp}_{\rho, 0}^\nu(\langle \zeta_1 \rangle).
\]

By definition this gives that

\[
\text{TYPE}_n(v_1, \ldots, v_{n+1}, c_\alpha, c_\kappa) \in \text{tp}_{\kappa, \rho}^\kappa(z^\rho(\langle \zeta_0 \rangle)).
\]

Another application of Lemma 6.3.12 then gives that \( a = \text{tp}_{\kappa, \zeta_0}^\kappa \).

(c) Let \( z \) be a in the proof of (b). Let \( \delta < \kappa \). Property (3) of \( \langle \zeta_0, \zeta_1, \rho \rangle \), Lemma 6.3.13, and the fact that \( \kappa + \zeta_0 = \zeta_0 \), yield the following chain of equivalences:

\[
\begin{align*}
\delta & \text{ is } \zeta_0\text{-reflecting in } z \text{ relative to } \kappa \\
\iff & \text{REFL}_n(v_1, \ldots, v_{n+1}, c_\delta, d) \in \text{tp}_{\kappa, \rho}^{\delta+1}(z^-(\langle \zeta_0 \rangle)) \\
\iff & \text{REFL}_n(c_{z_1}, \ldots, c_{z_n}, v_1, c_\delta, c_\kappa) \in \text{tp}_{\rho, 0}^\nu(\langle \zeta_0 \rangle) \\
\iff & \text{REFL}_n(c_{z_1}, \ldots, c_{z_n}, v_1, c_\delta, c_\kappa) \in \text{tp}_{\rho, 0}^\nu(\langle \zeta_1 \rangle) \\
\iff & \text{REFL}_n(v_1, \ldots, v_{n+1}, c_\delta, d) \in \text{tp}_{\kappa, \rho}^{\delta+1}(z^-(\langle \zeta_1 \rangle)) \\
\iff & \delta \text{ is } \zeta_1\text{-reflecting in } z \text{ relative to } \kappa
\end{align*}
\]

The proof of the following lemma will show how we can construct infinite alternating chains and how we can make the branch Even illfounded.

**Lemma 8.2.4.** Assume that \( \kappa \) is Woodin. Then there is a an infinite alternating chain \( C \) on \( V \) such that \( C \in V_\kappa \) and such that the branch Even of \( C \) is not wellfounded.

**Proof.** Let \( \gamma < \kappa \) be such that \( T \in V_\gamma \). Let \( \delta_0 > \gamma \) be \( (\zeta_0 + 1)\)-reflecting in \( \emptyset \) relative to \( \kappa \). Let \( \beta_0 = \zeta_0 \). Let \( k < n \) and assume inductively that we have an alternating chain \( C_{2k} \) of length \( 2k + 1 \) on \( V \). As in the proof of Lemma 8.2.1 let \( C_{2k} \) have extenders \( E_m \in V_\kappa \), \( m < 2k \), models \( M_m \), \( m \leq 2k \), and embeddings \( j_{m, m'} \), \( M \models m' \leq 2k \). Assume also that there is an ordinal \( \delta_{2k} \) such that \( \gamma < \delta_{2k} < \kappa \) and that there are ordinals \( \beta_m \), \( m \leq k \), satisfying the following conditions:
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(i) $M_{2k}$ and $M_{2k-1}$ agree through $\delta_{2k} + 1$;
(ii) $\text{tp}_{\kappa, \beta_k+1}^{\delta_{2k}}(\emptyset) = \text{tp}_{\kappa, 0}^{\delta_{2k}+1}(\emptyset)$;
(iii) $\delta_{2k}$ is $(\beta_k + 1)$-reflecting in $\emptyset$ relative to $\kappa$ in $M_{2k}$;
(iv) $(\forall m < k)(\forall m' \leq k)(m < m' \rightarrow \beta_{m'} < j_{2m, 2m'}(\beta_m))$.

Since all the $j_{m, m'}$ fix $\kappa$, $\kappa$ is Woodin in $M_{2k}$. Thus the hypotheses of the One-Step Lemma hold for $\kappa$ with

\[
\begin{align*}
M &= M_{2k}; \\
N &= M_{2k-1}; \\
\delta &= \delta_{2k}; \\
\eta &= \delta_{2k}; \\
\beta &= \beta_k + 1; \\
\xi &= \beta_k; \\
\beta' &= \zeta_0 + 1; \\
x &= \emptyset; \\
y &= \emptyset; \\
x' &= \emptyset; \\
\chi(v) &= "\kappa + v is the greatest ordinal."
\end{align*}
\]

Let $\lambda$ and $E$ be given by the One-Step Lemma. As in the analogous step in the proof of Lemma 8.2.1, $\prod_{E}^{M_{2k-1}}(M_{2k-1}; \in)$ is wellfounded. Let then $\delta^*$, $\xi^*$, and $y^*$ be given by the One-Step Lemma. By clause $(2^*)$ of the One-Step Lemma, $y^* = \emptyset$. By clause $(4^*)$, $\xi^* = \zeta_0$. Extend $C_k$ to an alternating chain $C_{2k+1}$ of length $2k + 2$ by setting $E_{2k} = E$. Let $\delta_{2k+1} = \delta^*$. We have then that

(a) $M_{2k+1}$ and $M_{2k}$ agree through $\delta_{2k+1} + 1$;
(b) $\text{tp}_{\kappa, \zeta_0}^{\delta_{2k+1}}(\emptyset) = \text{tp}_{\kappa, \delta_k}^{\delta_{2k+1}}(\emptyset)$;
(c) $\delta_{2k+1}$ is $\zeta_0$-reflecting in $\emptyset$ relative to $\kappa$ in $M_{2k+1}$.

By (b) and (c) together with parts (b) and (c) of Lemma 8.2.3, we have that

(b') $\text{tp}_{\kappa, \zeta_1}^{\delta_{2k+1}}(\emptyset) = \text{tp}_{\kappa, \delta_k}^{\delta_{2k+1}}(\emptyset)$;
(c') $\delta_{2k+1}$ is $\zeta_1$-reflecting in $\emptyset$ relative to $\kappa$ in $M_{2k+1}$. 
Since $\kappa$ is Woodin in $M_{2k+1}$, the hypotheses of the One-Step Lemma hold for $\kappa$ with

\[
\begin{align*}
M &= M_{2k+1}; \\
N &= M_{2k}; \\
\delta &= \delta_{2k+1}; \\
\eta &= \delta_{2k+1}; \\
\beta &= \zeta_1; \\
\xi &= \zeta_0 + 1; \\
\beta' &= \beta_k; \\
x &= \emptyset; \\
y &= \emptyset; \\
x' &= \emptyset; \\
\chi(v) &= "v = v." 
\end{align*}
\]

Let $\lambda$ and $E$ be given by the One-Step Lemma. By Theorem 7.3.2, the model $\prod E^{M_{2k}}(M_{2k}; \in)$ is wellfounded. Let then $\delta^*, \xi^*$, and $y^*$ be given by the One-Step Lemma. Once more, $y^* = \emptyset$. Extend $C_{2k+1}$ to an alternating chain $C_{2k+2}$ of length $2k + 3$ by setting $E_{2k+1} = E$. Let $\delta_{2k+2} = \delta^*$. Let $\beta_{k+1} = \xi^*$. The inequality $\xi^* < i_E^N(\beta')$ of the One-Step Lemma gives us that

\[
\beta_{k+1} = \xi^* < i_E^{M_{2k}}(\beta_k) = j_{2k+2}(\beta_k).
\]

From this and induction hypothesis (iv) for $k$ we get hypothesis (iv) for $k + 1$. Thus all our induction hypotheses hold for $k + 1$.

Let $C$ be the alternating chain of length $\omega$ whose restrictions are the $C_k$. We must verify that the model $\tilde{M}_{\text{Even}}^C$ is not wellfounded. But condition (iv) implies that

\[
(\forall m \in \omega)(\forall n \in \omega)(m < n \rightarrow j_{2n, \text{Even}}^C(\beta_n) < j_{2m, \text{Even}}^C(\beta_m)).
\]

Thus the $j_{2n, \text{Even}}^C(\beta_n)$ are an infinite descending sequence of ordinals of $\tilde{M}_{\text{Even}}^C$.

\[\square\]

We now know how to build infinite alternating chains. Moreover we know a way to arrange that the Even branch of our chains is illfounded. But how are we going to build chains $C_x$ whose Even branches are illfounded if and
only if \([U(x)] \neq \emptyset\) ? How are we going to make our illfoundedness construction work only if \([U(x)] \neq \emptyset\) ? And how are we to guarantee that the branch Even is wellfounded in the case \([U(x)] = \emptyset\)?

To solve this last problem, we will make sure that \(C_x\) is continuously illfounded off Even if \([U(x)] = \emptyset\). Lemma 8.2.5 below asserts that every alternating chain is plus one. Therefore Corollary 7.4.6 will imply that Even is wellfounded.

Making an alternating chain continuously illfounded off Even is equivalent with making Odd illfounded. (See Exercise 8.2.1.) To make Odd illfounded when \([U(x)] = \emptyset\), we will arrange that \(J_{0,\mathrm{Odd}}(x)\) is illfounded for every \(x\). At the beginning of \(k\)th stage of the construction of \(C_x\) we will have chosen an element \(s_k\) of \(J_{0,2k-1}(x)\]. During the \(k\)th stage, we will choose an element \(s_{k+1}\) of \(J_{0,2k+1}(x)\] such that \(s_{k+1} \supseteq J_{2k-1,2k+1}(s_k)\). The \(J_{2k-1,Odd}(s_k)\) will thus witness the illfoundedness of \(J_{0,Odd}(x)\). Moreover the \(s_k\) will directly provide us with ordinals witnessing that \(C\) is continuously illfounded off Even.

We will get the \(s_k\) via the One-Step Lemma. Assume that \(\langle U_p \mid p \in T\rangle\) witnesses that \(U\) is \(\kappa^+\)-homogeneous. Given \(k\), consider \(\mathrm{Ult}(V;U_{\kappa|k})\). This is the same as \(i_{\kappa,x|k}(V)\), where the \(i_{p,q}\), \(p \subseteq q \in T\), are defined as on page 200. In this model there is a canonical element of \(i_{\emptyset,x|k}(U[x|k])\], namely

\[
s_{x|k} = \pi_{U_{\kappa|k}}([id]_{U_{\kappa|k}})\).
\]

From \(s_{x|k}\), we get

\[
J_{0,2k}(s_{x|k}),
\]

an element of \(J_{0,2k}(i_{\emptyset,x|k}(U[x|k]))\). With the aid of the One-Step-Lemma, we will arrange inductively that, for some ordinal \(\beta_k\), the type

\[
(tp_{\kappa,\beta_k+1})^{J_{0,2k}(i_{\emptyset,x|k}(V))}(J_{0,2k}(i_{\emptyset,x|k}(U)))^{-J_{0,2k}(s_{x|k})}
\]

is the same as

\[
(tp_{\kappa,\beta_k+1})^{M_{2k+1}}(J_{0,2k+1}(U))^{-s_k}.
\]

We will also arrange inductively that \(\beta_k\) is \((\beta_k + 1)\)-reflecting in the finite sequence \(J_{0,2k}(i_{\emptyset,x|k}(U)))^{-J_{0,2k}(s_{x|k})}\) relative to \(\kappa\) in \(J_{0,2k}(i_{\emptyset,x|k}(V))\).

The ordinals \(\beta_k\) will play a role analogous to the role of the \(\beta_k\) appearing in the proof of Lemma 8.2.4. But, whereas the latter gave an infinite descending sequence of ordinals of \(\mathcal{M}_{Even}^C\), the \(\beta_k\) of our new construction will give instead an infinite descending sequence of ordinals of
\[(i_\emptyset^C_{0,\text{Even}}(V))( \hat{\mathcal{M}}_{C_{x,0},\text{Even}}) = (i_\emptyset^C_{0,\text{Even}}(V)).\] When \(\hat{\mathcal{M}}_{C_{x,0},\text{Even}}\) is wellfounded, the sequence will show that \(j_{C_{x,0},\text{Even}}(V)\) is illfounded. By absoluteness and the elementarity of \(j_{C_{x,0},\text{Even}}\), \(M_x = i_\emptyset^C(V)\) will be illfounded and so we will have \([U(x)] \neq \emptyset\). The \(\beta_m\) will also give us ultrafilters that will witness the homogeneity of \(U^\dagger\). For each \(k \in \omega\), the finite sequence, \(\langle j_{C_{x,2k},2k}^C(\beta_m) \mid m < k \rangle\) will be an element of \(j_{0,2k}(U^\dagger[x \upharpoonright k])\). This finite sequence and Lemma 6.1.1 will yield an ultrafilter on \(U^\dagger[x \upharpoonright k]\).

In the preceding discussion, we have been dealing with a fixed \(x \in [T]\). The following theorem and its proof will involve all such \(x\) simultaneously, so we will have to modify some of our notation. Thus \(s_k\) will become \(s_{x,k}\), \(\delta_{2k}\) will become \(\delta_{x,k}\), \(\beta_k\) will become \(\beta_{x,k}\), \(M_{2k}\) will now become \(M_{x,k}\), etc.

Before proceeding to the theorem, let us first verify that alternating chains are all plus one.

**Lemma 8.2.5.** Every alternating chain is plus one.

**Proof.** By the definition (given on page 395, an iteration tree \(\mathcal{T} = (\langle \mathcal{M}, T, \langle E_\alpha \mid \alpha + 2 < \ell \text{h}(\mathcal{T}) \rangle)\) is plus one if and only if, for all \(\beta\) such that \(\beta + 2 < \ell \text{h}(\mathcal{T})\),

\[\mu^\mathcal{T}(\beta) < \text{strength}^{\mathcal{M}_\beta}(E_\beta),\]

where

\[\mu^\mathcal{T}(\beta) = \sup \{\text{crit } (E_\alpha) \mid (\alpha + 1)^{\mathcal{T}} < \beta < \alpha\}.\]

Part (b) of Corollary 7.2.6 implies that, for \(\alpha + 1 < \ell \text{h}(\mathcal{T})\) and \((\alpha + 1)^{\mathcal{T}} < \beta < \alpha\),

\[\text{crit } (E_\alpha) < \text{strength}^{\mathcal{M}_\beta}(E_\beta).\]

It follows that \(\mu^\mathcal{T}(\beta) < \text{strength}^{\mathcal{M}_\beta}(E_\beta)\) for every \(\beta\) such that \(\{\alpha \mid (\alpha + 1)^{\mathcal{T}} < \beta < \alpha\}\) is finite. But, for \(m\) and \(n \in \omega\),

\[(m + 1)^C < n < m \leftrightarrow m = n + 1,\]

where \(C\) is the tree ordering of alternating chains. \(\square\)

Recall that, for \(\kappa\) a cardinal number, a set is \((< \kappa)\)-homogeneously Souslin if it is \(\gamma\)-homogeneously Souslin for every \(\gamma < \kappa\). If \(T\) is a game tree, \(Y\) is a set, and \(U\) is a tree on field \((T) \times Y\), then let us say that \(U\) is \((< \kappa)\)-homogeneous for \(T\) if \(U\) is \(\gamma\)-homogeneous for \(T\) for every \(\gamma < \kappa\).
8.2. PROJECTIVE DETERMINACY

**Theorem 8.2.6.** Assume that $\kappa$ is Woodin and that $\langle \mathcal{U}_p \mid p \in T \rangle$ witnesses that $U$ is $\kappa^+$-homogeneous for $T$. Let $U^+ = U^+(\langle \mathcal{U}_p \mid p \in T \rangle)$. Then, for every sufficiently large ordinal $\alpha$, $U^+ \upharpoonright \alpha$ is $(<\kappa)$-homogeneous for $T$.

**Proof.** Let $\gamma < \kappa$ be such that $T \in V_\gamma$.

Define the embeddings $i_{p,q}$, $p \subseteq q \in T$, the models $\mathcal{M}_x$, $x \in [T]$, and the embeddings $i^*_p$, $p \in T$ and $p \subseteq x \in [T]$, as on page 200. For $p \in T$ let $s_p = \pi_{t_{\ell_p}}([id]_{t_{\ell_p}})$.

We will define, by induction on $p \in T$, objects $\delta_p$, $\beta_p$, $C_p$, and $s_p$. Both $\delta_p$ and $\beta_p$ will be ordinals, with $\delta_p < \kappa$. $s_p$ will be a sequence with $\ell h(s_p) = \ell h(p)$, $C_p$ will be an alternating chain of length $2\ell h(p) + 1$ on $V$. Its extenders will be $E^p_m$, $m < 2\ell h(p)$, its models will be $M^p_m$, $m < 2\ell h(p)$, and its embeddings will be $j^p_{m,n}$, $mCn < 2\ell h(p)$. Whenever $p \subseteq q \in T$ then we will have $C_p = C_q \upharpoonright (2\ell h(p) + 1)$.

To avoid having excessively cumbersome notation, let us make the following definitions. Let $p$, $q$, and $q'$ be elements of $T$ with $q \subseteq q'$. Let $m \leq \ell h(p)$. Set

$$
\begin{align*}
\check{\gamma}^{p}_{q,q'} &= j^{p}_{0,2\ell h(p)}(i_{q,q'});
\check{N}^{p}_{q} &= \check{\gamma}^{p}_{\emptyset,q}(M^{p}_{2\ell h(p)});
\check{U}^{p}_{q} &= j^{p}_{0,2\ell h(p)}(i_{\emptyset,q}(U));
\check{\beta}^{p}_{m} &= j^{p}_{2m,2\ell h(p)}(\beta^{p}_{m}).
\end{align*}
$$

The embedding $\check{\gamma}^{p}_{q,q'}$ is the image of $i_{q,q'}$ in $M^{p}_{2\ell h(p)}$. The class model $\check{N}^{p}_{q}$ is the image of $\text{Ult}(V; \mathcal{U}_q)$ in $M^{p}_{2\ell h(p)}$. In other words,

$$
\check{N}^{p}_{q} = j^{p}_{0,2\ell h(p)}(i_{\emptyset,q}(V)).
$$

The tree $\check{U}^{p}_{q}$ is the image of $i_{\emptyset,q}(U)$ in $M^{p}_{2\ell h(p)}$; i.e., it is the image of $U$ in $\check{N}^{p}_{q}$.

For $p = \emptyset$ we have only to define $\delta_{\emptyset}$ and $\beta_{\emptyset}$. Choose $\delta_{\emptyset} > \gamma$ to be $\zeta_0 + 1$-reflecting in $\langle U \rangle$ relative to $\kappa$. Let $\beta_{\emptyset} = \zeta_0$.

For the induction step of our definition, let $p \in T$. Let $k = \ell h(p)$. Assume that $\delta_{p'}$, $\beta_{p'}$, $C_{p'}$, and $s_{p'}$ are defined for all $p' \subseteq p$ so as to satisfy the conditions stated above and also the following conditions:

(i) $M^{p}_{2k}$ and $M^{p}_{2k-1}$ agree through $\delta_p + 1$;

(ii) $(tp^{\delta_p}_{\kappa,\beta_{p} + 1})_{\check{\gamma}^{p}_{\emptyset},\check{U}^{p}_{p}}(j^{p}_{0,2k}(s_p)) = (tp^{\delta_p}_{\kappa,\zeta_0 + 1})_{M^{p}_{2k-1}}((j^{p}_{0,2k-1}(U))^{-s_p})$.


(iii) $\delta_p$ is $(\beta_p + 1)$-reflecting in $(\hat{U}_p^p) \sim j_{0,2k}(s_p)$ relative to $\kappa$ in $\hat{N}_p^p$;

(iv) for all $m$ and $m'$ with $m < m' \leq k$, $\overline{\beta}_{m'}^p < \overline{\beta}_{m'}^p(\beta_{m'}^p)$;

(v) $s_p$ belongs to $j_{0,2k-1}(U[p])$, and, for all $m \leq k$,

$$j_{2m-1,2k-1}^p(s_p|m) \subseteq s_p;$$

(vi) for all $m < k$, $\gamma < \delta_p|m = \text{crit}(\hat{E}_m^p) < \text{crit}(\hat{E}_{m+1}^p) < \delta_p|m+1$;

Note that these conditions all hold for $p = \emptyset$.

Remarks:

(a) Condition (i) and the fact that $\text{crit}(\hat{i}_p^0) \geq \text{crit}(i_{\emptyset,p}) > \kappa$ guarantee that $\hat{N}_p^p$ and $M_{2k-1}^p$ agree through $\delta_p + 1$. (That they agree through $\delta_p$ is also implied by condition (ii).)

(b) Conditions (i), (ii), and (iii) are to make possible our applications of the One-Step Lemma. Condition (iv) will yield that $\hat{M}_{\text{Even}}^C$ is illfounded whenever $[U(x)] \neq \emptyset$, and it will allow us to use the $\beta_p$ to define ultrafilters witnessing homogeneity. Condition (v) will yield that $\hat{M}_{\text{Odd}}^C$ is illfounded whenever $[U(x)] = \emptyset$. Condition (vi) guarantees that $T$, all members of $T$, and all members of $[T]$, are fixed by the embeddings of our alternating chains.

Let $q$ be any element of $T$ such that $p \subseteq q$ and $\ell h(q) = k + 1$.

By (i) and the fact that $\text{crit}(i_{\emptyset,q}^p) > \kappa$, it follows that $\hat{N}_q^p$ and $M_{2k-1}^p$ agree through $\delta_p + 1$.

Note that

$$\hat{N}_q^p = \overline{\varphi}_{\emptyset,q}^p(M_{2k}^p);$$

$$\hat{U}_q^p = j_{0,2k}(i_{\emptyset,q}(U));$$
and that
\[ j_{0,2k}^p(i_{p,q}(s_p)) = (j_{0,2k}^p(i_{p,q})) (j_{0,2k}^p(s_p)) = v_{p,q}^p (j_{0,2k}^p(s_p)). \]

The fact that \( \text{crit}(\check{\nu}_{p,q}) > \kappa \), together with the facts just mentioned, gives that
\[
\begin{align*}
(\text{tp}_{\eta,\beta_{p+1}}^p(\check{\nu}_{p,q}))^{S_p^p} (\check{U}_p^p - j_{0,2k}^p (s_p)) \\
= v_{p,q}^p (\text{tp}_{\eta,\beta_{p+1}}^p(\check{\nu}_{p,q}))^{S_p^p} (\check{U}_p^p - j_{0,2k}^p (s_p)) \\
= (\text{tp}_{\eta,\beta_{p+1}}^p(\check{\nu}_{p,q}))^{S_p^p} (\check{U}_p^p - j_{0,2k}^p (i_{p,q}(s_p))).
\end{align*}
\]
and so by (ii) this last is the same as \( (\text{tp}_{\eta,\beta_{p+1}}^p(\check{\nu}_{p,q}))^{S_p^p} (\check{U}_p^p - j_{0,2k}^p (i_{p,q}(s_p))) \).

From (iii) it similarly follows that \( \delta_p \) is \( (\text{tp}_{\eta,\beta_{p+1}}^p(\check{\nu}_{p,q}))^{S_p^p} \)-reflecting in the finite sequence \( (\check{U}_p^p - j_{0,2k}^p (i_{p,q}(s_p))) \) relative to \( \kappa \) in \( \check{N}_{q}^p \).

Since \( j_{0,2k}^p \) and \( \check{\nu}_{p,q}^p \) fix \( \kappa \), we have that \( \kappa \) is Woodin in \( \check{N}_{q}^p \).

Thus the hypotheses of the One-Step Lemma hold for \( \kappa \) with
\[
\begin{align*}
M & = \check{N}_{q}^p, \\
N & = M_{2k-1}^p, \\
\delta & = \delta_p; \\
\eta & = \delta_p; \\
\beta & = \check{\nu}_{p,q}^p(\beta_{p}) + 1; \\
\xi & = \check{\nu}_{p,q}^p(\beta_{p}); \\
\beta' & = \xi_0 + 1; \\
x & = (\check{U}_q^p - j_{0,2k}^p (i_{p,q}(s_p))); \\
y & = (j_{0,2k}^p (s_q) (k)); \\
x' & = (j_{0,2k-1}^p (U))^{-s_p}; \\
\chi(v) & = “\kappa + v is the greatest ordinal.”
\end{align*}
\]

Let \( \lambda \) and \( E \) be given by the One-Step Lemma. Since \( \check{\nu}_{p,q}^p \) fixes \( \lambda, E, \) and \( \delta_p \), it follows that \( E \) is a \( (\delta_p, \lambda) \)-extender in \( M_{2k}^p \). Thus Theorem 7.3.2 guarantees that \( \prod_{E}^M M_{2k-1}^p (M_{2k-1}^p; \in) \) is wellfounded. Let then \( \delta^*, \xi^*, \) and \( y^* \) be given by the One-Step Lemma. By clause (4*) of the One-Step Lemma, \( \xi^* = \xi_0 \). Extend \( C_{p} \) to an alternating chain that will be \( C_{q} \upharpoonright 2k + 2 \) by setting \( E_{2k}^p = E \). The ordinal \( \delta^* \) we will call \( \check{\delta}_{q}^* \). Set \( s_q = (j_{2k-1,2k+1}^q (s_p))^{-y^*}. \)
We will use without comment in the sequel the facts $M_n^q = M_n^p, E_n^q = E_n^p,$ and $j_{m,n}^q = j_{m,n}^p$ whenever these equations make sense.

By the elementarity of $j_{0,2k}^q$ and the definition of the $s_p$, we have

\[ x \sim y = \langle \bar{U}_q^p \rangle \sim j_{0,2k}^p (i_{p,q} (s_p)) \sim (j_{0,2k}^p (s_q))(k) \]

\[ = \langle \bar{U}_q^p \rangle \sim j_{0,2k}^p (i_{p,q} (s_p)) \sim (s_q(k)) \]

\[ = \langle \bar{U}_q^p \rangle \sim j_{0,2k}^p (s_q \upharpoonright k) \sim (s_q(k)) \]

\[ = \langle \bar{U}_q^p \rangle \sim j_{0,2k}^p (s_q). \]

Since $j_{0,2k}^p (s_q)$ is an element of $\mu^p_{\emptyset,q} (U) = \bar{U}_q^p$, clause (2*) of the One-Step Lemma implies that $\bar{s}_q \in j_{0,2k+1}^q (U[q])$. Thus the first clause of condition (v) holds for $q$. Since $j_{2k+1,2k+1}^q (s_p) \subseteq \bar{s}_q$, the second clause of condition (v) holds for $q$ in the case $m = k$.

We have that

(a) $M_{2k+1}^q$ and $\bar{N}_q^p$ agree through $\delta_{q}^0 + 1$;

(b) $\langle \mu_{\delta_{q}^0}^p \rangle M_{2k+1}^q (j_{0,2k+1}^q (U)) \sim \bar{s}_q$

\[ = (\mu_{\delta_{q}^0}^p) M_{2k+1}^q (j_{0,2k}^p (s_q)); \]

(c) $\delta_{q}^0$ is $\zeta_0$-reflecting in $\langle j_{0,2k+1}^q (U) \rangle \sim \bar{s}_q$ relative to $\kappa$ in $M_{2k+1}^q$.

By (a) and the fact that $\text{crit}(\mu_{\emptyset,q}^p) > \delta_{q}^0$, it follows that $M_{2k+1}^q$ and $M_{2k}^q$ agree through $\delta_{q}^1 + 1$.

By (b) and (c) together with parts (b) and (c) of Lemma 8.2.3, we have that

(b') $\langle \mu_{\delta_{q}^0}^p \rangle M_{2k+1}^q (j_{0,2k+1}^q (U)) \sim \bar{s}_q$

\[ = (\mu_{\delta_{q}^0}^p) M_{2k+1}^q (j_{0,2k}^p (s_q)); \]

(c') $\delta_{q}^0$ is $\zeta_1$-reflecting in $\langle j_{0,2k+1}^q (U) \rangle \sim \bar{s}_q$ relative to $\kappa$ in $M_{2k+1}^q$.

Let $f : j_{0,2k}^p (U[p]) \to \text{Ord}$ belong to $M_{2k}^p$ and be such that

\[ \beta_p = \pi_{j_{0,2k}^p (U[p])} (\bigcup_{j_{0,2k}^p (U[p])} M_{2k}^p). \]

In other words, let $f$ be such that $\beta_p = (\mu_{\emptyset,p}^f (f))(j_{0,2k}^p (s_p))$. By (b') and the fact that $\text{crit}(\mu_{\emptyset,p}^f) > \delta_{q}^0$, there is a set $X \in j_{0,2k}^p (U[p])$ such that, for all $t \in X,$
Choose any element \( t \) of \( X \). Since \( \kappa \) is Woodin in \( M_{2k+1}^\kappa \), the hypotheses of the One-Step Lemma hold for \( \kappa \) with

\[
\begin{align*}
M &= M_{2k+1}^\kappa; \\
N &= M_{2k}^\kappa; \\
\delta &= \delta'; \\
\eta &= \delta'; \\
\beta &= \zeta_1; \\
\xi &= \zeta_0 + 1; \\
\beta' &= f(t \upharpoonright k); \\
x &= \langle j^q_{0,2k+1}(U) \rangle s_q; \\
y &= \emptyset; \\
x' &= \langle j^p_{0,2k}(U) \rangle t; \\
\chi(v) &= \text{``}v = v.'' 
\end{align*}
\]

Let \( \lambda \) and \( E \) be given by the One-Step Lemma. By Theorem 7.3.2, the model \( \prod_{E_{2k}}^\lambda (M_{2k}^\kappa; \in) \) is wellfounded. Let then \( \delta^*, \xi^* \), and \( y^* \) be given by the One-Step Lemma. (We will make no use of \( \xi^* \) and \( y^* \).)

For all elements \( u \) of \( X \),

\[
(tp^\delta_{i_E^{2k}}(j^q_{0,2k+1}(U)) \upharpoonright u) = (tp^\delta_{i_E^{2k}}(j^p_{0,2k}(U)) \upharpoonright t).
\]

Thus the elementarity of \( i_{E^{2k}}^{M_{2k}^\kappa} \) gives that, for all \( u \in i_{E^{2k}}^{M_{2k}^\kappa}(X) \),

\[
\begin{align*}
(tp^\delta_{i_E^{2k}}(j^q_{0,2k+1}(U)) \upharpoonright u)_{\Ult(M^{\kappa}_{2k}; E)}(i_{E^{2k}}^{M_{2k}^\kappa}(j^p_{0,2k}(U)) \upharpoonright t) \\
= (tp^\delta_{i_E^{2k}}(j^q_{0,2k+1}(U)) \upharpoonright i_{E^{2k}}^{M_{2k}^\kappa}(f))_{\Ult(M^{\kappa}_{2k}; E)}(i_{E^{2k}}^{M_{2k}^\kappa}(j^p_{0,2k}(U)) \upharpoonright t).
\end{align*}
\]

Since \( \delta^* < i_{E^{2k}}^{M_{2k}^\kappa}(j^q_{0,2k+1}(U)) \), for all \( u \in i_{E^{2k}}^{M_{2k}^\kappa}(X) \) we have in particular that

\[
\begin{align*}
(tp^{\delta^*+1}_{i_E^{2k}}(j^q_{0,2k+1}(U)) \upharpoonright u)_{\Ult(M^{\kappa}_{2k}; E)}(i_{E^{2k}}^{M_{2k}^\kappa}(j^p_{0,2k}(U)) \upharpoonright t) \\
= (tp^{\delta^*+1}_{i_E^{2k}}(j^q_{0,2k+1}(U)) \upharpoonright i_{E^{2k}}^{M_{2k}^\kappa}(f))_{\Ult(M^{\kappa}_{2k}; E)}(i_{E^{2k}}^{M_{2k}^\kappa}(j^p_{0,2k}(U)) \upharpoonright i_{E^{2k}}^{M_{2k}^\kappa}(t)).
\end{align*}
\]
For each $u \in \mathcal{E}_{2k}^P(X)$, we make an application of the last part of the One-Step Lemma, with $z = \langle i_E^P(j_{0,2k}^P(U)) \rangle \cup u$ and with $\alpha = (i_E^P(f))(|u \setminus (2k))$. If $\xi$ and $\bar{y}$ are as given by this application, then clause (2) of the One-Step Lemma implies that $\check{\xi} = \emptyset$ and $\check{\bar{y}}$ is a successor ordinal. Let $g(u)$ be the least ordinal $\mu$ such that clauses (2) and (3) of the One-Step Lemma hold with $\xi = \mu + 1$ (and $\bar{y} = \emptyset$).

Observe that the function $g : i_E^P(X) \rightarrow \text{Ord}$ belongs to $\text{Ult}(\mathcal{E}_{2k}^P; E)$. We finish the definition of $\mathcal{E}_0$ by setting $E_{2k+1}^q = E$. Thus $E_{2k}^q = \text{Ult}(\mathcal{E}_{2k}^P; E)$ and $j_{2k,2k+2}^q = i_E^P$. Clause (1*) of the One-Step Lemma gives inductive condition (i) for $q$.

Let $\delta_q = \delta^*$. Clauses (2) and (3) of the One-Step Lemma give that, for all $u \in j_{2k,2k+2}^q(X)$,

1. $(\check{0}) (\check{\text{tp}}_{\check{\kappa}, \check{\gamma}(u)} M_{2k+1}^q ((j_{0,2k+2}^q(U))^\check{u}) \setminus \check{u}) = (\check{\text{tp}}_{\check{\kappa}, \check{\gamma}(u)} M_{2k+1}^q ((j_{0,2k+1}^q(U))^\check{u}) \setminus \check{u})$;
2. $\delta_q$ is $(g(u) + 1)$-reflecting in $(j_{0,2k+2}^q(U))^\check{u}$ relative to $\kappa$ in $M_{2k+2}^q$.

The set $j_{2k,2k+2}^q(X)$ belongs to $j_{0,2k+2}^q(U_q)$. This fact allows us to complete our definitions by setting

$$\beta_q = \pi^2_{M_{2k+2}^q j_{0,2k+2}^q(U_q)} ([g] M_{2k+2}^q j_{0,2k+2}^q(U_q)).$$

Using Los’s Theorem in $M_{2k+2}^q$ and using the fact that $j_{0,2k+2}^q(s_q) = \pi_{M_{2k+2}^q j_{0,2k+2}^q(U_q)} ([\text{id}] M_{2k+2}^q j_{0,2k+2}^q(U_q))$, we see that (2) and (3) imply that

1. $(\check{\text{tp}}_{\check{\kappa}, \check{\gamma}(u)} N_q ((j_{0,2k+2}^q(i_{0,1}^q(U))) \setminus \check{u}) = (\check{\text{tp}}_{\check{\kappa}, \check{\gamma}(u)} N_q ((j_{0,2k+1}^q(U))) \setminus \check{u})$;
2. $\delta_q$ is $(\beta_q + 1)$-reflecting in $(j_{0,2k+2}^q(i_{0,1}^q(U))) \setminus j_{0,2k+2}^q(s_q)$ relative to $\kappa$ in $N_q^q$.

(iii') and (iii') are just our inductive conditions (ii) and (iii) for $q$.

The inequality $\check{\xi} < \alpha$ of the One-Step Lemma gives us that

$$\forall u \in j_{0,2k+2}^q(X)) g(u) + 1 < (j_{2k,2k+2}^q(f))(u | k).$$

This in turn implies that

$$\beta_q + 1 < \check{\gamma}_{P,q}^q (j_{2k,2k+2}^q(\beta_p)).$$
Since $\overline{\beta}_{k+1} = \beta_q$ and $\overline{\beta}_k = j_{2k,2k+2}^q(\beta_p)$, condition (iv) for $q$ holds in the case $m = k$ and $m' = k + 1$. To verify condition (iv) for $q$ for $m < m' \leq k$, note that $j_{2k,2k+2}^p(\overline{\beta}_m) = \overline{\beta}_m$, $j_{2k,2k+2}^p(\overline{\beta}_{m'}) = \overline{\beta}_{m'}$, and $j_{2k,2k+2}^p(\overline{\beta}_m) = j_{2k,2k+2}^q(\overline{\beta}_m)$.

By these facts, by condition (iv) for $p$, and by the elementarity of $j_{2k,2k+2}^q$, \[ \overline{\beta}_{m'} = j_{2k,2k+2}^q(\overline{\beta}_{m'}) < j_{2k,2k+2}^q(\overline{\beta}_m) = j_{2k,2k+2}^q(\overline{\beta}_m). \]

Condition (iv) for $q$ holds for $m < k$ and $m' = k + 1$ because \[ \overline{\beta}_{k+1} < j_{p,k,p+1}(\overline{\beta}_k) < j_{p,k,p+1}(\overline{\beta}_m) = j_{p,k,p+1}(\overline{\beta}_m). \]

We have already verified for $q$ the first clause of condition (v) and the case $m = k$ of the second clause of that condition. The other cases of the second clause follow easily from the corresponding cases for $p$ and the case $m = k$ for $q$.

Because the One-Step Lemma gives that $\eta < \delta^*$, we have that \[ \delta_p < \delta_q' < \delta_q. \]

Now \( \text{crit} (E_{2k}^q) = \delta_p \) and \( \text{crit} (E_{2k+1}^q) = \delta_q' \). Since condition (vi) holds for $p$ and (in the case $p = \emptyset$) since $\gamma < \delta_0$, it follows that condition (vi) holds for $q$.

This completes our construction and the verification that it has the desired properties.

We will show that the system \( ((M_2^p | p \in T), (j_{m,2m}^q | p \in T \land m < \ell h(p) \in T)) \) gives an embedding normal form for the $T$-projection of $U^\dagger$.

Fix $x \in [T]$. Let $C_x$ be the alternating chain of length $\omega$ whose restrictions are the $C_{x|n}$.

We must show that $[U^\dagger(x)] \neq \emptyset$ if and only if $\tilde{\mathcal{M}}_{\text{Even}}^C$ is wellfounded.

Assume first that $[U^\dagger(x)] \neq \emptyset$. By Theorem 8.1.1, $[U(x)] = \emptyset$, i.e., $U(x)$ is a wellfounded tree. For each $k \in \omega$, let \[ \xi_{2k} = j_{0,2k}^C(\|s_0\| U(x)); \]
\[ \xi_{2k+1} = \|s_{x|k+1}\| j_{0,2k+1}^\dagger(U(x)). \]
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Condition (v) of our construction implies that, for all \( m \) and \( n \in \omega \) with \( m < n \),

\[
\begin{align*}
J_2^{c_x} \langle \xi_{2m-1}, \xi_{2n-1} \rangle &= \|J_2^{c_x} \langle \xi_{2m-1}, \xi_{2n-1} \rangle ||^{c_x}_{0,2n-1}(U(x)) \\
&> \|\xi_{2n-1}||^{c_x}_{0,2n-1}(U(x)) \\
&= \xi_{2n-1}.
\end{align*}
\]

Thus the sequence \( \langle \xi_n \mid n \in \omega \rangle \) witnesses that \( C_x \) is continuously illfounded off Even. Since Lemma 8.2.5 implies that \( C_x \) is plus one, Corollary 7.4.6, gives that \( \tilde{M}_{\text{Even}} \) is wellfounded.

Now assume that \( \tilde{M}_{\text{Even}} \) is wellfounded. For \( m \in \omega \), \( p \in T \) with \( p \subseteq x \), and \( q \subseteq q' \in T \), let

\[
\begin{align*}
\bar{\beta}_m &= J_0^{c_x}(\beta|x|_m); \\
\bar{r}_q &= J_0^{c_x}(i_{q,q'}); \\
\bar{i}_p &= J_0^{c_x}(i_x|_p).
\end{align*}
\]

Condition (iv) gives that, for all \( m < m' \leq k \in \omega \),

\[
\bar{\beta}_{m'} < \bar{r}_{x|m'|}^{c_x}(\bar{\beta}_m^{c_x}).
\]

Applying \( J_2^{c_x} \text{Even} \) to both sides of this inequality, we find that

\[
\bar{\beta}_{m'} < \bar{r}_{x|m|'}^{c_x}(\bar{\beta}_m^{c_x}).
\]

Applying \( \bar{r}_{x|m'}^{c_x} \) to both sides, we get that

\[
\bar{r}_{x|m'}^{c_x}(\bar{\beta}_{m'}) < \bar{r}_{x|m}^{c_x}(\bar{\beta}_m^{c_x}).
\]

Thus the \( \bar{r}_{x|m}^{c_x}(\bar{\beta}_m^{c_x}), m \in \omega \), form an infinite descending chain in the ordinals of the model \( J_0^{c_x} \tilde{\mathcal{M}}_x \), and so that model is illfounded. It follows by absoluteness that \( \mathcal{M}_x \) is illfounded. This means that \( [U(x)] = \emptyset \). Therefore \( [U^\dagger(x)] \neq \emptyset \).

Remark. We did not really have to give the argument of the preceding paragraph, for our proof of the theorem will not use the fact that \( [U^\dagger(x)] \neq \emptyset \) when \( \tilde{M}_{\text{Even}} \) is wellfounded. The converse—proved two paragraphs ago—will, however, be an essential ingredient in our proof of the theorem.
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To complete the proof of the theorem, let \( \alpha \) be any ordinal such that \( \beta_p < \alpha \) for all \( p \in T \).

For \( p \in T \), set

\[
\mathcal{V}_p = \{ X \subseteq (U^\uparrow \restriction \alpha)[p] \mid \langle \bar{\beta}_m^p \mid m < \ell h(p) \rangle \in j_{0,2\ell h(p)}^p(X) \}.
\]

We will show that \( \langle \mathcal{V}_p \mid p \in T \rangle \) witnesses that \( U^\uparrow \restriction \alpha \) is \( \gamma \)-homogenous.

We first prove that clause (1) of the definition of homogeneity holds—that each \( \mathcal{V}_p \) is a \( \gamma \)-complete ultrafilter on \( (U^\uparrow \restriction \alpha)[p] \). Let \( p \in T \). We begin by proving that \( (U^\uparrow \restriction \alpha)[p] \in \mathcal{V}_p \), i.e., that

(i) \( \forall m < \ell h(p) \) \( \bar{\beta}_m^p < j_{0,2\ell h(p)}^p(\alpha) \);
(ii) \( \langle \bar{\beta}_m^p \mid m < \ell h(p) \rangle \in j_{0,2\ell h(p)}^p(U^\uparrow[p]) \).

For (i), let \( m < \ell h(p) \). Then\( \bar{\beta}_m^p = j_{2m,2\ell h(p)}^p(\beta_{p|m}) < j_{2m,2\ell h(p)}^p(\alpha) \leq j_{0,2\ell h(p)}^p(\alpha) \).

By the definition of \( U^\uparrow \), what we must show to prove (ii) is that, for all \( m \) and \( m' \) such that \( m < m' < \ell h(p) \),

\[
\bar{\beta}_{m'}^p < (j_{0,2\ell h(p)}^p(i_{p|m|m'}))(\bar{\beta}_m^p).
\]

But this follows directly from condition (iv) of our construction.

Since \( (U^\uparrow \restriction \alpha)[p] \in \mathcal{V}_p \), it follows by Lemma 6.1.1 that \( \mathcal{V}_p \) is an ultrafilter on \( (U^\uparrow \restriction \alpha)[p] \).

By condition (vi), \( \gamma \leq \text{crit}(j_{0,2\ell h(p)}^p) \). Hence Lemma 6.1.1 yields that \( \mathcal{V}_p \) is \( \gamma \)-complete.

To check clause (2) in the definition of homogeneity, let \( p \subseteq q \in T \). We must verify that

\[
(\forall X \in \mathcal{V}_q)(\{ t \in U^\uparrow[q] \mid t \restriction \ell h(p) \in X \} \in \mathcal{V}_p).
\]

Let \( X \subseteq U^\uparrow[p] \). Then

\[
\{ t \in U^\uparrow[q] \mid t \restriction \ell h(p) \in X \} \in \mathcal{V}_q
\leftrightarrow \langle \bar{\beta}_m^p \mid m < \ell h(q) \rangle \restriction \ell h(p) \in j_{0,2\ell h(q)}^q(X)
\leftrightarrow \langle j_{2\ell h(p),2\ell h(q)}^p(\bar{\beta}_m^p) \mid m < \ell h(p) \rangle \in j_{2\ell h(p),2\ell h(q)}^q(j_{0,2\ell h(p)}^p(X))
\leftrightarrow \langle \bar{\beta}_m^p \mid m < \ell h(p) \rangle \in j_{0,2\ell h(p)}^p(X)
\leftrightarrow X \in \mathcal{V}_p.
\]
To complete the proof, we verify clause (3') in the definition of homogeneity. For \( p \subseteq q \in T \), let \( i^1_{p,q} : \text{Ult}(V; \mathcal{V}_p) \prec \text{Ult}(V; \mathcal{V}_q) \) be the canonical elementary embedding. Fix \( x \in [T] \). Let \( (\mathcal{M}^1_x, \langle i^n_{x,n} \mid n \in \omega \rangle) \) be the direct limit of the system \( (\text{Ult}(V; \mathcal{V}_{x^n}) \mid n \in \omega), \langle i^n_{x^n,m,x^n} \mid m \leq n \in \omega \rangle) \).

Assume that \( x \) belongs to the \( T \)-projection of \( U^\dagger \). We have already shown that \( \mathcal{M}_{\text{Even}}^C \) is wellfounded for any such \( x \). We must show that \( \mathcal{M}^1_x \) is wellfounded. It will be sufficient for us to prove that \( \mathcal{M}^1_x \) can be elementarily embedded into \( \mathcal{M}_{\text{Even}}^C \).

For \( m \in \omega \) and \( \pi_{\mathcal{V}_{x\mid n}}([f]_{\mathcal{V}_{x\mid n}}) \in \text{Ult}(V; \mathcal{V}_{x\mid n}) \), set

\[
k_n(\pi_{\mathcal{V}_{x\mid n}}([f]_{\mathcal{V}_{x\mid n}})) = (j^C_{0,2n}(f))(\langle \beta^x_m \mid m < n \rangle).
\]

To see that \( k_n \) is well-defined, assume that \( [f]_{\mathcal{V}_{x\mid n}} = [g]_{\mathcal{V}_{x\mid n}} \). Then

\[
\{ t \in U^\dagger \mid x \mid n \mid f(t) = g(t) \} \in \mathcal{V}_{x\mid n},
\]

and the definition of \( \mathcal{V}_{x\mid n} \) gives that

\[
(j^C_{0,2n}(f))(\langle \beta^x_m \mid m < n \rangle) = (j^C_{0,2n}(g))(\langle \beta^x_m \mid m < n \rangle),
\]

and so that \( k_n(\pi_{\mathcal{V}_{x\mid n}}([f]_{\mathcal{V}_{x\mid n}})) = \pi_{\mathcal{V}_{x\mid n}}([g]_{\mathcal{V}_{x\mid n}}) \). A similar argument shows that \( k_n : \text{Ult}(V; \mathcal{V}_{x\mid n}) \prec \mathcal{M}_{2n}^C \). Furthermore, if \( m \leq n \in \omega \) and \( \pi_{\mathcal{V}_{x\mid m}}([f]_{\mathcal{V}_{x\mid m}}) \in \text{Ult}(V; \mathcal{V}_{x\mid m}) \), then

\[
j^C_{2m,2n}(k_n(\pi_{\mathcal{V}_{x\mid m}}([f]_{\mathcal{V}_{x\mid m}}))) = (j^C_{2m,2n}(j^C_{0,2m}(f)))(\langle \beta^x_{m'} \mid m' < m \rangle)
\]

\[
= (j^C_{0,2m}(f))(\langle \beta^x_{m'} \mid m' < m \rangle)
\]

\[
= (j^C_{0,2n}(f))(\langle \beta^x_{m'} \mid m' < n \mid m \rangle)
\]

\[
= k_n(i^n_{x\mid m,x\mid n}(\pi_{\mathcal{V}_{x\mid m}}([f]_{\mathcal{V}_{x\mid m}}))).
\]

This argument shows that

\[
j^{C}_{2m,2n} \circ k_n = k_n \circ i^1_{x\mid m,x\mid n}.
\]

Thus we can define an elementary embedding \( k : \mathcal{M}^1_x \prec \mathcal{M}_{\text{Even}}^C \) by setting

\[
k(i^n_{x\mid n}(z)) = j^C_{n,\text{Even}}(k_n(z))
\]

for \( z \in \text{Ult}(V; \mathcal{V}_{x\mid m}) \).
We now turn to the $U^+$ construction.

Assume that $\langle U_{p,r} \mid \langle p, r \rangle \in T \otimes \langle \omega \rangle \rangle$ witnesses that $U$ is weakly $\kappa^+$-homogeneous for $T$. Let $\langle \pi_{p,r} \mid \pi, r \rangle \in T \otimes \langle \omega \rangle$, $\langle \langle i_{p,r}, q, s \rangle \mid \langle p, r \rangle \in T \otimes \langle \omega \rangle \rangle$, and $\langle M_{x,y}, \langle i(y,n,g) \rangle \mid n \in \omega \rangle \rangle$ be as on page 421. Let $i \mapsto r_k$ be the function introduced on page 428. Let $U^+ = U^+\langle \langle U_{p,r} \mid \langle p, r \rangle \in T \otimes \langle \omega \rangle \rangle \rangle$.

To show that the $T$-projection of $U^+$ has an embedding normal form (indeed that $U^+$ is homogeneous), we will build, for each $x \in |T|$, a plus one iteration tree $S_x$ of length $\omega$. The set of all branches of $S_x$ will be $\{\text{Even}\} \cup \{b_y \mid y \in \langle \omega \rangle \rangle\}$, where $\text{Even} = \{2m \mid m \in \omega\}$ as before and

$$b_y = \{2k - 1 \mid k \in \omega \land r_k \subseteq y\}.$$

We will arrange that $\mathcal{M}_{\text{Even}}^{S_x}$ is wellfounded if and only if $[U(x)] = \emptyset$.

To guarantee that $\mathcal{M}_{\text{Even}}^{S_x}$ is wellfounded when $[U(x)] = \emptyset$, we will make sure that $S_x$ is continuously illfounded off Even if $[U(x)] = \emptyset$. To do this we will make sure, in a sufficiently continuous fashion, that the trees $j_{0,2k}^{S_x}(U(x))$ are illfounded for every $x \in |T|$ and every $y \in \langle \omega \rangle \rangle$. This will be done with the aid of objects $s_p$, $p \in T$, that will play a role similar to the role played by the $s_p$ of the proof of Theorem 8.2.6. Each $s_{x,k}$ will belong to $j_{0,2k-1}^{S_x}(U[x \upharpoonright \text{th}(r_k)])$, and whenever $r_k \subseteq r_k'$ then we will have $j_{2k-1,2k'}^{S_x} \subseteq s_{x,k'}$. To get the $s_p$, we will use elements $s_{p,r} \in j_{0,2k-1,2k'}^{S_x}(U[x \upharpoonright \text{th}(r_k)])$, and from these elements we will get the $s_p$ with the aid of the One-Step Lemma.

To arrange that when $\mathcal{M}_{\text{Even}}^{S_x}$ is wellfounded then $[U(x)] = \emptyset$, we will use ordinals $\beta_p$, $p \in T$. For each $y \in \langle \omega \rangle \rangle$, the $\beta_{x,k}$, $r_k \subseteq y$, will give rise to an infinite descending chain of ordinals of $j_{0,\text{Even}}^{S_x}(\langle \omega \rangle \rangle(V))$. When $\mathcal{M}_{\text{Even}}^{S_x}$ is wellfounded, these chains will show that $\mathcal{M}_{x,y}$ is illfounded for every $y \in \langle \omega \rangle \rangle$ and hence that $[U(x)] = \emptyset$.

The $\beta_n$ will generate ultrafilters witnessing the homogeneity of $U^+$ in pretty much the same way that the corresponding ordinals performed the analogous task in the proof of Theorem 8.2.7.

**Theorem 8.2.7.** Assume that $\kappa$ is Woodin and that $\langle U_{p,r} \mid \langle p, r \rangle \in T \otimes \langle \omega \rangle \rangle$ witnesses that $U$ is $\kappa^+$-homogeneous for $T$. Let $U^+ = U^+\langle \langle U_{p,r} \mid \langle p, r \rangle \in T \otimes \langle \omega \rangle \rangle \rangle$ Then, for every sufficiently large ordinal $\alpha$, $U^+ \upharpoonright \alpha$ is $(< \kappa)$-homogeneous for $T$. 
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Proof. Let $\gamma < \kappa$ be such that $T \in V_\gamma$.

Let $\langle p, r \rangle \in T \otimes <^\omega \omega$, $\langle i_{\gamma, r}, q, s \rangle \in T \otimes <^\omega \omega$, $(M_{x,y}, \langle i_{x[n,y,n]} \mid n \in \omega \rangle)$, and $i \mapsto r_i$ be as in the discussion preceding the statement of the theorem. For $\{p, r\} \in T \otimes <^\omega \omega$, let $s_{p,r} = \pi_{U_{p,r}}([id]_{U_{p,r}})$.

Let $S$ be the tree ordering of $\omega$ defined as follows:

1. $0 S n$ for every $n > 0$
2. $2m S 2n$ if $m < n$
3. $2m + 1 S 2n + 1$ if $r_{m+1} \nsubseteq r_{n+1}$
4. $m S n$ only if (i), (ii), or (iii) requires that $m S n$

Note that the branches of an iteration tree whose tree ordering is $S$ are just $\langle \delta, \beta \rangle_{\xi \in \omega}$. The embedding $\check{\mathcal{N}}$ is defined above.

We will define, by induction on $p \in T$, objects $\delta_p$, $\beta_p$, $S_p$, and $s_p$. Both $\delta_p$ and $\beta_p$ will be ordinals, with $\delta_p < \kappa$. $s_p$ will be a sequence such that $\ell h(s_p) = \ell h(r_{\ell h(p)})$. $S_p$ will be an iteration tree of length $2\ell h(p) + 1$ on $V$. Its tree ordering will be the restriction of $S$. Its extenders will be $E^p_m$, $m < 2\ell h(p)$, its models will be $M^p_m$, $m \leq 2\ell h(p)$, and its embeddings will be $j^p_{m,n}$, $m S n \leq 2\ell h(p)$, Whenever $p \leq q \in T$ then we will have $S_p = S_q \upharpoonright 2\ell h(p) + 1$.

To simplify notation, we make some definitions. Let $p$ and $q$ belong to $T$ and let $m \leq n \leq \ell h(q)$ such that $r_m \subseteq r_n$. Let

$$
\begin{align*}
i_{q;m, n} &= i_{q[\ell h(r_m), r_n], (q[\ell h(r_n), r_n])}; \\
i^p_{q;m, n} &= j^p_{0,2\ell h(p)}(i_{q;m, n}); \\
\check{N}^p_{q;m} &= \check{\gamma}^p_{q;0,m}(M^p_{2\ell h(p)}); \\
\check{U}^p_{q;m} &= j^p_{0,2\ell h(p)}(i_{q;0,m}(U)); \\
\beta^q_m &= j^q_{2m,2\ell h(q)}(\beta_{q,m}); \\
s^q_m &= s_{q[\ell h(r_m), r_n]}.
\end{align*}
$$

The embedding $\check{\gamma}^p_{q;m,n}$ is the image of $i_{q;m,n}$ in $M^p_{2\ell h(p)}$. The class model $\check{N}^p_{q;m}$ is the image of $\text{Ult}(V; U_{q[\ell h(r_m), r_n]})$ in $M^p_{2\ell h(p)}$. In other words, $\check{N}^p_{q;m} = j^p_{0,2\ell h(p)}(i_{q;0,m}(V))$.

The tree $\check{U}^p_{q;m}$ is the image of $i_{q;0,m}(U)$ in $M^p_{2\ell h(p)}$; i.e., it is the image of $U$ in $\check{N}^p_{q;m}$.
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For \( p = \emptyset \) we have only to define \( \delta_\emptyset \) and \( \beta_\emptyset \). (Note that \( s_\emptyset = \emptyset \); for \( \ell h(s_\emptyset) = \ell h(0) = \ell h(\emptyset) \), because of the property of \( i \mapsto r_i \) stated on page 428.) Choose \( \delta_\emptyset > \gamma \) to be \( \zeta_0 + 1 \)-reflecting in \( \langle U \rangle \) relative to \( \kappa \). Let \( \beta_\emptyset = \zeta_0 \).

Let \( p \in T \). Let \( k = \ell h(p) \). Assume that \( \delta_p \), \( \beta_p \), \( S_p \), and \( s_p \) are defined for all \( p' \subseteq p \) so as to satisfy the conditions stated above and also the following conditions:

(i) for all \( m \leq k \), \( M^p_{2m} \) and \( M^p_{2m+1} \) agree through \( \delta_{p|m} + 1 \);

(ii) for all \( m \leq k \), the type \( (\text{tp}_{\kappa, \beta_m+1})_{\mathcal{N}^p_{p,m}}(\langle \hat{U}^p_{p,m} \rangle - j^p_{0,2k}(\mathcal{s}_m^p)) \) is the same as \( (\text{tp}_{\kappa, \zeta_0+1})_{\mathcal{M}^p_{2m-1}}(\langle j^p_{0,2m-1}(U) \rangle - s_{p|m}) \);

(iii) for all \( m \leq k \), \( \delta_{p|m} \) is \( (\bar{\beta}_m+1) \)-reflecting in \( \langle \hat{U}^p_{p,m} \rangle - j^p_{0,2k}(\bar{s}_m^p) \) relative to \( \kappa \) in \( \mathcal{N}^p_{p,m} \);

(iv) for all \( m \) and \( m' \) with \( m < m' \leq k \), if \( r_m \subseteq r_{m'} \) then

\[
\bar{\beta}_{p|m'} < \bar{\gamma}_{p,m,m'}(\bar{\beta}_m);
\]

(v) for all \( m \leq k \), \( s_{p|m} \) belongs to \( (j^p_{0,2m-1})(U[p \upharpoonright \ell h(r_m)]) \), and

\[
r_m \subseteq r_k \Rightarrow j^p_{2m-1,2k-1}(s_{p|m}) \subseteq s_p;
\]

(vi) for all \( m < k \),

\[
\gamma < \delta_{p|m} < \text{crit } (E^p_{2m+1}) < \delta_{p|m+1},
\]

and \( \text{crit } (E^p_{2m}) = \delta_{p|m} \), where \( 2\bar{m} - 1 = (2m + 1)^- \).

Note that these conditions all hold for \( p = \emptyset \).

Remark. The first seven conditions have pretty much the same roles as in the proof of Theorem 8.2.6. Indeed, the present construction can be thought of as a whole tree of constructions, each one like the construction of the proof of that lemma. Condition (vi), besides doing the work of the old condition (vi), will guarantee that these constructions do not conflict with one another. Condition (vi) will also be used in proving that our iteration trees \( S_x \) are plus one.

Let \( q \) be any element of \( T \) such that \( p \subseteq q \) and \( \ell h(q) = k + 1 \).
Let $n$ be the largest number $\leq k$ such that $r_n \subseteq r_{k+1}$. Let $e = \ell h(r_n)$. By the property of $i \mapsto r_i$, stated on page 428, $\ell h(r_{k+1}) = e + 1$.

By (i) and the fact that crit ($\check{\ddot{\eta}}_{q,0,k+1}$) $> \kappa$, it follows that $\check{\ddot{N}}_{q,k+1}^p$ and $M_{2n+1}^p$ agree through $\delta_p|n + 1$.

Note that

$$\check{\ddot{N}}_{q,k+1}^p = \check{\ddot{\eta}}_{q,0,k+1}^p(M_{2k}^p)$$
$$= \check{\ddot{\eta}}_{q,n,k+1}^p(M_{2k}^p)$$
$$= \check{\ddot{\eta}}_{q,n,k+1}^p(\check{\ddot{N}}_{p,n}^p),$$

that

$$\check{\ddot{U}}_{q,k+1}^p = \check{\ddot{j}}_{0,2k}^p(\check{\ddot{\eta}}_{q,0,k+1}^p(U))$$
$$= \check{\ddot{j}}_{0,2k}^p(\check{\ddot{\eta}}_{q,n,k+1}^p(U))$$
$$= (\check{\ddot{j}}_{0,2k}^p(\check{\ddot{\eta}}_{q,n,k+1}^p(U)))$$
$$= \check{\ddot{j}}_{q,n,k+1}^p(\check{\ddot{U}}_{p,n}^p),$$

and that

$$\check{\ddot{j}}_{0,2k}^p(\check{\ddot{\eta}}_{q,n,k+1}^p(s_n^p)) = (\check{\ddot{j}}_{0,2k}^p(\check{\ddot{\eta}}_{q,n,k+1}^p))(\check{\ddot{j}}_{0,2k}^p(s_n^p))$$
$$= \check{\ddot{j}}_{q,n,k+1}^p(\check{\ddot{j}}_{0,2k}^p(s_n^p)).$$

Since crit ($\check{\ddot{\eta}}_{q,n,k+1}^p$) $> \kappa$, it follows that

$$(\check{\ddot{j}}_{q,n,k+1}^p)\check{\ddot{N}}_{q,n}^p((\check{\ddot{U}}_{p,n}^p)\check{\ddot{j}}_{0,2k}^p(s_n^p))$$
$$= \check{\ddot{j}}_{q,n,k+1}^p((\check{\ddot{j}}_{q,n,k+1}^p)\check{\ddot{N}}_{q,n}^p((\check{\ddot{U}}_{p,n}^p)\check{\ddot{j}}_{0,2k}^p(s_n^p)))$$
$$= (\check{\ddot{U}}_{q,k+1}^p)\check{\ddot{j}}_{0,2k}^p(i_{q,n,k+1}(s_n^p))$$

and so by (ii) this last is the same as $(\check{\ddot{j}}_{q,n,k+1}^p(M_{2n+1}^p))(\check{\ddot{j}}_{0,2k}^p(s_{p|n})).$

From (iii) it similarly follows that $\delta_p|n$ is $(\check{\ddot{j}}_{q,n,k+1}^p(M_{p+1}^p))$-reflecting in the sequence $(\check{\ddot{U}}_{q,k+1}^p)\check{\ddot{j}}_{0,2k}^p(i_{q,n,k+1}(s_n^p))$ relative to $\kappa$ in $\check{\ddot{N}}_{q,k+1}^p$.

Since $\check{\ddot{j}}_{0,2k}$ and $\check{\ddot{j}}_{q,n,k+1}$ fix $\kappa$, we have that $\kappa$ is Woodin in $\check{\ddot{N}}_{q,k+1}^p$.

Thus the hypotheses of the One-Step Lemma hold for $\kappa$ with

$$M = \check{\ddot{N}}_{q,k+1}^p;$$
$$N = M_{2n+1}^p;$$
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\[ \delta = \delta_{p|n}; \]
\[ \eta = \delta_p; \]
\[ \beta = \overset{p}{q}_{n,k+1}(\bar{\beta}_n) + 1; \]
\[ \xi = \overset{p}{q}_{n,k+1}(\bar{\beta}_n); \]
\[ \beta' = \zeta_0 + 1; \]
\[ x = (\bar{U}^p_{q,k+1})^{-j} 0,2k(i_{q,n,k+1}(s^p_n)); \]
\[ y = (j^p_{0,2k}(s^q_{k+1})(e)); \]
\[ x' = (j^p_{0,2n-1}(U))^{-s_p|n}; \]
\[ \chi(v) = "\kappa + v is the greatest ordinal." \]

Let \( \lambda \) and \( E \) be given by the One-Step Lemma. Since \( \overset{p}{q}_{0,k+1} \) fixes \( \lambda \), \( E \), and \( \delta_p \), it follows that \( E \) is a \((\delta_{p|n}, \lambda)\)-extender in \( M^p_{2k} \). Thus Theorem 7.3.2 gives that \( \prod_{E} M^p_{2n-1}(M^p_{2n-1}; \in) \) is wellfounded. Let then \( \delta^*, \xi^*, \) and \( y^* \) be given by the One-Step Lemma. By clause (4*) of the One-Step Lemma, \( \xi^* = \zeta_0 \).

Extend \( S_p \) to an alternating chain that will be \( S_q \mid 2k+2 \) by setting \( \overset{q}{s}_{0,2k+1}(s^p_n) \) whenever these equations make sense. We will use these identities without comment in the sequel.

We have that
\[ x^-y = (\bar{U}^p_{q,k+1})^{-j} 0,2k(i_{q,n,k+1}(s^p_n)) - (j^p_{0,2k}(s^q_{k+1})(e)). \]

Observe that
\[ i_{q,n,k+1}(s^p_n) = i_{q,n,k+1}(s_{p|e,r_n}) = s_{q|e+1,k+1} \mid e = s^q_{k+1} \mid e. \]

Thus \( x^-y \) is the concatenation of \( (\bar{U}^p_{q,k+1}) \) and
\[ j^p_{0,2k}(s^q_{k+1} \mid e) - (j^p_{0,2k}(s^q_{k+1})(e)). \]

By the elementarity of \( j^p_{0,2k} \), we finally get that
\[ x^-y = (\bar{U}^p_{q,k+1})^{-j} 0,2k(s^q_{k+1}). \]
Now $s^q_k + 1 = s^q_{k+1}$ belongs to $i^q_{0,k+1}(U[q \upharpoonright e + 1])$. Therefore the sequence $j_{0,2k}(s^q_k) + 1$ belongs to $j_{0,2k}(i^q_{0,k+1}(U[q \upharpoonright e + 1])) = U_{0,k+1}$. It follows by clause (2) of the One-Step Lemma that $s_q \in j_{0,2k}(U[q \upharpoonright e + 1])$, and so the first clause of condition (v) holds for $q$. Since $j_{2n-1,2k+1}(s_q) \subseteq s_q$, the second clause of condition (v) holds for $q$ the case $m = n$.

We have that

(a) $M^q_{2k+1}$ and $N^p_{q,k+1}$ agree through $\delta'_q + 1$;

(b) $(t^p_{\delta'_q})^M_{2k+1}((j^q_{0,2k+1}(U)) \sim s_q) = (t^p_{\delta'})^q_{n,n,k+1}(\beta^p_n))^q_{q,k+1}((\beta^q_{q,k+1}) \sim j^p_{0,2k}(s^q_{k+1}))$;

(c) $\delta'_q$ is $\zeta_0$-reflecting in $(j^q_{0,2k+1}(U)) \sim s_q$ relative to $\kappa$ in $M^q_{2k+1}$.

By (b) and (c) together with parts (b) and (c) of Lemma 8.2.3, we have that

(b') $(t^p_{\delta'_q})^M_{2k+1}((j^q_{0,2k+1}(U)) \sim s_q) = (t^p_{\delta'})^q_{n,n,k+1}(\beta^p_n))^q_{q,k+1}((\beta^q_{q,k+1}) \sim j^p_{0,2k}(s^q_{k+1}))$;

(c') $\delta'_q$ is $\zeta_1$-reflecting in $(j^q_{0,2k+1}(U)) \sim s_q$ relative to $\kappa$ in $M^q_{2k+1}$.

Let $f : j^p_{0,2k}(U[p \upharpoonright e]) \to \text{Ord}$ belong to $M^p_{2k}$ and be such that

$$\tilde{\beta}_n = \pi^M_{j^p_{0,2k}(U[p \upharpoonright e])}(\bigcup_{j^p_{0,2k}(U[p \upharpoonright e])}^M_{M^p_{2k}}).$$

In other words, let $f$ be such that $\tilde{\beta}_n = (t^p_{p,0,n}(f))(j^p_{0,2k}(s^q_n))$. By (b') and the fact that crit $(s^q_{q,0,n,k+1}) > \delta'_q$, there is a set $X \in j^p_{0,2k}(U[p \upharpoonright e \upharpoonright k+1])$ such that, for all $t \in X$,

(b'') $(t^p_{\delta'_q})^M_{2k+1}((j^q_{0,2k+1}(U)) \sim s_q) = (t^p_{\delta'_q})^M_{j^p_{0,2k}(U[p \upharpoonright e])}.\sim t$.

Choose any element $t$ of $X$. Since $\kappa$ is Woodin in $M^q_{2k+1}$, the hypotheses of the One-Step Lemma hold for $\kappa$ with

$$M = M^q_{2k+1};$$

$$N = M^p_{2k};$$

$$\delta = \delta'_q;$$

$$\eta = \delta'_q;$$
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\[ \begin{align*}
\beta &= \zeta_1; \\
\xi &= \zeta_0 + 1; \\
\beta' &= f(t \upharpoonright e); \\
x &= \langle j^q_0,2k+1(U) \rangle \sim s_q; \\
y &= \emptyset; \\
x' &= \langle j^p_0,2k(U) \rangle \sim t; \\
\chi(v) &= \text{“}v = v,\text{”}
\end{align*} \]

Let \( \lambda \) and \( E \) be given by the One-Step Lemma. By Theorem 7.3.2, the model \( P_{E \mathcal{M}_k}(M_{E \mathcal{M}_k}; e) \) is wellfounded. Let then \( \delta^*, \xi^*, \) and \( Y^* \) be given by the One-Step Lemma. (We will make no use of \( \xi^* \) and \( y^* \).)

For all elements \( u \) of \( X \),

\[ (\text{tp}_{\kappa, f(u \upharpoonright e)})^{M_{E \mathcal{M}_k}}_{M_{E \mathcal{M}_k}}(\langle j^p_0,2k(U) \rangle \sim u) = (\text{tp}_{\kappa, f(t \upharpoonright e)})^{M_{E \mathcal{M}_k}}_{M_{E \mathcal{M}_k}}(\langle j^p_0,2k(U) \rangle \sim t). \]

Thus the elementarity of \( i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}} \) gives that, for all \( u \in i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(X) \),

\[ (\text{tp}_{\kappa, i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(f)(u \upharpoonright e)})^{\text{Ult}(M_{E \mathcal{M}_k}; E)}(\langle i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(j^p_0,2k(U)) \rangle \sim u) = (\text{tp}_{\kappa, i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(f)(i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(t)) \upharpoonright e})^{\text{Ult}(M_{E \mathcal{M}_k}; E)}(\langle i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(j^p_0,2k(U)) \rangle \sim i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(t)). \]

Since \( \delta^*_q < i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(\delta^*_q) \), for all \( u \in i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(X) \) we have that

\[ (\text{tp}_{\kappa, i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(f)(u \upharpoonright e)})^{\text{Ult}(M_{E \mathcal{M}_k}; E)}(\langle i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(j^p_0,2k(U)) \rangle \sim u) = (\text{tp}_{\kappa, i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(f)(i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(t)) \upharpoonright e})^{\text{Ult}(M_{E \mathcal{M}_k}; E)}(\langle i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(j^p_0,2k(U)) \rangle \sim i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(t)). \]

Let \( u \in i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(X) \). We make an application of the last part of the One-Step Lemma, with \( z = \langle i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(j^p_0,2k(U)) \rangle \sim u \) and with \( \alpha = (i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}} f)(u \upharpoonright e) \). If \( \hat{\xi} \) and \( \hat{y} \) are as given by this application, then clause (2) of the One-Step Lemma implies that \( \hat{y} = \emptyset \) and \( \hat{\xi} \) is a successor ordinal. Let \( g(u) \) be the least ordinal \( \nu \) such that clauses (2) and (3) of the One-Step Lemma hold with \( \hat{\xi} = \nu + 1 \) and \( \hat{y} = \emptyset \).

Observe that the function \( g : i_{E \mathcal{M}_k}^{M_{E \mathcal{M}_k}}(X) \to \text{Ord} \) belongs to \( \text{Ult}(M_{E \mathcal{M}_k}; E) \).
We finish the definition of $S_q$ by setting $E_{2k+1}^{q} = E$. Thus $E^{q}_{2k+2} = \text{Ult}(M^{p}_{2k}; E)$ and $j^{q}_{2k,2k+2} = j^{p}_{E2k}$. Clause (1*) of the One-Step Lemma gives the case $k + 1$ of inductive condition (i) for $q$.

Let $\delta_q = \delta^*$. Clauses (2) and (3) of the One-Step Lemma give that, for all $u \in j^{q}_{2k,2k+2}(X)$,

- (2) $(tp_{\kappa,\xi,u(+)1})^{M^{2k+2}_{0,2k+2}}((j^{q}_{0,2k+2}(U))\upharpoonright u) = (tp_{\kappa,\xi,0+1})^{M^{2k+1}_{0,2k+1}}((j^{q}_{0,2k+1}(U))\upharpoonright s_q)$;
- (3) $\delta_q$ is $(\xi(u) + 1)$-reflecting in $(j^{q}_{0,2k+2}(U))\upharpoonright u$ relative to $\kappa$ in $M^{q}_{2k+2}$.

The set $j^{q}_{2k,2k+2}(X)$ belongs to $j^{q}_{0,2k+2}(U_{q|\xi+1,r_k+1})$. This fact allows us to complete our definitions by setting

$$\beta_q = \pi^{M^{2k+2}_{0,2k+2}}_{j^{q}_{0,2k+2}(U_{q|\xi+1,r_k+1})}((j^{q}_{0,2k+2}(U_{q|\xi+1,r_k+1})))$$

Using Loš’s Theorem in $M^{q}_{2k+2}$ and using the fact that

$$s^{q}_{k+1} = \pi^{M^{2k+2}_{0,2k+2}}_{j^{q}_{0,2k+2}(U_{q|\xi+1,r_k+1})}((\text{id})^{M^{2k+2}_{0,2k+2}}_{j^{q}_{0,2k+2}(U_{q|\xi+1,r_k+1})})$$

we see that (2) and (3) imply that

- (ii') $(tp_{\kappa,\beta_q+1})^{N^{q}_{q; q, k+1}}((j^{q}_{0,2k+2}(v^{q}_{0,k+1}(U)))\upharpoonright j^{q}_{0,2k+2}(s_q))$
  $$= (tp_{\kappa,\xi,0+1})^{M^{2k+1}_{0,2k+1}}((j^{q}_{0,2k+1}(U))\upharpoonright s_q)$;  
- (iii') $\delta_q$ is $(\beta_q + 1)$-reflecting in $(j^{q}_{0,2k+2}(v^{q}_{0,q}(U)))\upharpoonright j^{q}_{0,2k+2}(s_q)$ relative to $\kappa$ in $N^{q}_{q; k+1}$.

Observe that (ii') and (iii') are just the case $k + 1$ of our inductive conditions (ii) and (iii) for $q$.

Let us now verify our all our inductive conditions for $q$. To verify (i), (ii), and (iii) for $q$, let $m \leq k + 1$ be arbitrary.

We have already noted that condition (i) for the case $m = k + 1$ follows from clause (1*) of the One-Step Lemma. Suppose that $m \leq k$. By condition (vi) for $p$, we have that $\delta_{q|m} \leq \delta_p$. Because the One-Step Lemma gives that $\eta < \delta^*$, we also have that

$$\delta_p < \delta_q' < \delta_q.$$  

By condition (i) for $p$, $M^{q}_{2k}$ and $M^{q}_{2m+1}$ agree through $\delta_{q|m} + 1$. Since $M^{q}_{2k+1}$ and $M^{q}_{2k}$ agree through $\delta_q' + 1$, it follows that $M^{q}_{2k+1}$ and $M^{q}_{2m+1}$ agree through
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\[\delta_{p|m} + 1.\] Since \(M^q_{2k+2}\) and \(M^q_{2k+1}\) agree through \(\delta_q + 1\), we get that \(M^q_{2k+2}\) and \(M^q_{2m-1}\) agree through \(\delta_{p|m} + 1 = \delta_{q|m} + 1\).

We have already checked that (ii) and (iii) hold for the case \(m = k + 1\). Now let \(m \leq k\). By the fact that \(\text{crit} (j^q_{2k,2(k+1)}) = \delta_q > \delta_{q|m}\) and by our definitions, we have that

1. \(j^q_{2k,2(k+1)}(\delta_{q|m}) = \delta_{q|m}\);
2. \(j^q_{2k,2(k+1)}(\bar{\beta}_m) = \bar{\beta}_m\);
3. \(j^q_{2k,2(k+1)}(\bar{N}_p) = \bar{N}_q\);
4. \(j^q_{2k,2(k+1)}(\bar{U}_p) = \bar{U}_q\).

The fact that \(\text{crit} (j^q_{2k,2(k+1)}) > \delta_{q|m}\) also implies that

\[\left(\text{tp}_{\delta_{p|m},k,\bar{\beta}_m+1}\right)(j^q_{2k,2(k+1)}(s^q_m)) < (\text{tp}_{\delta_{q|m},q,\bar{\beta}_m+1})(j^p_{0,2k}(s^p_m)).\]

Hence (ii) for \(q\) follows from (ii) for \(p\). Similarly case \(m\) of (iii) for \(q\) follows from (iii) for \(p\).

The inequality \(\hat{\xi} < \alpha\) of the One-Step Lemma gives us that

\[(\forall u \in j^q_{2k,2(k+1)}(X)) g(u) + 1 < (j^q_{2k,2(k+1)}(f))(u \upharpoonright k).\]

This in turn implies that

\[\beta_q + 1 < \bar{v}^q_{q,n,k+1}(j^q_{2k,2k+2}(\bar{\beta}_p)).\]

Since \(\beta_q = \bar{\beta}_q\) and \(j^q_{2k,2(k+1)}(\bar{\beta}_p) = \bar{\beta}_n\), we have condition (iv) for \(q\) in the case \(m = n\). Assume that \(m < m' \leq k\) and that \(r_m \subseteq r_{m'}\). Recall that \(j^p_{2k,2k+2}(\bar{\beta}_m) = \bar{\beta}_m\) and \(j^p_{2k,2k+2}(\bar{\beta}_p) = \bar{\beta}_m\), and observe that \(j^p_{2k,2k+2}(\bar{v}_{p,m,m'}) = \bar{v}^q_{q,m,m'}\). By these facts, condition (iv) for \(p\), and the elementarity of \(j^q_{2k,2k+2}\),

\[\bar{\beta}^q_{m'} = j^q_{2k,2k+2}(\bar{\beta}_p) < j^q_{2k,2k+2}(\bar{v}_{p,m,m'})(\bar{\beta}_p) = \bar{v}^q_{q,m,m'}(\bar{\beta}^q_{m'}).\]
The remaining case of condition (iv) for $q$ is $m \neq n$ and $m' = k + 1$. Assume that $m \leq k$, $m \neq n$, and $r_m \subseteq r_{k+1}$. By the definition of $n$, we must have $m < n$ and $r_m \subseteq r_n$. But then

$$\bar{\beta}_{k+1}^q < \gamma_{q,m,k+1}^q (\bar{\beta}_k^q) < \gamma_{q,n,k+1}^q (\gamma_{q,m,n}(\bar{\beta}_m^q)) = \gamma_{q,m,k+1}^q (\bar{\beta}_m^q).$$

We have already verified for $q$ the first clause of condition (v) and the case $m = k$ of the second clause of that condition. The other cases of the second clause follow easily from the corresponding cases for $p$ and the case $m = k$ for $q$.

We have noted that $\delta_p < \delta'_q < \delta_q$. Note also that $2n - 1 = (2k + 1)_{\bar{s}}$. Now $\text{crit} (E_{2k}^q) = \delta_{q|m}$ and $\text{crit} (E_{2k+1}^q) = \delta'_q$. Since condition (vi) holds for $p$ and since $\gamma < \delta_0$, it follows that condition (vi) holds for $q$.

This completes our construction and the verification that it has the desired properties.

We will show that the system

$$(\langle M_p^{2\ell(h(p))} \mid p \in T \rangle, \langle j_{m,2\ell(h(p))}^p \mid p \in T \land m < \ell(h(p) \in T) \rangle)$$

gives an embedding normal form for the $T$-projection of $U^\dagger$.

Fix $x \in [T]$. Let $S_x$ be the iteration tree of length $\omega$ whose restrictions are the $S_{x|n}$.

We will need to know that $S_x$ is a plus one tree. To prove this, let $n \in \omega$. It is not true for $S$, as it was for the alternating chain ordering $C$, that the set

$$\{m \mid (m + 1)_{\bar{s}} \leq n < m\}$$

is finite. But the first part of the proof of Lemma 8.2.5 shows that it suffices to prove the weaker fact that, for every $k \in \omega$, the set

$$\{\text{crit} (E_m^{S_x}) \mid (m + 1)_{\bar{s}} \leq n < m\}$$

is finite. Condition (vi) of our construction implies that, for all $m$ and $m' \in \omega$,

$$(m + 1)_{\bar{s}} = (m' + 1)_{\bar{s}} \rightarrow \text{crit} (E_m^{S_x}) = \text{crit} (E_{m'}^{S_x}),$$

and there are only finitely many numbers $\leq n$ of the form $(m + 1)_{\bar{s}}$.

We must show that $[U^\dagger(x)] \neq \emptyset$ if and only if $\hat{M}_{\text{Even}}^{S_x}$ is wellfounded.
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Assume first that \([U^+(x)] \neq \emptyset\). By Theorem 8.1.8, \([U(x)] = \emptyset\), i.e., \(U(x)\) is a wellfounded tree. For each \(k \in \omega\), let
\[
\xi_{2k} = j^\mathcal{S}_x_{2k}(\|s_0\|^{U(x)});
\]
\[
\xi_{2k+1} = \|s_{x|k+1}\|^\mathcal{S}_x_{0,2k+1}(U(x)).
\]
Using condition (v), we get that, for all \(m, n \in \omega\) with \(r_m \subseteq r_n\),
\[
j^\mathcal{S}_x_{2m-1,2n-1}(\xi_{2m-1}) = \|j^\mathcal{S}_x_{2m-1,2n-1}(s_{x|m})\|^\mathcal{S}_x_{0,2n-1}(U(x)) > \|s_{x|n}\|^\mathcal{S}_x_{0,2n-1}(U(x)) = \xi_{2n-1}.
\]
Thus the sequence \(\langle \xi_n \mid n \in \omega \rangle\) witnesses that \(\mathcal{S}_x\) is continuously illfounded off Even. Since \(\mathcal{S}_x\) is plus one, Corollary 7.4.6 gives that \(\mathcal{M}^{\mathcal{S}_x}_{\text{Even}}\) is wellfounded.

Now assume that \(\mathcal{M}^{\mathcal{S}_x}_{\text{Even}}\) is wellfounded. For elements \(m, m' \in \omega\) with \(m \leq m'\), let
\[
\bar{\beta}_{x}^{m} = j^\mathcal{S}_x_{0,\text{Even}}(\beta_{x|m});
\]
\[
\bar{\gamma}_{m,m'}^{x} = j^\mathcal{S}_x_{0,\text{Even}}(i_{x|n;m;n})
\]
\[
\bar{\bar{\gamma}}_{m}^{x} = j^\mathcal{S}_x_{0,\text{Even}}(\bar{i}_{x|\text{fh}(r_m),r_m}).
\]
Let \(y \in ^\omega \omega\) be arbitrary. For each \(n \in \omega\), let \(m_n\) be such that \(y \upharpoonright n = r_{m_n}\). Let \(n < n' \in \omega\) and let \(k \in \omega\) be such that \(m_{n'} \leq k\). By condition (iv), we have that
\[
\bar{\beta}_{x}^{m_{n'}}^{k} < \bar{\gamma}_{x|k;m_n,m_{n'}}^{x}(\bar{\beta}_{m_n}^{x}).
\]
Applying \(j^\mathcal{S}_x_{2k,\text{Even}}\) to both sides of this inequality, we find that
\[
\bar{\beta}_{x}^{m_{n'}}^{k} < \bar{\gamma}_{x|k;m_n,m_{n'}}^{x}(\bar{\beta}_{m_n}^{x}).
\]
Applying \(\bar{\gamma}_{m_{n'}}^{x}\) to both sides, we get that
\[
\bar{\gamma}_{m_{n'}}^{x}(\bar{\beta}_{m_n}^{x}) < \bar{\gamma}_{m_{n}}^{x}(\bar{\beta}_{m_n}^{x}).
\]
This argument shows that the \(\bar{\gamma}_{m_n}^{x}(\bar{\beta}_{m_n}^{x})\), \(n \in \omega\), form an infinite descending chain in the ordinals of the model \(j^\mathcal{S}_x_{0,\text{Even}}(\mathcal{M}_{x,y})\), and so that model is illfounded. It follows by absoluteness that \(\mathcal{M}_{x,y}\) is illfounded. Since \(y\) was an arbitrary element of \(\omega\), this means that \([U(x)] = \emptyset\). Therefore \([U^+(x)] \neq \emptyset\).
Remark. As in the proof of Theorem 8.2.6, the second half of the proof of embedding normal form is not needed for the proof of the theorem.

To complete the proof of the theorem, let $\alpha$ be any ordinal such that $\beta_p < \alpha$ for all $p \in T$.

For $p \in T$, set

$$\mathcal{V}_p = \{ X \subseteq (U^+ \downarrow \alpha)[p] \mid \langle \tilde{\beta}_m \mid m < \ell h(p) \rangle \in j_{0,2\ell h(p)}^p(X) \}. $$

We will show that $\langle \mathcal{V}_p \mid p \in T \rangle$ witnesses that $U^+ \downarrow \alpha$ is $\gamma$-homogenous.

We first prove that clause (1) of the definition of homogeneity holds—that each $\mathcal{V}_\alpha$ is a $\gamma$-complete ultrafilter on $(U^+ \downarrow \alpha)[p]$. Let $p \in T$. We begin by proving that $(U^+ \downarrow \alpha)[p] \in \mathcal{V}_p$, i.e., that

(i) $(\forall m < \ell h(p)) \tilde{\beta}_m^p < j_{0,2\ell h(p)}^p(\alpha)$;

(ii) $\langle \tilde{\beta}_m \mid m < \ell h(p) \rangle \in j_{0,2\ell h(p)}^p(U^+[p])$.

For (i), let $m < \ell h(p)$. Then

$$\tilde{\beta}_m^p = j_{2m,2\ell h(p)}^p(\beta_{p|m}) < j_{2m,2\ell h(p)}^p(\alpha) \leq j_{0,2\ell h(p)}^p(\alpha).$$

By the definition of $U^+$, what we must show to prove (ii) is that, for all $m$ and $m'$ such that $m < m' < \ell h(p)$,

$$\tilde{\beta}_{m'}^p < (j_{0,2\ell h(p)}^p(i_{p|\ell h(r_m),r_m})(p|\ell h(r_{m'}),r_{m'}))\langle \tilde{\beta}_m^p \rangle,$$

i.e., that

$$\tilde{\beta}_{m'}^p < (j_{0,2\ell h(p)}^p(i_{p;m,m'}))(\tilde{\beta}_m^p).$$

But this follows directly from condition (iv) of our construction.

Since $(U^+ \downarrow \alpha)[p] \in \mathcal{V}_p$, it follows by Lemma 6.1.1 that $\mathcal{V}_p$ is an ultrafilter on $(U^+ \downarrow \alpha)[p]$.

By condition (vi), $\gamma \leq \text{crit}(j_{0,2\ell h(p)}^p)$. Hence Lemma 6.1.1 yields that $\mathcal{V}_p$ is $\gamma$-complete.

The verifications of clauses (2) and (3') in the definition of homogeneity is exactly like the corresponding verifications in the proof of Theorem 8.2.6, and we omit them.

\[ \Box \]

Corollary 8.2.8. Let $\kappa$ be a Woodin cardinal, let $T$ be a game tree such that $|T| < \kappa$, and let $A \subseteq [T]$ be such that $[T] \setminus A$ is weakly $\kappa^+$-homogeneously Souslin. Then $A$ is $(< \kappa)$-homogeneously Souslin.
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Proof. Let $Y$ be a set and $U$ a tree on field $(T) \otimes Y$ witnessing that $T$ is weakly $\kappa^+$-homogeneously Souslin. Let $\langle U_{p,r} \mid p \in T \land r \in \omega^\omega \rangle$ witness that $U$ is weakly $\kappa^+$-homogeneous for $T$. Let $U^\dagger = U^\dagger(\langle U_{p,r} \mid p \in T \land r \in \omega^\omega \rangle)$. By Theorem 8.2.7, let $\alpha \geq \max\{\omega, (2^{|Y|})^+\}$ be such that $U^\dagger \upharpoonright \alpha$ is $(< \kappa)$-homogeneous for $T$. By Theorem 8.1.8, $A$ is the $T$-projection of $U^\dagger \upharpoonright \alpha$, and so $A$ is $(< \kappa)$-homogeneously Souslin. □

Theorem 8.2.9. Let $T$ be a game tree and let $n \in \omega$. Let $\langle \kappa_i \mid i \leq n \rangle$ be a strictly increasing sequence of cardinals such that $|T| < \kappa_0$, $\kappa_n$ is measurable and, for $i < n$, $\kappa_i$ is Woodin. Then every $\Pi^1_{n+1}$ subset of $[T]$ is $(< \kappa_0)$-homogeneously Souslin.

Proof. We prove by induction on $m \leq n$ that, for every game tree $T'$ such that $|T'| \leq \max\{\aleph_0, |T|\}$, every $\Pi^1_{m+1}$ subset of $[T']$ is $(< \kappa_{n-m})$-homogeneously Souslin.

By Theorem 4.3.6, every $\Pi^1_1$ subset of such a $[T']$ is $\kappa_n$-homogeneously Souslin and so $(< \kappa_n)$-homogeneously Souslin.

Let $m < n$ and assume that what we want to prove holds of $m$. Let $T'$ be a game tree such that $|T'| \leq \max\{\aleph_0, |T|\}$. Let $A \subseteq [T]$ with $A \in \Pi^1_{m+2}$. Let $B \subseteq [T] \times \omega^\omega$ be such that $B \in \Pi^1_n$ and $A = [T] \setminus pB$. Let $B^* = \{\langle x, y \rangle \mid \langle x, y \rangle \in B\}$. Then $B^* \subseteq [T \otimes \omega^\omega]$ and $B^* \in \Pi^1_n$. Since $\kappa_{n-(m+1)} < \kappa_{n-m}$, we have that $B^*$ is $(\kappa_{n-(m+1)})^+$-homogeneously Souslin. But this means that $B$ is $(\kappa_{n-(m+1)})^+$-homogeneously Souslin. (See page 423.) By Theorem 8.1.3, $[T] \setminus A$ is weakly $\kappa$-homogeneously Souslin. Since $\kappa_{n-(m+1)}$ is Woodin, Corollary 8.2.8 implies that $A$ is $(< \kappa)$-homogeneously Souslin. □

Theorem 8.2.10. Let $T$ be a game tree and let $n \in \omega$. Assume that there are $n$ distinct Woodin cardinals all greater than $|T|$ and that there is a measurable cardinal greater than all of them. Then every $\Pi^1_{n+1}$ game in $T$ is determined.

Proof. The theorem follows directly from Theorems 8.2.9 and 4.3.5. □

Exercise 8.2.1. Let $b$ be one of the two branches of an alternating chain $C$ of length $\omega$. Assume that the other branch of $S$ is illfounded. Prove that $C$ is continuously illfounded off $b$.

Hint. Use Lemma 7.4.2.
Exercise 8.2.2. Let \( n \in \omega \). Assume that there is a Woodin cardinal. Prove that there is an infinite plus \( n \) alternating chain on \( V \).

Hint. The most natural way to proceed is to prove a variant of the One-Step Lemma which will let one build a plus \( n \) alternating chain by a construction just like that of Lemma 8.2.4. It is possible, nevertheless, to get by with the One-Step Lemma itself.

8.3 Variations

In this section we discuss variants of the construction of the proof of Theorem 8.2.7.

The first variant is a cleaned-up version of the construction of [Martin and Steel, 1989]. It has the advantage of being a little simpler than the construction of the proof of Theorem 8.2.7. Its disadvantage lies in its not immediately yielding homogeneity ultrafilters for \( U^\dagger \). After giving the construction, we prove the lemma needed to get the existence of these ultrafilters. We then prove a result of K. Windßus that allows one to sidestep the problem, propagating homogeneous Souslinness without proving the homogeneity of \( U^\dagger \).

The second variant construction is due to Itay Neeman. Rather than propagate homogeneous Souslinness, Neeman propagates what he calls the auxiliary game property. His method has various advantages, one of which is that it seems to yield sharper results.

In the proof in §8.2 of Theorem 8.2.7, we maintained inductively relations between the models

\[
\hat{N}_q^p = j^p_{0,2k}(\hat{N}_p^{p,0,m}(V))
\]

and the models \( M_{2n+1}^p = j^p_{0,2m+1}(V) \). The induction step of the construction involved successive applications of the One-Step Lemma. The first of these applications was to the models \( \hat{N}_q^p \) and \( M_{2n+1}^p \), for some \( n < k \). But the second application was not, as one might have anticipated, to \( M_{2n+1}^p \) and \( \hat{N}_{q,k+1}^p \). Instead we undid the embedding \( i^p_{q,k+1} \), made a whole set of applications of the second half of the One-Step Lemma to the models \( M_{2n+1}^p \) and \( M_{2k}^p \), and then applied \( i^q_{q,0,k+1} \) to the results of these applications. The reason for this round-about process was that the direct method would have yielded, for example, the model \( i^p_{E^q_{2k+1}}(\hat{N}_q^{p,k+1}) \) in place of \( \hat{N}_q^{q,k+1} \).

The analogous construction in [Martin and Steel, 1989] could—at least, after a mostly cosmetic rearrangement—be regarded as maintaining induc-
tively certain relations between the models

\[ i_{p,0,m}^p(j_{0,2k}^p(V)) \]

and the models \( M_{2m+1}^p \). The induction step could be seen as involving two straightforward applications of the One-Step Lemma. In [Martin and Steel, 1989] embedding normal form was not hard to demonstrate, but homogeneity of \( U^\dagger \) needed a substantial lemma.

Let us see how would work after the “cosmetic rearrangement.”

Let \( T, \kappa, Y, U, \nu, \zeta_0, \zeta_1, \rho, \langle U_{p,r}^p \mid \langle p, r \rangle \in T \otimes \langle \omega \rangle \rangle, U^\dagger, \gamma, \langle \pi_{p,r} \mid \langle p, r \rangle \in T \otimes \langle \omega \rangle \rangle, \langle i_{p,r},(q,s) \mid \langle p, r \rangle \subseteq \langle q, s \rangle \in T \otimes \langle \omega \rangle \rangle, (M_{x,y}; \langle i_{x,y}^{r,s} \mid n \in \omega \rangle), i \mapsto r_i, \langle s_{p,r} \mid \langle p, r \rangle \in T \otimes \langle \omega \rangle \rangle, \) and \( S \) be as in the proof of Theorem 8.2.7.

As in the proof of Theorem 8.2.7, we will define, by induction on \( p \in T \), objects \( \delta_p, \beta_p, S_p \), and \( s_p \). Both \( \delta_p \) and \( \beta_p \) will be ordinals, with \( \delta_p < \kappa \). \( s_p \) will be a sequence such that \( \ell h(s_p) = \ell h(r_i h(p)) \). \( S_p \) will be an iteration tree of length \( 2 \ell h(p) + 1 \) on \( V \). Its tree ordering will be the restriction of \( S \). Its extenders will be \( E^p_m, m < 2 \ell h(p) \), its models will be \( M^p_m, m \leq 2 \ell h(p) \), and its embeddings will be \( j^p_{m,n}, m S n \leq 2 \ell h(p) \). Whenever \( p \subseteq q \in T \) then we will have \( S_p = S_q \upharpoonright 2 \ell h(p) + 1 \).

We introduce some notation. Let \( p \) and \( q \) belong to \( T \). Let \( k \leq k' \leq 2 \ell h(p) \). Let \( m \leq n \leq \ell h(q) \) such that \( r_m \subseteq r_n \). Let

\[
\begin{align*}
  i_{q,m,n} &= i_{(q|h(r_m),r_m),(q|h(r_n),r_n)}; \\
  \tilde{j}_{p,k,k'}^m &= i_{q,0,m}(j_{k,k'}^p); \\
  \tilde{N}_p^{q,m} &= i_{q,0,m}(M^p_{2\ell h(p)}); \\
  \tilde{\bar{U}}_p^{q,m} &= i_{q,0,m}(j_{0,2\ell h(p)}(U)); \\
  \tilde{\beta}_m &= \tilde{j}_{p,2m,2\ell h(p)}(\beta_{p,m}); \\
  s^q_m &= s_{q|h(r_m),r_m}.
\end{align*}
\]

The embedding \( \tilde{j}_{p,k,k'}^{q,m} \) is the image of \( j_{k,k'}^p \) in \( \text{Ult}(V; U_{q|h(r_m),r_m}) = i_{q,0,m}(V) \).

The class model \( \tilde{N}_p^{q,m} \) is the image of \( M^p_{2\ell h(p)} \) in \( i_{q,0,m}(V) \). In other words,

\[ \tilde{N}_p^{q,m} = i_{q,0,m}(j_{0,2\ell h(p)}(V)). \]

The tree \( \tilde{\bar{U}}_p^{q,m} \) is the image of \( j_{0,2\ell h(p)}^p(U) \) in \( i_{q,0,m}(V) \); i.e., it is the image of \( U \) in \( \tilde{N}_p^{q,m} \).
For $p = \emptyset$ we proceed exactly as in the proof of Theorem 8.2.7. We choose $\delta_0 > \gamma$ to be $\zeta_0 + 1$-reflecting in $\langle U \rangle$ relative to $\kappa$, and we let $\beta_0 = \zeta_0$.

Let $p \in T$. Let $k = \ell h(p)$. Assume that $\delta_{p'}, \beta_{p'}, S_{p'}$, and $s_{p'}$ are defined for all $p' \subseteq p$ so as to satisfy the conditions stated above and also the following conditions:

(i) for all $m \leq k$, $M^p_{2k}$ and $M^p_{2k-1}$ agree through $\delta_{p|m} + 1$;

(ii) for all $m \leq k$, the type $(\text{tp}_{\delta_{p|m}}^{\delta_{p|m}+1}) \hat{N}_{2m}^p \langle \hat{U}^p_{p}\rangle S_{p|m}$ is the same as $(\text{tp}_{\delta_{p|m}}^{\delta_{p|m}+1}) M^p_{2k-1} \langle \hat{U}^p_{p}\rangle s_{p|m}$;

(iii) for all $m \leq k$, $\delta_{p|m}$ is $(\beta_{p|m} + 1)$-reflecting in $\langle U^p_{p}\rangle S_{p|m}$ relative to $\kappa$ in $\hat{N}_{2m}^p$;

(iv) for all $m$ and $m'$ with $m \leq m'$, if $r_m \subseteq r_{m'}$ then

$$\beta_{m'}^p < i_{p,m,m'}(\beta_{m}^p);$$

(v) for all $m \leq k$, $s_{p|m}$ belongs to $\langle j_{0,2m-1}^p (U[p] \ell h(r_m)) \rangle$, and

$$r_m \subseteq r_k \rightarrow j_{2m-1,2k-1}^p (s_{p|m}) \subseteq s_p;$$

(vi) for all $m < k$,

$$\gamma < \delta_{p|m} < \text{crit}(E_{2m+1}^p) < \delta_{p|m+1},$$

and $\text{crit}(E_{2m}^p) = \delta_{p|m}$, where $2m + 1 = (2m + 1)_{\mathbb{S}}$.

Note that these conditions all hold for for $p = \emptyset$.

Let $q$ be any element of $T$ such that $p \subseteq q$ and $\ell h(q) = k + 1$.

Let $n$ be the largest number $\leq k$ such that $r_n \subseteq r_{k+1}$. Let $e = \ell h(r_n)$. Thus $\ell h(r_{k+1}) = e + 1$.

By (i) and the fact that $\text{crit}(i_{q,0,k+1}) > \kappa$, it follows that $\hat{N}_{p}^{q,k+1}$ and $M^p_{2k+1}$ agree through $\delta_{p|n} + 1$.

One readily computes that

$$\hat{N}_{p}^{q,k+1} = i_{q,n,k+1}(\hat{N}_{p}^{n});$$

$$\hat{U}_{p}^{q,k+1} = i_{q,n,k+1}(\hat{U}_{p}^{n});$$

$$j_{p,0,2k}^{q,k+1}(i_{q,n,k+1}(s_{p}^{n})) = i_{q,n,k+1}(j_{p,0,2k}^{p,n}(s_{p}^{n})).$$
Since $\text{crit}(i_{q,n,k+1}) > \kappa$, it follows that
\[
(tp_{\delta p/1}^{\delta p/1} N^p_{q,k+1} ((\hat{U}_p^q) - \gamma_{p,n,k} \langle s_n^p \rangle))
= i_{q,n,k+1}(tp_{\delta p/1}^{\delta p/1} N^p_{q,k+1} ((\hat{U}_p^q) - \gamma_{p,n,k} \langle s_n^p \rangle))
= (tp_{\delta p/1}^{\delta p/1} N^p_{q,k+1} ((\hat{U}_p^q) - \gamma_{p,n,k+1} \langle s_n^p \rangle))
\]
and so by (ii) this last is the same as $(tp_{\delta p/1}^{\delta p/1} M^p_{k+1} ((\hat{U}_p^q) - s_{p,n,n}))$.

From (iii) it similarly follows that $\delta_{p/1}$ is $(i_{q,n,k+1}((\beta_n^p) + 1))$-reflecting in the sequence $(\hat{U}_p^q + 1) - \gamma_{p,n,k+1} \langle i_{q,n,k+1} \langle s_n^p \rangle \rangle$ relative to $\kappa$ in $N^p_{q,k+1}$.

Since $j_{p,0,n}^q$ and $i_{q,0,n+1}$ fix $\kappa$, we have that $\kappa$ is Woodin in $N^p_{q,k+1}$.

Thus the hypotheses of the One-Step Lemma hold for $\kappa$ with
\[
M = \hat{N}^p_{q,k+1};
N = M^p_{2n-1};
\delta = \delta_{p/1};
\eta = \delta_p;
\beta = i_{q,n,k+1}(\beta_n^p) + 1;
\xi = i_{q,n,k+1}(\beta_n^p);\]
\[
\beta' = \zeta_0 + 1;
1,
\gamma = (U_p^q + 1) - j_{p,0,k}^q \langle i_{q,n,k+1} \langle s_n^p \rangle \rangle;
\gamma' = (j_{p,0,n}^q (U) - s_{p,n});
\chi(v) = "\kappa + v is the greatest ordinal."
\]

Let $\lambda$ and $E$ be given by the One-Step Lemma. Since $i_{q,n,k+1}$ fixes $\lambda$, $E$, and $\delta_p$, it follows that $E$ is a $(\delta_{p/1}, \lambda)$-extender in $M^p_{2k}$. Thus Theorem 7.3.2 gives that $\prod_{E^p_{2n-1}}(M^p_{2n-1}; \in)$ is wellfounded. Let then $\delta^*$, $\xi^*$, and $\gamma^*$ be given by the One-Step Lemma. By clause $(4^*)$ of the One-Step Lemma, $\xi^*$ = $\zeta_0$. Extend $S_p$ to an alternating chain that will be $S_q \upharpoonright 2k + 2$ by setting $E^q_{2k} = E$.

The ordinal $\delta^*$ we will call $\delta^*_q$. Set $s_q = (j_{2n+1,2k+1}^q(s_{p,n})) - y^*$. Set $s_q = (j_{2n-1,2k+1}^q(s_{p,n})) - y^*$.

Note that $M^q_m = M^p_m$, $E^q_m = E^p_m$, $j_{m,m'}^q = j_{m,m'}^p$, and $i_{m,m'}^q = i_{m,m'}^p$ whenever these equations make sense. We will use these identities without comment in the sequel.

A computation like the analogous one in the proof of Theorem 8.2.7 gives that
\[
x^* - y^* = (U_p^q + 1) - j_{p,0,k}^q (s_{k+1}^q).
\]
Since \( s^q_{k+1} \) belongs to \( i_{q,0,k+1}(U[q \upharpoonright e + 1]) \), the sequence \( j^q_{p,0,2k}(s^q_{k+1}) \) belongs to \( j^q_{p,0,2k}(i_{0,0,k+1}(U[q \upharpoonright e + 1])) = i_{q,0,k+1}(j^q_{0,2k})(i_{0,0,k+1}(U[q \upharpoonright e + 1])) = i_{q,0,k+1}(j^q_{0,2k}(U[q \upharpoonright e + 1])) = \hat{U}^q_{p,k+1} \). It follows by clause \((2^*)\) of the One-Step Lemma that \( s_q \in j^q_{0,2k+1}(U[q \upharpoonright e + 1]) \), and so the first clause of condition \((v)\) holds for \( q \). Since \( j^q_{2n-1,2k+1}(s_{q,n}) \subseteq s_q \), the second clause of condition \((v)\) holds for \( q \) the case \( m = n \).

We have that

(a) \( M^q_{2k+1} \) and \( \hat{N}^q_{p,k+1} \) agree through \( \delta_q' + 1 \);

(b) \( \left( \text{tp}_{\delta_q'}M^q_{2k+1}\right)(\langle j^q_{0,2k+1}(U) \rangle - s_q) = \left( \text{tp}_{\delta_p}M^q_{p,k+1}\right)(\langle \hat{U}^q_{p,k+1} \rangle - j^q_{p,0,2k}(s^q_{k+1})) \);

(c) \( \delta_q' \) is \( \zeta_0 \)-reflecting in \( \langle j^q_{0,2k+1}(U) \rangle - s_q \) relative to \( \kappa \) in \( M^q_{2k+1} \).

By (b) and (c) together with parts (b) and (c) of Lemma 8.2.3, we have that

\[
(b') \left( \text{tp}_{\delta_q'}M^q_{2k+1}\right)(\langle j^q_{0,2k+1}(U) \rangle - s_q) = \left( \text{tp}_{\delta_p}M^q_{p,k+1}\right)(\langle \hat{U}^q_{p,k+1} \rangle - j^q_{p,0,2k}(s^q_{k+1})) ;
\]

\[
(c') \delta_q' \) is \( \zeta_1 \)-reflecting in \( \langle j^q_{0,2k+1}(U) \rangle - s_q \) relative to \( \kappa \) in \( M^q_{2k+1} \).
\]

Since \( \kappa \) is Woodin in \( M^q_{2k+1} \), the hypotheses of the One-Step Lemma hold for \( \kappa \) with

\[
\begin{align*}
M &= M^q_{2k+1}; \\
N &= \hat{N}^q_{p,k+1}; \\
\delta &= \delta_q'; \\
\eta &= \delta_q'; \\
\beta &= \zeta_1; \\
\xi &= \zeta_0 + 1; \\
\beta' &= i_{q,n,k+1}(\hat{\beta}'_n); \\
x &= \langle j^q_{0,2k+1}(U) \rangle - s_q; \\
y &= \emptyset; \\
x' &= \langle \hat{U}^q_{p,k+1} \rangle - j^q_{p,0,2k}(s^q_{k+1}); \\
\chi(v) &= \text{“}v = v\text{”}.
\end{align*}
\]
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Let λ and E be given by the One-Step Lemma. From (a) above and the fact that \( \text{crit}(q;0,k+1) > \kappa > \delta'_q + 1 \), it follows that that \( M^q_{2k+1} \) and \( M^q_{2k} \) agree through \( \delta'_q + 1 \). Thus the model \( \prod E (M^q_{2k}; \in) \) is wellfounded. By the elementarity of \( q;0,k+1 \), \( \prod i;0,k+1(M^q_{2k}; \in) \) is wellfounded. Let then \( q^* \), \( \xi^* \), and \( y^* \) be given by the One-Step Lemma. Clause (2*) of the One-Step Lemma implies that \( y^* = \emptyset \) and that \( \xi^* \) is a successor ordinal. We finish the definition of \( S_q \) by setting \( E^q_{2k+1} = E \). Let \( \delta_q = \delta^* \). Let \( \beta_q \) be such that \( \beta_q + 1 = \xi^* \).

Observe that

\[
\text{Ult}(\hat{N}^{q,k+1}_q, E) = \text{Ult}(i_{q;0,k+1}(M^p_{2k}); E) = i_{q;0,k+1}(\text{Ult}(M^p_{2k}; E)) = i_{q;0,k+1}(M^q_{2k+2}) = \hat{N}^{q,k+1}_q
\]

and that

\[
i^{\hat{N}^{q,k+1}_q}_E = i_{q;0,k+1}(i^{M^p_{2k}}_E) = i_{q;0,k+1}(j^{q}_{2k,2k+2}) = j^{q}_{2k,2k+2}.
\]

Thus

\[
i^{\hat{N}^{q,k+1}_q}_E (\hat{U}^{q,k+1}_p) = ((i_{q;0,k+1}(j^{q}_{2k,2k+2}))(i_{q;0,k+1}(j^{p}_{0,2h(p)}(U))) = i_{q;0,k+1}(j^{q}_{0,2k+2}(U)) = \hat{U}^{q,k+1}_q
\]

and

\[
i^{\hat{N}^{q,k+1}_q}_E (j^{q}_{p;0,2k}(s^{q}_{k+1})) = j^{q,k+1}_{p;0,2k+2}(s^{q}_{k+1}) = j^{q,k+1}_{p;0,2k+2}(s^{q}_{k+1}).
\]

To verify inductive conditions (i)–(iii) for \( q \), let \( m \leq k + 1 \) be arbitrary.

Since \( \text{Ult}(\hat{N}^{q,k+1}_p; E) = i_{q;0,k+1}(M^q_{2k+2}) \), condition (i) for the case \( m = k + 1 \) follows from clause (1*) of the One-Step Lemma and the fact that \( \text{crit}(i_{q;0,k+1}) > \kappa \). The verification of (i) for \( m \leq k \) is exactly like the corresponding step in the proof of Theorem 8.2.7.

Clauses (2*) and (3*) of the One-Step Lemma and the identities above give that
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(ii') \((\text{tp}_{\delta q, \beta q + 1}^\kappa) \tilde{N}_q^{q,k+1} (\langle \hat{U}_q^{q,k+1} \rangle - \tilde{J}_q^{q,k+1}(s_q)) = (\text{tp}_{\delta q, \beta q + 1}^\kappa) \hat{N}_q^{q,k+1} (\langle \hat{J}_0^{q,0,2k+2}(U) \rangle - s_q)\);

(iii') \(\delta_q\) is \((\beta_q + 1)\)-reflecting in \(\langle \hat{U}_q^{q,k+1} \rangle - \tilde{J}_q^{q,k+1}(s_q)\) relative to \(\kappa\) in \(\hat{N}_q^{q,k+1}\).

(ii') and (iii') are just our inductive conditions (ii) and (iii) for \(q\).

Now let \(m \leq k\). By the fact that \(\text{crit} (\hat{\jmath}_q^{q,m}) = \delta_q > \delta_q|m\) and by our definitions, we have that

\begin{align*}
(1) & \quad \hat{J}_p^{q,m}(\delta_q|m) = \delta_q|m; \\
(2) & \quad \hat{J}_p^{q,m}(\hat{\jmath}_m) = \beta_q^m; \\
(3) & \quad \hat{J}_p^{q,m}(\hat{N}_p^{p,m}) = \hat{N}_q^{q,m}; \\
(4) & \quad \hat{J}_p^{q,m}(\hat{U}_p^{q,m}) = \hat{U}_q^{q,m}. \\
\end{align*}

The fact that \(\text{crit} (\hat{J}_p^{q,m}(\delta_q|m)) > \delta_q|m\) also implies that

\[(\text{tp}_{\delta_q|m}^{\hat{\jmath}_p^{q,m}, \beta_q^{m} + 1}) \hat{N}_p^{p,m} (\langle \hat{U}_p^{p,m} \rangle - \tilde{J}_p^{p,m}(s_q^p))\]

is fixed by \(\hat{J}_p^{q,m}(\delta_q|m)\). Thus we get condition (ii) for \(q\) just as we got the corresponding fact in the proof of Theorem 8.2.7. Condition (iii) for \(q\) follows similarly.

The inequality \(\xi^* < i_E^N\) of the One-Step Lemma gives us that

\[\beta_q < \xi^* < i_E^N (i_q^{q,n,k+1}(\beta_n^p)) = (i_q^{q,n,k+1}(\beta_n^p))(i_{q,n,k+1}(\tilde{\beta}_n^p)) = i_{q,n,k+1}(i_q^{q,n}(\beta_{2k,2}(k+1)))(\beta_n^p) = i_{q,n,k+1}(\beta_{qm,n,k+1}(\beta_n^p)) = i_{q,n,k+1}(\hat{\jmath}_q^{q,m}(\beta_n^p)).\]

Since \(\beta_q = \hat{\jmath}_q^{q,k+1}\), this gives us condition (iv) for \(q\) in the case \(m = n\). The other cases are handled as were the corresponding cases in the proof of Theorem 8.2.7.
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Assume that $m < m' \leq k$ and that $r_m \subseteq r_{m'}$. Note that $j_{q;2k,2k+2}^{m,m'}(\beta_m^q)$. Using condition (iv) for $p$, we get that

$$\beta_{m'}^q = j_{q;2k,2k+2}^m(\beta_m^p)$$

This in turn gives that $\beta_{m'}^q = i_{q;m,m'}(j_{q;2k,2k+2}^m(\beta_m^p)) = i_{q;m,m'}(j_{q;2k,2k+2}^m(\beta_m^p)) = i_{q;m,m'}(j_{q;2k,2k+2}^m(\beta_m^p))$.

The remaining case of condition (iv) for $q$ is $m \not= n$ and $m' = k + 1$. The proof for this case is analogous to the corresponding step in the proof of Theorem 8.2.7.

The proofs of conditions (v) and (vi) for $q$ are like the corresponding steps in the proof of Theorem 8.2.7.

We will show that the system

$$(\langle M_{2/n}^p \mid p \in T \rangle, \langle j_{m,2/n}^{p,h(p)} \mid p \in T \land m < lh(p) \in T \rangle)$$

gives an embedding normal form for the $T$-projection of $U^\dagger$.

Fix $x \in [T]$. Let $S_x$ be the iteration tree of length $\omega$ whose restrictions are $S_{x \mid n}$.

The proof that $M_{2/n}^{S_x}$ is wellfounded if $[U^\dagger(x)] \not= \emptyset$, is exactly like the corresponding step in the proof of Theorem 8.2.7.

Now assume that $[U^\dagger(x)] = \emptyset$. Thus $[U(x)] \not= \emptyset$. By Lemma 8.1.2, let $y \in ^{\omega} \omega$ be such that $M_{x,y}$ is wellfounded. For each $n \in \omega$, let $y \upharpoonright n = r_{m_n}$. Let $n < n' \in \omega$. Applying condition (iv) with $p = x \upharpoonright m_n'$, we get that

$$\beta_{m_n'}^x < i_{x \mid m_n',m_{n},m_{n'}}(\beta_{m_n'}^x).$$

Unpacking our definitions, we find that this means that

$$\beta_{x \mid m_n'} < i_{x \mid n,g(n),x \upharpoonright n',y \upharpoonright n'}(j_{2/m_n,2/m_{n'}}^x(i_{x \mid n,g(n),x \upharpoonright n',y \upharpoonright n'}(\beta_{x \mid m_n})).$$

This in turn gives that

$$\beta_{x \mid m_n'} < (i_{x \mid n,g(n),x \upharpoonright n',y \upharpoonright n'}(j_{2/m_n,2/m_{n'}}^x(i_{x \mid n,g(n),x \upharpoonright n',y \upharpoonright n'}(\beta_{x \mid m_n})).$$

Applying $i_{x \mid n,g(n),x \upharpoonright n'}$ to both sides of this inequality, we get that

$$i_{x \mid n,g(n),x \upharpoonright n'}(\beta_{x \mid m_n}) < (i_{x \mid n,g(n),x \upharpoonright n',y \upharpoonright n'}(j_{2/m_n,2/m_{n'}}^x(i_{x \mid n,g(n),x \upharpoonright n',y \upharpoonright n'}(\beta_{x \mid m_n})).$$
Applying \( i^{\bar{x},\bar{y}}_{(\bar{0},\bar{0})} (\mathcal{J}^{\bar{S}}_{2m,\alpha,\text{Even}}) \) to both sides, we get that

\[
(i^{\bar{x},\bar{y}}_{(\bar{0},\bar{0})} (\mathcal{J}^{\bar{S}}_{2m,\alpha,\text{Even}}))(i^{\bar{x},\bar{y}}_{(\bar{0},\bar{0})} (\mathcal{J}^{\bar{S}}_{2m,\alpha,\text{Even}})) < (i^{\bar{x},\bar{y}}_{(\bar{0},\bar{0})} (\mathcal{J}^{\bar{S}}_{2m,\alpha,\text{Even}}))(i^{\bar{x},\bar{y}}_{(\bar{0},\bar{0})} (\mathcal{J}^{\bar{S}}_{2m,\alpha,\text{Even}})).
\]

Thus the \( (i^{\bar{x},\bar{y}}_{(\bar{0},\bar{0})} (\mathcal{J}^{\bar{S}}_{2m,\alpha,\text{Even}}))(i^{\bar{x},\bar{y}}_{(\bar{0},\bar{0})} (\mathcal{J}^{\bar{S}}_{2m,\alpha,\text{Even}})), n \in \omega, \) form an infinite descending chain in the ordinals of the model \( i^{\bar{x},\bar{y}}_{(\bar{0},\bar{0})} (\mathcal{M}^{\bar{S}}_{\text{Even}}). \) This implies that \( \mathcal{M}^{\bar{S}}_{\text{Even}} \) is illfounded.

There are two ways to use our construction and the result just proved about it to get the homogeneous Souslinness of the \( T \)-projection of \( U^+ \). One way is to use it to get the homogeneity of \( U^+ \). We now show how this can be done.

For \( p \in T \) and \( m \leq \ell(h(p)) \), define \( \bar{\beta}^p_m \) as in the earlier construction: \( \bar{\beta}^p_m = j^{\bar{P}}_{2m,2\ell(h(p))}(\beta^p_m). \) Let \( \alpha > \beta^p \) for all \( p \). We can use the ordinals \( \bar{\beta}^p_m \) to define ultrafilters \( \mathcal{V}_p \) on \( \ell(h(p)) \alpha_\). For \( X \subseteq \ell(h(p)) \alpha_\), let

\[
X \in \mathcal{V}_p \leftrightarrow \langle \bar{\beta}^p_m \mid m < \ell(h(p)) \rangle \in j^{\bar{P}}_{0,2\ell(h(p))}(X).
\]

The arguments of the proofs of Theorems 8.2.6 and 8.2.7 show that the \( \mathcal{V}_p \) are compatible, that each \( \mathcal{V}_p \) is a \( \gamma \) complete ultrafilter on \( \ell(h(p)) \alpha \), and that whenever \( \mathcal{M}^{\bar{S}}_{\text{Even}} \) is wellfounded then the direct limit model for the system given by the \( \mathcal{V}_x \mid n_\), \( n \in \omega \), is wellfounded. Since \( \mathcal{M}^{\bar{S}}_{\text{Even}} \) is wellfounded whenever \( x \) belongs to the \( T \)-projection of \( U^+ \), we can verify homogeneity condition (3').

There is only one problem in showing that \( \langle \mathcal{V}_p \mid p \in T \rangle \) witnesses that \( U^+ \) is homogeneous for \( T \): we must prove \( (U^+ \upharpoonright \alpha)[p] \in \mathcal{V}_p \) for each \( p \). If we can do this, then we will have shown that each \( \mathcal{V}_p \) induces an ultrafilter on \( (U^+ \upharpoonright \alpha)[p] \) and that the corresponding system of ultrafilters witnesses the \( \gamma \)-homogeneity of \( U^+ \upharpoonright \alpha \).

To prove that \( (U^+ \upharpoonright \alpha)[p] \in \mathcal{V}_p \), we need to show that, for all \( m \) and \( m' \) such that \( m < m' < \ell(h(p)) \) and \( r_m \subseteq r_{m'} \),

\[
\bar{\beta}^p_m < (j^{\bar{P}}_{0,2\ell(h(p))}(i_{p,m,m'}))(\bar{\beta}^p_{m'}).
\]

To see in simpler terms what we need to show, fix \( m \) and \( m' \) with \( m < m' < \ell(h(p)) \) and \( r_m \subseteq r_{m'} \). By the definitions of the ordinals \( \bar{\beta}^p_n \), what we must show is that

\[
j^{\bar{P}}_{2m',2\ell(h(p))}(\bar{\beta}^p_{m'}) < (j^{\bar{P}}_{0,2\ell(h(p))}(i_{p,m,m'}))(j^{\bar{P}}_{2m,2\ell(h(p))}(\beta^p_m)).
\]
By the elementarity of \( j^p_{2m',2\ell h(p)} \), this is equivalent with the assertion that
\[
\beta_{p|m'} < (j^p_{0,2m'}(i_{p,m,m'}))(j^p_{2m,2m'}(\beta_{p|m})).
\]

Condition (iv) of our construction gives that
\[
(i_{p,0,m'}(j^p_{2m',2\ell h(p)}))(\beta_{p|m'}) < (i_{p,m,2m'}(j^p_{2m,2m'}(\beta_{p|m}))).
\]

By the elementarity of \( i_{p;m,m'} \) and of \( i_{0,0,m'}(j^p_{2m',2\ell h(p)}) \), this is equivalent with the assertion that
\[
\beta_{p|m'} < (i_{p,0,m'}(j^p_{2m,2m'}))(i_{p,m,m'}(\beta_{p|m})).
\]

We will prove a theorem showing that that, on the ordinals,

\[\begin{align*}
\text{a)} & \quad j^p_{0,2m'}(i_{p|m,m'}) \text{ agrees with } i_{p|m,m'}; \\
\text{b)} & \quad i_{0,p,m'}(j^p_{2m',2m'}) \text{ agrees with } j^p_{2m,2m'}; \\
\text{c)} & \quad i_{p|m,m'} \circ j^p_{2m,2m'} \text{ agrees with } j^p_{2m,2m'} \circ i_{p|m,m'}. 
\end{align*}\]

It will be easy to see that these facts give the equivalence of the two inequalities above.

Instead of proceeding directly to the proof of this result, we first illustrate the ideas in a simplified form.

**Lemma 8.3.1.** Let \( \kappa \) be a strong limit cardinal and let \( \mathcal{U} \) be a \( \kappa \)-complete ultrafilter on some set \( X \). Let \( \mathcal{V} \in V_\kappa \) be an ultrafilter a some set \( \bar{X} \) (which must also belong to \( V_\kappa \)). Then

\[\begin{align*}
\text{(a)} & \quad i_{\mathcal{V}}(i_{\mathcal{U}}) = i_{\mathcal{U}} \upharpoonright \text{Ult}(V; \mathcal{V}); \\
\text{(b)} & \quad i_{\mathcal{U}}(i_{\mathcal{V}}) = i_{\mathcal{V}} \upharpoonright \text{Ult}(V; \mathcal{U}); \\
\text{(c)} & \quad i_{\mathcal{U}} \circ i_{\mathcal{V}} \upharpoonright \text{ON} = i_{\mathcal{V}} \circ i_{\mathcal{U}} \upharpoonright \text{ON}.
\end{align*}\]

**Proof.** Let \( j = i_{\mathcal{V}} \) and \( i = i_{\mathcal{U}} \).

(a) We will prove two technical facts and deduce (a) from them.

(i) If \( Y \in \text{Ult}(V; \mathcal{V}) \) and \( Y \subseteq j(X) \), then
\[
Y \in j(\mathcal{U}) \iff \{z \in X \mid j(z) \in Y\} \in \mathcal{U}.
\]
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To prove (i), first let \( Y \) be an element of \( j(X) \). Let \( f : \bar{X} \to \mathcal{P}(X) \) be such that \( Y = \pi_\mathcal{V}([f]_\mathcal{V}) \). Let

\[
K = \{ x \in \bar{X} \mid f(x) \in \mathcal{U} \}.
\]

By Theorem 3.2.5, \( K \in \mathcal{V} \). Let \( Z = \bigcap_{x \in K} f(x) \).

The \( \kappa \)-completeness of \( \mathcal{U} \) implies that \( Z \in \mathcal{V} \). The following chain of implications show that \( j''Z \subseteq Y \).

\[
z \in Z \rightarrow (\forall x \in K) z \in f(x) \\
\rightarrow \{ x \mid z \in f(x) \} \in \mathcal{V} \\
\rightarrow j(z) \in \pi_\mathcal{V}([f]_\mathcal{V}) \\
\rightarrow j(z) \in Y.
\]

Now let \( Y \in \text{Ult}(\mathcal{V}; \mathcal{V}), Y \subseteq j(X), \text{ and } Y \notin j(\mathcal{U}) \). Then \( j(X) \setminus Y \in j(\mathcal{U}) \). By what we have already proved, \( \{ z \in X \mid j(z) \notin Y \} \in j(\mathcal{U}) \). Thus \( \{ z \in X \mid j(z) \in Y \} \notin \mathcal{U} \).

(ii) For each \( F \in \text{Ult}(\mathcal{V}; \mathcal{V}) \cap j(X)\text{Ult}(\mathcal{V}; \mathcal{V}) \), define \( \Phi(F) : X \to \text{Ult}(\mathcal{V}; \mathcal{V}) \) by

\[
(\Phi(F))(z) = F(j(z)).
\]

Then for each \( G \in ^X\text{Ult}(\mathcal{V}; \mathcal{V}) \) there is an \( F \in \text{Ult}(\mathcal{V}; \mathcal{V}) \cap j(X)\text{Ult}(\mathcal{V}; \mathcal{V}) \) such that \( F : j(X) \to \text{Ult}(\mathcal{V}; \mathcal{V}) \) and

\[
[\Phi(F)]_\mathcal{U} = [G]_\mathcal{U}.
\]

Let \( G \in ^X\text{Ult}(\mathcal{V}; \mathcal{V}) \). Choose for each \( z \in X \) an \( f_z \) such that \( G(z) = \pi_\mathcal{V}([f_z]_\mathcal{V}) = G(z) \). Define \( h : \bar{X} \to ^X\mathcal{V} \) by

\[
(h(x))(z) = f_z(x).
\]

Let

\[
F = \pi_\mathcal{V}([h]_\mathcal{V}).
\]
Clearly $F \in \text{Ult}(V; \mathcal{V}) \cap j(X)\text{Ult}(V; \mathcal{V})$. For any $z \in X$, 

$$
F(j(z)) = (\pi_{\mathcal{V}}([h]_{\mathcal{V}}))(j(z)) = \pi_{\mathcal{V}}([f]_{\mathcal{V}}) = G(z).
$$

To prove (a), let $\Phi$ be as in (ii). For $F \in \text{Ult}(V; \mathcal{V}) \cap j(X)\text{Ult}(V; \mathcal{V})$, set

$$
\Phi^*([F]_j) = [\Phi(F)]_U.
$$

To see that $\Phi^*$ is well-defined, note that

\[ [F_1]_j = [F_2]_j \iff \{ y \in j(X) \mid F_1(y) = F_2(y) \} \in j(U) \]
\[ \iff \{ z \in X \mid F_1(j(z)) = F_2(j(z)) \} \in U \]
\[ \iff \{ z \in X \mid (\Phi(F_1))(z) = (\Phi(F_2))(z) \} \in U \]
\[ \iff [\Phi(F_1)]_U = [\Phi(F_2)]_U. \]

The second of the biconditionals is a consequence of (i). A similar chain of equivalences shows that

$$
\Phi^* : \prod_{j \in U} \text{Ult}(V; \mathcal{V}) \prec \prod_U \text{Ult}(V; \mathcal{V}).
$$

(See the proof of Lemma 8.3.3 below.) By (i), the elementary embedding $\Phi^*$ is a surjection, and so it is an isomorphism. This in turn gives that $\pi_U \circ \Phi^* \circ (\pi^U_{\text{Ult}(V; \mathcal{V})})^{-1}$ is an isomorphism between $(j(i))(\text{Ult}(V; \mathcal{V}))$ and $i(\text{Ult}(V; \mathcal{V}))$. Since these two classes are transitive, they must be identical, and the isomorphism must be the identity. If $w \in \text{Ult}(V; \mathcal{V})$, we have

\[ (j(i))(w) = \pi^U_{\text{Ult}(V; \mathcal{V})}(\text{Ult}(V; \mathcal{V})) \]
\[ = \pi_U(\Phi^*([c_w]_j)) \]
\[ = \pi_U([c_w]_U) \]
\[ = i(w). \]

Here we have ambiguously used “$c_w$” for two distinct functions with constant value $w$.

The proof of (a) is now complete.
(b) The proof of (b) is simpler. Since $\mathcal{U}$ is $\kappa$-complete and since $X \in V_\kappa$, Lemma 3.2.11 implies that $V$ and $\text{Ult}(V; \mathcal{U})$ have the same functions $f : X \to \text{Ult}(V; \mathcal{U})$. Since $i(\mathcal{V}) = \mathcal{V}$, we have that $\prod_{(i(\mathcal{V}))}^\mathcal{U} \text{Ult}(V; \mathcal{U})$ and $\prod_{\mathcal{V}} \text{Ult}(V; \mathcal{U})$ are identical and that $i(j) \upharpoonright \text{Ult}(V; \mathcal{U}) = j \upharpoonright \text{Ult}(V; \mathcal{U})$.

(c) Let $\alpha \in \text{ON}$. By (a) and by the elementarity of $j$, we have that $i(j(\alpha)) = j(i(\alpha)) = j(i(\alpha))$.

(We could just as well have deduced (c) from (b) and the elementarity of $i$.)

\[ \square \]

From now through Theorem 8.3.6, let $\kappa$ be a strong limit cardinal and let $T \in V_\kappa$ be an iteration tree of length $\leq \omega$ on $V$ with tree ordering $T$, extenders $E_n$, $n + 1 < \ell h(T)$, models $M_n$, $n < \ell h(T)$, and embeddings $j_{m,n}$, $m T n < \ell h(T)$. For $n + 1 < \ell h(T)$, let $\delta_n$ and $\lambda_n$ be such that $E_n$ is a $(\delta_n, \lambda)$-extender in $M_n$.

The following lemma is a generalization of assertion (i) of the proof of Lemma 8.3.1, with the embedding $i_\mathcal{V}$ replaced by the embeddings $j_{0,n}$. (See Exercise 8.3.1, however.) Because the $j_{0,n}$ are the embeddings of an iteration as opposed to a single ultrapower, our proof is by induction. The individual steps are similar to the proof of Lemma 8.3.1. No extra problems are caused by (1) the fact that the $j_{m+1,T,m+1}$ come from extenders rather than ultrafilters or (2) the fact that the ultrapowers are of models to which the extenders to not belong.

**Lemma 8.3.2.** Let $\mathcal{U}$ be a $\kappa$-complete ultrafilter on a set $X$. Let $n < \ell h(T)$. If $Y \in M_n$ and $Y \subseteq j_{0,n}(X)$, then

\[ Y \in j_{0,n}(\mathcal{U}) \leftrightarrow \{ z \in X \mid j_{0,n}(z) \in Y \} \in \mathcal{U}. \]

**Proof.** We prove the lemma (for fixed $\mathcal{U}$) by induction. It trivially holds for $n = 0$. Suppose that it holds for all $n' \leq n$. Let $m = (n + 1)\overline{T}$.

First let $Y$ be any element of $j_{0,n+1}(\mathcal{U})$. Let $a$ and $f$ be such that $Y = \pi_{E_n}^M([a, f]_{E_n}^M)$. Let

\[ K = \{ x \in [\delta_n]^a \mid f(x) \in j_{0,m}(\mathcal{U}) \}. \]
By Theorem 6.3.15, we have that \( K \in (E_n)_a \). By our induction hypothesis for \( m \), choose for each \( x \in K \) a set \( Z_x \subseteq X \) such that

\[
Z_x \in \mathcal{U} \land (\forall z \in Z_x) \ j_{0,m}(z) \in f(x).
\]

Let \( Z = \bigcap_{x \in K} (Z_x) \). The \( \kappa \)-completeness of \( \mathcal{U} \) implies that \( Z \in \mathcal{U} \). The following chain of implications shows that \( j_{0,n+1}^\mathcal{U} Z \subseteq Y \).

\[
z \in Z \rightarrow (\forall x \in K) z \in Z_x
\rightarrow (\forall x \in K) j_{0,m}(z) \in f(x)
\rightarrow \{ x \mid j_{0,m}(z) \in f(x) \} \in (E_n)_a
\rightarrow j_{m,n+1}(j_{0,m}(z)) \in \pi_{E_n}^m([a,f]_{E_n}^m)
\rightarrow j_{0,n+1}(z) \in Y.
\]

Now let \( Y \in M_n \), \( Y \subseteq j_{0,n+1}(x) \), and \( Y \notin j_{0,n+1}(\mathcal{U}) \). Then \( j_{0,n+1}(X) \setminus Y \in j_{0,n+1}(\mathcal{U}) \). By what we have already proved, \( \{ z \in X \mid j_{0,n+1}(z) \notin Y \} \) belongs to \( \mathcal{U} \). Thus \( \{ z \in X \mid j_{0,n+1}(z) \in Y \} \notin \mathcal{U} \). □

The next lemma generalizes assertion (ii) of the proof of Lemma 8.3.1 in the way that Lemma 8.3.2 generalizes (i). The proof is in two ways more complicated than the proof of Lemma 8.3.1: Like the proof of Lemma 8.3.2, it proceeds by induction. (But see Exercise 8.3.1.) The fact that the \( E_n \) are extenders rather than ultrafilters occasions an additional use of the \( \kappa \)-completeness of \( \mathcal{U} \).

**Lemma 8.3.3.** Let \( \mathcal{U} \) be a \( \kappa \)-complete ultrafilter on a set \( X \). Let \( n < \text{th}(\mathcal{T}) \). For each \( F \in M_n \cap j_{0,n}(X)M_n \), define \( \Phi_n(F) : X \rightarrow M_n \) by

\[
(\Phi_n(F))(z) = F(j_{0,n}(z)).
\]

Then for each \( G \in X M_n \) there is an \( F \in M_n \cap j_{0,n}(X)M_n \) such that \( F : j_{0,n}(X) \rightarrow M_n \) and

\[
[\Phi_n(F)]_{j_\mathcal{U}} = [G]_{j_\mathcal{U}}.
\]

**Proof.** We prove the lemma by induction. The case \( n = 0 \) is trivial. Assume that the lemma holds for \( \mathcal{U} \) for all \( n' \leq n \). Let \( m = (n + 1)_T \).

Let \( G \in X M_{n+1} \). Choose for each \( z \in X \) a pair \( \langle a_z, f_z \rangle \) such that \( G(z) = \pi_{E_n}^m([a_z, f_z]_{E_n}^m) \). By the \( \kappa \)-completeness of \( \mathcal{U} \), let \( a \in [\lambda_n]^{<\omega} \) be such that \( \{ z \mid a_z = a \} \in \mathcal{U} \). Thus

\[
\{ z \mid G(z) = \pi_{E_n}^m([a, f_z]_{E_n}^m) \} \in \mathcal{U}.
\]
Define $G^* : X \to M_m$ by

$$G^*(z) = f_z.$$  

By our induction hypothesis for $M$, let $F^* \in M_m \cap \beta_{0,m}(X)$ be such that

$$[\Phi_m(F^*)]_U = [G^*]_U.$$  

Define $h : [\delta_n]^{[a]} \to M_m$ by letting $h(x) \in j_{0,m}(X)M_m$ be given by

$$(h(x))(y) = (F^*y)(x).$$  

Let $F = \pi_{E_n}^M([a, h]_E^M)$. Clearly $F$ belongs to $M_{n+1} \cap j_{0,n}(X)M_{n+1}$. We must show that

$$\{z \in X | F(j_{0,m}(z)) = G(z)\} \in U.$$  

We know that

$$\{z \in X | \{z | G(z) = \pi_{E_n}^M([a, f_z]_E^M)\} \in U \land F^*(j_{0,m}(z)) = G^*(z)\} \in U.$$  

Let $z$ be any member of this set. For any $x \in j_{0,m}(X)$, we have that

$$(h(x))(j_{0,m}(z)) = (F^*(j_{0,m}(z)))(x) = (G^*(z))(x) = f_z(x).$$  

Hence

$$F(j_{0,n+1}(z)) = F(j_{m,n+1}(j_{0,m}(z))) = \pi_{E_n}^M([a, f_z]_E^M) = G(z).$$

\[ \square \]

The next lemma and its corollary generalize part (a) of Lemma 8.3.1. Their proof is essentially the same as the proof of Lemma 8.3.1 from (i) and (ii).

**Lemma 8.3.4.** Let $U$ be a $\kappa$-complete ultrafilter on a set $X$. Let $i = i_U$. Let $n < \ell h(T)$. Define $\Phi_n$ as in the statement of Lemma 8.3.3. For $F \in j_{0,n}(X)M_n$, set

$$\Phi^*_n([F]_{j_{0,n}(U)}) = [\Phi_n(F)]_U.$$
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Then $\Phi^*_n$ is well-defined and

$$\Phi^*_n : \prod_{j_0, n(\mathcal{U})} M_n \cong \prod_{\mathcal{U}} M_n.$$ 

**Proof.** Let $\varphi(v_1, \ldots, v_k)$ be a formula of the language of set theory and let $F_1, \ldots, F_k$ be elements of $M_n \cap j_0, n(X)M_n$. We have the following chain of equivalences, where Lemma 8.3.2 is used to get the third line from the second:

$$\prod_{j_0, n(\mathcal{U})} M_n \models \varphi[[F_1]_{j_0, n(\mathcal{U})}, \ldots, [F_k]_{j_0, n(\mathcal{U})}]$$

$$\iff \{ y \in j_0, n(X) \ | \ M_n \models \varphi[F_1(y), \ldots, F_k(y)] \} \in j_0, n(\mathcal{U})$$

$$\iff \{ z \in X \ | \ M_n \models \varphi[(\Phi_n(F_1))(z), \ldots, (\Phi_n(F_k))(z)] \} \in \mathcal{U}$$

$$\iff \prod_{\mathcal{U}} M_n \models \varphi[[\Phi_n(F_1)]_{\mathcal{U}}, \ldots, [\Phi_n(F_k)]_{\mathcal{U}}].$$

Taking $v_1 = v_2$ for $\varphi$, we see that $\Phi^*_n$ is well-defined. Taking $\varphi$ as arbitrary, we then see that $\Phi^*_n : \prod_{j_0, n(\mathcal{U})} M_n \prec \prod_{\mathcal{U}} M_n$.

By Lemma 8.3.3, the elementary embedding $\Phi^*_n$ is a surjection, and so it is an isomorphism.

**Corollary 8.3.5.** Let $\mathcal{U}$ be a $\kappa$-complete ultrafilter on a set $X$. Let $i = i_\mathcal{U}$. Let $n < \ell(\mathcal{T})$. Then

(i) $(j_{0, n}(i))(M_n) = i(M_n)$;

(ii) $j_{0, n}(i) = i \upharpoonright M_n$.

**Proof.** (i) follows immediately from Lemma 8.3.4. For (ii), note that, for $w \in M_n$,

$$((j_{0, n}(i))(w) = \pi_{j_0, n(\mathcal{U})}^M([c_w]_{j_0, n(\mathcal{U})})$$

$$= \pi_{\mathcal{U}}(\Phi_n^([c_w]_{j_0, n(\mathcal{U})}))$$

$$= \pi_{\mathcal{U}}([c_w]_{\mathcal{U}})$$

$$= i(w),$$

where we have once more ambiguously used "$c_w$" for two distinct constant functions.

$\square$
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The theorem that follows is analogous to Lemma 8.3.1. Besides the extra complications due to the replacement of the \( V \) of Lemma 8.3.1 by the \( j_{0,n} \), there are other complications. Part (b) of the theorem asserts that \( i \) acts trivially on the \( j_{m,n} \), not just on the \( j_{0,n} \). Part (a) involves not just \( \mathcal{U} \) but also another ultrafilter \( \mathcal{U}' \) that projects to \( \mathcal{U} \). It says that the \( j_{0,n} \) act trivially on the canonical embedding \( i^* : \text{Ult}(V; \mathcal{U}) \prec \text{Ult}(V; \mathcal{U}') \).

**Theorem 8.3.6.** Let \( \mathcal{U} \) and \( \mathcal{U}' \) be \( \kappa \)-complete ultrafilters on sets \( X \) and \( X' \) respectively. Let \( i = i_{\mathcal{U}} \) and let \( i' = i_{\mathcal{U}'} \). Assume that \( \mathcal{U}' \) projects to \( \mathcal{U} \) by \( \chi : X' \to X \). (See page 199.) Let \( i^* = i_{\mathcal{U}';\mathcal{U}} : \text{Ult}(V; \mathcal{U}) \prec \text{Ult}(V; \mathcal{U}') \). Let \( mTn < \ell \h_\ell(T) \). Then

\[
\begin{align*}
(a) \ j_{0,n}(i^*) & = i^* \upharpoonright j_{0,m}(\text{Ult}(V; \mathcal{U})) = i^* \upharpoonright i(M_n); \\
(b) \ i(j_{m,n}) & = j_{m,n} \upharpoonright i(M_m) = j_{m,n} \upharpoonright j_{0,m}(\text{Ult}(V; \mathcal{U})); \\
(c) \ i^* \circ j_{m,n} \upharpoonright \text{ON} & = j_{m,n} \circ i^* \upharpoonright \text{ON}.
\end{align*}
\]

**Proof.** Note first that

\[
\begin{align*}
 j_{0,n}(\text{Ult}(V; \mathcal{U})) & = j_{0,n}(i(V)) \\
 & = (j_{0,n}(i))(j_{0,n}(V)) \\
 & = (j_{0,n}(i))(M_n) \\
 & = i(M_n),
\end{align*}
\]

where the last line follows from its predecessor by Corollary 8.3.5.

(a) Define \( \Phi_n \) as in the statement of Lemma 8.3.3. Similarly define \( \Phi'_n \) from \( \mathcal{U}' \). Let \( x \in j_{0,n}(\text{Ult}(V; \mathcal{U})) = i(M_n) \). By Let \( F \in j_{0,n}(X)M_n \) be such that \( x = \pi_{j_{0,n}(\mathcal{U})}^M([F]_{j_{0,n}(\mathcal{U})}) \). By Lemma 8.3.4,

\[
\pi_{j_{0,n}(\mathcal{U})}^M([F]_{j_{0,n}(\mathcal{U})}) = \pi_{\mathcal{U}}([\Phi(F)]_{\mathcal{U}}).
\]

We have that

\[
(j_{0,n}(i^*))(\pi_{j_{0,n}(\mathcal{U})}^M(x)) = (j_{0,n}(i^*))(\pi_{j_{0,n}(\mathcal{U})}^M([F]_{j_{0,n}(\mathcal{U})})) = \pi_{j_{0,n}(\mathcal{U})}^M([F \circ j_{0,n}(\chi)]_{j_{0,n}(\mathcal{U}')}).
\]

But we also have that

\[
i^*(x) = i^*\pi_{\mathcal{U}}([\Phi(F)]_{\mathcal{U}})
\]
\[ \pi_U([\Phi(F) \circ \chi]_{U'}) = \pi_U([F \circ j_{0,n} \circ \chi]_{U'}) = \pi_U([\Phi'(F \circ j_{0,n}(\chi))]_{U'}) = \pi_{M_{j_{0,n}(U')}}([F \circ j_{0,n}(\chi)]_{M_{j_{0,n}(U')}}). \]

(b) We proceed by induction on \( n \). Suppose (b) holds for all \( m' \) and \( n' \) such that \( m'Tn' \leq n \). Let \( \bar{m} = (n + 1)\bar{\pi} \). Since \( U \) is \( \kappa \)-complete, it follows from Lemma 3.2.11 that \( V \) and \( \text{Ult}(V;U) \) have the same functions from \( ^{< \omega} \kappa \) to \( \text{Ult}(V;U) \). By the elementarity of \( j_{0,\bar{m}}, \) this implies that \( M_{\bar{m}} \) and \( j_{0,\bar{m}}(\text{Ult}(V;U)) \) have the same functions from \( ^{< \omega} j_{0,\bar{m}}(\kappa) \) to \( j_{0,\bar{m}}(\text{Ult}(V;U)) \), i.e., to \( i(M_{\bar{m}}) \). It follows that \( M_{\bar{m}} \) and \( i(M_{\bar{m}}) \) have the same functions from \( ^{< \omega} \delta_n \) to \( i(M_{\bar{m}}) \). Since \( i(E_n) = E_n \) and since \( i(M_{\bar{m}}) \) is the domain of \( i(j_{m,n+1}) \), we get that
\[
i(j_{\bar{m},n+1}) = j_{\bar{m},n+1} \upharpoonright i(M_{\bar{m}}).
\]
To finish the proof of (b) for \( n+1 \), let \( mTn+1 \). By our induction hypothesis, we have that
\[
i(j_{m,\bar{m}}) = j_{m,\bar{m}} \upharpoonright i(M_m).
\]
Thus
\[
i(j_{m,n+1}) = i(j_{m,n+1} \circ j_{m,\bar{m}}) = i(j_{m,n+1}) \circ i(j_{m,\bar{m}}) = (j_{m,n+1} \upharpoonright i(M_{\bar{m}})) \circ i(j_{m,\bar{m}}) = (j_{m,n+1} \upharpoonright i(M_{\bar{m}})) \circ (j_{m,\bar{m}} \upharpoonright i(M_m)) = (j_{m,n+1} \circ j_{m,\bar{m}}) \upharpoonright i(M_{\bar{m}}) = j_{m,n+1} \upharpoonright i(M_m).
\]

(c) Let \( \alpha \in \text{ON} \). By two instances of (a) and the elementarity of \( j_{m,n} \), we have that
\[
i^*(j_{m,n}(\alpha)) = (j_{0,n}(i^*) (j_{m,n}(\alpha))) = j_{m,n}(i^*(\alpha)) = j_{m,n}(i^*(\alpha)).
\]
\[\square\]
Let us now apply Theorem 8.3.6 to show that the construction of the beginning of this section gives the homogeneity of $U^\dagger$. In the notation of that construction and the subsequent discussion, let $p \in T$ and let $m < m' < \ell h(p)$ with $r_m \subseteq r_{m'}$. By the discussion on page 471, it the $\gamma$-homogeneity of $U \upharpoonright \alpha$ will follow if we can prove that

$$\beta_{p|m'} < (j_{0,2m'}^p(i_{p;m,m'}))(j_{2m,2m'}^p(\beta_{p|m})).$$

By condition (iv) of our construction and by the discussion on page 471, we have that

$$\beta_{p|m'} < (i_{p:0,m'}(j_{2m,2m'}^p))(i_{p;m,m'}(\beta_{p|m})).$$

By part (b) of Theorem 8.3.6, we get that

$$\beta_{p|m'} < j_{2m,2m'}^p(i_{p;m,m'}(\beta_{p|m})).$$

By part (c) of Theorem 8.3.6, we get that

$$\beta_{p|m'} < i_{p;m,m'}(j_{2m,2m'}^p(\beta_{p|m})).$$

Finally we apply part (a) of Theorem 8.3.6 to get the desired inequality.

The second way to use our construction to get the homogeneous Souslin-ness of $U^\dagger$ is due to Katrin Windßus.

Let $T$ be any game tree and let $A \subseteq [T]$. Let $\lambda$ be an infinite cardinal number. Say that $A$ of $U^\dagger$ has a $\lambda$-closed embedding normal form, if there is a system

$$((M_p \mid p \in T), (k_{p_1;p_2} \mid p_1 \subseteq p_2 \in T))$$

witnessing that $A$ has an embedding normal form and such that

$$(\forall p \in T)^\lambda M_p \subseteq M_p.$$ 

Here is Windßus’ theorem:

**Theorem 8.3.7.** Let $T$ be a game tree, let $A \subseteq [T]$, and let

$$((M_p \mid p \in T), (k_{p_1;p_2} \mid p_1 \subseteq p_2 \in T))$$

witness that $A$ has a $2^{\aleph_0}$-closed embedding normal form. Let $\gamma$ be a cardinal number such that $\gamma \geq \text{crit } k_{p_1,p_2}$ for all $p_1$ and $p_2 \in T$ with $p_1 \subseteq p_2$.

Then $A$ is $\gamma$-homogeneously Souslin.
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Proof. For \( p \in T \), let \( A_p \) be the \( T \)-projection of \( T[p] \). Let \( \alpha \in \text{Ord} \). Let \( U_\alpha \) be the set of all \( \{p,u\} \) such that \( p \in T \), \( u \in \ell h(p)V \), and, for all \( m \) and \( n \) smaller than \( \ell h(p) \),

(a) \( u(m) : [T[x \upharpoonright m]] \setminus A_{p|m} \rightarrow \alpha \);

(b) \( m < n \rightarrow (\forall x \in [T[x \upharpoonright n]] \setminus A_{p|n})(u(n))(x) < (u(m))(x) \).

We first show how to define a tree whose \( T \)-projection is \( A \). This construction will make no use of the given embedding normal form.

Let \( \alpha \) be infinite. Let \( \langle x,f \rangle \in [U_\alpha] \). We show that \( A \) is the \( T \)-projection of \( U_\alpha \). Let \( x \in [T] \). First assume \( \langle x,f \rangle \in [U] \) and that \( x \in [T] \setminus A \). Then \( x \in \text{domain} f(n) \) for every \( n \in \omega \) and \( (f(n))(x) < (f(m))(x) \) whenever \( m < n \in \omega \). This is a contradiction. Now assume that \( x \in A \). For \( n \in \omega \) and \( y \in [T[x \upharpoonright m]] \setminus A_{p|m} \), let \( (f(m))(y) \) be \( n - m \), where \( n \) is the least number such that \( x(n) \neq y(n) \). Then \( \langle x,f \rangle \in [U] \).

We now use our embedding normal form to prove that \( U_\alpha \) is homogeneous.

For each \( x \in [T] \), let \( (\mathcal{M}_x, \langle k^x_{z|m} \mid m \in \omega \rangle) \) be the direct limit of \( (\langle M_{z|m} \mid M \in \omega \rangle, k_{z|m,x|n}) \) whenever \( m < n \in \omega \).

Let \( X \in [T] \setminus A \). Let \( \langle z_n \mid n \in \omega \rangle \) be an infinite descending sequence of ordinals of the illfounded model \( \mathcal{M}_x \). For each \( n \in \omega \), let \( \beta^x_n \) and \( k_n \) be such that \( z_n = k^x_{z|k_n}(\beta^x_n) \). We may assume that the sequence \( \langle k_n \mid n \in \omega \rangle \) is strictly increasing. Multiplying the given \( \beta^x_n \) by \( \omega \) if necessary, we may assume that each \( \beta^x_n \) is a limit ordinal. Filling in if necessary, we may assume that \( k_n = n \) for all \( n \). Thus the sequence \( \beta^x_n \) has the property that

\[
(\forall m \in \omega)(\forall n \in \omega)(m < n \rightarrow \beta^x_n < k^x_{z|m,x|n}(\beta^x_n)).
\]

Let \( \alpha \) be an ordinal larger than all the ordinals \( \beta^x_n \).

For each \( p \in T \), let \( g_p : [T[p]] \setminus A \rightarrow \text{Ord} \) be defined by setting

\[
g_p(x) = \beta^x_{\ell h(p)}
\]

for all \( x \in [T[p]] \setminus A \). Since \( 2^\alpha M_p \subseteq M_p \), the function \( g_p \) belongs to \( M_p \).

For \( p \in T \), let

\[
U_p = \{X \subseteq (U_\alpha)[p] \mid \langle k_{p|m,p}(g_{p|m}) \mid m < \ell h(p) \rangle \in k_{p,\ell h(p)}(X)\}. 
\]

To see that \( U[p] \in U_p \), observe that, for \( m \) and \( n \) smaller than \( \ell h(p) \) and \( x \in [T[p]] \setminus A \),

\[
m < n \rightarrow \beta^x_n < k^x_{z|m,x|n}(\beta^x_m) 
\]

\[
\rightarrow k_{p|m,p}(\beta^x_n) < k_{p|m,p}(\beta^x_m) 
\]

\[
\rightarrow k_{p|m,p}(g_{p|m}(x)) < k_{p|m,p}(g_{p|m}(x)). 
\]
By arguments like those in the last parts of the proofs of Theorems 8.2.6 and 8.2.7, one can show that \( \langle U_p \mid p \in T \rangle \) witnesses that \( U_\alpha \) is \( \gamma \)-homogeneous.

Using Theorem 6.3.7, it is easy to prove a strengthened One-Step Lemma in which it is demanded that if the models \( \gamma \leq \delta, \quad 2^{\aleph_0} M \subseteq M, \) and \( 2^{\aleph_0} N \subseteq N, \) then \( 2^{\aleph_0} (\text{Ult}(N; E)) \subseteq \text{Ult}(N; E). \) (In [Martin and Steel, 1989], this is done, but with the unimportant difference that \( \aleph_1 \) replaces \( 2^{\aleph_0}. \)) If we do the construction of the beginning of this section using this version of the One-Step Lemma, then all the \( \mathcal{M}^S_p \) will satisfy \( 2^{\aleph_0} \mathcal{M}^S_p \subseteq \mathcal{M}^S_p. \) Thus the construction will yield a \( 2^{\aleph_0} \)-closed embedding normal form for the \( T \)-projection of \( U^\dagger. \) Thus Theorem 8.3.7 will yield Theorems 8.2.8 and 8.2.9.

In [Koepke, 1998], the Windßus’ construction is used to prove the results of this chapter without any use of homogeneous trees. This is done by propagating directly the property of having an embedding normal form.

The hypothesis of \( 2^{\aleph_0} \)-closure cannot be omitted from Theorem 8.3.7. This seems to have been noted by several people, including Menachem Magidor and Koepke. See Exercise 8.3.2.

In [Neeman, 1995], [Neeman, 2002], [Neeman, 2004], [Neeman, 2010], and [Neeman, 2007], Neeman develops machinery that starts with something like the Martin-Steel construction but goes far beyond it. With this machinery, he is able to prove determinacy for large classes of games. These include not just larger classes of ordinary \( \omega \)-length games, but also classes of games with transfinitely many moves, even certain games of length \( \omega_1. \) Furthermore his theorems have large-cardinal hypotheses that are provably optimal.

Exercise 8.3.1. Let \( T \) be an iteration tree on \( M \) and let \( n \in \omega. \) Prove that there is an extender \( E \) in the sense of Exercise 6.1.2 such that \( E \in V_\kappa, \) and \( j_{T,n}^E = i_E. \) Using this result one can eliminate the use of induction from Lemmas 8.3.2 and 8.3.3.

Hint. Let \( \lambda \) be such that each \( E^T_m, \quad m < n, \) is in \( M^T_m \) a \( (\delta, \lambda') \) extender for some \( \lambda' \leq \lambda. \) Prove inductively that every element of \( M_n \) is of the form \( (j_{0,n}^T (f))(a) \) for some \( a \in [\lambda]^{<\omega}. \) (The basic fact is that

\[
[b, g]_{M^T_n}^{(m+1)} = (j^T_{(m+1),n}) (g)(b).
\]

Use this fact to deduce that \( j_{0,n} = i_E, \) where \( E \) the (generalized) \( (\text{crit} (j_{0,n}), \lambda)- \) extender derived from \( j_{0,n}. \)
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Exercise 8.3.2. Assume that there is a measurable cardinal and prove that every $A \subseteq \omega \omega$ has an embedding normal form.

Hint. (The following example is from [Koepke, 1998]. The result was probably first proved by Magidor.) Let $\alpha \mapsto x_\alpha$ be a one-one correspondence between $2^{\aleph_0}$ and $\omega \omega$. Let $A \subseteq \omega \omega$. Let $\kappa$ be measurable and let $U$ be a uniform normal ultrafilter on $\kappa$. Let $i = i_U$. Let $N = \text{Ult}_{2^{\aleph_0}}(V; U)$. For $s \in <\omega \omega$, and $\alpha \leq 2^{\aleph_0}$, define $j^s_{\alpha, \beta} : N \prec N$ inductively as follows. Let $j^s_0$ be the identity. Let

$$j^s_{\alpha, \alpha + 1} = \begin{cases} i_{\omega \alpha} & \text{if } s \subseteq x_\alpha \text{ and } x_\alpha \not\in A; \\ \text{the identity} & \text{otherwise.} \end{cases}$$

(See §3.3 for definitions.) For $\gamma < \alpha$ let

$$j^s_{\gamma, \alpha + 1} = j^s_{\alpha, \alpha + 1} \circ j^s_{\gamma, \alpha}.$$ 

For limit ordinals $\lambda \leq 2^{\aleph_0}$, let

$$(N, \{j^s_{\alpha, \lambda} \mid \alpha < \lambda\})$$

be the direct limit of the system

$$(\langle N \mid \alpha < \lambda\rangle, \{j^s_{\alpha, \beta} \mid \alpha \leq \beta < \lambda\}),$$

which one can verify inductively to be directed. Now let $M_0 = V$ and $M_s = N$ for $s \in <\omega \omega \setminus \emptyset$. For $s \subseteq t \in <\omega \omega$ with $\ell h(t) = \ell h(s) + 1$, let

$$k^s_{s, t} = \begin{cases} i_{0, 2^{\aleph_0}} & \text{if } s = \emptyset; \\ j^s_{0, 2^{\aleph_0}} & \text{if } s \not= \emptyset. \end{cases}$$

Define $k^s_{s, t}$ for other $s \subseteq t \in <\omega \omega$ by commutativity. Show that

$$(\langle M_s \mid s \in <\omega \omega\rangle, \{j^s_{s, t} \mid s \subseteq t \in <\omega \omega\})$$

gives an embedding normal form for $A$.

Exercise 8.3.3. Recall that a cone of degrees of unsolvability is the set of all degrees of unsolvability above some particular degree. For any degree of unsolvability $d$, let $T(d)$ be the set of sentences true in class models $(L[x])[H]$ where $x$ has degree $d$ and $H$ is $\text{Coll}(\omega, \omega_1^{L[x]})$-generic over $L[x]$. (See page 537.)

Assume that, for each positive integer $n$, there is a transitive proper class $M$ and a countable ordinal $\kappa$ such that
(1) $M$ is a model of $ZF + V = L(V_\kappa) + DC_\kappa$ + “There is no $\kappa' < \kappa$ such that $\kappa'$ is Woodin” + “$\kappa$ is Woodin”; i.e., $M$ is as in Exercises 7.4.7, 7.4.8, 7.4.9, 7.4.10, and 7.4.13 and $\kappa$ is Woodin in $M$;

(2) $M$ is $n$-iterable.

(See Exercise 7.4.10.) Use the result of Exercise 7.4.13 to show that $T(d)$ is constant on a cone of degrees.

This result is due to Woodin, who has also proved that if $T(d)$ is constant on a cone of degrees, then all $\Pi^1_2$ games in $<_\omega$ are determined. Using the fact that Exercise 7.4.10 and a simple forcing argument show that the consistency of $ZFC + "There is a Woodin cardinal"$ gives the consistency of the hypothesis of the present exercise, Woodin also deduced the following theorem:

If $ZFC + "There is a Woodin cardinal"$ is consistent, then so is $ZFC + "All $\Pi^1_2$ games in $<_\omega$ are determined."

Woodin has proved the converse of this theorem, so it is an equiconsistency result. (See part (4) of the hint to Exercise 9.6.4.)

Hint. Fix $n \in \omega$, and let $M$ be as given by the assumption for $n + 1$.

Let

$$A_n = \{ x \in \kappa \mid \kappa < \omega_1^{L[x]} \land V^M_\kappa \in L[x] \}.$$ 

It suffices to show that, for $H$ as above, the set of $\Sigma_n$ sentences true in $L[x]$ is constant on $A_n$. The set $A_n$ of degrees of members of $A_n$ contains a cone. Hence $\bigcap_n A_n$ contains a cone, and we will have that $T(d)$ is constant on $\bigcap_n A_n$.

Argue as follows. For $x \in A$, let $T_x$ be the iteration tree given by Exercise 7.4.13. Let $H$ be as above. Since $|P^{M(T_x), L(T_x)}|^{M(T_x)} = \omega_1^{L[x]}$, it follows by Lemma 25.11 of Jech [1978] that there is an $H'$ such that $H'$ is $Coll(\omega, \omega_1^{L[x]})$-generic over $M(T_x)$ and

$$(L[x])[H] = (M(T_x))[G_x^{M(T_x), L(T_x)}][H] = (M(T_x))[H'].$$

Hence for $x \in A$ the set of $\Sigma_n$ sentences true in $(L[x])[H]$ depends only on the set of $\Sigma_n$ sentences true in $M(T_x)$. Since $M$ is $(n+1)$-iterable, this is the same as the set of $\Sigma_n$ sentences true in $M$. 

$$\text{CHAPTER 8. PROJECTIVE GAMES}$$
Chapter 9

Games in $L(\mathcal{R})$

In this chapter we will present work of Hugh Woodin which, combined with the results of Chapter 8, allows one to prove from Woodin cardinals the determinacy of a much wider class of games than the projective.

Though Woodin has determinacy theorems for even larger classes, we will restrict ourselves to the games $G(A; <_{\omega}\omega)$ for $A \in L(\mathcal{R})$. The class $L(\mathcal{R})$ can be described as the smallest class model of ZF containing all reals and all ordinals. If we take $\mathcal{R}$ to be the reals, then this characterization of $L(\mathcal{R})$ follows easily from the definition in §3.4 of the classes $L(a)$. Since we have no reason to deal with actual real numbers, we take the slightly artificial step of construing “$\mathcal{R}$” as a synonym for “$\omega\omega$.” This of course leaves $L(\mathcal{R})$ unchanged. From time to time we will mention the “reals.” On such occasions, the reader may take us to be referring either to the real reals or to $\omega\omega$.

One reason for studying games in $L(\mathcal{R})$ is that if all such games are determined then the Axiom of Determinacy (AD) holds in $L(\mathcal{R})$. Thus our determinacy results will imply that AD is consistent relative to our large cardinal hypotheses.

In §9.1 we introduce and study a general concept of stationary set. Sections 9.2–9.5 are devoted to the study of Woodin’s stationary tower forcing and the associated generic ultrapowers. In §9.6 we use the stationary tower and the results of Chapter 8 to prove, from slightly more than the existence of infinitely many Woodin cardinals, the determinacy of all games in $L(\mathcal{R})$. We also prove that if ZFC + “There are infinitely many Woodin cardinals” is consistent then so is ZF + AD. This is half of an equiconsistency result due to Woodin.
From §9.2 on we assume that the reader is familiar with forcing. A good source for this material is Kunen [1980]. For one or two results, we will appeal to the more comprehensive Jech [1978]. Just as we have been informal and a bit careless in dealing with proper classes throughout the book, we will now deal very informally with forcing extensions of the universe. We will talk freely of objects $G$ that are $P$-generic over $V$ and of the classes $V[G]$ that result from adjoining such objects to the set-theoretic universe $V$, even though no set can be $P$-generic over $V$ unless the partial ordering $P$ has atoms. We trust that the reader knows how to interpret such talk in terms, e.g., of Boolean valued models.

Since this chapter is devoted primarily to work of Woodin, we make the following convention: Any theorem, lemma, etc. that appears in this chapter without attribution either is folklore (mainly in §9.1) or is due to Woodin.

9.1 Stationary Sets

In this section, we introduce a fairly general notion, due to Woodin, of stationary set. Before doing so, we review the standard concept of a stationary subset of an ordinal number.

Let $\lambda$ be a limit ordinal. A subset $C$ of $\lambda$ is closed in $\lambda$ if $\alpha \in C$ whenever $\alpha < \lambda$ is a limit ordinal such that $C$ is unbounded in $\alpha$. A subset $S$ of $\lambda$ is stationary in $\lambda$ if $S \cap C \neq \emptyset$ for every $C$ that is closed and unbounded in $\lambda$. (These notions were introduced in Exercise 3.2.7.)

**Lemma 9.1.1.** Let $\lambda$ be a limit ordinal of uncountable cofinality. Any countable intersection of sets each of which is closed and unbounded in $\lambda$ is closed and unbounded in $\lambda$.

**Proof.** Let $C_i$, $i \in \omega$, be closed and unbounded in $\lambda$.

To see that $\bigcap_{i \in \omega} C_i$ is closed, let $\alpha < \lambda$ be a limit ordinal such that $\bigcap_{i \in \omega} C_i$ is unbounded in $\alpha$. Then each $C_i$ is unbounded in $\alpha$. Hence $\alpha \in C_i$ for all $i \in \omega$, i.e., $\alpha \in \bigcap_{i \in \omega} C_i$.

To see that $\bigcap_{i \in \omega} C_i$ is unbounded in $\lambda$, let $\beta < \lambda$. Let $\beta_0 = \beta$. Assume that $\beta_m$, $m \leq n$, are defined and that

(i) $\beta_0 < \cdots < \beta_n < \lambda$;

(ii) $(\forall m < n)(\forall i \in \omega)(\exists \gamma \in C_i) \beta_m < \gamma \leq \beta_{m+1}$.
For each $i \in \omega$, let $\gamma_{n,i}$ be such that $
abla \beta_n < \gamma_{n,i} < \lambda$ \land \gamma_{n,i} \in C_i$.

Let 
\[ \beta_{n+1} = \sup\{\gamma_{n,i} \mid i \in \omega\} \].

Since $\text{cf}(\lambda) > \omega$, we have that $\beta_{n+1} < \lambda$. Set $\alpha = \sup\{\beta_n \mid n \in \omega\}$. Since $\text{cf}(\lambda) > \omega$, we have that $\alpha < \lambda$. By (i), $\alpha$ is a limit ordinal. By (ii), each $C_i$ is unbounded in $\alpha$. Thus $\beta < \alpha$ and $\alpha \in \bigcap_{i \in \omega} C_i$. 

**Remark.** The proof of Lemma 9.1.1 trivially adapts to show that the intersection of fewer than $\text{cf}(\lambda)$ sets each of which is closed and unbounded in $\lambda$ is closed and unbounded in $\lambda$, provided that $\text{cf}(\lambda) > \omega$.

For $\lambda$ a limit ordinal of uncountable cofinality, Lemma 9.1.1 implies that the collection of sets closed and unbounded in $\lambda$ generates a countably complete filter $\mathcal{F}_{\text{club}}^\lambda$ on $\lambda$:

\[ \mathcal{F}_{\text{club}}^\lambda = \{ B \subseteq \lambda \mid (\exists C)(C \text{ is closed and unbounded in } \lambda \land C \subseteq B) \} \].

This filter is called the *closed, unbounded filter* on $\lambda$, or simply the *club filter* on $\lambda$.

If $\mathcal{F}$ is a filter on a set $A$, then a subset $B$ of $A$ is $\mathcal{F}$-positive if $A \setminus B \not\in \mathcal{F}$. Note that the $\mathcal{F}_{\text{club}}^\lambda$-positive sets are simply the sets stationary in $\lambda$.

A filter $\mathcal{F}$ on a limit ordinal $\lambda$ is *normal* if, for all $\mathcal{F}$-positive sets $B$ and all $f : B \rightarrow \lambda$, if $f(\alpha) < \alpha$ for all $\alpha \in B$ then $f$ is constant on an $\mathcal{F}$-positive set.

The following basic result comes from Fodor [1956].

**Lemma 9.1.2 (Fodor’s Theorem)** *If $\lambda$ is an uncountable regular cardinal, then the club filter on $\lambda$ is normal.*

**Proof.** The proof of is similar to that of Lemma 9.1.1. Let $S \subset \lambda$ be stationary in $\lambda$ and let $f : S \rightarrow \lambda$ be such that $f(\alpha) < \alpha$ for all $\alpha \in S$. Assume for a contradiction that there is no set stationary in $\lambda$ on which $f$ is constant. For $\xi < \lambda$, let $C_\xi$ be closed and unbounded in $\lambda$ and such that

\[ \{\alpha \mid f(\alpha) = \xi\} \cap C_\xi = \emptyset. \]

We define by transfinite induction a sequence $\langle \beta_\eta \mid \eta < \lambda \rangle$ of ordinals. Our definition will guarantee that, for every $\eta < \lambda$,
(i) $\beta_\eta < \lambda$;

(ii) $\forall \xi < \beta_\eta \exists \gamma \in C_\xi \beta_\eta < \gamma \leq \beta_{\eta+1}$;

(iii) if $\eta$ is a limit ordinal, then $\beta_\eta = \sup_{\xi < \eta} \beta_\xi$.

Let $\beta_0 = 0$. Let $\eta < \lambda$ and assume that $\beta_\eta$ has been defined and $\beta_\eta < \lambda$. For each $\xi < \beta_\eta$, let $\gamma_{\xi,\eta}$ be the least ordinal such that
\[ \beta_\eta < \gamma_{\xi,\eta} < \lambda \wedge \gamma_{\xi,\eta} \in C_\xi. \]

Let
\[ \beta_{\eta+1} = \sup \{ \gamma_{\xi,\eta} \mid \xi < \beta_\eta \}. \]

Since $\lambda$ is regular, we have that $\beta_{\eta+1} < \lambda$. Now let $\eta < \lambda$ be a limit ordinal, and assume that $\beta_\xi$ has been defined and is smaller than $\lambda$ for each $\xi < \eta$. The regularity of $\lambda$ implies that $\sup_{\xi < \eta} \beta_\xi < \lambda$. Thus we can define $\beta_\eta$ so as to satisfy (i) and (iii).

The set
\[ \{ \beta_\eta \mid \eta < \lambda \wedge \eta \text{ is a limit ordinal} \} \]
is closed and unbounded in $\lambda$. Since $S$ is stationary in $\lambda$, let $\eta < \lambda$ be a limit ordinal with $\beta_\eta \in S$. By (ii) and (iii), $C_\xi$ is unbounded in $\beta_\eta$ for each $\xi < \eta$. Thus $\beta_\eta \in \bigcap_{\xi < \beta_\eta} C_\xi$. Now $f(\beta_\eta) < \beta_\eta$. But this gives the contradiction that $\beta_\eta \in C_{f(\beta_\eta)}$. \( \square \)

We now turn to the general concept of a stationary set. Instead of talking about stationary subsets of a limit ordinal $\lambda$, we want to talk of stationary subsets of $\mathcal{P}(X)$ for $X$ an arbitrary set. (The case of limit ordinals is indeed an example of this, since $\lambda \subseteq \mathcal{P}(\lambda)$ for limit ordinals $\lambda$.) Thus we wish to define the notion of a set’s being stationary in $\mathcal{P}(X)$ for arbitrary sets $X$. One can do this directly, but we follow Woodin [1988] and take a slightly different route. The direct definition would have the consequence that if $S$ is stationary in $\mathcal{P}(X)$ then $X = \bigcup S$. Thus the parameter $X$ is redundant, and we might as well define the concept of a set’s being stationary absolutely.

A non-empty set $S$ is stationary if
\[ (\forall f : <\omega \bigcup S \rightarrow \bigcup S)(\exists Y \in S) f''<\omega Y \subseteq Y. \]

There is one trivial way for a set $S$ to be stationary: if $\bigcup S \in S$ then $S$ is stationary. Let us call a set $S$ trivially stationary if $\bigcup S \in S$. The following lemma shows that, for countable $\bigcup S$, being trivially stationary is the only way for $S$ to be stationary.
Lemma 9.1.3. Let $S$ be a non-empty set with $\bigcup S$ countable. Then $S$ is stationary if and only if $S$ is trivially stationary.

**Proof.** If $\bigcup S = \emptyset$ then, since $S$ is non-empty, $S = \{\emptyset\} = \{\bigcup S\}$.

Assume then that $\bigcup S \neq \emptyset$. Let $a_i, \ i \in \omega$, be all the members of $\bigcup S$. (The $a_i$ need not be distinct.) Let $f : \omega \to \bigcup S$ be such that $f((a_i \mid i < n)) = a_n$ for each $n \in \omega$. If $Y \subseteq \bigcup S$ and $f'' \omega \subseteq Y$, then $Y = \bigcup S$. □

The following lemma gives four model-theoretic equivalents of the property of being stationary.

Lemma 9.1.4. For any non-empty set $S$, the following are equivalent:

1. $S$ is stationary.
2. Let $M = (M; \bigcup S, \ldots)$ be a model for a countable language $L$. (Here $\bigcup S$ is the interpretation in $M$ of a one-place relation symbol; hence $\bigcup S \subseteq M$.) Then there is an elementary submodel $(M'; \bigcup S \cap M', \ldots)$ of $M$ such that $M' \cap \bigcup S \in S$.
3. Let $M$ be any set such that $\bigcup S \cup \{\bigcup S\} \subseteq M$. Let $Z$ be a countable subset of $M$. Then there is an $(M'; \in) \prec (M; \in)$ such that $Z \cup \{\bigcup S\} \subseteq M'$ and $M' \cap \bigcup S \in S$.
4. Let $\beta$ be an ordinal such that $\bigcup S \subseteq V_\beta$. Let $\alpha > \beta$. Then for any countable subset $Z$ of $V_\alpha$ there is an $(M'; \in) \prec (V_\alpha; \in)$ with $Z \cup \{\bigcup S\} \subseteq M'$ and $M' \cap \bigcup S \in S$.
5. There is limit ordinal $\beta$ with $\bigcup S \subseteq V_\beta$ and there is an ordinal $\alpha > \beta$ such that, for any countable subset $Z$ of $V_\alpha$, there is an $(M'; \in) \prec (V_\alpha; \in)$ with $Z \cup \{\bigcup S\} \subseteq M'$ and $M' \cap \bigcup S \in S$.

**Proof.** We first show that (1) implies (2). Assume that $S$ is stationary and let $M = (M; \bigcup S, \ldots)$ and $L$ be as in (2). Let $M$ be the universe of $M$. Choose a set of Skolem functions for $M$; i.e., for each formula $\varphi(v_0, \ldots, v_n)$ of $L$ with $v_n$ occurring free in $\varphi$, let $f_\varphi : nM \to M$ be such that, for all $x_1, \ldots, x_n \in M$,

$$(\exists x_0 \in M) M \models \varphi[x_0, \ldots, x_n] \to M \models \varphi[f_\varphi(x_1, \ldots, x_n), x_1, \ldots, x_n].$$

Let $\langle g_i : nM \to M \mid i \in \omega \rangle$, enumerate all compositions of the $f_\varphi$ in such a manner that each $n_i \leq i$. We may assume that $\bigcup S \neq \emptyset$. Fix $a \in \bigcup S$ and
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define, for each $i \in \omega$, a function $g'_i : n_i \cup S \to \bigcup S$ by

$$g'_i(x_1, \ldots, x_{n_i}) = \begin{cases} g_i(x_1, \ldots, x_{n_i}) & \text{if } g_i(x_1, \ldots, x_{n_i}) \in \bigcup S; \\ a & \text{otherwise.} \end{cases}$$

Define $h : {^\omega}M \to M$ by

$$h(x_1, \ldots, x_i) = g'_i(x_1, \ldots, x_{n_i}).$$

Since $S$ is stationary, let $Y \in S$ be such that $h''{^\omega}Y \subseteq Y$. Let $M'$ be the closure of $Y$ under the $f_x$. We have that $(M'; \bigcup S \cap M', \ldots) \prec (M; \bigcup S, \ldots)$ and that $M' \cap \bigcup S = Y \in S$.

Next we show that (2) implies (3). Assume that (2) holds. Let $M$ and $Z$ be as in the statement of (3). Let $\mathcal{L}$ be the result of adding to the language of set theory constants $c_a$, $a \in Z$, and a one-place relation symbol $P$. Expand $(M; e)$ to a model $\bar{M}$ for $\mathcal{L}$ by interpreting each $c_a$ by $a$ and interpreting $P$ by membership in $\bigcup S$. Let $\bar{M}'$ be given by (2). The reduct $M'$ of $\bar{M}'$ to the language of set theory is as required for (3).

To see that (3) implies (4), note that, for $\beta$ and $\alpha$ as in the statement of (4), $\bigcup S \cup \{\bigcup S\} \subseteq V_\alpha$.

It is immediate that (4) implies (5).

Finally we show that (5) implies (1). Let $\beta$ and $\alpha$ witness that (5) holds. Let $f : {^\omega} \bigcup S \to \bigcup S$. Since $\beta$ is a limit ordinal, we have that $f \subseteq V_\beta$ and so that $f \in V_\alpha$. By (5) let $(M'; e) \prec (V_\alpha; e)$ with $\{f, \bigcup S\} \subseteq M'$ and $M' \cap \bigcup S \in S$. Let $Y = M' \cap \bigcup S$. Then $Y$ is as required by (1). □

Remark. The requirement in (5) that $\beta$ be a limit ordinal can be dropped. We included this requirement in order to avoid unnecessary technical details in the proof that (5) implies (1).

For sets $X$ and $S$, we say that $S$ is stationary in $\mathcal{P}(X)$ if $S$ is stationary and $\bigcup S = X$.

The notion a set’s being stationary in a limit ordinal $\lambda$ is defined from the notion of a set’s being closed and unbounded in $\lambda$. One can also define our general notion of a stationary set in this way. Say that a non-empty set $C$ is club if there is a function $f : {^\omega} \bigcup C \to \bigcup C$ such that

$$C = \{Y \subseteq \bigcup C \mid f''{^\omega}Y \subseteq Y\}.$$  

For sets $X$, say that a set $C$ is club in $\mathcal{P}(X)$ if $C$ is club and $\bigcup C = X$. Note that a set $C$ is club in $\mathcal{P}(X)$ if and only if there is some $f : {^\omega}X \to X$ such that $C = \{Y \subseteq X \mid f''{^\omega}Y \subseteq Y\}$. 
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Remark. The word “club” is a standard abbreviation for “closed and unbounded.” For our general notion, we will not use “closed” or “unbounded” but only “club.”

The following lemma follows immediately from the definitions.

**Lemma 9.1.5.** A non-empty set $S$ is stationary if and only if $S \cap C \neq \emptyset$ for every $C$ that is club in $\mathcal{P}(\bigcup S)$.

**Lemma 9.1.6.** For any non-empty $X$, the collection of all sets that are club in $\mathcal{P}(X)$ is closed under countable intersections.

**Proof.** For $i \in \omega$, let $C_i$ be club in $\mathcal{P}(X)$. For each $i$, let $f_i : \subseteq X \to X$ witness that $C_i$ is club in $\mathcal{P}(X)$. Define $g : \subseteq X \to X$ as follows. For $u \in \subseteq X$, let $i_u$ be the least number $i$ such that the $i + 1$st prime $p_i$ divides $\ell h(u) + 2$. Let $n_u + 1$ be the exponent of $p_i$ in the prime factorization of $\ell h(u) + 2$. Set $g(u) = f_{i_u}(u | n_u)$.

If $Y \subseteq X$, then $Y$ is closed under $g$ if and only if $Y$ is closed under $f_i$ for every $i \in \omega$. Thus $g$ witnesses that $\bigcap_{i \in \omega} C_i$ is club. \hfill \square

For any set $X$, let

$$\mathcal{F}_{\text{club}}^{\mathcal{P}(X)} = \{ A \subseteq \mathcal{P}(X) \mid (\exists C)(C \text{ is club in } \mathcal{P}(X) \land C \subseteq A) \}.$$ 

By Lemma 9.1.6, $\mathcal{F}_{\text{club}}^{\mathcal{P}(X)}$ is a countably complete filter on $\mathcal{P}(X)$. We call it the **club filter on $\mathcal{P}(X)$**.

The club filter on $\mathcal{P}(X)$ is just the filter on $\mathcal{P}(X)$ generated by the sets club in $\mathcal{P}(X)$. The following lemma gives what are in effect alternative notions of club, all generating the same club filter.

**Lemma 9.1.7.** Let $X$ be any set. For each of the following conditions on subsets $C$ of $\mathcal{P}(X)$, the collection of subsets of $\mathcal{P}(X)$ satisfying the condition generates the club filter on $\mathcal{P}(X)$.

1. $C$ is club in $\mathcal{P}(X)$.
2. There is a model $\mathcal{M} = (M; X, \ldots)$ for a countable language $\mathcal{L}$ such that $C = \{ M' \cap X \mid (M'; X \cap M', \ldots) \prec (M; X, \ldots) \}$.
(3) For some set $M \supseteq X \cup \{X\}$ and some countable subset $Z$ of $M$,

$$C = \{M' \cap X \mid Z \cup \{X\} \subseteq M' \land (M'; \in) \prec (M; \in)\}.$$  

(4) There is an ordinal $\beta$ with $X \subseteq V_\beta$, there is an $\alpha > \beta$, and there is a countable subset $Z$ of $V_\alpha$ such that

$$C = \{M' \cap X \mid Z \cup \{X\} \subseteq M' \land (M'; \in) \prec (V_\alpha; \in)\}.$$  

(5) For every limit ordinal $\beta$ with $X \subseteq V_\beta$ and for every $\alpha > \beta$, there is a countable subset $Z$ of $V_\alpha$ such that

$$C = \{M' \cap X \mid Z \cup \{X\} \subseteq M' \land (M'; \in) \prec (V_\alpha; \in)\}.$$  

**Proof.** The proof of Lemma 9.1.4 is essentially a proof of the present lemma. □

**Corollary 9.1.8.** Let $1 \leq k \leq 5$. A non-empty set $S$ is stationary if and only if $S \cap C \neq \emptyset$ for every $C$ such that $\bigcup C = \bigcup S$ and $C$ satisfies clause $(k)$ of Lemma 9.1.7.

For any stationary set $S$, say that a set $C$ is *club in* $S$ if $C$ is the intersection with $S$ of some set club in $P(\bigcup S)$. By Lemma 9.1.6, the collection of sets that are club in $S$ generates a countably complete filter $F_S^{\text{club}}$ on $S$, which we call the *club filter on* $S$.

For any stationary set $S$, call a subset $S'$ of $S$ *stationary in* $S$ if $S'$ is stationary and $\bigcup S' = \bigcup S$. Equivalently, $S' \subseteq S$ is stationary in $S$ if $S' \cap C \neq \emptyset$ for every $C$ that is club in $S$.

For limit ordinals $\lambda$ we may have two competing notions of *stationary in* $\lambda$ and the club filter on $\lambda$. The next lemma implies that in each case the competing notions coincide whenever both are defined.

**Lemma 9.1.9.** Let $\lambda$ be a limit ordinal.

(a) If $\lambda$ is not an uncountable regular cardinal, then $\lambda$ is nonstationary.

(b) Assume that $\lambda$ is an uncountable regular cardinal. Then $\lambda$ is stationary. Moreover the stationary subsets of $\lambda$ in our new sense are the same as the stationary subsets of $\lambda$ in our first sense. Thus our new $F_\lambda^{\text{club}}$ is the same as our original $F_\lambda^{\text{club}}$. 

**Proof.** (a) Let $f : \text{cf}(\lambda) \to \lambda$ have unbounded range. By case (4) of Corollary 9.1.8, every stationary subset of $\mathcal{P}(\lambda) (= \mathcal{P}(\bigcup \lambda))$ meets 

$$C = \{ M \cap \lambda \mid M < V_{\lambda+1} \land \omega \cup \{ f, \text{cf}(\lambda) \} \subseteq M \}.$$ 

If $\text{cf}(\lambda) = \omega$, then every member of $C$ is unbounded in $\lambda$, and so $\lambda \cap C = \emptyset$. If $\text{cf}(\lambda) < \lambda$ and $\alpha \in C$ is an ordinal, then $\text{cf}(\lambda) < \alpha$; hence $\alpha$ is unbounded in $\lambda$, and so $\lambda \leq \alpha$. Thus $\text{cf}(\lambda) < \lambda$ also implies that $\lambda \cap C = \emptyset$.

(b) We first prove that every set stationary in $\lambda$ in our first sense is stationary in $\mathcal{P}(\lambda)$. Since $\lambda$ is stationary in $\mathcal{P}(\lambda)$ in our first sense, this will show that $\lambda$ is stationary. It will also show that every subset of $\lambda$ stationary in $\lambda$ in the first sense is stationary in $\lambda$ in the new sense.

Suppose that $S$ is stationary in $\lambda$ in our first sense. It follows that $S$ is unbounded in $\lambda$ and so that $\bigcup S = \lambda$. Let $f : <\omega \lambda \to \lambda$. Let $C$ be the set of all ordinals $< \lambda$ that are closed under $f$. $C$ is evidently closed in $\lambda$. Since $\lambda$ is regular and uncountable, $C$ is unbounded in $\lambda$. Let $\alpha \in C \cap S$. If $Y = \alpha$, then $Y \in S$ and $f^{<\omega}Y \subseteq Y$.

Now suppose that $S$ is stationary in $\lambda$ in the new sense, i.e., is stationary in $\mathcal{P}(\lambda)$. Let $C$ be closed and unbounded in $\lambda$. Let

$$C^* = \{ M \cap \lambda \mid M < V_{\lambda+1} \land C \subseteq M \}.$$ 

By case (4) of Corollary 9.1.8, let $M$ witness that $\alpha \in S \cap C^*$. Since $V_{\lambda+1} \models "C"$ is unbounded in $\lambda,"$ it follows that $M$ also satisfies “$C$ is unbounded in $\lambda.”$ This means that $C$ is unbounded in $M \cap \lambda$, i.e., that $C$ is unbounded in $\alpha$. Since $C$ is closed, we get that $\alpha \in C$ and hence that $\alpha \in S \cap C$. □

**Remarks:**

(a) One can just as easily prove Lemma 9.1.9 directly from the definition of stationary set (without using Corollary 9.1.8). The proof given, however, illustrates a very useful technique.

(b) We have been talking about the “general” notion of stationary set. As part (a) of Lemma 9.1.9 shows, this notion is not completely general, in that it fails to cover some cases of a set’s being stationary in an ordinal number.

(c) Another standard notion of “stationary” is that in Jech [1973] of a stationary subset of $\mathcal{P}_\kappa(X)$, for $\kappa$ an uncountable regular cardinal. Exercise 9.1.2 relates this notion to that of a set’s being stationary in $\mathcal{P}_\kappa(X)$ in our sense.

A filter $\mathcal{F}$ on a set $A$ is fine if, for all $a \in \bigcup A$, $\{ Y \in A \mid a \in Y \} \in \mathcal{F}$. 

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**Lemma 9.1.10.** For any stationary set $S$, the club filter on $S$ is fine.

**Proof.** Let $S$ be stationary, and let $a \in \bigcup S$. Let $c_a : \omega \rightarrow \bigcup S \rightarrow \bigcup S$ be the constant function with value $a$. Then the club set $\{ Y \in S \mid c_a'' Y \subseteq Y \}$ is just $\{ Y \in S \mid a \in Y \}$. \hfill $\square$

A filter $\mathcal{F}$ on a set $A$ is **normal** if, for all $\mathcal{F}$-positive sets $D$ and all $f : D \rightarrow \bigcup A$, if $f(Y) \in Y$ for all $Y \in D$ then $f$ is constant on an $\mathcal{F}$-positive set.

Let $X$ be a set. If $\langle B_a \mid a \in X \rangle$ is such that each $B_a \in \mathcal{F}$, then the diagonal intersection $\Delta_{a \in X} B_a$ is defined by

$$\Delta_{a \in X} B_a = \{ Y \mid (\forall a \in Y) Y \in B_a \}.$$ 

The next lemma generalizes Lemma 3.1.5.

**Lemma 9.1.11.** Let $\mathcal{F}$ be a filter on a set $A$. Then $\mathcal{F}$ is normal if and only if $\mathcal{F}$ is closed under diagonal intersections.

**Proof.** Assume first that $\mathcal{F}$ is normal. Let $\langle B_a \mid a \in X \rangle$ be such that each $B_a \in \mathcal{F}$. Without loss of generality, we may assume that $X = \bigcup A$. Suppose that $\Delta_{a \in X} B_a \notin \mathcal{F}$. Then $D = A \setminus \Delta_{a \in X} B_a$ is $\mathcal{F}$-positive. Let $f : D \rightarrow \bigcup A$ be such that $(\forall Y \in D)(f(Y) \in Y \land Y \notin B_f(Y))$.

By normality, let $a \in \bigcup A$ be such that $f(Y) = a$ for all members $Y$ of some $\mathcal{F}$-positive subset $E$ of $D$. Since $E \cap B_a = \emptyset$ and $B_a \in \mathcal{F}$, we have a contradiction.

Now assume that $\mathcal{F}$ is closed under diagonal intersections. Let $D$ be $\mathcal{F}$-positive and let $f : D \rightarrow \bigcup A$ be such that $f(Y) \in Y$ for all $Y \in D$. For $a \in \bigcup A$, let $B_a = \{ Y \in A \mid Y \in D \setminus f(Y) \neq a \}$. Since $D \cap \Delta_{a \in A} B_a = \emptyset$, we have that $\Delta_{a \in A} B_a \notin \mathcal{F}$. From closure under diagonal intersections, we get an $a \in \bigcup A$ such that $B_a \notin \mathcal{F}$. But then $A \setminus B_a$ is an $\mathcal{F}$-positive subset of $D$ on which $f$ takes the constant value $a$. \hfill $\square$

The following lemma generalizes Fodor’s Theorem (Theorem 9.1.2) to the present context.

**Lemma 9.1.12.** For any stationary set $S$, the club filter on $S$ is normal.
Proof. Let $S' \subseteq S$ be $\mathcal{F}^\text{club}_S$-positive, i.e., such that $S'$ is stationary in $S$. Let $f : S' \to \bigcup S$ be such that $f(Y) \in Y$ for all $Y \in S'$. Let $\beta$ be a limit ordinal with $\bigcup S \subseteq V_\beta$. Assume for a contradiction that there is no set stationary in $S$ on which $f$ is constant. For $a \in \bigcup S$, we use (5) of Lemma 9.1.7 (or, more accurately, of Corollary 9.1.8) to get a countable subset $g(a)$ of $V_{\beta+1}$ such that, for all $M \prec V_{\beta+1}$,

\[(M \prec V_{\beta+1} \land g(a) \cup \{\bigcup S\} \subseteq M) \to f(M \cap \bigcup S) \neq a.\]

Now $g$ is an element of $V_{\beta+4}$. Let

\[C = \{M \cap \bigcup S \mid M \prec V_{\beta+4} \land \{g, \bigcup S\} \subseteq M\}.\]

Since $S'$ is stationary in $S$, case (4) of Corollary 9.1.8 gives us a $Y \in S' \cap C$. Let $M \prec V_{\beta+4}$ be such that $Y = M \cap \bigcup S$ and such that $\{g, \bigcup S\} \subseteq M$. Let $a = f(Y)$. Since $a \in Y \subseteq M$ and $g \in M$, we have that $g(a) \in M$. Since $g(a)$ is countable and $M \prec V_{\beta+4}$, there is a surjection $h : \omega \to g(a)$ such that $h \in M$. Since every member of $\omega$ belongs to $M$, we get that $g(a) \subseteq M$. But then $g(a) \subseteq M \cap V_{\beta+1}$. The fact that $M \prec V_{\beta+4}$ implies that $M \cap V_{\beta+1} \prec V_{\beta+1}$. This is a contradiction. \[\square\]

Exercise 9.1.1. Show that every successor ordinal is stationary. Prove that the stationary subsets of a successor ordinal are precisely the subsets of the ordinal that are trivially stationary.

Exercise 9.1.2. Let $\kappa$ be an uncountable regular cardinal and let $X$ be a set. In the original definition of Jech [1973], a subset $C$ of $\mathcal{P}_\kappa(X)$ is said to be closed and unbounded if

(i) $(\forall x \in X)(\exists Y \in C) x \in Y$;

(ii) if $\gamma < \kappa$ and $\langle Y_\alpha \mid \alpha < \gamma \rangle$ is an increasing sequence of subsets of $C$, then $\bigcup_{\alpha < \gamma} Y_\alpha \in C$.

In this terminology, a subset $S$ of $\mathcal{P}_\kappa(X)$ is called stationary if it meets every closed and unbounded subset of $\mathcal{P}_\kappa(X)$.

Let $\kappa$ be an uncountable regular cardinal and let $X$ be a set.

(a) Prove that clause (ii) of the definition above, “increasing sequence” can be replaced by “directed set” without changing the concept defined.
(b) Prove that $\mathcal{P}_\kappa(X)$ is stationary. Assume that $\kappa \subseteq X$ and prove that $\mathcal{P}_\kappa(X) \cap \{Y \mid Y \cap \kappa \in \kappa\}$ is stationary.

(c) Assume that $\kappa \subseteq X$. Prove that the stationary subsets of $\mathcal{P}_\kappa(X)$ in the sense just defined are just the sets stationary in $\mathcal{P}_\kappa(X) \cap \{Y \mid Y \cap \kappa \in \kappa\}$ in our official sense.

(d) Assume that $\kappa = \aleph_1$. Prove that the closed and unbounded subsets of $\mathcal{P}_\kappa(X)$ in the sense defined above are the same as the sets club in $\mathcal{P}_\kappa(X)$ in our official sense.

Some of these results are due to Kueker [??].

9.2 The Stationary Tower

In this section we introduce one of Woodin’s two main stationary towers, the full tower. We begin with a discussion of generic ultrapowers associated with ideals. Then we introduce and study the full stationary tower and the generic ultrapowers associated with it.

From now on, we assume that the reader knows the basic facts about forcing. Sources for this information are Kunen [1980] and Jech [1978]. The former is more introductory and the latter more encyclopedic.

For the purposes of fixing terminology, we now give a few of the basic definitions. In the context of forcing, we say that $(P; \leq)$ is a partial ordering if $P$ is a nonempty set and $\leq$ is a transitive reflexive relation in $P$. In other words, we do not forbid distinct elements $p$ and $q$ of $P$ such that $p \leq q \land q \leq p$.

We often say “$P$” when we mean “$(P; \leq)$.” A subset $D$ of $P$ is dense in $P$ if for any $p \in P$ there is a $q \in D$ with $q \leq p$. If $M$ is a class, then a subset $G$ of $P$ is $P$-generic over $M$ if

(a) $(\forall p \in G)(\forall q \in P)(p \leq q \rightarrow q \in G)$;

(b) $(\forall p \in G)(\forall q \in G)((\exists r \in G)(r \leq p \land r \leq q))$;

(c) if $D$ is any dense subset of $P$ such that $D \in M$, then $D \cap G \neq \emptyset$.

We will talk of objects that are $P$-generic over $M$ even when they do not exist (in $V$).

An ideal on a set $A$ is any set $\mathcal{I}$ of the form

$$\mathcal{I} = \{A \setminus B \mid B \in \mathcal{F}\},$$

where $\mathcal{F}$ is a filter on $A$. For such $\mathcal{I}$ and $\mathcal{F}$, we say that $\mathcal{I}$ is the dual ideal of $\mathcal{F}$ and that $\mathcal{F}$ is the dual filter of $\mathcal{I}$. We denote the dual ideal of the club
filter on a set $Z$ (a stationary set or a limit ordinal of uncountable cofinality) by $\mathcal{I}_Z^n$. We say that an ideal is principal [countably complete, normal, etc.] just in case its dual filter is principal [countably complete, normal, etc.]. In addition we say that a subset $B$ of $A$ is $\mathcal{I}$-positive just in case $B$ is $\mathcal{F}$-positive for $\mathcal{F}$ the dual filter of $\mathcal{I}$, in other words, just in case $B \notin \mathcal{I}$.

Remark. There is no serious reason for discussing ideals in addition to filters. It so happens that certain concepts are, as a matter of tradition, formulated in terms of ideals rather than in terms of filters.

Let $\mathcal{I}$ be an ideal on a set $A$. Let $\mathbf{P}(\mathcal{I})$ be the set of all $\mathcal{I}$-positive subsets of $A$. Partially order $\mathbf{P}(\mathcal{I})$ by

$$B \leq B' \leftrightarrow B \setminus B' \in \mathcal{I}.$$ 

We will not distinguish notationally between the set $\mathbf{P}(\mathcal{I})$ and the associated partial ordering.

Suppose that $G$ is $\mathbf{P}(\mathcal{I})$-generic over $V$. Then $G$ extends the dual filter of $\mathcal{I}$. Moreover $G$ is (in $V[G]$) a $V$-ultrafilter on $A$, i.e.,

(a) $A \in G$ and $\emptyset \notin G$;
(b) $(\forall B \in G \cap V)(\forall B' \in G \cap V) B \cap B' \in G$;
(c) $(\forall B \in G)(\forall B' \in V)(B \subseteq B' \subseteq A \rightarrow B' \in G)$;
(d) $(\forall B \in V)(B \subseteq A \rightarrow (B \in G \vee A \setminus B \in G))$.

In $V[G]$ we can form the generic ultrapower $\prod_G(V; \in)$. For $f$ and $g$ functions with domain $A$ that belong to $V$, define

$$f \sim_G g \leftrightarrow \{a \in A \mid f(a) = g(a)\} \in G.$$

Let $[f]_G$ be the set of all elements of minimal rank of the equivalence class of $f$ with respect to $\sim_G$. We define a relation $\in_G$ by

$$[f]_G \in_G [g]_G \leftrightarrow \{a \in A \mid f(a) \in g(a)\} \in G.$$

This gives us $\prod_G(V; \in)$ in the usual way.

Remark. It would be more consistent with the notation of earlier chapters to write, e.g., $\prod^V_G(V; \in)$ instead of $\prod_G(V; \in)$. We do not do so because it would make our notation a little more cumbersome and because there is no danger of confusion.

We omit the proof of the following variant of the Theorem of Loś:
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**Theorem 9.2.1.** Let $\mathcal{I}$ be an ideal on a set $A$. Let $G$ be $\mathcal{P}(\mathcal{I})$-generic over $V$. Let $\varphi(v_1, \ldots, v_n)$ be any formula of the language of set theory. Let $f_1, \ldots, f_n : A \to V$ with each $f_i \in V$. Then

$$\prod_G(V; \in) \models \varphi[\{f_1\}_G, \ldots, \{f_n\}_G] \iff \{a \in A \mid (V; \in) \models \varphi[f_1(a), \ldots, f_n(a)]\} \in G.$$ 

Define $i'_G : (V; \in) \prec \prod_G(V; \in)$ by setting $i'_G(x) = [c_x]_G$, where $c_x$ is the constant function with value $x$.

An ideal $\mathcal{I}$ on a set $A$ is called *precipitous* if the maximal element $A$ of $\mathcal{P}(\mathcal{I})$ forces $\prod_G(V; \in)$ to be wellfounded. (This is expressible by a formula of the language of set theory. See Exercise 9.2.1.) The dual ideal of a countably complete ultrafilter is of course precipitous. For another way to get precipitous ideals, see Exercise 9.2.2.

If $G$ is $\mathcal{P}(\mathcal{I})$-generic over $V$ and $\prod_G(V; \in)$ is wellfounded, then

$$\pi_G([\text{id}]_G) = i''_G \bigcup A.$$ 

**Proof.** If $a \in \bigcup A$ then, by fineness of $\mathcal{I}$,

$$\{Y \in A \mid a \in Y\} \in G;$$

hence

$$[c_a]_G \in G [\text{id}]_G$$

and so

$$i_G(a) \in \pi_G([\text{id}]_G).$$
Thus $i_G'' \cup A \subseteq \pi_G([\text{id}]_G)$. To establish the reverse inclusion, let

$$[f]_G \sqsubseteq G [\text{id}]_G.$$ 

Then $E \in G$, where

$$E = \{Y \in A \mid f(Y) \in Y\}.$$ 

It suffices to show that the set

$$\{B \in P(I) \mid (\exists a \in \bigcup A) B \forces [f]_G = [c_a]_G\}$$

is dense below $E$ in $P(I)$. (Here $G$ is the standard $P(I)$-name for the generic object.) Let $D \in P(I)$ with $D \leq E$. Without loss of generality, we may assume that $D \subseteq E$. By the normality of $I$, there is an $a \in \bigcup A$ such that \{\{Y \in D \mid f(Y) = a\} is $I$-positive. This set is the desired $B \leq D$. \hfill \Box

Just as the notion of ultrapowers with respect to ultrafilters can be generalized to that of ultrapowers with respect to extenders, so the notion of generic ultrapowers with respect to ideals can be extended to that of generic ultrapowers with respect to towers of ideals. Instead of doing this in generality, we proceed directly to the special cases given by the stationary tower.

First we prove some simple facts about the relation between the club and stationary subsets of $P(X)$ and $P(X')$ when $X \subseteq X'$.

If $A$ and $X$ are sets with $X \subseteq \bigcup A$, then let

$$A \res X = \{Y \cap X \mid Y \in A\}.$$ 

If $A$ and $X$ are sets with $\bigcup A \subseteq X$, then let

$$A \ext X = \{Y \subseteq X \mid Y \cap \bigcup A \in A\}.$$ 

(One might read “res” as “restricted to” and “ext” as “extended to.”) Note that $(A \ext X) \res \bigcup A = A$.

**Lemma 9.2.3.** Let $C$ be club.

(a) If $X \subseteq \bigcup C$ then $C \res X \in F^{\text{club}}_{P(X)}$.

(b) If $\bigcup C \subseteq X$ then $C \ext X$ is club in $P(X)$. 


Proof. (a) The proof is similar to a portion of the proof of Lemma 9.1.4. Let \( f : \langle \omega \cup C \rightarrow \bigcup C \) be such that \( C = \{ Y \subseteq \bigcup C \mid f'' Y \subseteq Y \} \). We may assume that \( X \neq \emptyset \). Let \( \langle g_i : n_i \bigcup C \rightarrow \bigcup C \mid i \in \omega \rangle \) enumerate all compositions of the \( f \upharpoonright n \bigcup C \), \( n \in \omega \), in such a manner that each \( n_i \leq i \). Fix \( a \in X \) and define, for each \( i \in \omega \), a function \( g'_i : n_i X \rightarrow X \) by

\[
g'_i(x_1, \ldots, x_{n_i}) = \begin{cases} g_i(x_1, \ldots, x_{n_i}) & \text{if } g_i(x_1, \ldots, x_{n_i}) \in X; \\ a & \text{otherwise.} \end{cases}
\]

Define \( h : \langle \omega X \rightarrow X \) by

\[
h(x_1, \ldots, x_i) = g'_i(x_1, \ldots, x_{n_i}).
\]

Clearly \( C \res X \supseteq \{ Y \subseteq X \mid h'' Y \subseteq Y \} \).

(b) For \( u \in \langle \omega X \), let \( k_u = |\{ n < \chi h(u) \mid u(n) \in \bigcup C \}| \) and let \( \rho_u : k_u \rightarrow \{ n < \chi h(u) \mid u(n) \in \bigcup C \} \) be order preserving. Define \( g : \langle \omega \bigcup C \rightarrow \langle \omega \bigcup C \) by setting \( g(u) = u \circ \rho_u \). If \( f : \langle \omega \bigcup C \rightarrow \bigcup C \) witnesses that \( C \) is club, then \( f \circ g \) witnesses that \( C \ext X \) is club in \( P(X) \). \( \square \)

Corollary 9.2.4. Let \( S \) be stationary.

(a) If \( X \subseteq \bigcup S \) then \( S \res X \) is stationary in \( P(X) \).

(b) If \( \bigcup S \subseteq X \) then \( S \ext X \) is stationary in \( P(X) \).

Proof. (a) follows easily from part (b) of Lemma 9.2.3 and (b) follows easily from part (a) of that lemma. \( \square \)

Let \( \rho \) be a limit ordinal. We now define the full stationary tower of height \( \rho \). Let

\[
P_\rho = \{ S \in V_\rho \mid S \text{ is stationary} \}.
\]

Partially order \( P_\rho \) by setting \( S \leq S' \) just in case both of the following hold:

1. \( \bigcup S \supseteq \bigcup S' \);
2. \( S' \setminus (S' \ext \bigcup S) \) is not stationary in \( P(\bigcup S) \).

It is routine to check that \( \leq \) partially orders \( P_\rho \). We will not distinguish notationally between the set \( P_\rho \) and the partial ordering \( (P_\rho; \leq) \).

Remark. What we call \( P_\rho \) Woodin [1988] calls \( P_{<\rho} \). Similarly Woodin calls \( Q_{<\rho} \) what we will later call \( Q_\rho \).
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The next lemma gives some simple facts about the partially ordered set $P$. For $S$ and $S'$ belonging to $P$, let us say that $S \leq \omega S'$ if whenever $S^* \leq S$ then $S^*$ is compatible with $S'$ (i.e., $S^*$ and $S'$ have a common lower bound—a common extension). If $S \leq \omega S'$, then $S' \in G$ whenever $G$ is $P$-generic over $V$ and $S \in G$.

**Lemma 9.2.5.** Let $\rho$ be a limit ordinal. Let $X \in V_\rho$. Let $S$ and $S'$ be elements of $P$.  

1. If $X \subseteq \cup S$, then $S \leq \omega S X$.
2. If $\cup S \subseteq X$, then $S \omega X S \leq \omega S \omega X$.  
3. If $\cup S = \cup S'$ and $S$ and $S'$ are compatible, then $S \cap S'$ is stationary in $\mathcal{P}(\cup S)$ and $S \cap S'$ is a greatest lower bound of $S$ and $S'$ in $P$.  

**Proof.** (a) and the first part of (b) follow immediately from the definition of $\leq$.

For the second part of (b) assume that $\cup S \subseteq X$ and let $S^* \leq S$. Let

$$ \tilde{S} = S^* \omega X \omega X^* .$$

By part (b) of Corollary 9.2.4, $\tilde{S} \in P$. By the first part of the present (b), $\tilde{S} \leq S^* \leq S$. To show that $\tilde{S} \leq \omega S X$, we must prove that $\tilde{S} \omega (\omega S X \omega X^*)$ is not stationary in $\mathcal{P}(X \cup X^*)$, i.e., that $(\omega S \omega X \cup X^*) \setminus (\omega S \omega X \cup X^*)$ is not stationary in $\mathcal{P}(X \cup X^*)$. If it were stationary in $\mathcal{P}(X \cup X^*)$, then part (a) of Corollary 9.2.4 would give that $S^* \omega (\omega S X \omega X^*)$ is stationary in $\mathcal{P}(X^*)$. This would contradict the hypothesis that $S^* \leq S$.

For (c), assume that $\cup S = \cup S'$ and let $S^* \leq S$ and $S^* \leq S'$. Thus $\cup S \subseteq \cup S^*$ and neither $S^* \setminus (\omega S \omega X \omega X^*)$ nor $S^* \setminus (\omega S' \omega X \omega X^*)$ is stationary in $\mathcal{P}(\cup S^*)$. Thus $S^* \setminus (\omega S \omega X \cup X^*)$ is not stationary in $\mathcal{P}(\cup S)$. By part (a) of Corollary 9.2.4, $S \cap S'$ is stationary in $\mathcal{P}(\cup S)$. By the definition of $\leq$, $S^* \leq (S \cap S')$. \hfill \Box

If $G$ is $P$-generic over $V$ and if $X \in V_\rho$, then let $G_X = \{ A \in G \mid \cup A = X \}$.

**Lemma 9.2.6.** Let $\rho$ be a limit ordinal and let $G$ be $P$-generic over $V$. For $X \in V_\rho$, the set $G_X$ is a $V$-ultrafilter on $\mathcal{P}(X)$ (i.e., on $\mathcal{P}(X)$) extending $\mathcal{F}_{P(X)}$. Moreover the $G_X$ are compatible: if $X \subseteq X' \in V_\rho$ then, for all $A \subseteq \mathcal{P}(X)$ and all $A' \subseteq \mathcal{P}(X')$,
functions $f$ define the generic ultrapower $P$. In doing something slightly different but equivalent.

It follows that $A \in G_X \to A \text{ext } X' \in G_X$.

**Proof.** Fix $X \in V$. The restriction of $\leq$ to $\{S \mid \bigcup S = X\}$ is just our standard partial ordering of $P(I_{\mathcal{P}(X)})$. Since $S \in G$ and $S \leq S'$ implies $S' \in G$, it follows that, if $S \in G_X$ and $S \leq S'$ in the sense of $P(I_{\mathcal{P}(X)})$, then $S' \in G_X$. In particular, if $S' \supseteq S \in G_X$ then $S' \in G_X$.

Let $S$ and $S'$ be members of $G_X$ that are stationary in $\mathcal{P}(X)$. By part (c) of Lemma 9.2.5 and the fact that $S$ and $S'$ have a common extension that belongs to $G$, we get that $S \cap S' \in G$ and so that $S \cap S' \in G_X$.

To show that $G_X$ is a $V$-ultrafilter on $\mathcal{P}(X)$, it is enough to show that, for all $A \subseteq \mathcal{P}(X)$ and for all $S \in P_{\rho}$, at least one of $A$ and $\mathcal{P}(X) \setminus A$ belongs to $P_{\rho}$ and is compatible with $S$.

Let then $A \subseteq \mathcal{P}(X)$ and let $S \in P_{\rho}$. Let $X' = \bigcup S$. By Corollary 9.2.4 we have that $S \text{ext } X \cup X'$ is stationary in $\mathcal{P}(X \cup X')$. Thus at least one of $A \text{ext } X \cup X'$ and $(\mathcal{P}(X) \setminus A) \text{ext } X \cup X'$ must have an intersection with $S \text{ext } X \cup X'$ that is stationary in $\mathcal{P}(X \cup X')$. Assume for definiteness that

$$(A \text{ext } X \cup X') \cap (S \text{ext } X \cup X') \text{ is stationary in } \mathcal{P}(X \cup X').$$

It follows that $A \text{ext } X \cup X'$ is stationary in $\mathcal{P}(X \cup X')$. Since $A = (A \text{ext } X \cup X') \text{res } X$, it follows that $A$ is stationary in $\mathcal{P}(X)$. Moreover

$$(A \text{ext } X \cup X') \cap (S \text{ext } X \cup X') \subseteq A \text{ext } X \cup X' \subseteq A;$$

$$(A \text{ext } X \cup X') \cap (S \text{ext } X \cup X') \subseteq S \text{ext } X \cup X' \subseteq S.$$

To show that $G_X$ extends $\mathcal{F}_{\mathcal{P}(X)}^{\text{club}}$, let $C$ be club in $\mathcal{P}(X)$. We repeat the argument of the last paragraph, with $A = C$. Since $C \text{ext } X \cup X'$ is club in $\mathcal{P}(X \cup X')$, it follows that $(C \text{ext } X \cup X') \cap (S \text{ext } X \cup X')$ is stationary in $\mathcal{P}(X \cup X')$. Thus $C$ and $S$ have a common extension.

Assertions (i) and (ii) follow from parts (a) and (b) respectively of Lemma 9.2.5.

$\square$

**Remark.** The $G_X$ need not be $P(I_{\mathcal{P}(X)})$-generic over $V$.

Let $\rho$ be a limit ordinal and let $G$ be $P_{\rho}$-generic over $V$. In $V[G]$ we define the generic ultrapower $\prod_G(V;\in)$. One way to do this would be to use functions $f : \mathcal{P}(X) \to V$ with $f \in V$ and $X \in V_{\varphi}$. We follow Woodin [1988] in doing something slightly different but equivalent.
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Suppose that $S \in \mathbf{P}_\rho$ and that $f : S \to V$ belongs to $V$. If $\bigcup S \subseteq X$ then define $f^X : S \text{ext} X \to V$ by

$$f^X(Y) = f(Y \cap \bigcup S).$$

Suppose $f : S \to V$ and $f' : S' \to V$ are functions belonging to $V$ whose domains are elements of $\mathbf{G}$. Let $X = \bigcup(S \cup S')$. We define

$$f \sim \mathbf{G} f' \leftrightarrow \{Y \in (S \text{ext} X \cap S' \text{ext} X) \mid f^X(Y) = f'^X(Y)\} \in \mathbf{G}.$$ 

Let $[f]_\mathbf{G}$ be the set of all elements of minimal rank of the equivalence class of $f$ with respect to $\sim$. We define a relation $\in \mathbf{G}$ as follows. Given $f : S \to V$ and $f' : S' \to V$, both belonging to $V$, let $X = \bigcup(S \cup S')$ and set

$$[f]_\mathbf{G} \in \mathbf{G} [f']_\mathbf{G} \leftrightarrow \{Y \in (S \text{ext} X \cap S' \text{ext} X) \mid f^X(Y) \in f'^X(Y)\} \in \mathbf{G}.$$ 

This gives us $\prod_\mathbf{G}(V; \in)$ in the usual way.

We have the following Loś-type theorem:

**Theorem 9.2.7.** Let $\rho$ be a limit ordinal and let $\mathbf{G}$ be $\mathbf{P}_\rho$-generic over $V$. Let $\varphi(v_1, \ldots, v_n)$ be any formula of the language of set theory. Let $S_1, \ldots, S_n$ be elements of $\mathbf{G}$. For $1 \leq i \leq n$, let $f_i : S_i \to V$ with each $f_i \in V$. Let $X = \bigcup_{1 \leq i \leq n} S_i$. Let $S^* = \bigcap_{1 \leq i \leq n}(S_i \text{ext} X)$. Then

$$\prod_\mathbf{G}(V; \in) \models \varphi([f_1]_\mathbf{G}, \ldots, [f_n]_\mathbf{G}) \leftrightarrow \{Y \in S^* \mid (V; \in) \models \varphi[f_1^X(Y), \ldots, f_n^X(Y)]\} \in \mathbf{G}.$$ 

If $\mathbf{G}$ is $\mathbf{P}_\rho$-generic over $V$, define $i^*_\mathbf{G} : (V; \in) \prec \prod_\mathbf{G}(V; \in)$ by setting

$$i^*_\mathbf{G}(x) = [c^S_x]_\mathbf{G},$$

where $c^S_x : \{0\} \to \{x\}$. (Any other constant function $c^S_x$ whose domain $S$ belongs to $\mathbf{G}$ would do as well as $c^S_x$.)

If $\prod_\mathbf{G}(V; \in)$ is wellfounded, let $\pi_\mathbf{G} : \prod_\mathbf{G}(V; \in) \cong (\text{Ult}(V; \mathbf{G}); \in)$ and $i_\mathbf{G} : V \prec \text{Ult}(V; \mathbf{G})$ be defined in the usual manner.

Even if $\prod_\mathbf{G}(V; \in)$ is not wellfounded, the ordinal wf ord ($\prod_\mathbf{G}(V; \in)$) (defined on page 406) may be fairly large. In particular, it turns out that $\rho \leq \text{wf ord}(\prod_\mathbf{G}(V; \in))$ always. (See Lemma 9.2.12.) It will therefore be convenient to construe $\pi_\mathbf{G}$ and $i_\mathbf{G}$ as functions that always exist but whose domains may be proper subsets of the universe of $\prod_\mathbf{G}(V; \in)$ and $V$ respectively. The definitions that follow accomplish this.

If $(N; E)$ is a class model for the language of set theory and $x \in N$, say that $x$ belongs to WFP$(N; E)$ just in case the class model $(\{y \mid y \in \text{tclos}^{(N; E)}(x)\}; E)$
is wellfounded, where tclos\(^{(N;E)}\)(x) is the transitive closure of x in the sense of (N; E). Equivalently, \(x \in \text{WFP}(N; E)\) just in case there are no infinite descending E-chains beginning with x. If WFP(N; E) is nonempty then (WFP(N; E); E) is a wellfounded model. Hence if it is set-like and satisfies Extensionality then it is isomorphic by a unique isomorphism to a unique transitive class. The ordinals of this class are exactly the ordinals smaller than wford (N; E).

For \(\rho\) a limit ordinal and for \(G\) that is \(P_\rho\)-generic over \(V\), let us always denote by \(\pi_G\) the unique isomorphism of WFP\((\prod_G(V; \in))\) onto a transitive class. Let us always denote by \(i_G\) the composition \(\pi_G \circ i'_G\), which is defined on the class of all \(x \in V\) such that \(i'_G(x) \in \text{WFP}(\prod_G(V; \in))\).

**Lemma 9.2.8.** Let \(\rho\) be a limit ordinal and let \(G\) be \(P_\rho\)-generic over \(V\). Let \(S\) and \(S'\) belong to \(G\). For all \(f : S' \to V\) with \(f \in V\),

\[
[f]_G \in_G [\text{id}_S]_G \leftrightarrow (\exists a \in \bigcup S) [f]_G = i'_G(a),
\]

where \(\text{id}_S\) is the identity function on \(S\).

**Proof.** Suppose first that \(a \in \bigcup S\). Let \(S^* = \{Y \in S \mid a \in Y\}\). Since \(S^*\) is club in \(S\), we have that \(S^* \in G\). Thus

\[
\{Y \subseteq \bigcup S \mid \text{c}_S(Y) \in \text{id}_S(Y)\} \subseteq G.
\]

Hence \(i'_G(a) \in_G [\text{id}_S]_G\).

Now suppose that \(f : S' \to V\) belongs to \(V\) and is such that \([f]_G \in_G [\text{id}_S]_G\). Since \([f]_G = [f^{\bigcup(S \cup S')}]_G\), we may assume that \(\bigcup S \subseteq \bigcup S'\). Since \((S' \cap (S \text{ ext } \bigcup S')) \in G\) by part (c) of Lemma 9.2.5, we may assume that \(S' \subseteq S \text{ ext } \bigcup S'\). By the definition of \(\in_G\), we have that \(S \subseteq G\), where

\[
\tilde{S} = \{Y \subseteq \bigcup S' \mid f(Y) \in Y \cap \bigcup S\}.
\]

By Lemma 9.1.12, every set stationary in \(\tilde{S}\) has a subset stationary in \(\tilde{S}\) on which \(f\) is constant. By parts (a) and (b) of Lemma 9.2.5, it follows that the set of all \(\tilde{S} \subseteq \tilde{S}\) such that

\[
(\exists a \in \bigcup S) (\forall Y \in \tilde{S}) f(Y \cap \bigcup S') = a
\]

is dense below \(\tilde{S}\) in \(P_\rho\). Thus some such \(\tilde{S}\) belongs to \(G\), and \(\tilde{S} \text{ ext } \bigcup S'\) witnesses that \([f]_G = i'_G(a)\) for the corresponding \(a \in \bigcup S\). \(\square\)
Corollary 9.2.9. Let \( \rho \) be a limit ordinal and let \( \mathbf{G} \) be \( \mathbf{P}_\rho \)-generic over \( V \). Let \( S \in \mathbf{G} \) be such that \( i'^*_\mathbf{G}'' \cup S \subseteq \text{WFP}(\Pi_\mathbf{G}(V; \in)) \). Then \( \langle \text{id}_S \rangle_\mathbf{G} \in \text{WFP}(\Pi_\mathbf{G}(V; \in)) \) and
\[
i'^*_\mathbf{G}'' \cup S = \pi_\mathbf{G}(\langle \text{id}_S \rangle_\mathbf{G}).
\]

Proof. This follows from Lemma 9.2.8 \( \square \)

Lemma 9.2.10. Let \( \rho \) be a limit ordinal and let \( \mathbf{G} \) be \( \mathbf{P}_\rho \)-generic over \( V \) and such that \( \Pi_\mathbf{G}(V; \in) \) is wellfounded. Let \( S \in \mathbf{P}_\rho \). Then
\[
S \in \mathbf{G} \iff i'^*_\mathbf{G}'' \cup S \in i_\mathbf{G}(S).
\]

Proof. Trivially range \( (\text{id}_S) \subseteq S \). Thus if \( S \in \mathbf{G} \) then \( \pi_\mathbf{G}(\langle \text{id}_S \rangle_\mathbf{G}) \) belongs to \( i_\mathbf{G}(S) \). By Corollary 9.2.9, \( i'^*_\mathbf{G}'' \cup S \in i_\mathbf{G}(S) \).

If \( S \notin \mathbf{G} \), then \( (\mathcal{P}(\bigcup S) \setminus S) \in \mathbf{G} \). Thus
\[
i'^*_\mathbf{G}'' \cup S = i'^*_\mathbf{G}'' \cup (\mathcal{P}(\bigcup S) \setminus S)
\in i_\mathbf{G}(\mathcal{P}(\bigcup S) \setminus S).
\]
Thus \( i'^*_\mathbf{G}'' \cup S \notin i_\mathbf{G}(S) \). \( \square \)

Part (b) of the next lemma is the analogue for the stationary tower of Lemma 6.1.12. Part (a) is the “nonwellfounded” version of the lemma.

Lemma 9.2.11. Let \( \rho \) be a limit ordinal and let \( \mathbf{G} \) be \( \mathbf{P}_\rho \)-generic over \( V \). Let \( S \in \mathbf{G} \) and let \( f : S \to V \) with \( f \in V \). Then
\[
\begin{align*}
(a) \ & \Pi_\mathbf{G}(V; \in) \models (i'_\mathbf{G}(f))(\langle \text{id}_S \rangle_\mathbf{G}) = \langle f \rangle_\mathbf{G}; \\
(b) \ & \text{if } i'_\mathbf{G}(f) \in \text{WFP}(\Pi_\mathbf{G}(V; \in)), \text{ then } (i_\mathbf{G}(f))(i'^*_\mathbf{G}'' \cup S) = \pi_\mathbf{G}(\langle f \rangle_\mathbf{G}).
\end{align*}
\]

Proof. By Theorem 9.2.7, the structure \( \Pi_\mathbf{G}(V; \in) \) satisfies
\[
(\langle c^S_\mathbf{G} \rangle_\mathbf{G})(\langle \text{id}_S \rangle_\mathbf{G}) = \langle f \rangle_\mathbf{G}.
\]
This gives (a). (b) then follows by Corollary 9.2.9. \( \square \)

If \( a \) is a transitive set, then it is clear that the set of all \( Y \subseteq a \) such that \( (Y; \in) \) satisfies Extensionality belongs to \( \mathcal{F}_{P(a)}^{\text{club}} \). For any set \( Y \) such that \( (Y; \in) \) satisfies Extensionality, let \( \text{trcoll} (Y) \) be the unique transitive set
isomorphic to $Y$. For all other sets $Y$ let $\text{trcoll}(Y)$ be $Y$. For $Y$ satisfying extensionality, let $\pi_Y : (Y; \in) \cong (\text{trcoll}(Y); \in)$ be the unique isomorphism. For other sets $Y$, let $\pi_Y : (Y; \in) \cong (\text{trcoll}(Y); \in)$ be the identity. For every set $x$ and every transitive $a$ with $x \in a$, define a function $\text{prj}_x^a$ with domain $\mathcal{P}(a)$ by setting

$$\text{prj}_x^a(Y) = \begin{cases} \pi_Y(x) & \text{if } x \in Y; \\ \emptyset & \text{otherwise.} \end{cases}$$

**Lemma 9.2.12.** Let $\rho$ be a limit ordinal and let $G$ be $P_\rho$-generic over $V$.

1. For every transitive $a \in V_\rho$,
   
   (a) $[\text{trcoll}^a \restriction \mathcal{P}(a)]_G \in \text{WFP}(\prod G(V; \in))$;
   
   (b) $a = \pi_G([\text{trcoll}^a \restriction \mathcal{P}(a)]_G)$;
   
   (c) $(\forall x \in a) x = \pi_G([\text{prj}_x^a]_G)$.

2. $\rho \leq \text{wford}(\prod G(V; \in))$.

**Proof.** For (1), fix $a \in V_\rho$ with $a$ transitive. Let $t = \text{trcoll} \restriction \mathcal{P}(a)$.

By Theorem 9.2.7 (the Loś theorem) and the paragraph preceding the statement of the lemma, $\prod G(V; \in) \models "(i'_G(t))(\text{id}_{\mathcal{P}(a)})_G\ is\ transitive"$ and

$$\prod G(V; \in) \models \pi_{[\text{id}_{\mathcal{P}(a)}]_G} : (i'_G(t))(\text{id}_{\mathcal{P}(a)})_G \cong [\text{id}_{\mathcal{P}(a)}]_G.$$ 

In particular, this implies that the models

$$(\{[f]_G \mid [f]_G \in G \ (i'_G(t))(\text{id}_{\mathcal{P}(a)})_G\}; \in_G)$$

and

$$(\{[f]_G \mid [f]_G \in G \ [\text{id}_{\mathcal{P}(a)}]_G\}; \in_G)$$

are isomorphic.

By the transitivity of $a$ and by Lemma 9.2.8 with $S = \mathcal{P}(a)$, there is a unique isomorphism

$$\hat{\pi} : (\{[f]_G \mid [f]_G \in G \ [\text{id}_{\mathcal{P}(a)}]_G\}; \in_G) \cong (a; \in),$$

and $\hat{\pi}$ is defined by setting

$$\hat{\pi}(i'_G(x)) = x.$$
By the transitivity of isomorphism, we have that
\[(\{[f]_G \mid [f]_G \in G (i'_G(t))(\id_{P(a)})_G\}; \in_G) \cong (a; \in).\]
This means that
\[(i'_G(t))(\id_{P(a)})_G) \in \WFP(G; [V; \in]),\]
and that
\[\pi_G((i'_G(t))(\id_{P(a)})_G)) = a.\]
By part (a) of Lemma 9.2.11,
\[(i'_G(t))(\id_{P(a)})_G) = [t]_G.\]
Thus (1)(a) and (1)(b) follow.

The uniqueness of \(\hat{\pi}\) implies that
\[\prod_G(V; \in) \models \pi_{[\id_{P(a)}]_G}(i'_G(x)) = \pi^{-1}_G(x),\]
i.e., that
\[\prod_G(V; \in) \models (i'_G(\text{pr}^a_x))(\id_{P(a)})_G) = \pi^{-1}_G(x).\]
By part (a) of Lemma 9.2.11, we get that
\[[\text{pr}^a_x]_G = \pi^{-1}_G(x).\]
Applying \(\pi_G\) to both sides of this equation gives (1)(c).

(2) follows from (1). \(\square\)

The next lemma is a technical consequence of the preceding lemmas that we will have need of later.

**Lemma 9.2.13.** Let \(\rho\) be a limit ordinal and let \(G\) be \(P_\rho\)-generic over \(V\). Let \(\beta\) and \(\alpha\) be ordinals smaller than \(\rho\) with \(\beta\) a limit ordinal.

(a) There is a function \(h_{\beta,\alpha} \in V\) such that domain \((h_{\beta,\alpha}) \in G\) and such that, for all \(f : S' \to V\) with \(S' \in G\) and \(f \in V\),
\[\[f\]_G \in_G [h_{\beta,\alpha}]_G\]
\[\leftrightarrow (\exists S \in G \cap V_\beta)(\exists g \in V) (g : S \to V_\alpha \land [f]_G = [g]_G).\]
(b) If \( \prod_{G}(V; \in) \) is wellfounded then
\[
\{ \pi_{G}(\lfloor f \rfloor_{G}) \mid (\exists S \in G \cap V_{\beta})(f \in V \land f : S \rightarrow V_{\alpha}) \} \in \text{Ult}(V; G).
\]

**Proof.** To make the proof of (a) more comprehensible, we first prove (b).
Assume that \( \prod_{G}(V; \in) \) is wellfounded. By Lemma 9.2.12, we have that \( V_{\rho} \subseteq \text{Ult}(V; G) \).
For \( \gamma < \rho \), let \( i_{\gamma} = i_{G} \mid V_{\gamma} \).
Corollary 9.2.9 gives that
\[
i_{G}''V_{\gamma} = \pi_{G}(\lfloor \text{id}_{P(V_{\gamma})} \rfloor_{G}) \in \text{Ult}(V; G).
\]
Hence \( i_{\gamma} \in \text{Ult}(V; G) \).
Let \( \gamma = \max\{\alpha, \beta\} + 3 \).
Part (b) of Lemma 9.2.11 implies that our desired set is
\[
\{ z \mid (\exists X \in V_{\beta})(\exists f \in V_{\gamma})(f : P(X) \rightarrow V_{\alpha} \land z = (i_{G}(f))(i_{G}''X)) \},
\]
But this is the same as
\[
\{ z \mid (\exists X \in V_{\beta})(\exists f \in V_{\gamma})(f : P(X) \rightarrow V_{\alpha} \land z = (i_{\gamma}(f))(i_{\gamma}''X)) \},
\]
which in turn is the same as
\[
\{ z \mid (\exists X \in V_{\beta})(\exists f \in V_{\gamma})(f : P(X) \rightarrow V_{\alpha} \land z = (i_{\gamma}(f))(i_{\gamma}''X)) \},
\]
and this set belongs to \( \text{Ult}(V; G) \).
For the proof of (a), first note that, by part (1)(a) of Lemma 9.2.12, \( V_{\rho} \) is a subset of the range of \( \pi_{G} \).
For \( x \in V_{\rho} \), let \( x^{*} = \pi_{G}^{-1}(x) \).
For \( \gamma < \rho \), \( i'_{G} \mid V_{\gamma} \) “belongs to” \( \prod_{G}(V; \in) \) in a sense we make precise as follows.
Recall the functions \( \pi_{Y} \) and \( \text{prj}_{x}^{V} \) defined just before Lemma 9.2.12.
For \( \gamma < \rho \), define a function \( f_{\gamma} \) with domain \( P(V_{\gamma}) \) by setting \( f_{\gamma}(Y) = \pi_{Y}^{-1} \).
Note that, for all all \( Y \subseteq V_{\gamma} \) and all \( x \in Y \),
\[
(f_{\gamma}(Y))^{-1}(x) = \text{prj}_{x}^{V_{\gamma}}(Y).
\]
Let \( i_{\gamma} = \lfloor f_{\gamma} \rfloor_{G} \).
Part (1)(b) of Lemma 9.2.12 implies that \( \prod_{G}(V; \in) \models \text{“} i_{\gamma} \text{ is a function whose domain is } (V_{\gamma})^{*} \text{.”} \)
For \( x \in V_{\gamma} \), Theorem 9.2.7 and the preceding displayed equation give that
\[
\prod_{G}(V; \in) \models i_{\gamma}^{-1}(i'_{G}(x)) = [\text{prj}_{x}^{V_{\gamma}}]_{G}.
\]
By part (1)(c) of Lemma 9.2.12, \([\text{proj}^V_G]_G = x^*\). Therefore
\[
\prod_G (V; \in) \models \tilde{i}_\gamma(x^*) = i'_G(x),
\]
for all \(x \in V_\gamma\).

For \(\gamma < \rho\) and \(S \in G \cap V_\gamma\), Lemma 9.2.8 gives us that
\[
\prod_G (V; \in) \models [\text{id}_S]_G = \tilde{i}_\gamma''(\bigcup S)^*.
\]
In particular, we have for \(X \in V_\gamma\) that
\[
\prod_G (V; \in) \models [\text{id}_{P(X)}]_G = \tilde{i}_\gamma'' X^*.
\]

Now let \(\gamma \geq \max\{\beta, \alpha\} + 3\) and let \(h_{\beta, \alpha}\) give \([h_{\beta, \alpha}]_G\) such that \(\prod_G (V; \in)\) satisfies the formula
\[
(\forall z)(z \in [h_{\beta, \alpha}]_G \leftrightarrow (\exists x)(\exists y \in (V_\beta)^*)(x : \mathcal{P}(y) \rightarrow (V_\alpha)^* \land z = (\tilde{i}_\gamma(x))(\tilde{i}_\gamma'' y))).
\]
By part (a) of Lemma 9.2.11, \([h_{\beta, \alpha}]_G\) is as required for (a). \(\square\)

Remarks:

(1) In the both the proof of (b) and that of (a), we made implicit use of the fact that \([f]_G = [f']_G\) when \(f \subseteq f'\), domain \((f) = S \in G\), and domain \((f') = \mathcal{P}(\bigcup S)\). Instead we could, in proving (b), have shown \(G \cap V_\gamma \in \text{Ult}(V; G)\) for \(\gamma < \rho\), and we could have established an analogous fact in proving (a).

(2) The assumption that \(\beta\) is a limit ordinal is not really necessary.

We finish this section by investigating possible fixed points of the embeddings \(i_G\).

An cardinal \(\kappa > \aleph_1\) is completely Jonsson if for every stationary set \(S\) such that \(\bigcup S \in \kappa\), the following set is stationary in \(\mathcal{P}(\kappa)\):
\[
\{Y \subseteq \kappa \mid \kappa = |Y| \land Y \cap \bigcup S \in S\}.
\]

Lemma 9.2.14. Let \(\kappa\) be a measurable cardinal and let \(\mathcal{U}\) be a uniform normal ultrafilter on \(\kappa\). Then \(\kappa\) is completely Jonsson. Moreover the set of all completely Jonsson cardinals smaller than \(\kappa\) belongs to \(\mathcal{U}\).
**Proof.** The first assertion implies the second, since it implies that $\kappa$ is completely Jonsson in $\text{Ult}(V; U)$.

For the first assertion, let $S$ be stationary with $\bigcup S = \gamma < \kappa$. Let $f : <^\omega \kappa \to \kappa$. We must find a $Y \subseteq \kappa$ with $|Y| = \kappa$, $Y \cap \bigcup S \in S$, and $f'' < \omega Y \subseteq Y$.

Let $g_i$, $i \in \omega$, be all compositions of the functions $f \restriction n \kappa$, $n \in \omega$. If $g_i : n \kappa \to \kappa$ and if $p$ is a permutation of $\{1, \ldots, n_i\}$, then let $g^p_i : [\kappa]^{n_i} \to \kappa$ be defined by setting

$$g^p_i(z) = g_i((z_{p(1)}, \ldots, z_{p(n_i)})),$$

where, by our usual convention $z = \{z_1, \ldots, z_{n_i}\}$ and $z_1 < \cdots < z_{n_i}$. For each $i$ and each $p$, let $h^p_i : [\kappa]^{n_i} \to \gamma$ be given by

$$h^p_i(z) = \begin{cases} g^p_i(z) & \text{if } g^p_i(z) < \gamma; \\ 0 & \text{otherwise.} \end{cases}$$

For each $i$, for each $j \leq n_i$, for each permutation $p$ of $\{1, \ldots, n_i\}$, and for each $x \in [\gamma]^j$, there is a set $Z^p_{i,x} \in \mathcal{U}$ such that $Z^p_{i,x} \cap \gamma = \emptyset$ and such that $h^p_i(x \cup z)$ is constant for $z \in [Z^p_{i,x}]^{n_i-j}$. Let $Z$ be the intersection of all the $Z^p_{i,x}$. Then $Z \in \mathcal{U}$ and so $|Z| = \kappa$.

For each $i$, $j$, and $p$, let $\bar{h}^p_{i,j} : [\gamma]^j \to \gamma$ be given by setting $\bar{h}^p_{i,j}(x) = h^p_i(x \cup z)$ for $z \in [Z]^{n_i-j}$. Using the fact that $S$ is stationary, one can easily show that there is an $X \in S$ with $X$ closed under the $\bar{h}^p_{i,j}$. Let $Y$ be the closure of $X \cup Z$ under $f$. Then $Z \subseteq Y$ and $Y \cap \gamma = X \in S$. □

**Remark.** See Exercise 9.2.4 for a slightly different proof of the lemma, with a weaker hypothesis for the first assertion.

**Lemma 9.2.15.** Let $\kappa$ be completely Jonsson.

(a) If $\kappa > 2^{\aleph_0}$ then $\kappa$ is a strong limit cardinal.

(b) If $\kappa \geq |V_{\omega_1}|$ then $|V_\kappa| = \kappa$.

**Proof.** Let $\gamma < \kappa$. The set $\mathcal{P}_{\omega_1}(\gamma)$ is stationary in $\mathcal{P}(\gamma)$. Since $\kappa$ is completely Jonsson, it follows with the aid of condition (4) of Lemma 9.1.4 that there is an $X \prec V_{\kappa+1}$ such that $|X \cap \kappa| = \kappa$, $\gamma \in X$, and $X \cap \gamma$ is countable. Let $\pi : (X; \in) \cong (M; \in)$ with $M$ transitive. Then $\pi(\kappa) = \kappa$ and $\pi(\gamma) < \omega_1$. 


To prove (a), suppose that $2^{\lvert\gamma\rvert} \geq \kappa$. Then $M \models 2^{\pi(\gamma)} \geq \kappa$, and so some sequence of $\kappa$ distinct subsets of $\pi(\gamma)$ belongs to $M$. Hence $\kappa \leq 2^{\aleph_0}$.

To prove (b), suppose that $\lvert V_\gamma \rvert \geq \kappa$. Then some sequence of $\kappa$ distinct elements of $V^M_{\pi(\gamma)}$ belongs to $M$. Hence $\kappa \leq \lvert V_{\pi(\gamma)} \rvert < \lvert V_{\omega_1} \rvert$. □

The following lemma follows easily from part (b) of Lemma 9.2.15 and the definition of “completely Jonsson.”

**Lemma 9.2.16.** Let $\kappa \geq \lvert V_{\omega_1} \rvert$ be completely Jonsson. Let $\gamma < \kappa$ and let $S$ be stationary in $\mathcal{P}(V_\gamma)$. Let

$$\tilde{S} = \{Y \subseteq \kappa \cup V_\gamma \mid \kappa = \lvert Y \cap \kappa \rvert \land Y \cap V_\gamma \in S\}.$$ 

Then $\tilde{S}$ is stationary in $\mathcal{P}(\kappa \cup V_\gamma)$.

**Lemma 9.2.17.** Let $\kappa$ be a cardinal and let $\rho > \kappa$ be limit ordinal. Suppose that $G$ is $\mathcal{P}_\rho$-generic over $V$ and that $\{Y \subseteq \kappa \mid \lvert Y \rvert = \kappa\} \in G$. Then $i_G(\kappa) = \kappa$.

**Proof.** By Lemma 9.2.12,

$$\kappa = \pi_G([\text{trcoll } \mathcal{P}(\kappa)]_G).$$

But $\text{trcoll } (Y) = \kappa$ for all $Y \subseteq \kappa$ such that $\lvert Y \rvert = \kappa$. Thus

$$\kappa = \pi_G([c_\kappa]_G) = i_G(\kappa).$$

□

**Lemma 9.2.18.** Let $\kappa \geq \lvert V_{\omega_1} \rvert$ be completely Jonsson and let $\rho > \kappa$ be a limit ordinal. Let $\gamma < \kappa$ and let $S$ be stationary in $\mathcal{P}(V_\gamma)$. Then there is a $G$ that is $\mathcal{P}_\rho$-generic over $V$ and such that $S \in G$ and $i_G(\kappa) = \kappa$.

**Proof.** Let $\tilde{S}$ be given by Lemma 9.2.16. Let $G$ be $\mathcal{P}_\rho$-generic over $V$ with $\tilde{S} \in G$. Then $\tilde{S} \leq \hat{S} \res V_\gamma \leq S \in G$. Lemma 9.2.17 implies that $i_G(\kappa) = \kappa$. □

Thinking of Ord as an ordinal number, we can regard $\mathcal{P}_{\text{Ord}}$ as defined by our definition of the $\mathcal{P}_\kappa$. Exercise 9.2.7 gives sufficient conditions for $\prod_G(V; \varepsilon)$ to be wellfounded and isomorphic to $V[G]$ for every $G$ that is $\mathcal{P}_{\text{Ord}}$-generic over $V$. 

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To prove (a), suppose that $2^{\lvert\gamma\rvert} \geq \kappa$. Then $M \models 2^{\pi(\gamma)} \geq \kappa$, and so some sequence of $\kappa$ distinct subsets of $\pi(\gamma)$ belongs to $M$. Hence $\kappa \leq 2^{\aleph_0}$.

To prove (b), suppose that $\lvert V_\gamma \rvert \geq \kappa$. Then some sequence of $\kappa$ distinct elements of $V^M_{\pi(\gamma)}$ belongs to $M$. Hence $\kappa \leq \lvert V_{\pi(\gamma)} \rvert < \lvert V_{\omega_1} \rvert$. □
Exercise 9.2.1. Show how to define *precipitous* by a formula of the language of set theory.

*Hint.* For $G$ that is $P(I)$-generic over $V$ and for each ordinal $\alpha$, let $\prod_G(V_\alpha; \in)$ be defined in the obvious way. Show that $\prod_G(V; \in)$ is wellfounded if and only if $\prod_G(V(2^\kappa)^+; \in)$ is wellfounded.

For a more combinatorial equivalent of precipitousness, see Jech–Prikry [1976], where the concept was introduced, or §35 of Jech [1978].

Exercise 9.2.2. Suppose that $\kappa$ is a measurable cardinal. Let $U$ be a uniform normal ultrafilter on $\kappa$. Let $G$ be $\text{Coll}(\omega, < \kappa)$-generic over $V$. (See page 538.) Prove that in $V[G]$ the set $\mathcal{I}_{\text{ns}}$ generates a precipitous ideal on $\omega_1$. This is a theorem of Jech–Magidor–Mitchell–Prikry [1980].

Exercise 9.2.3. For cardinals $\lambda$, an ideal $I$ on a set $A$ is $\lambda$-*saturated* if $P(I)$ has the $\lambda$-chain condition.

Suppose that $I$ is a $\kappa^+$-saturated, $\kappa$-complete ideal on an uncountable regular cardinal $\kappa$. Prove that $I$ is precipitous.

*Hint.* First show that if $\{B_x \mid x \in X\}$ is any antichain in $P(I)$ then there are $\{B'_x \mid x \in X\}$ such that the symmetric difference of $B_x$ and $B'_x$ belongs to $I$ for each $x$ and such that the $B_x$ are pairwise disjoint.

Exercise 9.2.4. A cardinal $\kappa$ is *Ramsey* if, for every $f : [\kappa]^{<\omega} \to \{0, 1\}$, there is a subset $Z$ of $\kappa$ such that $|Z| = \kappa$ and such that $f$ is constant on $[Z]^{<\omega}$. Clearly every measurable cardinal is Ramsey.

(1) Let $U$ be a uniform normal ultrafilter on a cardinal $\kappa$. Prove that $\{\alpha < \kappa \mid \alpha$ is Ramsey\} belongs to $U$.

(2) Prove that every Ramsey cardinal is completely Jonsson.

Exercise 9.2.5. Prove that no singular completely Jonsson cardinal has cofinality greater than $\omega$.

Exercise 9.2.6. A cardinal $\kappa$ is *Jonsson* if every model of size $\kappa$ for a countable language has a proper elementary submodel. An uncountable cardinal $\kappa$ is *Rowbottom* if, for every $\gamma < \kappa$ and every model $(\kappa; \gamma, \ldots)$ for a countable language, there is an $(A; B, \ldots) \prec (\kappa; \gamma, \ldots)$ such that $|A| = \kappa$ and such that $B$ is countable. Prove that every completely Jonsson cardinal is Rowbottom and that every Rowbottom cardinal is Jonsson.
Exercise 9.2.7. Assume that there is a proper class of completely Jonsson cardinals. Let $G$ be $\text{P}_{\text{Ord}}$-generic over $V$, in the sense that $G$ meets every dense subset of $\text{P}_{\text{Ord}}$. Let $V[G] = \bigcup_{\kappa \in \text{Ord}} L(V, G \cap V_\kappa)$. Then $\prod_G(V; \in)$ is wellfounded and $\text{Ult}(V; G) = V[G]$.

9.3 $\text{P}_\kappa$ for $\kappa$ Woodin

If $\kappa$ is Woodin and $G$ is $\text{P}_\kappa$-generic over $V$, then the generic ultrapower coming from $G$ is wellfounded and has closure properties in the forcing extension $V[G]$. Moreover, for ordinals $\rho > \kappa$, there are objects $G$ that are $\text{P}_\rho$-generic over $V$ and such that $G \cap \text{P}_\kappa$ is $\text{P}_\kappa$-generic over $V$. The business of this section is to prove these facts.

We begin by introducing some technical concepts.

Let $\rho$ be a limit ordinal and let $A \subseteq \text{P}_\rho$. A set $X$ captures $A$ if there is an $S \in A \cap X$ such that $X \cap \bigcup S \in S$.

In our main applications of our results about capturing subsets of $\text{P}_\rho$, we will be dealing with sets $A$ such that $A$ is an antichain in $\text{P}_\rho$, i.e., such that the members of $A$ are pairwise incompatible. The following lemma shows that for such an $A$ there is a large class of sets $X$ such that the witness, if any, to $X$’s capturing $A$ is unique.

Lemma 9.3.1. Let $\rho$ be a limit ordinal and let $A \subseteq \text{P}_\rho$. Let $\lambda$ be an ordinal $\geq \rho$ and let $X \prec V_\lambda$. If both $S$ and $S'$ witness that $X$ captures $A$, then $S$ and $S'$ are compatible elements of $\text{P}_\rho$.

Proof. For any limit ordinal $\gamma$, or for $\gamma = \text{Ord}$, and for any incompatible elements $S$ and $S'$ of $\text{P}_\gamma$, there is a set $C$ club in $\mathcal{P}(\bigcup(S \cup S'))$ such that

$$(\forall Y \in C)(Y \cap \bigcup S \notin S \lor Y \cap \bigcup S' \notin S').$$

The formal version of the preceding sentence, without “or for $\gamma = \text{Ord}$” if $\lambda$ is not a limit ordinal, is true in $V_\lambda$. Since $X \prec V_\lambda$, it is also true in $X$, and the witnesses $C$ to the truth in $X$ of instances of it are witnesses to the truth in $V$ of these instances.

Assume that $S$ and $S'$ are incompatible elements of $\text{P}_\rho$, both of which witness that $X$ captures $A$. Since $\lambda \geq \rho$ and since both $S$ and $S'$ belong to $X$, it follows that there is a $C \in X$ witnessing as above (in $X$ and in $V$) the
incompatibility of $S$ and $S'$. Fix such a $C \subseteq X$. Let $f : \omega^\omega(S \cup S') \rightarrow \bigcup(S \cup S')$ be such that
\[ C = \{ Y \subseteq \bigcup(S \cup S') \mid f^n Y \subseteq Y \}. \]
The set $X \cap \bigcup(S \cup S')$ is closed under $f$. Thus $X \cap \bigcup(S \cup S') \in C$. Because $C$ witnesses the incompatibility of $S$ and $S'$,
\[ ((X \cap \bigcup(S \cup S')) \cap \bigcup S \notin S) \lor ((X \cap \bigcup(S \cup S')) \cap \bigcup S' \notin S'). \]
In other words,
\[ X \cap \bigcup S \notin S \lor X \cap \bigcup S' \notin S'. \]
But this contradicts the assumption that $S$ and $S'$ witness that $X$ captures $A$. \[ \square \]

Say that a set $W$ is an end extension of a set $Z$ if $W \cap \bigcup Z \subseteq Z$.

Recall that a subset $D$ of a partially ordered set $P$ is predense if each element of $P$ is compatible with an element of $D$.

Let $\kappa$ be an uncountable strong limit cardinal. A predense subset $A$ of $P_\kappa$ is semiproper if there are arbitrarily large ordinals $\lambda$ such that, for every $X \prec V_\lambda$ such that $|X| < \kappa$ and $A \in X$, there is a $Y$ such that
\begin{enumerate}
  \item $Y \prec V_\lambda$;
  \item $X \subseteq Y$;
  \item $Y$ captures $A$;
  \item $Y \cap V_\kappa$ is an end extension of $X \cap V_\kappa$.
\end{enumerate}

Remark. If we omitted the word “predense” from the definition of semiproper, then we would obtain a coextensive concept, for it would be provable that every semiproper set is predense. (Exercise 9.3.1.)

If $A$ is a predense subset of $P_\kappa$ and $X$ is a set, let us say that a set $Y$ is $(A, X)$-good if clauses (ii), (iii), and (iv) above hold.

Lemma 9.3.2. Let $\kappa$ be an uncountable strong limit cardinal and let $A$ be a predense subset of $P_\kappa$. The following are equivalent:

1. $A$ is semiproper.
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(2) The set of all $X \in \mathcal{P}_\kappa(V_{\kappa+1})$ such that there is a $Y \prec V_{\kappa+1}$ that is $(A, X)$-good belongs to $\mathcal{F}_{\mathcal{P}_\kappa}^{\text{club}}(V_{\kappa+1})$.

(3) Let $M$ be a transitive set such that $V_\kappa M \subseteq M$ and $V_{\kappa+2} \subseteq M$. Then, for any $X \prec M$ such that $|X| < \kappa$ and $\{\kappa, A\} \subseteq X$, there is a $Y \prec M$ such that $Y$ is $(A, X)$-good.

(4) For every limit ordinal $\lambda$ of cofinality greater than $|V_\kappa|$ and for every $X \prec V_\lambda$ such that $|X| < \kappa$ and $\{\kappa, A\} \subseteq X$, there is a $Y \prec V_\lambda$ such that $Y$ is $(A, X)$-good.

Proof. To show that (1) implies (2), assume that $A$ is semiproper. Let $\lambda > \kappa$ be given by the fact that $A$ is semiproper. By (4) of Lemma 9.1.7, the set

$$
\{X \in \mathcal{P}_\kappa(V_{\kappa+1}) \mid (\exists M \prec V_\lambda)(|M| < \kappa \land A \in M \land X = M \cap V_{\kappa+1})\}
$$

belongs to $\mathcal{F}_{\mathcal{P}_\kappa}^{\text{club}}(V_{\kappa+1})$. Let $X$ belong to this set and let $M$ witness that it belongs. The semiproperness of $A$ gives us a $Y \prec V_\lambda$ that is $(A, M)$-good.

We next prove that (2) implies (3). Assume that (2) holds. Let $M$ be as in the statement of (3). Let $X \prec M$ be such that $|X| < \kappa$ and $\{\kappa, A\} \subseteq X$. By (2), there is $f : \langle \omega V_{\kappa+1} \rightarrow V_{\kappa+1}$ such that

$$
(\forall Z \in \mathcal{P}_\kappa(V_{\kappa+1}))(f'' \subseteq Z \subseteq Z \rightarrow (\exists Y \prec V_{\kappa+1}) Y \text{ is } (A, Z)-\text{good}).
$$

Any such $f$ is coded by an element of $V_{\kappa+2}$ and therefore belongs to $M$. Since $X \prec M$ and $\{\kappa, A\} \subseteq X$, some such $f$ belongs to $X$. For this $f$, $X \cap V_{\kappa+1}$ is closed under $f$. Thus there is a $Y \prec V_{\kappa+1}$ that is $(A, X \cap V_{\kappa+1})$-good.

Let

$$
Y^* = \{g(u) \mid g \in X \land g : \langle \omega V_\kappa \rightarrow M \land u \in \langle \omega (Y \cap V_\kappa)\}
$$

The constant functions witness that $X \subseteq Y^*$.

Suppose that $y \in Y^* \cap V_\kappa$. It is easily seen that there are $g$ and $u$ witnessing $y \in Y^*$ with $g : \langle \omega V_\kappa \rightarrow V_\kappa$. For such a $g$, $g \in X \cap V_{\kappa+1}$, and so $g \in Y$. Since $u \in Y$, the fact that $Y \prec V_{\kappa+1}$ implies that $y \in Y$. We have thus shown that $Y^* \cap V_\kappa \subseteq Y$. Clearly $Y \cap V_\kappa \subseteq Y^* \cap V_\kappa$. Hence $Y^* \cap V_\kappa = Y \cap V_\kappa$. Therefore $Y^*$ captures $A$ and $Y^* \cap V_\kappa$ is an end extension of $X \cap V_\kappa$. 
We have shown that $Y^*$ is $(A, X)$-good. It remains to show that $Y^* \prec M$.
Let $\varphi(v_0, \ldots, v_n)$ be a formula and let $y_1, \ldots, y_n$ be elements of $Y^*$. Assume that

$$(\exists y_0 \in M) \ M \models \varphi[y_0, y_1, \ldots, y_n].$$

We must prove that there is a $y_0 \in Y^*$ such that $M \models \varphi[y_0, y_1, \ldots, y_n]$.

For $1 \leq i \leq n$, let $g_i$ and $u_i$ witness that $y_i \in Y^*$. There is a function $h : \omega \rightarrow V_\kappa$ such that for all $u'_1, \ldots, u'_n \in V_\kappa$ with $\ell h(u'_i) = \ell h(u_i)$,

$$(\exists y \in M) \ M \models \varphi[y, g_1(u'_1), \ldots, g_n(u'_n)] \rightarrow M \models \varphi[h(u'_1 \cdots u'_n), g_1(u'_1), \ldots, g_n(u'_n)].$$

Since $V_\kappa \subseteq M$, any such $h$ belongs to $M$. Now $X \prec M$, and all the $g_i$ belong to $X$. Hence there is such an $h \in X$. Our desired $y_0$ is $h(u_1 \cdots u_n)$.

To see that (3) implies (4), note that the hypotheses of (3) hold for $M = V_\lambda$ for every limit $\lambda$ such that $\text{cf}(\lambda) > |V_\kappa|$.

(4) trivially implies (1). \[\square\]

Remark. At the cost of a little more work in the proof, we could have dropped the condition that $\kappa \in X$ from (3) and (4) of the lemma. The fact that $\kappa \in X$ could be deduced, when needed, from the fact that $A \in X$.

We are now ready for the main technical theorem about $P_\kappa$ for Woodin $\kappa$.

**Theorem 9.3.3.** Let $\kappa$ be a Woodin cardinal. Let $\langle A_\alpha | \alpha < \kappa \rangle$ be a sequence of predense subsets of $P_\kappa$. Then the set of inaccessible cardinals $\delta < \kappa$ such that

$$(\forall \alpha < \delta) \ A_\alpha \cap P_\delta \ is \ predense \ in \ P_\delta \ and \ is \ semiproper$$

is stationary in $\kappa$.

**Proof.** Let

$$D = \{ \delta < \kappa \mid \delta \ is \ a \ limit \ \wedge \ (\forall \alpha < \delta) \ A_\alpha \cap P_\delta \ is \ predense \ in \ P_\delta \}. $$

For $\alpha < \kappa$ and $S \in P_\kappa$, let $\beta(\alpha, S)$ be the least limit $\beta$ such that $S \in P_\beta$ and some $A \in A_\alpha \cap P_\beta$ is compatible with $S$ in $P_\kappa$. If $S$ and $A$ are any compatible elements of $P_\kappa$, then parts (b) and (c) of Lemma 9.2.5 imply that they have a common extension whose union is $\bigcup(S \cup A)$. Thus $\beta(\alpha, S)$ is the least
limit $\beta$ such that $S \in P_\beta$ and some $A \in A_\alpha \cap P_\beta$ is compatible with $S$ in $P_\beta$. Evidently
\[ D = \{ \delta < \kappa \mid (\forall \alpha < \delta)(\forall S \in P_\delta) \beta(\alpha, S) \leq \delta \}. \]
Hence $D$ is club in $\kappa$.

Assume that the conclusion of the theorem does not hold. Let $C \subseteq D$ be club in $\kappa$ and such that
\[ (\forall \delta \in C)(\delta \text{ is inaccessible} \rightarrow (\exists \alpha < \delta) A_\alpha \cap P_\delta \text{ is not semiproper}). \]
Define $f : \kappa \rightarrow \kappa$ by setting
\[ f(\delta) = \mu \gamma (\delta < \gamma \land \gamma \in C). \]
For $\delta < \kappa$, let $g(\delta) = f(\delta) + \omega$. By Theorem 6.3.8, let $\delta$ and $j : V \prec M$ be such that $\delta$ and $j$ witness that $\kappa$ is Woodin for $g$ and such that $<^M \delta M \subseteq M$.

Since $\delta$ is closed under $g$ and hence under $f$, we have that $\delta \in C$. Moreover $\delta$ is measurable and therefore inaccessible. This gives us an $\alpha < \delta$ such that $A_\alpha \cap P_\delta$ is predense in $P_\delta$ but not semiproper.

Let $S$ be the set of all $X \in P_\delta(V_{\delta+1})$ such that there is no $Y \prec V_{\delta+1}$ that is $(A_\alpha \cap P_\delta, X)$-good. Since $A_\alpha \cap P_\delta$ is not semiproper, clause (2) of Lemma 9.3.2 shows that $S$ is stationary in $P_\delta(V_{\delta+1})$. Because $V_{\delta+2}^M = V_{\delta+2}$, it is also true in $M$ that $S$ is stationary in $P_\delta(V_{\delta+1})$.

We will reach a contradiction by building an $X \in j(S)$ and a $Y \prec V_{j(\delta)+1}^M$ such that $Y$ is $(j(A_\alpha \cap P_\delta), X)$-good in $M$.

The ordinal $(j(f))(\delta)$ belongs to $j(C)$ and therefore to $j(D)$. Since $\alpha = j(\alpha) < \delta < (j(f))(\delta)$, it follows from the elementarity of $j$ that $j(A_\alpha) \cap P_{(j(f))(\delta)}^M$ is predense in $P_{(j(f))(\delta)}^M$. But $V_{(j(f))(\delta)}^M = V_{(j(f))(\delta)}$, and so $P_{(j(f))(\delta)}^M = P_{(j(f))(\delta)}$ and $j(A_\alpha) \cap P_{(j(f))(\delta)}$ is predense in $P_{(j(f))(\delta)}$ in $V$.

Let then $A \in j(A_\alpha) \cap P_{(j(f))(\delta)}$ be such that $A$ and $S$ have a common extension in $P_{(j(f))(\delta)}$ and let $\tilde{S}$ be such a common extension. Shrinking $\tilde{S}$ by a nonstationary set if necessary, we may assume that
\[ (\forall X \in \tilde{S})(X \cap V_{\delta+1} \in S \land X \cap \bigcup A \in A). \]
Let $h : <^\omega(V_{j(\delta)+1}^M) \rightarrow V_{j(\delta)+1}^M$ belong to $M$ and be such that any nonempty subset of $V_{j(\delta)+1}^M$ closed under $h$ is an elementary submodel of $V_{j(\delta)+1}^M$. Such an $h$ can be gotten in $M$ from a set of Skolem functions for $V_{j(\delta)+1}^M$.

Since $\tilde{S}$ is stationary, there is (in $V$) a set $Z \subseteq V_{j(\delta)+1}$ such that
(a) $Z$ is closed under $h$;

(b) $A \in Z$ and $Z$ is closed under $j \upharpoonright V_{\delta+1}$;

(c) $Z \cap \bigcup \tilde{S} \in \tilde{S}$.

Consider the sets $Z \cap V_{\delta+1}$, $j''(Z \cap V_{\delta+1})$, and $Z \cap \bigcup \tilde{S}$. The first is an element of $S$, and so the first two have size $< \delta$, and the third belongs to $V_{j(j(f))}(\delta)$. Therefore all three belong to $M$. In addition, all three are subsets of $Z$.

Let $Y$ be the closure under $h$ of

$$(Z \cap V_{\delta+1}) \cup j''(Z \cap V_{\delta+1}) \cup (Z \cap \bigcup \tilde{S}) \cup \{A\}.$$ 

It is immediate that $Y \in M$, $Y \subseteq Z$, $Y \prec V^M_{j(\delta)+1}$, $A \in Y$, and $Y \cap \bigcup \tilde{S} = Z \cap \bigcup \tilde{S} \in \tilde{S}$. The last two facts imply that $A \in (j(A_\alpha) \cap P_{j(f)}(\delta)) \cap Y \subseteq j(A_\alpha \cap P_\delta) \cap Y$ and that $Y \cup A \in A$ respectively. Thus $Y$ captures $j(A_\alpha \cap P_\delta)$.

Let $X = j(Y \cap V_{\delta+1})$. Now $Y \cap V_{\delta+1} = Z \cap V_{\delta+1} \subseteq S$. This implies directly that $X \in j(S)$. But it also implies that $|Y \cap V_{\delta+1}| < \delta$. Hence $X = j''(Y \cap V_{\delta+1}) \subseteq Y$.

To show that $Y \cap V^M_{j(\delta)}$ is an end extension of $X \cap V^M_{j(\delta)}$, let $a$ be any element of $X \cap V^M_{j(\delta)}$. We must show that $a \cap Y \subseteq X$. There is a $b \in Y \cap V_{\delta+1}$ such that $a = j(b)$. This $b$ must belong to $V_\delta$, and so $a = b$. But $X \cap V_\delta = j(Y \cap V_\delta) = Y \cap V_\delta$. Thus $a \cap Y = a \cap X$.

We have proved that, in $M$, $Y$ is good for $(j(A_\alpha \cap P_\delta), X)$. This, together with the facts that $Y \prec V^M_{j(\delta)+1}$ and $X \in j(S)$, contradicts the definition of $S$. \hfill \Box

If $\rho$ is a limit ordinal and $A$ is predense in $P_\rho$, then an element $S$ of $P_\rho$ seals off $A$ if $\bigcup S$ is transitive and every member of $S$ captures $A$. Suppose that $\delta \leq \rho$ and that $\langle A_\alpha \mid \alpha < \delta \rangle$ is a sequence of predense subsets of $P_\rho$. Then an element $S$ of $P_\rho$ seals off $\langle A_\alpha \mid \alpha < \delta \rangle$ if $\bigcup S$ is transitive and, for all $X \in S$ and for all $\alpha \in X \cap \delta$, $X$ captures $A_\alpha$.

Remark. An alternative definition of sealing off for sequences would require that $\delta < \rho$ and that $\delta \subseteq \bigcup S$. Suppose we adopted this definition. In the proof of the modified Theorem 9.3.5, we would have to choose $\delta$ so that $\delta \leq \delta$. This would be offset by our not having to specifically require, in the proof of Theorem 9.3.8, that $\delta \subseteq \bigcup S$.

Lemma 9.3.1 has the following consequences for sealing off.
Lemma 9.3.4. Let $\rho$ be a limit ordinal and let $S \in P_\rho$.

(a) Let $A$ be a maximal antichain in $P_\rho$. Assume that $S$ seals off $A$. Then there is an $S' \in P_\rho$ such that

(i) $\bigcup S = \bigcup S'$, $S' \subseteq S$, and $S \backslash S'$ is not stationary in $S$ (and so $S \leq S'$ and $S' \leq S$);

(ii) for all $X \in S'$, there is a unique $A \in A \cap X$ such that $X \cap \bigcup A \in A$.

Proof. (a) Let $S' = S \cap \{ Y \cap \bigcup S \mid Y \prec V_\rho \}$. (i) is satisfied. For (ii), let $X \in S'$. Let $Y \prec V_\rho$ be such that $X = Y \cap \bigcup S$. The definition of sealing off requires that $\bigcup S$ be transitive, so $Y$ is an end extension of $X$. If $A$ is a witness to $X$’s capturing $A$, then $A \in \bigcup S$, and so $\bigcup A \subseteq \bigcup S$. Hence $Y \cap \bigcup A = X \cap \bigcup A$. Therefore every witness to $X$’s capturing $A$ is also a witness to $Y$’s capturing $A$. By Lemma 9.3.1, there is at most one such $A$.

(b) The proof of (b) is similar, and we omit it. \qed

The following theorem is a consequence of Theorem 9.3.3. Parts of its proof will be ingredients in the proofs of further consequences of that theorem.

Theorem 9.3.5. Let $\kappa$ be a Woodin cardinal, let $\bar{\delta} \leq \kappa$, and let $\langle A_\alpha \mid \alpha < \bar{\delta} \rangle$ be a sequence of predense subsets of $P_\kappa$. Then the set of $S \in P_\kappa$ such that $S$ seals off $\langle A_\alpha \mid \alpha < \bar{\delta} \rangle$ is dense in $P_\kappa$.

Proof. Without loss of generality, we may assume that $\bar{\delta} = \kappa$.

Let $S \in P_\kappa$. By Theorem 9.3.3, let $\delta < \kappa$ be inaccessible and such that $S \in P_\delta$ and, for all $\alpha < \delta$, $A_\alpha \cap P_\delta$ is predense in $P_\delta$ and semiproper.

Let

$$S' = \{ X \in P_\delta(V_\delta) \mid X \cap \bigcup S \in S \land (\forall \alpha \in X \cap \delta) X \text{ captures } A_\alpha \cap P_\delta \}.$$ 

We will show that $S'$ is stationary in $P_\delta(V_\delta)$. This will show that $\bigcup S' = V_\delta$, a transitive set, and so it will follow from the definition of $S'$ that $S'$ seals off $\langle A_\alpha \mid \alpha < \kappa \rangle$. Since $S' \res \bigcup S \subseteq S$, it will follow also that $S' \leq S$. Thus the theorem will be proved.
Let \( f : \omega V_\delta \to V_\delta \). We must find a \( Z \in S' \) such that \( Z \) is closed under \( f \). By (4) of Lemma 9.3.2, let \( \lambda > \delta + 1 \) be such that, for all \( \alpha < \delta \) and all \( X < V_\lambda \) with \( A_\alpha \cap P_\delta \in X \) and \( |X| < \delta \), there is a \( Y < V_\lambda \) such that \( Y \) is \((A_\alpha, X)\)-good.

We will construct an elementary chain \( \langle X_\beta \mid \beta \leq \gamma \rangle \) for some \( \gamma < \delta \). Each \( X_\beta \) will be an element of \( \mathcal{P}_\delta(V_\lambda) \) such that \( X_\beta < V_\lambda \). For \( \beta' < \beta \leq \gamma \), \( X_\beta \cap V_\delta \) will be an end extension of \( X_{\beta'} \cap V_\delta \).

To begin the construction, let \( X < V_\lambda \) be such that \( \{ \delta, f, S, \langle A_\alpha \cap P_\delta \mid \alpha < \delta \rangle \} \subseteq X \) and \( X \cap \bigcup S \in S \). Such an \( X \) exists by part (4) of Lemma 9.1.4, since \( S \) is stationary. Now let \( X_0 \) be any elementary submodel of \( X \) with \( |X_0| < \delta \) and \( \{ \delta, f, S, \langle A_\alpha \cap P_\delta \mid \alpha < \delta \rangle \} \cup (X \cap \bigcup S) \subseteq X_0 \).

For limit \( \beta < \delta \), set \( X_\beta = \bigcup_{\gamma < \beta} X_\gamma \).

Suppose that \( \beta < \delta \) and \( \langle X_\gamma \mid \gamma \leq \beta' \leq \beta \rangle \) is defined and has the required properties. Since \( \bigcup S \subseteq X_0 \cap V_\delta \) and \( X_\beta \cap V_\delta \) is an end extension of \( X_0 \cap V_\delta \), we have that \( X_\beta \cap \bigcup S = X_0 \cap \bigcup S \subseteq S \). If \( X_\beta \cap V_\delta \subseteq S' \), let \( \gamma = \beta \). Otherwise there is some \( \alpha \in X_\beta \cap \delta \) such that \( X_\beta \) does not capture \( A_\alpha \cap P_\delta \). Let \( \xi_\beta \leq \beta \) be the least \( \xi \) with such an \( \alpha \) belonging to \( X_\xi \). Let \( \alpha_\beta \) be the least such \( \alpha \in X_{\xi_\beta} \). Let \( Y < V_\lambda \) be \((A_{\alpha_\beta} \cap P_\delta, X_\beta)\)-good. Let \( A \in A_{\alpha_\beta} \cap Y \) be such that \( Y \cap \bigcup A \in A \). Let \( X_{\beta+1} \) be such that \( X_\beta \cup \{ A \} \cup (Y \cap \bigcup A) \subseteq X_{\beta+1} \prec Y \) and \( |X_{\beta+1}| < \delta \). Note that \( X_\beta < X_{\beta+1} \) and that \( X_{\beta+1} \) is \((A_{\alpha_\beta} \cap P_\delta, X_\beta)\)-good.

Assume for a contradiction that \( X_\beta \) is defined for every \( \beta < \delta \). Since \( \xi_\beta \) is defined for limit \( \beta \), there is an \( \eta < \delta \) and there is a set \( S \) stationary in \( \delta \) such that \( \xi_\beta = \eta \) for all \( \beta \in S \). Since \( |X_\eta| < \delta \), there must be ordinals \( \beta_1 \) and \( \beta_2 \) belonging to \( S \) such that \( \beta_1 < \beta_2 \) and \( \alpha_{\beta_1} = \alpha_{\beta_2} \). But \( X_{\beta_1+1} \) captures \( A_{\alpha_{\beta_1}} \cap P_\delta \) and \( X_{\beta_2} \cap V_\delta \) is an end extension of \( X_{\beta_1+1} \cap V_\delta \). Hence \( X_{\beta_2} \) captures \( A_{\alpha_{\beta_2}} \cap P_\delta \). This contradicts the definition of \( \alpha_{\beta_2} \).

If we set \( Z = X_\gamma \cap V_\delta \), then \( Z \) is our desired element of \( S' \). \qed

The proof of Theorem 9.3.5 is a basic ingredient in the proof of our next result, Theorem 9.3.6. We will not actually use Theorem 9.3.6 and Corollary 9.3.7 in proving our determinacy results. Rather we will use (strengthenings of) Theorem 9.4.17 and Corollary 9.4.18, the analogues of these results for the partial ordering \( Q_\alpha \) introduced in the next section. For this reason, and because the proof of Theorem 9.4.17 is simpler than that of Theorem 9.3.6, the reader may want to skip ahead to Theorem 9.3.8.

Theorem 9.3.6 is due to Steel. The analogous Theorem 9.4.17 was proved earlier by Woodin.
Theorem 9.3.6. Let $\kappa$ be a Woodin cardinal and let $S \in \mathcal{P}_\kappa$. Let $S^*$ be the set of $X \in \mathcal{P}_\kappa(V_{\kappa+1})$ such that $X \cap \bigcup S \in S$ and such that $X$ captures every $A \in X$ that is predense in $\mathcal{P}_\kappa$. Then $S^*$ is stationary in $\mathcal{P}_\kappa(V_{\kappa+1})$.

Proof. Let $f : \omega V_{\kappa+1} \rightarrow V_{\kappa+1}$. Let $\lambda$ be a limit ordinal with $\text{cf}(\lambda) > \kappa$. We will construct a set $X \prec V_{\lambda}$ such that $X \cap V_{\kappa+1}$ belongs to $S^*$ and is closed under $f$. This will show that $S^*$ is stationary in $\mathcal{P}_\kappa(V_{\kappa+1})$.

We would like to construct an elementary chain $\langle X_n \mid n \in \omega \rangle$ of elementary submodels of $V_{\lambda}$ and to define ordinals $\delta_n, n \in \omega$, such that $\{f, S\} \subseteq X_0$, such that $X_0 \cap \bigcup S \in S$, and such that each $X_{n+1} \cap V_{\delta_n}$ is an end extension of $X_n \cap V_{\delta_n}$ and captures $A \cap \mathcal{P}_{\delta_n}$ for every predense $A \in X_n$. To get $X_{n+1}$ from $X_n$ as in the proof of Lemma 9.3.5, we have to make sure that $\delta_n \in X_n$. This is needed to apply (4) of Lemma 9.3.2. The extra complexities of the following construction are introduced to deal with this problem. (There will be no such complexities in the proof of the analogous Theorem 9.4.17, whose construction will be similar to a length $\omega$ version of that of the proof of Theorem 9.3.5.)

Let $\bar{\delta} < \kappa$ be such that $S \in V_{\bar{\delta}}$.

We define inductively

1. $\delta_n, n \in \omega$, such that
   
   (a) for every $n$, $\delta_n$ is inaccessible;
   (b) $\bar{\delta} < \delta_0$;
   (c) $\delta_0 < \delta_1 < \cdots$;
   (d) $(\forall n) \delta_n < \kappa$;

2. $X_n, n \in \omega$, such that
   
   (a) $X_0 \prec X_1 \prec \cdots$;
   (b) $\{\bar{\delta}, f, \kappa, S\} \subseteq X_0$;
   (c) $(\forall n) \delta_n \in X_n$;
   (d) $(\forall n) X_n < V_{\lambda}$;
   (e) $|X_0| \leq \bar{\delta}$;
   (f) $(\forall n) |X_{n+1}| \leq \delta_n$;
   (g) for every $n$, $X_{n+1} \cap V_{\delta_n}$ is an end extension of $X_n \cap V_{\delta_n}$;

3. $X_{n,i}$ and $Y_{n,i}, n \in \omega$ and $i \in \omega$, such that
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(a) \((\forall n)(\forall i)(\forall j)(i \leq j \rightarrow X_{n,i} \subseteq X_{n,j});\)

(b) \((\forall i)(\forall m)(\forall n)(m \leq n \rightarrow X_{m,i} \subseteq X_{n,i});\)

(c) \((\forall n)\bigcup_{i \in \omega} X_{n,i} = X_n;\)

(d) \((\forall n)(\forall i)X_{n,i} \subseteq Y_{n,i} \cap X_n \text{ and } Y_{n,i} \in X_n;\)

(e) \((\forall n)(\forall i)|Y_{n,i}| < \kappa;\)

(f) for every \(n, X_{n+1} \cap V_{\delta_n} \text{ captures } A \cap P_{\delta_n} \text{ for every } A \in X_{n,n} \text{ that is predense in } P_n.\)

Let \(Y_{0,i}, i \in \omega, \text{ be such that}\)

(i) \(Y_{0,0} \prec Y_{0,1} \prec \cdots \text{ and each } Y_{0,i} \prec V_{\lambda};\)

(ii) each \(|Y_{0,i}| < \kappa;\)

(iii) \(V_{\delta} \cup \{f, \kappa\} \subseteq Y_{0,0};\)

(iv) \(Y_{0,i} \in Y_{0,i+1} \text{ for each } i.\)

Let \(Y_0 = \bigcup_{i \in \omega} Y_{0,i}.\)

Since \(S\) is stationary, we can use the Löwenheim–Skolem Theorem to get \(X_0 \prec Y_0\) be such that \(|X| \leq \delta, \text{ such that } \{\delta, f, \kappa, S\} \cup \{Y_{0,i} \mid i \in \omega\} \subseteq X_0,\)
and such that \(X_0 \cap \bigcup S \subseteq S.\) Let \(X_{0,i} = X_0 \cap Y_{0,i}.\) Note that the relevant parts of (1)–(3) hold for \(n = 0.\)

Suppose that \(\langle \delta_m \mid m < n \rangle, \langle X_m \mid m \leq n \rangle, \langle X_{m,i} \mid m \leq n \land i \in \omega \rangle,\)
and \(\langle Y_{m,i} \mid m \leq n \land i \in \omega \rangle\) are defined and satisfy the relevant parts of (1)–(3).

Let \(A_n\) be the set of all predense subsets of \(P_n\) belonging to \(Y_{n,n}.\) By (3)(e), \(|A_n| < \kappa;\) so there is an inaccessible \(\delta\) such that

(i) \(\delta < \kappa;\)

(ii) \(\bar{\delta} < \delta;\)

(iii) \(\delta_{n-1} < \delta \text{ if } n > 0;\)

(iv) for all \(A \in A_n, A \cap P_{\delta} \text{ is predense in } P_{\delta} \text{ and semiproper.}\)

Because \(X_n \prec V_{\lambda} \text{ and because } \{\delta, \delta_0, \ldots, \delta_{n-1}, \kappa, Y_{n,n}, S\} \subseteq X_n, \) some \(\delta\) with these properties belongs to \(X_n.\) Let \(\delta_n\) be the least such \(\delta.\)

We next assert that there is a \(Z \prec V_{\lambda} \text{ such that } Z \cap V_{\delta_n} \text{ is an end extension of } X_n \cap V_{\delta_n} \text{ and such that } Z \text{ captures } A \cap P_{\delta_n} \text{ for every } A \in A_n \cap X_n.\) We can build such a \(Z\) by a construction like the one in the proof of Theorem 9.3.5, the construction that started with the \(X_0\) of that proof and built the set \(X_{\gamma}.\)

By an analogous construction, we can start with our present \(X_n\) and build \(Z\)
as desired. The main difference between that situation and the present one is a simplification: we now have fewer than \( \kappa \) as desired. The main difference between that situation and the present one is 9.3.

\( \mathcal{P}_\kappa \) leave the details to the reader.

We do not have to worry about our construction's terminating. We leave the details to the reader.

Let

\[
X_{n+1} = \{g(u) \mid g \in X_n \land g : \langle \omega \nu \delta \rangle \to \lambda \land u \in \langle \omega \nu \delta \rangle \cap Z \}.
\]

Observe that \( |X_{n+1}| \leq \delta_n \). By an argument like one occurring in the proof of Lemma 9.3.2, the facts that \( \delta_n \in X_n \), that \( X_n \prec \lambda \), and that \( \text{cf}(\lambda) > \kappa \) guarantee that \( X_n \subseteq X_{n+1} \), that \( X_{n+1} \cap \delta \lambda = Z \cap \delta \lambda \), and that \( X_{n+1} \prec \lambda \). Thus the relevant parts of (1), (2), and (3)(f) hold.

For \( i \in \omega \), let

\[
X_{n+1,i} = \{g(u) \mid g \in X_{n,i} \land g : \langle \omega \nu \delta \rangle \to \lambda \land u \in \langle \omega \nu \delta \rangle \cap Z \};
\]

\[
Y_{n+1,i} = \{g(u) \mid g \in Y_{n,i} \land g : \langle \omega \nu \delta \rangle \to \lambda \land u \in \langle \omega \nu \delta \rangle \}.
\]

It is easy to verify that clauses (a)–(c) and (e) of (3) hold for \( n+1 \). For (d) fix \( i \) and note first that \( Y_{n+1,i} \) is definable in \( \lambda \) from \( Y_{n,i} \) and \( \delta \). Since both \( Y_{n,i} \) and \( \delta \) belong to \( X_{n+1} \) and since \( X_{n+1} \prec \lambda \), it follows that \( Y_{n+1,i} \) belongs to \( X_{n+1} \). Obviously \( X_{n+1,i} \subseteq Y_{n+1,i} \cap X_{n+1} \).

Let \( X = \bigcup_{n<\omega} X_n \). Then \( |X| < \kappa \) and \( X \prec \lambda \). Since \( f \in X \), it follows \( X \cap V_{\kappa+1} \) is closed under \( f \). By (1)(b), \( \bigcup S \in V_{\delta} \subseteq V_{\delta_0} \). Since \( \bigcup S \in X_{0} \cap V_{\delta_0} \) and \( X \cap V_{\delta_0} \) is an end extension of \( X_0 \cap V_{\delta_0} \), we have that \( X \cap \bigcup S \in S \). Thus \( (X \cap V_{\kappa+1}) \cap \bigcup S \subseteq S \). Let \( A \in X \) be predense in \( \mathcal{P}_\kappa \). Let \( n \) be such that \( A \in X_n \). By (3)(e) and (3)(a), let \( i \geq n \) be such that \( A \in X_{n,i} \). By (3)(b), \( A \in X_{i,i} \). By (3)(f), \( X_{i+1} \cap V_{\delta_0} \) captures \( A \cap \mathcal{P}_{\delta_0} \). By (2)(g), \( X \) captures \( A \). We have thus shown that \( X \cap V_{\kappa+1} \) is closed under \( f \) and belongs to \( S^* \).

The following corollary shows that if we force (with respect to a larger \( \mathcal{P}_\rho \) below one of the sets given by Theorem 9.3.6, then the intersection of the generic object with \( \mathcal{P}_\kappa \) is \( \mathcal{P}_\kappa \)-generic.

**Corollary 9.3.7.** Let \( \kappa \) be a Woodin cardinal. Let \( \hat{S} \) be the set of all \( X \in \mathcal{P}_\kappa(V_{\kappa+1}) \) such that \( X \) captures every \( A \in X \) that is predense in \( \mathcal{P}_\kappa \). Let \( \rho > \kappa \) be a limit ordinal and let \( G \) be \( \mathcal{P}_\rho \)-generic over \( V \) with \( \hat{S} \subseteq G \). Then \( G \cap \mathcal{P}_\kappa \) is \( \mathcal{P}_\kappa \)-generic over \( V \).

**Proof.** Let \( S \in \mathcal{P}_\rho \) with \( S \leq \hat{S} \). Let \( D \) be dense in \( \mathcal{P}_\kappa \). We must show that there is an \( \hat{S} \leq S \) such that, for some \( D \subseteq \hat{S} \), \( \hat{S} \leq D \). This will complete
the proof. We may assume that
\[(\forall Z \in S)(Z \cap V_{\kappa+1} \subseteq \tilde{S} \wedge D \in Z).\]
Thus each member of \(S\) captures \(D\). In other words, for each \(Z \in S\) there is a \(D_Z \in D \cap Z\) such that \(Z \cap \bigcup D_Z \in D_Z\). By the normality of \(\mathcal{F}_S^{\text{club}}\), there is a \(D \in D\) such that \(D_Z = D\) for all \(Z\) in a stationary subset \(\tilde{S}\) of \(S\). Thus \(\tilde{S} \leq D\) and \(\tilde{S} \leq S\), as required. \(\square\)

We next make the main application of Theorem 9.3.5.

**Theorem 9.3.8.** Let \(\kappa\) be a Woodin cardinal. Let \(G\) be \(P_\kappa\)-generic over \(V\). Then \(\prod_G(V;\in)\) is wellfounded. Moreover \(<\kappa\Ult(V;G) \subseteq \Ult(V;G)\).

**Proof.** Let \(\delta < \kappa\). In \(V[G]\) let \(\langle x_\alpha \mid \alpha < \delta \rangle\) be a sequence of elements of \(\prod_G(V;\in)\). Let \(\tau \in V\) be a \(P_\kappa\)-name such that \(\tau_G\) is the given sequence. Without loss of generality, we may assume that \(\{\emptyset\}\) forces that \(\tau\) names a function from \(\delta\) to \(\prod_G(V;\in)\).

In \(V\) let \(\langle A_\alpha \mid \alpha < \delta \rangle\) and \(\langle f^a_\alpha \mid \alpha < \delta \wedge A \in A_\alpha \rangle\) be such that each \(A_\alpha\) is a maximal antichain in \(P_\kappa\), such that each \(f^a_\alpha : A \rightarrow V\), and such that
\[(\forall \alpha < \delta)(\forall A \in A_\alpha) A \models \tau(\check{\alpha}) = \check{[f^\alpha_\alpha]}_G.
\]
(Note that if \(A \models \tau(\check{\alpha}) = \check{f}_G\) for some \(f : B \rightarrow V\), then we must have \(A \leq_w B\). Hence we may assume that \(B = A\).)

By Theorem 9.3.5, let \(S \in G\) be such that \(\delta \subseteq \bigcup S\) and such that \(S\) seals off \(\langle A_\alpha \mid \alpha < \delta \rangle\).

By Lemma 9.3.4, we may without loss of generality assume that, for each \(X \in S\) and for each \(\alpha \in X \cap \delta\), there is a unique \(A \in A_\alpha \cap X\) such that \(X \cap \bigcup A \in A\). Let \(A_X^\alpha\) be this unique \(A\).

Let \(\alpha < \delta\). Since \(\delta \subseteq \bigcup S\), the set \(\tilde{S}\) of all \(X \in S\) such that \(\alpha \in X\) is club in \(S\). Hence \(S \leq_w \tilde{S}\) and so \(\tilde{S} \subseteq G\). Since \(A_X^\alpha\) is in \(X\) for all \(X \in \tilde{S}\), it follows from the normality of \(\mathcal{F}_{\tilde{S}}^{\text{club}}\) that there is an \(A_\alpha \in A_\alpha\) such that \(S_\alpha \in G\), where
\[S_\alpha = \{X \in \tilde{S} \mid A_X^\alpha = A_\alpha\}.
\]
For \(X \in S_\alpha\), we have that \(X \cap \bigcup A \in A_\alpha\). This means that \(S_\alpha \leq A_\alpha\) and so that \(A_\alpha \in G\).

For \(\alpha < \delta\), let \(h_\alpha : S \rightarrow V\) be such that
\[h_\alpha(X) = f^a_\alpha(X \cap \bigcup A_X^\alpha).\]
for $X \in \tilde{S}_\alpha$. The preceding paragraph shows that, for each $\alpha$ and each $X \in S_\alpha$,

$$f_\alpha^a(X \cap \bigcup A_\alpha) = f_\alpha^a(X \cap \bigcup A_\alpha) = h_\alpha(X).$$

Thus

$$x_\alpha = [f_\alpha^a]_G = [h_\alpha]_G.$$

Suppose first that $\delta = \omega$ and assume that $\langle x_n \mid n \in \omega \rangle$ is an infinite descending $\in_G$-sequence. Then $B_n \in G$ for each $n \in \omega$, where

$$B_n = \{ X \in S \mid h_{n+1}(X) \in h_n(X) \}.$$

Since $G$ is generic over $V$ and $\langle B_n \mid n \in \omega \rangle \in V$, there must be a $B \in G$ such that $B \leq_w B_n$ for all $n \in \omega$; for otherwise the set of all elements of $P_\kappa$ incompatible with some $B_n$ would be dense in $P_\kappa$ below some member of $G$. The existence of $B$ implies that

$$\{ X \in S \mid (\forall n \in \omega) h_{n+1}(X) \in h_n(X) \} \in G.$$

Since every member of $G$ is nonempty, this is a contradiction, and we have shown that $\prod_G(V; \in)$ is wellfounded.

Now consider the case of arbitrary $\delta < \kappa$.

For $X \in S$, let $\rho_X : \text{ot}(X \cap \delta) \to (X \cap \delta)$ be the order preserving bijection. Define $g : S \to V$ by

$$g(X) = \langle h_{\rho_X(\beta)}(X) \mid \beta \in \text{ot}(X \cap \delta) \rangle.$$

By Lemma 9.2.12,

$$\prod_G(V; \in) \models "[g]_G \text{ is a function with domain } \pi^{-1}_G(\delta)."$$

Moreover if $\alpha < \delta$ then

$$\prod_G(V; \in) \models [g]_G(\pi^{-1}_G(\alpha)) = [h_\alpha]_G.$$

In other words $(\pi_G([g]_G))_{\alpha} = x_\alpha$ for all $\alpha < \delta$. □

Remark. We did not need to give a separate proof of wellfoundedness. The proof of closure proves wellfoundedness; for it shows that any infinite descending $\in_G$-sequence would “belong to” $\prod_G(V; \in)$, and this is impossible.
Exercise 9.3.1. Show that if the requirement that $A$ be predense in $P_\kappa$ were omitted from the definition of “semiproper,” it would still follow that all semiproper sets are predense.

Exercise 9.3.2. Show that, in clause (4) of Lemma 9.3.2, the phrase “for every limit ordinal of cofinality greater than $|V_\kappa|$” can be replaced by “for every ordinal $\lambda \geq \kappa + 2$ that is not singular of cofinality $\leq |V_\kappa|$.”

9.4 The Tower $Q_\rho$

For our applications, we will need not only the full stationary tower $P_\rho$ but also a certain subset $Q_\rho$ of $P_\rho$. In this section we indicate how to adapt definitions results of the preceding two sections to $Q_\rho$. We also prove, for $\kappa$ Woodin and for $G$ that is $Q_\kappa$-generic over $V$, that $i_G(\omega_1) = \kappa$.

For limit ordinals $\rho$, let

$$Q_\rho = \{ S \in P_\rho \mid (\forall Y \in S) |Y| \leq \aleph_0 \}.$$  

We can also describe $Q_\rho$ as the set of all sets $S \in V_\rho$ such that $S$ is stationary in $P_{\aleph_1}(\bigcup S)$. We partially order $Q_\rho$ by the restriction of the partial ordering $\leq$ of $P_\rho$.

We will verify that the results we have proved for $P_\rho$ go through with only small changes for $Q_\rho$.

We first need to define a variant of the ext operation appropriate to $Q_\rho$. If $A$ and $X$ are sets with $\bigcup A \subseteq X$ and $A \subseteq P_{\aleph_1}(\bigcup A)$, then let

$$A^{\text{ext}}_0 X = \{ Y \subseteq X \mid |Y| \leq \aleph_0 \wedge Y \cap \bigcup A \in A \}.$$  

Note that $(A^{\text{ext}}_0 X) \res \bigcup A = A$.

We omit the proof of the following lemma, since it is essentially the same as the proof of Lemma 9.2.3.

Lemma 9.4.1. Let $C$ be club in $\mathcal{P}_{\aleph_1}(\bigcup C)$.

(a) If $X \subseteq \bigcup C$ then $C \res X \in \mathcal{P}_{\aleph_1}^{\text{club}}(X)$.

(b) If $\bigcup C \subseteq X$ then $C^{\text{ext}}_0 X$ is club in $\mathcal{P}_{\aleph_1}(X)$.

Corollary 9.4.2. Let $S$ be stationary in $\mathcal{P}_{\aleph_1}(\bigcup S)$.
(a) If $X \subseteq \bigcup \mathcal{S}$ then $\mathcal{S} \underset{\text{res}}{=} X$ is stationary in $\mathcal{P}_{\aleph_1}(X)$.
(b) If $\bigcup \mathcal{S} \subseteq X$ then $\mathcal{S} \underset{\text{ext}}{=} 0 X$ is stationary in $\mathcal{P}_{\aleph_1}(X)$.

The corollary follows directly from Lemma 9.4.1, just as Corollary 9.2.4 followed from Lemma 9.2.3.

The next lemma is an analogue for $\mathcal{Q}_{\rho}$ of Lemma 9.2.5, and its proof is just like the proof of the earlier lemma.

**Lemma 9.4.3.** Let $\rho$ be a limit ordinal. Let $X \in V_\rho$. Let $\mathcal{S}$ and $\mathcal{S}'$ be elements of $\mathcal{Q}_{\rho}$.

(a) If $X \subseteq \bigcup \mathcal{S}$, then $\mathcal{S} \leq \mathcal{S} \underset{\text{res}}{=} X$.
(b) If $\bigcup \mathcal{S} \subseteq X$, then $\mathcal{S} \underset{\text{ext}}{=} 0 X \leq \mathcal{S} \leq w \mathcal{S} \underset{\text{ext}}{=} 0 X$.
(c) If $\bigcup \mathcal{S} = \bigcup \mathcal{S}'$ and $\mathcal{S}$ and $\mathcal{S}'$ are compatible in $\mathcal{Q}_{\rho}$, then $\mathcal{S} \cap \mathcal{S}'$ is stationary in $\mathcal{P}_{\aleph_1}(\bigcup \mathcal{S})$ and $\mathcal{S} \cap \mathcal{S}'$ is a greatest lower bound of $\mathcal{S}$ and $\mathcal{S}'$ in $\mathcal{Q}_{\rho}$.

If $G$ is $\mathcal{Q}_{\rho}$-generic over $V$ and if $X \in V_\rho$, then let $G_X = \{ A \in G \mid \bigcup A = X \}$. By a proof just like that of Lemma 9.2.6, we get the following lemma.

**Lemma 9.4.4.** Let $\rho$ be a limit ordinal and let $G$ be $\mathcal{Q}_{\rho}$-generic over $V$. For $X \in V_\rho$, the set $G_X$ is a $V$-ultrafilter on $\mathcal{P}_{\aleph_1}(X)$ extending $\mathcal{F}^{\text{club}}_{\mathcal{P}_{\aleph_1}(X)}$. Moreover the $G_X$ are compatible: if $X \subseteq X' \in V_\rho$ then, for all $A' \in \mathcal{P}_{\aleph_1}(X')$ and all $A \in \mathcal{P}_{\aleph_1}(X)$,

(i) $A' \in G_{X'} \Rightarrow A' \underset{\text{res}}{=} X \in G_X$;
(ii) $A \in G_X \rightarrow A \underset{\text{ext}}{=} 0 X' \in G_{X'}$.

For $\rho$ a limit ordinal and $G$ that is $\mathcal{Q}_{\rho}$-generic over $V$, we define the generic ultrapower $\prod_G (V; \varepsilon)$, the associated functions $i^*_{G}$, $\pi_G$, and $i_G$, and—when it exists—the class $\text{Ult}(V; G)$ in complete analogy with the definition for $\mathcal{P}_{\rho}$.

We get a Łoś Theorem as usual:

**Theorem 9.4.5.** Theorem 9.2.7 continues to hold when “$\mathcal{P}_{\rho}$” is replaced by “$\mathcal{Q}_{\rho}$” in its statement.

The results for $\mathcal{P}_{\rho}$ about the objects represented by identity functions hold for $\mathcal{Q}_{\rho}$ with essentially the same proofs:
Lemma 9.4.6. Let $\rho$ be a limit ordinal and let $G$ be $Q_\rho$-generic over $V$. Let $S$ and $S'$ belong to $G$. For all $f : S' \rightarrow V$ with $f \in V$,

$$[[f]]_G \in G \leftrightarrow (\exists a \in \bigcup S) [[f]]_G = i'_G(a),$$

where $\text{id}_S$ is the identity function on $S$.

Corollary 9.4.7. Let $\rho$ be a limit ordinal and let $G$ be $Q_\rho$-generic over $V$. Let $S \in G$ be such that $i'_G \cup S \subseteq \text{WFP}(\prod G(V; \in))$. Then $[\text{id}_S]_G \in \text{WFP}(\prod G(V; \in))$ and

$$i''_G \cup S = \pi_G([\text{id}_S]_G).$$

The characterization of $G$ in terms of $i_G$ holds for $Q_\rho$:

Lemma 9.4.8. Let $\rho$ be a limit ordinal and let $G$ be $Q_\rho$-generic over $V$ and such that $\prod G(V; \in)$ is wellfounded. Let $S \in Q_\rho$. Then

$$S \in G \leftrightarrow i''_G \cup S \in i_G(S).$$

Lemma 9.2.11 holds for $Q_\rho$:

Lemma 9.4.9. Let $\rho$ be a limit ordinal and let $G$ be $Q_\rho$-generic over $V$. Let $S \in G$ and let $f : S \rightarrow V$ with $f \in V$. Then

(a) $\prod G(V; \in) \models (i'_G(f))([\text{id}_S]_G) = [[f]]_G$;
(b) if $i'_G(f) \in \text{WFP}(\prod G(V; \in))$, then $(i_G(f))(i''_G \cup S) = \pi_G([[f]]_G)$.

The result about the functions that represent elements of $V_\rho$ holds for $Q_\rho$ with the obvious modification:

Lemma 9.4.10. Let $\rho$ be a limit ordinal and let $G$ be $Q_\rho$-generic over $V$.

(1) For every transitive $a \in V_\rho$,

(a) $[\text{trcoll} \upharpoonright P_{\aleph_1}(a)]_G \in \text{WFP}(\prod G(V; \in))$;
(b) $a = \pi_G([\text{trcoll} \upharpoonright P_{\aleph_1}(a)]_G)$;
(c) $(\forall x \in a) x = \pi_G([\text{prj}_2^g]_G)$.

(2) $\rho \leq \text{wford}(\prod G(V; \in))$. 
The technical Lemma 9.2.13 holds for $Q_\rho$:

**Lemma 9.4.11.** Let $\rho$ be a limit ordinal and let $G$ be $Q_\rho$-generic over $V$. Let $\beta$ and $\alpha$ be ordinals smaller than $\rho$ with $\beta$ a limit ordinal.

(a) There is an $h_{\beta, \alpha}$ such that domain $(h) \in G$ and such that, for all $f : S' \to V$ with $S' \in G$ and $f \in V$,

$$[f]_G \in G \iff (\exists S \in V_\beta \cap G) (\exists g : S \to V_\alpha) [g]_G = [f]_G.$$

(b) If $\prod_G (V; \in)$ is wellfounded then

$$\{ \pi_G([f]_G) \mid (\exists S \in V_\beta \cap G) f : S \to V_\alpha \} \in \text{Ult}(V; G).$$

Since $Q_\rho \subseteq P_\rho$ for every $\rho$, the definition of capturing on page 513 applies to subsets of $Q_\rho$.

The proof of Lemma 9.3.1 works just as well for $Q_\rho$. Indeed, that lemma literally implies the following one:

**Lemma 9.4.12.** Let $\rho$ be a limit ordinal and let $A \subseteq Q_\rho$. Let $\lambda$ be an ordinal $\geq \rho$ and let $X \prec V_\lambda$. If both $S$ and $S'$ witness that $X$ captures $A$, then $S$ and $S'$ are compatible elements of $Q_\rho$.

The definition of *semiproper* for predense subsets of $Q_\kappa$ is like that for predense subsets of $P_\kappa$, except that “$|X| < \kappa$” is replaced by “$|X| = \aleph_0$.” The definition of $(A,X)$-good for predense subsets $A$ of $Q_\kappa$ and sets $X$ is just like the definition in the case of $P_\kappa$.

**Lemma 9.4.13.** Let $\kappa$ be an uncountable strong limit cardinal and let $A$ be a predense subset of $Q_\kappa$. The following are equivalent:

1. $A$ is semiproper.
2. The set of all $X \in \mathcal{P}_{\aleph_1}(V_{\kappa+1})$ such that there is a $Y \prec V_{\kappa+1}$ that is $(A,X)$-good belongs to $\mathcal{F}_{\text{club}}^{\mathcal{P}_{\aleph_1}(V_{\kappa+1})}$.
3. Let $M$ be a transitive set such that $V_\kappa \subseteq M$ and $V_{\kappa+2} \subseteq M$. Then, for any $X \prec M$ such that $|X| = \aleph_0$ and $\{\kappa, A\} \subseteq X$, there is a $Y \prec M$ such that $Y$ is $(A,X)$-good.
(4) For every limit ordinal $\lambda$ of cofinality greater than $|V_\kappa|$ and for every $X \prec V_\lambda$ such that $|X| = \aleph_0$ and $\{\kappa, A\} \subseteq X$, there is a $Y \prec V_\lambda$ such that $Y$ is $(A, X)$-good.

The proof of Lemma 9.4.13 is just like that of Lemma 9.3.2.

**Theorem 9.4.14.** Let $\kappa$ be a Woodin cardinal. Let $\langle A_\alpha \mid \alpha < \kappa \rangle$ be a sequence of predense subsets of $Q_\kappa$. Then the set of inaccessible cardinals $\delta < \kappa$ such that

$$(\forall \alpha < \delta) \ A_\alpha \cap Q_\delta \text{ is predense in } Q_\delta \text{ and is semiproper}$$

is stationary in $\kappa$.

**Proof.** The proof of Theorem 9.3.3 works if "$P$" is replaced everywhere by "$Q$," if "$P_\delta(V_{\delta+1})$" is replaced by "$P_{\aleph_1}(V_{\delta+1})$," and if the following additional change—actually, a simplification—is made. Replace the text from "Let $h : <\omega(V_{j(\delta)+1}) \to V_{j(\delta)+1}$ belong to $M$ and be such that . . . " through "Hence $X = j''(Y \cap V_{\delta+1}) \subseteq Y" by the following two paragraphs:

Since $\tilde{S}$ is stationary in $P_{\aleph_1}(\bigcup \tilde{S})$, there is a countable set $Y$ such that $Y \subseteq V_{j(\delta)+1}$ such that

(a) $Y \prec V_{j(\delta)+1}$;
(b) $A \in Y$ and $Y$ is closed under $j \restriction V_{\delta+1}$;
(c) $Y \cap \bigcup \tilde{S} \in \tilde{S}$.

From the first clause of (b) and from (c), we get that $A \in j(A_\alpha \cap Q_\delta) \cap Y$ and that $Y \cap \bigcup A \in A$ respectively. Thus $Y$ captures $j(A_\alpha \cap Q_\delta)$. Since $Y$ is countable, $Y$ belongs to $M$.

Let $X = j(Y \cap V_{\delta+1})$. Since $Y \cap V_{\delta+1} \in S$, we have that $X \in j(S)$. The countability of $Y$ implies that $X = j''(Y \cap V_{\delta+1}) \subseteq Y$.

**Remark.** Instead of deleting the old paragraphs and inserting the two new ones, we could also get a correct proof by making only trivial changes in the replaced paragraphs. Our wholesale replacement has the virtue of showing that it is slightly easier to prove Theorem 9.4.14 than it is to prove Theorem 9.3.3.
For $S \in Q_\rho$ and $A$ a predense subset of $Q_\rho$ or a sequence of such predense subsets, the definition of $S$ seals off $A$ is exactly like the corresponding definition for $P_\rho$. (See page 518.)

Lemma 9.3.4, the result about the uniqueness of witnesses to sealing off, holds for $Q_\rho$:

**Lemma 9.4.15.** Lemma 9.3.4 continues to hold when "$P_\rho$" is replaced by "$Q_\rho$" in its statement.

The proof is essentially the same as that of Lemma 9.3.4.

**Theorem 9.4.16.** Let $\kappa$ be a Woodin cardinal, let $\delta \leq \kappa$, and let $\langle A_\alpha \mid \alpha < \delta \rangle$ be a sequence of predense subsets of $Q_\kappa$. Then the set of $S \in Q_\kappa$ such that $S$ seals off $\langle A_\alpha \mid \alpha < \delta \rangle$ is dense in $Q_\kappa$.

**Proof.** The proof of Theorem 9.3.5 works when the following changes are made:

1. Replace "$P$" by "$Q$.''
2. Replace "$P_\delta$" by "$P_{\aleph_1}$.''
3. Replace all occurrences of "$< \delta$" from the one in "and $|X| < \delta$" to the end of the proof by occurrences of "$< \aleph_1$.''

**Theorem 9.4.17.** Let $\kappa$ be a Woodin cardinal and let $S \in Q_\kappa$. Let $S^*$ be the set of $X \in P_{\aleph_1}(V_{\kappa+1})$ such that $X \cap \bigcup S \in S$ and such that $X$ captures every $A \in X$ that is predense in $Q_\kappa$. Then $S^*$ is stationary in $P_{\aleph_1}(V_{\kappa+1})$.

**Proof.** Theorem 9.4.17 can be proved by a routine modification of the proof of Theorem 9.3.6. Nevertheless we give a detailed proof of Theorem 9.4.17, for this proof is simpler than that derived from the earlier proof—indeed it is simpler than the earlier proof itself.

Let $f : \omega V_{\kappa+1} \rightarrow V_{\kappa+1}$. Let $\lambda$ be a limit ordinal with $\text{cf}(\lambda) > \kappa$. We will prove that $S^*$ is stationary in $P_{\aleph_1}(V_{\kappa+1})$ by constructing an $X \in S^*$ that is closed under $f$.

We define inductively

1. $\delta_n, n \in \omega$, such that

   (a) for every $n$, $\delta_n$ is inaccessible;
(b) \( S \in V_{\delta_0}; \)
(c) \( \delta_0 < \delta_1 < \cdots; \)
(d) \( (\forall n) \delta_n < \kappa; \)

(2) \( X_n, n \in \omega, \) such that
(a) \( X_0 \prec X_1 \prec \cdots; \)
(b) \( \{f, \kappa, S\} \subseteq X_0; \)
(c) \( (\forall n) \delta_n \in X_n; \)
(d) \( (\forall n) X_n \prec V_{\lambda}; \)
(e) \( (\forall n) |X_n| = \aleph_0; \)
(f) for every \( n, \) \( X_{n+1} \cap V_{\delta_n} \) is an end extension of \( X_n \cap V_{\delta_n}. \)

If we satisfy clauses (2)(d) and (2)(e), then we can construct, as we proceed with the definition of the \( \delta_n \) and the \( X_n, \) a sequence \( \langle A_n \mid n \in \omega \rangle \) such that \( A_n \in X_n \) for each \( n \) and such that, for all objects \( A, \) \( A \) belongs to \( \{A_n \mid n \in \omega\} \) if and only if \( A \in \bigcup_{n \in \omega} X_n \) and \( A \) is predense in \( Q_\kappa. \) (Clause (2)(e) implies that the set of all such \( A \) is countable, and clause (2)(d) implies that its intersection with \( X_0 \) is non-empty.) We will arrange that

(3) for every \( n, \) \( X_{n+1} \cap V_{\delta_n} \) captures \( A_n \cap Q_{\delta_n}. \)

Using the fact that \( S \) is stationary and using the Löwenheim–Skolem Theorem, let \( X_0 \prec V_{\lambda} \) with \( \{f, \kappa, S\} \subseteq X_0, \) with \( |X_0| = \aleph_0, \) and with \( X_0 \cap \bigcup S \in S. \)

Assume that we have defined \( X_m \) for \( m \leq n \) and \( \delta_m \) for \( m < n \) so that the relevant parts of (1), (2), and (3) are satisfied. Note that by (2)(a) and (2)(c) it follows that \( \{\delta_0, \ldots, \delta_{n-1}\} \subseteq X_n. \)

By Theorem 9.4.14, there is an inaccessible \( \delta \) such that

(i) \( \delta < \kappa; \)
(ii) \( S \in V_{\delta}; \)
(iii) \( \delta_{n-1} < \delta \) if \( n > 0; \)
(iv) \( A_n \cap Q_{\delta} \) is predense in \( Q_{\delta} \) and semiproper.

Because \( X_n \prec V_{\lambda} \) and because \( \{\delta_0, \ldots, \delta_{n-1}, \kappa, S, A_n\} \subseteq X_n, \) some \( \delta \) with these properties belongs to \( X_n. \) Let \( \delta_n \) be the least such \( \delta. \)
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Using (4) of Lemma 9.4.13 and the L"{o}wenheim–Skolem Theorem, we get a countable $X_{n+1} \prec V_\lambda$ that is $(A_n \cap Q_{\delta_n}, X_n)$-good.

Let $X = \bigcup_{n \in \omega} X_n$. Then $|X| = \aleph_0$ and $X \prec V_\lambda$. Since $f \in X$, it follows $X \cap V_{\kappa+1}$ is closed under $f$. Since $\bigcup S \in X_0 \cap V_{\delta_0}$ and $X \cap V_{\delta_0}$ is an end extension of $X_0 \cap V_{\delta_0}$, we have that $X \cap \bigcup S \in S$. Let $A \in X$ be predense in $Q_\lambda$. Let $n$ be such that $A = A_n$. By (3), $X_{n+1} \cap V_{\delta_n}$ captures $A \cap Q_{\delta_n}$. Since $X \cap V_{\delta_n}$ is an end extension of $X_{n+1} \cap V_{\delta_n}$, it follows that $X$ captures $A$. We have thus shown that $X \cap V_{\kappa+1}$ is closed under $f$ and belongs to $S^*$. □

**Corollary 9.4.18.** Let $\kappa$ be a Woodin cardinal. Let $\hat{S}$ be the set of all $X \in P_{\aleph_1}(V_{\kappa+1})$ such that $X$ captures every $A \in X$ that is predense in $Q_\kappa$. Let $\rho > \kappa$ be a limit ordinal and let $G$ be $P_\rho$-generic or $Q_\rho$-generic over $V$ with $\hat{S} \in G$. Then $G \cap Q_\kappa$ is $Q_\kappa$-generic over $V$.

The proof of Corollary 9.4.18 is just like that of Corollary 9.3.7.

**Theorem 9.4.19.** Let $\kappa$ be a Woodin cardinal. Let $G$ be $Q_\kappa$-generic over $V$. Then $\prod_G (V; \in)$ is wellfounded. Moreover, $\lt \text{Ult}(V; G) \subseteq \text{Ult}(V; G)$.

The proof of Theorem 9.3.8 becomes a proof of Theorem 9.4.19 when “$P_\kappa$” is replaced by “$Q_\kappa$.”

The next theorem holds only for $Q_\kappa$, not for $P_\kappa$. (By an argument like that for Exercise 9.2.7, one can show that $i_G(\kappa) = \kappa$ if $\kappa$ is Woodin and $G$ is $P_\kappa$-generic over $V$.)

**Theorem 9.4.20.** Let $\kappa$ be a Woodin cardinal and let $G$ be $Q_\kappa$-generic over $V$. Then $i_G(\omega_1) = \kappa$.

**Proof.** Let $\alpha < \kappa$. By Lemma 9.4.10,

$$\alpha = \pi_G([\text{ot } P_{\omega_1}(\alpha)]_G).$$

Thus $\alpha < i_G(\omega_1)$.

Clearly $(\omega_1)^{\text{Ult}(V; G)} \leq (\omega_1)^{V[G]}$. (In fact, Lemma 9.4.19 implies that $(\omega_1)^{\text{Ult}(V; G)} = (\omega_1)^{V[G]}$.) Thus it suffices to show that $(\omega_1)^{V[G]} \leq \kappa$.

Suppose that $\tau$ is a $Q_\kappa$-name and that $\{0\} \models \tau : \check{\omega} \rightarrow \check{\kappa}$. For $n \in \omega$ let $A_n$ be a maximal antichain in $Q_\kappa$ such that for every member $A$ of $A_n$ there
is some \( \beta < \kappa \) such that \( A \models \tau(\bar{n}) = \bar{\beta} \). For \( n \in \omega \) and \( A \in A_n \), let \( \beta^A_n \) be the \( \beta \) such that \( A \models \tau(\bar{n}) = \bar{\beta} \).

By Theorem 9.4.16, let \( S \in G \) seal off \( \langle A_n \mid n \in \omega \rangle \). Lemma 9.4.15 allows us to assume that, for every \( X \in S \) and \( n \in \omega \), there is a unique \( A \in A_n \cap X \) such that \( X \cap \bigcup A \in A \). Let us denote this unique \( A \) by \( A(X, n) \). Let \( \beta(X, n) = \beta^A_n(X, n) \).

Fix \( n \in \omega \). Since \( A(X, n) \in X \) for all \( X \in S \), a normality argument gives us an \( A \in A_n \) such that

\[
\{ X \subseteq \bigcup S \mid A(X, n) = A \} \in G.
\]

This means that \( \tau_G(n) = \beta^A_n \), and so we have shown that there is an \( X \in S \) such that \( \tau_G(n) = \beta(X, n) \).

By the argument of the preceding paragraph,

\[
\text{range } \tau_G \subseteq \{ \beta(X, n) \mid X \in S \land n \in \omega \}.
\]

Since \( S \in V_\kappa \), \( |S| < \kappa \), and so the range of \( \tau_G \) is bounded in \( \kappa \).

\[
\square
\]

### 9.5 Infinitely Many Woodin Cardinals

In §9.6 we will use large cardinal hypotheses (1) to build by forcing a class model of AD and (2) to prove AD\(^L(\mathbb{R}) \). For (1) our hypothesis will be the existence of infinitely many Woodin cardinals and for (2) our hypotheses will be slightly more than the existence of infinitely many Woodin cardinals. In this section we prepare for §9.6 by studying the properties of stationary towers of height \( \rho > \kappa \) where \( \kappa \) is the supremum of \( \omega \) Woodin cardinals. We will do \textit{approximately} the following: (a) use a restricted generic ultrapower to construct a transitive class \( N \) and a function \( i \) such that \( \omega^N_1 = \kappa \) and \( i: V \prec N \); (b) show that the reals of \( N \) are the same as the reals in a model \( K \) obtained by a “symmetric” forcing that makes all cardinals less than \( \kappa \) countable; (c) show, for \( G \) generic over \( V \) for any partial ordering of size less than \( \kappa \), that the \( L(\mathbb{R}) \) of \( V, N, K \), and \( V[G] \) satisfy the same formulas with real parameters. This account is only “approximately” correct in that \( i \) will elementarily embed only parts of \( V \) into parts of \( N \) and, moreover, the formulas for which (c) is true will form a restricted class. Some of our results will use the existence of completely Jonsson or measurable cardinals between...
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κ and ρ and will give closer approximations to (a)–(c). A similar construction using stronger large cardinal hypotheses would make (a)–(c) literally true.

The following result has generalizations and analogues for $P_{\kappa}$ which we do not bother to state.

**Theorem 9.5.1.** Let $\langle \kappa_n \mid n \in \omega \rangle$ be a strictly increasing sequence of Woodin cardinals. Let $\kappa = \sup_{n \in \omega} \kappa_n$. Let $m \in \omega$ and let $S \in Q_{\kappa_m}$. Let $S^*$ be the set of $X \in P_{\kappa_1}(V_\kappa)$ such that $X \cap \bigcup S \in S$ and such that, for every $n \geq m$, $X$ captures every $A \in X$ that is predense in $Q_{\kappa_n}$. Then $S^*$ is stationary in $P_{\kappa_1}(V_\kappa)$.

**Proof.** We omit the proof. It is like that of Theorem 9.4.17, except that an elementary chain $\langle X_\beta \mid \beta < \omega^2 \rangle$ replaces the elementary chain $\langle X_n \mid n \in \omega \rangle$ of the earlier proof. \qed

**Corollary 9.5.2.** Let $\langle \kappa_n \mid n \in \omega \rangle$ be a strictly increasing sequence of Woodin cardinals. Let $\kappa = \sup_{n \in \omega} \kappa_n$. Let $\hat{S}$ be the set of all $X \in P_{\kappa_1}(V_\kappa)$ such that, for every $n \in \omega$, $X$ captures every $A \in X$ that is predense in $Q_{\kappa_n}$. Let $\rho > \kappa$ be a limit ordinal and let $G$ be $P_{\rho}$-generic or $Q_{\rho}$-generic over $V$ with $\hat{S} \in G$. Then, for each $n \in \omega$, $G \cap Q_{\kappa_n}$ is $Q_{\kappa_n}$-generic over $V$.

**Proof.** We omit the proof, which is similar to the proof of Corollary 9.4.18, i.e., to the proof of Corollary 9.3.7. \qed

For limit ordinals $\rho$, for ordinals $\gamma$ with $1 < \gamma \leq \rho$, and for objects $G$ that are $P_{\rho}$-generic or $Q_{\rho}$-generic over $V$, let

$$X_{G, \gamma} = \{[f]_G \mid (\exists S \in G \cap V_\gamma) f : S \rightarrow V\}.$$  

Then

$$i'_G : (V; \in) < (X_{G, \gamma}; \in_G)$$

and

$$(X_{G, \gamma}; \in_G) < \prod_G (V; \in).$$

Let $\pi_{G, \gamma} : (\text{WFP}(X_{G, \gamma}; \in_G); \in_G) \cong (N_{G, \gamma}; \in)$, with $N_{G, \gamma}$ transitive. Finally let $i_{G, \gamma}(x) = \pi_{G, \gamma}(i'_G(x))$ for every $x \in V$ such that $i'_G(x) \in \text{WFP}(X_{G, \gamma}; \in_G)$.

**Lemma 9.5.3.** Let $\rho$ be a limit ordinal, let $1 < \gamma \leq \rho$, and let $G$ be $P_{\rho}$-generic or $Q_{\rho}$-generic over $V$. 

(a) For any \( x \in V \), \( i_{G, \gamma}(x) \) is defined if \( i_G(x) \) is defined, i.e., if \( i'_G(x) \in \text{WFP}(\prod G(V; \in)) \);

(b) \( i_{G, \gamma} \mid V_\alpha \prec V_{i_{G, \gamma}(\alpha)} \) for every \( \alpha \in \text{domain}(i_{G, \gamma}) \);

(c) \( \gamma \subseteq N_{G, \gamma} \).

**Proof.** (a) and (b) are obvious. (c) follows from Lemmas 9.2.12 and 9.4.10.

**Lemma 9.5.4.** Let \( \langle \kappa_n \mid n \in \omega \rangle \) be a strictly increasing sequence of Woodin cardinals. Let \( \kappa = \sup_{n \in \omega} \kappa_n \). Let \( \hat{S} \) be as in the statement of Corollary 9.5.2. Let \( \rho > \kappa \) be a limit ordinal and let \( G \) be \( P_\rho \)-generic or \( Q_\rho \)-generic over \( V \) with \( \hat{S} \in G \). Then

1. \( i_{G, \kappa}(\omega_1) = \kappa \);
2. \( (\omega_\omega)^{N_{G, \kappa}} = \bigcup_{n \in \omega} (\omega_\omega)^{V[G \cap Q_{\kappa_n}]} \).

**Proof.** Clause (2) follows from the fact that \( (X_{G, \kappa}; \in_G) \) is just the union of the elementary chain of the \( \prod_{G \cap Q_{\kappa_n}}(V; \in) \). Since, by Theorem 9.4.20, \( i_{G \cap Q_{\kappa_n}}(\omega_1) = \kappa_n \) for each \( n \), clause (2) implies clause (1).

The next two lemmas will tell us that \( \text{Ord} \cap N_{G, \kappa} \) can be large enough for our purposes. The first of the lemmas will be used in the consistency proof for \( \text{AD}^L(\mathbb{R}) \), and the second will be used in the proof (from a stronger hypothesis) of \( \text{AD}^L(\mathbb{R}) \).

**Lemma 9.5.5.** Let \( \beta > 1 \) be an ordinal number, and let \( \rho \) be a limit ordinal greater than \( \beta \). Let \( G \) be \( P_\rho \)-generic or \( Q_\rho \)-generic over \( V \). Then \( \rho \subseteq \text{Ord} \cap N_{G, \beta} \).

**Proof.** Let \( \alpha < \rho \). Let

\[ X_{G, \beta, \alpha} = \{ [f]_G \in X_{G, \beta} \mid \text{range}(f) \subseteq V_\alpha \} \.

Let \( h_{\beta, \alpha} \) be given by Lemma 9.2.13 or Lemma 9.4.11, whichever applies. Then the objects that bear \( \in_G \) to \([h_{\beta, \alpha}]_G\) are exactly the elements of \( X_{G, \beta, \alpha} \). Let \([g_{\beta, \alpha}]_G\) be such that

\[ \prod G(V; \in) \models ([g_{\beta, \alpha}]_G \text{ is transitive} \land ([g_{\beta, \alpha}]_G; \in_G) \cong ([h_{\beta, \alpha}]_G; \in_G)) \]
Then
\[(\{[f]_G | [f]_G \in G [g_{\beta,\alpha}]_G; \in_G \} \cong (\{[f]_G | [f]_G \in G [h_{\beta,\alpha}]_G; \in_G \}).\]

Since \( \rho \subseteq \text{wford} (\prod_G (V; \in)) \), either \( (\{[f]_G | [f]_G \in G [g_{\beta,\alpha}]_G; \in_G \}) \) is well-founded or else \( \rho \subseteq \text{wford} ((\{[f]_G | [f]_G \in G [g_{\beta,\alpha}]_G; \in_G \}). \) From this follows the same assertion for \( (\{[f]_G | [f]_G \in G [h_{\beta,\alpha}]_G; \in_G \}) = (X_{G,\beta,\alpha}; \in_G) \). It is easy to see that \( \alpha \subseteq \text{wford} (X_{G,\beta,\alpha}; \in_G) \), and the ordinals of \( (X_{G,\beta,\alpha}; \in_G) \) are an initial segment of the ordinals of \( (X_{G,\beta}; \in_G) \). Hence \( \alpha \subseteq N_{G,\beta} \). Since \( \alpha < \rho \) was arbitrary, the lemma is proved.

If there is a completely Jonsson cardinal greater than \( \beta \), then we can get a stronger result, as the following lemma shows.

**Lemma 9.5.6.** Let \( \beta \) be an ordinal, let \( \kappa^* > \beta \) be a completely Jonsson cardinal, and let \( \rho \) be a limit ordinal greater than \( \kappa^* \). Let \( G \) be \( P_\rho \)-generic or \( Q_\rho \)-generic over \( V \) with \( \{X \subseteq \kappa^* | |X| = \kappa^* \} \in G \). Then \( \kappa^* \in \text{Ord} \cap N_{G,\beta} \) and
\[i_{G,\beta} | V_{\kappa^*} : V_{\kappa^*} < V_{\kappa^*}^{N_{G,\beta}}.\]

**Proof.** The first assertion follows from Lemma 9.5.5. For the second assertion, note that \( i_{G,\beta}(\kappa^*) \leq i_G(\kappa^*) \). Thus Lemma 9.2.17 implies that \( i_{G,\beta}(\kappa^*) = \kappa^* \).}

Our next goal is to show, for \( G \) and \( \kappa \) as in Lemma 9.5.4, that \( \omega \cap N_{G,\kappa} \) can be rearranged as \( \omega \cap K \), where \( K \) is obtained from \( V \) by collapsing, in a symmetric fashion, all ordinals smaller than \( \kappa \) to \( \omega \).

For any ordinal \( \beta > 0 \), \( \text{Coll}(\omega, \beta) \) is the set of all finite functions \( f \) such that domain \( (f) \subseteq \omega \) and range \( (f) \subseteq \beta \). Partially order \( \text{Coll}(\omega, \beta) \) by reverse inclusion. This partial ordering has appeared more than once in the exercises. If \( H \) is \( \text{Coll}(\omega, \beta) \)-generic over \( V \), then \( \bigcup H \) is a surjection of \( \omega \) onto \( \beta \). Note that \( \text{Coll}^M(\omega, \beta) = \text{Coll}(\omega, \beta) \) for any transitive class model \( M \) of \( \text{ZF} \)—or a large enough fragment of \( \text{ZF} \)—such that \( \beta \in M \).

We now recall some basic facts about \( \text{Coll}(\omega, \beta) \) for \( \beta \) uncountable.

**Lemma 9.5.7.** Let \( \beta \) be an uncountable ordinal. Let \( R \) be a partial ordering such that \( |R| = |\beta| \). Let \( J \) be \( R \)-generic over \( V \). Assume that \( V[J] \models \text{“} \beta \text{ is countable.”} \) Then in \( V[J] \) there is an \( H \) such that \( H \) is \( \text{Coll}(\omega, \beta) \)-generic over \( V \) and such that \( V[H] = V[J] \).
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For the proof of Lemma 9.5.7, see Lemma 25.11 of Jech [1978].

Part (a) of the next lemma is a general fact about forcing. Part (b) is a consequence of part (a) that concerns \( \text{Coll}(\omega, \beta) \).

**Lemma 9.5.8.** Let \( \kappa \) be an uncountable cardinal number. Let \( S \) be a partial ordering with \( |S| = \kappa \). Let \( K \) be \( S \)-generic over \( V \). Let \( X \in V[K] \) with \( X \subseteq V \).

(a) In \( V[X] \) there is a partial ordering \( T \) such that (i) \( |T|^{V[X]} \leq \kappa \) and (ii) in \( V[K] \) there is an \( H \) that is \( T \)-generic over \( V[X] \) with \( V[X][H] = V[K] \).

(b) If \( V[K] \models |\kappa| = \aleph_0 \) and \( V[X] \models |\kappa| > \aleph_0 \), then the \( T \) of (a) can be taken to be \( \text{Coll}(\omega, \beta) \) for any \( \beta \) such that \( |\beta| = \kappa \).

For what is essentially the proof of part (a), see Exercise 2.3.6 of Jech. Part (b) follows from (a) by applying Lemma 9.5.7 with \( V[X] \) in the role of \( V \).

The next lemma states a homogeneity property of \( \text{Coll}(\omega, \beta) \).

**Lemma 9.5.9.** Let \( \beta \) be an ordinal and let \( H \) and \( H' \) be \( \text{Coll}(\omega, \beta) \)-generic over \( V \). For any formula \( \varphi(v_1, \ldots, v_n) \) and any \( a_1, \ldots, a_n \) belonging to \( V \),

\[ V[H] \models \varphi[a_1, \ldots, a_n] \iff V[H'] \models \varphi[a_1, \ldots, a_n]. \]

**Proof.** Any two elements \( p_1 \) and \( p_2 \) of \( \text{Coll}(\omega, \beta) \) can be extended (in some forcing extension of \( V \) in which \( \beta^+ \) is countable) to \( \text{Coll}(\omega, \beta) \)-generic \( H_1 \) and \( H_2 \) respectively such that \( \cup H_1 \) and \( \cup H_2 \) are finitely different. For such \( H_1 \) and \( H_2 \), \( V[H_1] = V[H_2] \). Thus it is impossible that \( p_1 \models \neg \varphi[a_1, \ldots, a_n] \) while \( p_2 \models \neg \varphi[a_1, \ldots, a_n] \). \( \square \)

A partial ordering related to \( \text{Coll}(\omega, \beta) \) is the (Lévy collapse) ordering \( \text{Coll}(\omega, \beta) \). For any ordinal \( \beta > 0 \), \( \text{Coll}(\omega, \beta) \) is the set of all finite functions \( f \) such that domain \( (f) \subseteq \beta \times \omega \) and such that \( f(\alpha, m) < \alpha \) for all \( \alpha < \beta \) and all \( m \in \omega \). Partially order \( \text{Coll}(\omega, \beta) \) by reverse inclusion. Like \( \text{Coll}(\omega, \beta) \), \( \text{Coll}(\omega, \beta) \) is absolute for transitive class models of enough of ZF.
Let $\beta$ be a limit ordinal and let $H$ be $\text{Coll}(\omega, < \beta)$-generic over $V$. For $\gamma < \beta$, let $H_\gamma = \{ p \in H \mid \text{domain}(p) \subseteq \gamma \times \omega \}$. Let
\[ R_H^* = \bigcup_{\gamma < \beta} (\omega^\omega)^{V[H_\gamma]} \]
and let
\[ K = L(R_H^*). \]
Let us call $K$ the symmetric model defined from $H$.

Remark. There are other candidates for “the symmetric model defined from $H$.” An example is $\text{HOD}^{V[H]}_{R_H^* \cup \{ R_H^* \}}$, the class of all members $x$ of $V[H]$ such that, for all members $y$ of the smallest transitive set to which $x$ belongs, $y$ is definable in $V[H]$ from ordinals and members of $R_H^* \cup \{ R_H^* \}$. All the results we will prove in this chapter would hold if we used it instead of $L(R_H^*)$. Moreover there are contexts where it would be an adequate version of the symmetric model and $L(R_H^*)$ would not. For our limited purposes, however, the simpler $L(R_H^*)$ is perfectly satisfactory.

The last three lemmas have the following useful consequence.

**Lemma 9.5.10.** Let $\kappa$ be an uncountable limit cardinal. Let $H$ be $\text{Coll}(\omega, < \kappa)$-generic over $V$. Let $R$ be a partial ordering with $|R| < \kappa$. Let $J \in V[H]$ be $R$-generic over $V$. Then in $V[H]$ there is an $H'$ that is $\text{Coll}(\omega, < \kappa)$-generic over $V[J]$ and such that the symmetric model defined from $H'$ (with $V[J]$ in place of $V$) is the same as the symmetric model defined from $H$.

**Proof.** For ordinals $\gamma$ and $\delta > \gamma$, let $\text{Coll}_\gamma(\omega, < \delta)$ be the set of all $p \in \text{Coll}(\omega, < \delta)$ such that domain $(p) \cap \gamma = \emptyset$. Then
\[ \text{Coll}(\omega, < \delta) \cong \text{Coll}(\omega, < \gamma) \times \text{Coll}_\gamma(\omega, < \delta), \]
by the isomorphism $p \mapsto (p \upharpoonright \gamma \times \omega, p \upharpoonright (\delta \setminus \gamma) \times \omega)$.

Let $\gamma < \kappa$ be uncountable and such that $|R| < |\gamma|$. Let $H_{\gamma + 1}$ be as defined above, i.e., let $H_{\gamma + 1} = H \cap \text{Coll}(\omega, < \gamma + 1)$. By Lemma 9.5.8, let $H \in V[H_{\gamma + 1}]$ be $\text{Coll}(\omega, \gamma)$ generic over $V[J]$ and such that
\[ V[J][\hat{H}] = V[H_{\gamma + 1}]. \]
Now $\gamma$ is uncountable in $V[J]$ and countable in $V[J][\tilde{H}]$. Moreover $|\text{Coll}(\omega, < \gamma + 1)|^{V[J]} = |\gamma|$. Lemma 9.5.9 therefore allows us to apply Lemma 9.5.7 in reverse, so that we get an $\hat{H}$ that is $\text{Coll}(\omega, < \gamma + 1)$-generic over $V[J]$ with $V[J][\hat{H}] = V[J][\tilde{H}]$. Thus

$$V[J][\hat{H}] = V[H_{\gamma+1}].$$

Let $H'$ be the preimage of $\hat{H} \times (H \cap \text{Coll}_{\gamma+1}(\omega, < \kappa))$ under the isomorphism defined in the first paragraph of the proof. It is easily seen that $V[J][H'_\delta] = V[H_\delta]$ for every $\delta$ such that $\gamma < \delta \leq \kappa$. It follows that $H'$ is as required. \hfill \Box

The next lemma is the analogue of Lemma 9.5.9 for symmetric models.

**Lemma 9.5.11.** Let $\beta$ be a limit ordinal and let $H$ and $H'$ be $\text{Coll}(\omega, < \beta)$-generic over $V$. Let $K$ and $K'$ be the symmetric models defined from $H$ and $H'$ respectively. Then, for any formula $\varphi(v_1, \ldots, v_n)$ and any $a_1, \ldots, a_n$ belonging to $V$,

$$K \models \varphi[a_1, \ldots, a_n] \iff K' \models \varphi[a_1, \ldots, a_n].$$

**Proof.** The proof is like that of Lemma 9.5.9. Any two elements $p_1$ and $p_2$ of $\text{Coll}(\omega, < \beta)$ can be extended to generic $H_1$ and $H_2$ respectively such that $\bigcup H_1$ and $\bigcup H_2$ are finitely different. For such $H_1$ and $H_2$, the corresponding symmetric models $K_1$ and $K_2$ are identical. \hfill \Box

**Lemma 9.5.12.** Let $\beta$ be a limit ordinal, let $H$ be $\text{Coll}(\omega, < \beta)$-generic over $V$, and let $K$ be the symmetric model defined from $H$. Then

$$(\omega^\omega)^K = \mathcal{R}_H^\ast.$$  

**Proof.** Let $x \in (\omega^\omega)^K$. Then $x$ is ordinal definable in $K$ from members of $\mathcal{R}_H^\ast$. Let $\alpha < \beta$ be such that $x$ is ordinal definable in $K$ from members of $(\omega^\omega)^{V[H_\alpha]}$. Note that $K$ is also the symmetric model defined, with $V[H_\alpha]$ in the role of $V$, from $H \cap \text{Coll}_\alpha(\omega, < \beta)$. Applying Lemma 9.5.11 in $V[H_\alpha]$, we can deduce that $x \in V[H_\alpha]$. Hence $x \in \mathcal{R}_H^\ast$. \hfill \Box

We are now ready to prove our result about rearranging $\omega \cap N_{G,\kappa}$. 

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Lemma 9.5.13. Let $\langle \kappa_n \mid n \in \omega \rangle$ be a strictly increasing sequence of Woodin cardinals and let $\kappa = \sup_{n \in \omega} \kappa_n$. Let $\rho$ be a limit ordinal greater than $\kappa$. Let $G$ be $P_\rho$-generic or $Q_\rho$-generic over $V$ with $\mathcal{S} \in G$, where $\mathcal{S}$ is as in the statement of Corollary 9.5.2. In some forcing extension of $V[G]$, there is an $H$ that is $\text{Coll}(\omega, < \kappa)$-generic over $V$ and is such that, for $K$ the symmetric model defined from $H$,

$$(\omega^\omega)^{N_{G,\kappa}} = (\omega^\omega)^K.$$ 

Proof. In $V[G]$, define a partial ordering $(R; \leq)$ as follows. Elements of $R$ are elements $g$ of $N_{G,\kappa}$ such that $g$ is $\text{Coll}(\omega, < \gamma)$-generic over $V$ for some $\gamma < \kappa$. If $g_1$ and $g_2$ belong to $R$, then set $g_1 \leq g_2$ if and only if $g_1 \supseteq g_2$.

Working in some forcing extension of $V[G]$ in which which $R$ has only countably many subsets, let $H^*$ be $R$-generic over $V[G]$. Let $H = \bigcup H^*$.

To see that $H$ is $\text{Coll}(\omega, < \kappa)$-generic over $V$, let $D \in V$ be a dense subset of $\text{Coll}(\omega, < \kappa)$ and let $g \in R$. Let $\gamma$ be such that $g$ is $\text{Coll}(\omega, < \gamma)$-generic over $V$. The density of $D$ implies that $\{p \cap \text{Coll}(\omega, < \gamma) \mid p \in D\}$ is dense in $\text{Coll}(\omega, < \gamma)$. By the genericity of $g$, there is a $p \in D$ such that $p \cap \text{Coll}(\omega, < \gamma) \in g$. Let $\delta$ be such that $\gamma < \delta < \kappa$ and $p \in \text{Coll}(\omega, < \delta)$. Let $q = p \cap \text{Coll}_1(\omega, < \kappa) = p \cap \text{Coll}_1(\omega, < \delta)$. (See the proof of Lemma 9.5.10 for the definition of $\text{Coll}_1(\omega, < \kappa)$.) It follows from Lemmas 9.2.12 and 9.4.10 that $V_\kappa \subseteq N_{G,\kappa}$. By clause (1) of Lemma 9.5.4, $(\omega_1)^{N_{G,\kappa}} = \kappa$. Thus the set of dense subsets of $\text{Coll}_1(\omega, < \delta)$ that belong to $V[g]$ is countable in $N_{G,\kappa}$. It follows that there is in $N_{G,\kappa}$ a $\bar{g}$ such that $g$ is $\text{Coll}_1(\omega, < \delta)$-generic over $V[\bar{g}]$ and such that $q \in \bar{g}$. Let $g' = g \times \bar{g}$. Then $g'$ is $\text{Coll}(\omega, < \gamma) \times \text{Coll}_1(\omega, < \delta)$-generic over $V$, and so $g'$ gives a $\bar{g} \supseteq g$ that is $\text{Coll}(\omega, < \delta)$-generic over $V$ and such that $p \in \bar{g}$. Such a $\bar{g}$ belongs to $R \cap D$ and is $\leq g$. We have thus shown that the set of $\bar{g} \in R$ such that $\bar{g} \cap D \neq \emptyset$ is dense in $R$. It follows that $H$ meets this set.

For $n \in \omega$ let $G_n = G \cap Q_{\kappa_n}$. Each $G_n$ belongs to $N_{G,\kappa_n}$, and each $G_n$ is $Q_{\kappa_n}$-generic over $V$. Moreover every element of $(\omega^\omega)^{N_{G,\kappa_n}}$ belongs to $V[G_n]$ for some $n \in \omega$.

Suppose that $g \in R$ and $x \in (\omega^\omega)^{N_{G,\kappa_n}}$. Let $\gamma$ be such that $g$ is $\text{Coll}(\omega, < \gamma)$-generic over $V$. Since $\gamma$ is countable in $N_{G,\kappa_n}$, there is a $y \in (\omega^\omega)^{N_{G,\kappa_n}}$ such that both $x$ and $g$ belong to $V[y]$. Let $n \in \omega$ be such that $y \in V[G_n]$. Let $g' \in N_{G,\kappa_n}$ be $\text{Coll}(\omega, \kappa_n)$-generic over $V[G_n]$. By Lemma 9.5.8, there is an $h$ that is $\text{Coll}(\omega, \kappa_n)$-generic over $V[g]$ and such that $V[G_n][g'] = V[g][h]$. A reverse application of Lemma 9.5.7, like that in the proof of Lemma 9.5.10, gives us a $\bar{g}$ that is $\text{Coll}_1(\omega, < \kappa_n + 1)$-generic over $V[g]$ and such that...
$V[g][h] = V[g][g]$. This shows that there is a $g \in R$ with $g \leq g$ and $x \in V[g]$.

The argument just given shows that $(\omega)^{NG, \kappa} \subseteq K$. The reverse inclusion follows from Lemma 9.5.12.

Lemma 9.5.13 implies that certain formulas true in the $L(R)$ of the symmetric model defined from a Lévy collapse of $\kappa$ are also true in the $L(R)$ of $V$. Let $\xi(v)$ be some (natural) formula expressing “$v = \omega$.” Let us say that a formula is $\Sigma_1^1(R)$ if it is of the form $(\exists v_0)(\xi(v_0) \land \varphi(v_0, \ldots, v_n))$ with $\varphi$ a $\Sigma_1$ formula.

**Lemma 9.5.14.** Let $\langle \kappa_n \mid n \in \omega \rangle$ and $\kappa$ be as in Lemma 9.5.13. Let $H'$ be $\text{Coll}(\omega, < \kappa)$-generic over $V$ and let $K'$ be the symmetric model defined from $H'$. Let $\psi(v_1, \ldots, v_n)$ be a $\Sigma_1(R)$ formula and let $a_1, \ldots, a_n$ be elements of $(\omega)^V$. Then

$$(L(R))^{K'} \models \psi[a_1, \ldots, a_n] \rightarrow L(R) \models \psi[a_1, \ldots, a_n].$$

**Proof.** Assume that $(L(R))^{K'} \models \psi[a_1, \ldots, a_n]$. Let $\rho > \kappa$ be a limit ordinal such that

$$(L_{\rho}(R))^{K'} \models \psi[a_1, \ldots, a_n].$$

Let $G$ be $P_{\rho}$-generic or $Q_{\rho}$-generic over $V$ with $\hat{S} \in G$, where $\hat{S}$ is as in Corollary 9.5.2. Let $H$ and $K$ be given by Lemma 9.5.13. By Lemma 9.5.11,

$$(L_{\rho}(R))^K \models \psi[a_1, \ldots, a_n].$$

By Lemmas 9.5.5 and 9.5.13 and the fact that $\psi$ is $\Sigma_1(R)$,

$$(L(R))^{NG, \kappa} \models \psi[a_1, \ldots, a_n].$$

Since $i_{G, \kappa}(a_i) = a_i$ for each $i$ and since $\psi$ is $\Sigma_1(R)$, this implies that

$$(X_{G, \kappa}; \in_G) \models \psi[L(R)][i'_G(a_1), \ldots, i'_G(a_n)].$$

By the elementarity of $i'_G : (V; \in) \prec (X_{G, \kappa}; \in_G)$,

$L(R) \models \psi[a_1, \ldots, a_n].$

The next lemma will allow us to get stronger agreement with the aid of completely Jonsson cardinals.
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Lemma 9.5.15. Let \( \langle \kappa_n \mid n \in \omega \rangle \) and \( \kappa \) be as in Lemma 9.5.13. Let \( H' \) be \( \text{Coll}(\omega, < \kappa) \)-generic over \( V \) and let \( K' \) be the symmetric model defined from \( H' \). Let \( \kappa^* > \kappa \) be a completely Jonsson cardinal. Let \( \psi(v_1, \ldots, v_n) \) be a formula of set theory and let \( a_1, \ldots, a_n \) belong to \( (\omega^\omega)^V \). Then
\[
(L(R))^{K'} \models \psi[a_1, \ldots, a_n] \iff L(R) \models \psi[a_1, \ldots, a_n].
\]

Proof. Let \( \rho > \kappa^* \) be a limit ordinal. Let \( G \) be \( P_\rho \)-generic over \( V \) with \( \hat{S} \in G \), where \( \hat{S} \) is as in Corollary 9.5.2, and with \( \{ X \subseteq \kappa^* \mid |X| = \kappa^* \} \in G \). Let \( H \) and \( K \) be as given Lemma 9.5.13.

Since the \( \kappa_n \) are limits of measurable cardinals, a measurable cardinal exists. Thus Corollary 3.4.26 gives us that \( R^\# \) exists. By Corollary 3.4.27, we have that \( L_{\kappa^*}(R) \prec L(R) \).

Since \( R^\# \) exists, \( V_{\kappa^*} \models \text{“} R^\# \text{” exists.”} \) By the elementarity of \( i_{G, \kappa} \), \( V_{\kappa^*}^{N_{G, \kappa}} \models \text{“} R^\# \text{” exists.”} \) Thus we have, e.g. in \( V[G] \), the existence of \( (R^{N_{G, \kappa}})^\# \). Applying Corollary 3.4.27 again, we get that
\[
L_{\kappa^*}(R^{N_{G, \kappa}}) \prec L(R^{N_{G, \kappa}}).
\]

The facts just mentioned give us the fifth and the third equivalences in the following chain:
\[
L(R^{K'}) \models \psi[a_1, \ldots, a_n] \iff L(R^K) \models \psi[a_1, \ldots, a_n] \\
\iff L(R^{N_{G, \kappa}}) \models \psi[a_1, \ldots, a_n] \\
\iff L_{\kappa^*}(R^{N_{G, \kappa}}) \models \psi[a_1, \ldots, a_n] \\
\iff L_{\kappa^*}(R) \models \psi[a_1, \ldots, a_n] \\
\iff L(R) \models \psi[a_1, \ldots, a_n].
\]

The first equivalence is by Lemma 9.5.11, the second is by Lemma 9.5.13, and the fourth is by Lemma 9.5.6. \( \Box \)

We want to use the preceding lemmas to get agreement between the \( L(R) \) of the symmetric models and the \( L(R) \) of models gotten by forcing with partial orderings of size less than \( \kappa \). For this we need to show that Woodinness and measurability are preserved under small forcing. Such preservation is typical of large cardinal properties. Lemma 9.5.16, due to Lévy–Solovay [1967], is the basic example.
Lemma 9.5.16. Let $\kappa$ be a measurable cardinal. Let $R \in V_\kappa$ be a partial ordering and let $G$ be $R$-generic over $V$. Then $\kappa$ is measurable in $V[G]$.

**Proof.** Let $j : V \prec M$, with $M$ transitive and $\text{crit}(j) = \kappa$. In $V[G]$ define $j^* : V[G] \to M[G]$ by $j^*(\sigma_G) = (j(\sigma))_G$. It is easy to see that $j^*$ is well-defined and elementary and that $j^* \upharpoonright V = j$. Hence $j^*$ witnesses that $\kappa$ is measurable in $V[G]$. \hfill \Box

Lemma 9.5.17. Let $\kappa$ be a Woodin cardinal. Let $R \in V_\kappa$ be a partial ordering and let $G$ be $R$-generic over $V$. Then $\kappa$ is Woodin in $V[G]$.

**Proof.** In $V[G]$ let $f : \kappa \to \kappa$. We may assume that $R \in V_{f(0)}$. Let $f = \tau_G$. Let $r \in G$ be such that $r \parallel - \tau : \check{\kappa} \to \check{\kappa}$. For each $\alpha < \kappa$, let $g(\alpha) = \sup\{\gamma \mid (\exists p \in R)(p \leq r \land p \parallel - \tau(\check{\alpha}) = \check{\gamma})\}$. Then $g \in V$ and $g(\alpha) \geq f(\alpha)$ for all $\alpha < \kappa$.

Let $\delta$ and $j : V \prec M$ witness that $\kappa$ is Woodin for $g$. We must have $\delta > g(0) \geq f(0)$; hence $R \in V_\delta$. Define $j^* : V[G] \to M[G]$ by $j^*(\sigma_G) = (j(\sigma))_G$. It is easy to see that $j^*$ is well-defined and elementary and that $j^* \supseteq j$. Thus $\delta$ and $j^*$ witness that $\kappa$ is Woodin in $V[G]$ for $g$ and so for $f$. \hfill \Box

We now prove a pair of technical results that will provide the main applications of this section in Section 6.

Let $\kappa$ be an uncountable cardinal, let $\varphi$ be a formula, and let $a$ be a set. Say that $\langle \varphi, a \rangle$ is $\kappa$-generically absolute if, whenever $R_1$ and $R_2$ are partialy orderings of size $\leq \kappa$ and $J_i$, $1 \leq i \leq 2$, are $R_i$-generic over $V$ with $J_1 \in V[J_2]$, then, for all $x \in (^{\omega}\omega)^{V[J_1]}$, $V[J_1] \models \varphi[x, a] \iff V[J_2] \models \varphi[x, a]$.

Say that $\langle \varphi, a \rangle$ is ($< \kappa$)-generically absolute if $\langle \varphi, a \rangle$ is $\gamma$-generically absolute for every $\gamma < \kappa$.

**Lemma 9.5.18.** Let $\langle \kappa_n \mid n \in \omega \rangle$ and $\kappa$ be as in Lemma 9.5.13. Assume that there is a measurable cardinal greater than $\kappa$. Let $\psi(v_1, v_2)$ be a formula and let $\varphi$ be $\psi(L(R))$. Let $a \in \omega \omega$. Then $\langle \varphi, a \rangle$ is ($< \kappa$)-generically absolute.
Proof. Let $R_1$, $R_2$ be partial orderings of size less than $\kappa$. Let $J_1$ and $J_2$ be respectively $R_1$-generic and $R_2$-generic over $V$ and such that $V[J_1] \subseteq V[J_2]$. Let $x \in (\omega^\omega)^{V[J_1]}$.

Let $H_2$ be $\text{Coll}(\omega, < \kappa)$-generic over $V[J_2]$. Let $K_2$ be the symmetric model defined from $H_2$. Applying Lemma 9.5.10 in $V[J_1]$, let $H_1$ be $\text{Coll}(\omega, < \kappa)$-generic over $V[J_1]$ and such that $K_1 = K_2$, where $K_1$ is the symmetric model defined from $H_1$.

Let $\kappa^* > \kappa$ be measurable. By Lemma 9.5.16, $\kappa^*$ is measurable in both $V[J_1]$ and $V[J_2]$. Thus it is completely Jonsson in both models. Applying Lemma 9.5.15 in $V[J_1]$ and $V[J_2]$, where by Lemma 9.5.17 all but finitely many of the $\kappa_n$ are still Woodin, we get that

$$
(L(R))^V[J_1] \models \psi[x, a] \iff (L(R))^{K_1} \models \psi[x, a] \iff (L(R))^{K_2} \models \psi[x, a] \iff (L(R))^V[J_2] \models \psi[x, a].
$$

\[ \Box \]

Lemma 9.5.19. Let $\langle \kappa_n \mid n \in \omega \rangle$ and $\kappa$ be as in Lemma 9.5.13. Let $\psi(v_1, v_2)$ and $\psi'(v_1, v_2)$ be $\Sigma_1(R)$ formulas such that

$$
ZF \vdash (\forall v_1)(\forall v_2)(\neg \psi(v_1, v_2) \lor \neg \psi'(v_1, v_2)).
$$

Let $\phi(v_1, v_2)$ be $\psi^{L(R)}$. Let $a \in \omega^\omega$. Let $H$ be $\text{Coll}(\omega, < \kappa)$-generic over $V$ and let $K$ be the symmetric model defined from $H$. Assume that

$$
(\forall y \in (\omega^\omega)^{K})(L(R))^K \models \psi[y, a] \lor (L(R))^K \models \psi'[y, a]).
$$

Then $\langle \phi, a \rangle$ is $< \kappa$-generically absolute.

Proof. Let $R_1$, $R_2$, $J_1$, $J_2$, $x$, $K_1$, and $K_2$ be as in the proof of Lemma 9.5.18.

By Lemma 9.5.10, let $H'$ be $\text{Coll}(\omega, < \kappa)$-generic over $V$ and such that $K' = K_1$, where $K'$ is the symmetric model defined from $H'$. By Lemma 9.5.11, we get that

$$
(\forall y \in (\omega^\omega)^{K'})((L(R))^{K'} \models \psi[y, a] \lor (L(R))^{K'} \models \psi'[y, a]).
$$

If $(L(R))^{K'} \models \psi[x, a]$, then $(L(R))^{K_1} \models \psi[x, a]$ and $(L(R))^{K_2} \models \psi[x, a]$. Applying Lemma 9.5.14 in $V[J_1]$ and in $V[J_2]$, we get that

$$
(L(R))^V[J_1] \models \psi[x, a] \land (L(R))^V[J_1] \models \psi[x, a].
$$
If \((L(R))^{K'} \models \psi'[x,a]\), then we similarly get that
\[(L(R))^{V[J_1]} \models \psi'[x,a] \land (L(R))^{V[J_2]} \models \psi[x,a].\]
Since neither \((L(R))^{V[J_1]}\) nor \((L(R))^{V[J_2]}\) can satisfy both \(\psi[x,a]\) and \(\psi'[x,a]\), the lemma is proved. \(\square\)

**Exercise 9.5.1.** Let \(\langle \kappa_n \mid n \in \omega \rangle\) and \(\kappa\) be as in Lemma 9.5.13. Let \(H'\) be \(\text{Coll}(\omega, < \kappa)\)-generic over \(V\) and let \(K'\) be the symmetric model defined from \(H'\). Let \(\kappa^* > \kappa\) be a cardinal and let \(\rho > \kappa^*\) be a limit ordinal. Let \(G\) be \(P_{\rho^*}\)-generic over \(V\) with \(\hat{S} \in G\), where \(\hat{S}\) is as in Corollary 9.5.2, and with \(\{X \subseteq \kappa^* \mid |X| = \kappa^*\} \in G\). Let \(\psi(v_1, \ldots, v_n)\) be a formula of set theory and let \(a_1, \ldots, a_n\) be elements of \(L_{\kappa^*}(R)\) such that \(i_{G,\kappa}(a_i) = a_i\) for each \(i, 1 \leq i \leq n\). Show that
\[(L(R))^{K'} \models \psi[a_1, \ldots, a_n] \iff L(R) \models \psi[a_1, \ldots, a_n].\]

**Exercise 9.5.2.** Let \(\kappa\) be a measurable cardinal. Let \(R \in V_\kappa\) be a partial ordering and let \(G\) be \(R\)-generic over \(V\). Show that every uniform normal ultrafilter on \(\kappa\) in \(V\) generates a uniform normal ultrafilter on \(\kappa\) in \(V[G]\).

**Hint.** Let \(U\) be a uniform normal ultrafilter on \(\kappa\). Let \(\tau_G \subseteq \kappa\). For each \(p \in R\), let
\[X_p = \{\alpha < \kappa \mid p \models \bar{\alpha} \in \tau\}.\]
Show that either there is a \(p \in G\) such that \(X_p \in U\) or else there is a \(q \in G\) such that \(X_p \notin U\) for all \(p \leq q\). This implies that either \(\tau_G\) or its complement belongs to the filter generated in \(V[G]\) by \(U\).

**Exercise 9.5.3.** Let \(\langle \kappa_n \mid n \in \omega \rangle\) and \(\kappa\) be as in Lemma 9.5.13. Let \(H'\) be \(\text{Coll}(\omega, < \kappa)\)-generic over \(V\) and let \(K'\) be the symmetric model defined from \(H'\). Assume that there is a measurable cardinal greater than \(\kappa\). Let \(a\) be an ordered pair whose first component is a natural number and whose second component is an \(m\)-tuple of elements of the transitive closure of \(\omega\). (Such an object is a candidate for being a member of \(R^\#\).) Show that
\[(L(R))^{K'} \models a \in R^\# \iff L(R) \models a \in R^\#.\]

**Hint.** Let \(\kappa^* > \kappa\) be measurable and let \(U\) be a uniform normal ultrafilter on \(\kappa^*\). By Lemma 9.2.14, the set of completely Jonsson cardinals smaller than \(\kappa\) belongs to \(U\). Hence the intersection of this set with \(C^R\) belongs to \(U\). Let \(\rho\) and \(G\) be as in the proof of Lemma 9.5.15. Use Lemma 9.2.18 to deduce that the set of all elements of \(C^R\) that are fixed points of \(i_{G,\kappa}\) is unbounded in \(\kappa^*\). Use this fact and Exercise 9.5.1 to get the desired conclusion.
9.6. $\text{AD}^L(R)$

The proof, from a large cardinal hypothesis, of the determinacy of all games $G(A; \omega^\omega)$ with $A \in L(R)$ proceeds as follows. Say that $A \subseteq \omega^\omega$ is determined if $G(A; \omega^\omega)$ is determined. We first show that certain subsets of $\omega^\omega$ in $L(R)$ are $\kappa^+$-universally Baire for some Woodin cardinal $\kappa$. Next we show that every $\kappa^+$-universally Baire set is weakly $\kappa^+$-homogeneously Souslin. We then assume that there is a non-determined set in $L(R)$, and we deduce that some non-determined set is $\kappa$-universally Baire for some Woodin cardinal $\kappa$. With the aid of the results of Chapter 8, we derive a contradiction. The proof that AD holds in symmetric collapse models $K$ has a similar structure.

Since we will henceforth mainly be concerned with trees on sets of the form $\omega \times Y$, it will be convenient to have some notation for the $\omega^\omega$-projection of such trees. Let then

$$\text{proj} Z = \{x \in \omega^\omega | (\exists f \in \omega^Y) \langle x, f \rangle \in Z\}.$$ 

Thus $\text{proj} [U]$ is the $\omega^\omega$-projection of $U$ if $U$ is a tree on $\omega \times Y$.

For cardinal numbers $\kappa$, a subset $A$ of $\omega^\omega$ is $\kappa$-universally Baire if there are trees $U$ and $U'$, on $\omega \times Y$ and $\omega \times Y'$ respectively for some sets $Y$ and $Y'$, such that

(i) $A = \text{proj} [U]$;

(ii) $\neg A = \text{proj} [U']$;

(iii) for any partial ordering $R$ such that $|R| \leq \kappa$ and for any $H$ that is $R$-generic over $V$, the sets $(\text{proj} [U])^{V[H]}$ and $(\text{proj} [U'])^{V[H]}$ are complementary subsets of $(\omega^\omega)^{V[H]}$.

The definition of $(< \kappa)$-universally Baire is similar, except that in (iii) ”$|R| \leq \kappa$” is replaced by ”$|R| < \kappa$.” The notions of $\kappa$-universally Baire sets and of $(< \kappa)$-universally Baire sets come from Feng–Magidor–Woodin [1997]. The official definition of $\kappa$-universally Baire in that paper is as follows: $A$ is $\kappa$-universally Baire if for every topological space $X$ with a regular open base
proof. Since $\tilde{S} \in P_{\text{club}}(V_{\lambda})$, let $f: ^{<\omega}V_{\lambda} \rightarrow V_{\lambda}$ be such that every countable $X \subseteq V_{\lambda}$ that is closed under $f$ is an elementary submodel of $V_{\lambda}$ and is $(<\kappa)$-generically correct for $\langle \varphi, a \rangle$.

Let $\tilde{g}: \omega \rightarrow ^{<\omega}\omega$ be a surjection and such that range $(\tilde{g}(m)) \subseteq m + 1$ for all $m \in \omega$.

Let $U$ be the set of all $\{p, \{r, s\}\}$ such that, for some $k \in \omega$,

1. $p \in k\omega$;
2. $r \in kV_{\lambda}$;
3. for each $m$ such that $2m + 2 \leq k$, $r(2m+2) = f(\langle r((g(m))(i)) \mid i < \ell h(g(m))) \rangle)$;
4. if $k > 0$ then $r(0)$ is a partial ordering $(R; \leq)$ belonging to $V_{\kappa}$;
We prove (a); the proof of (b) is similar. Suppose that $x$ is (in the obvious sense), and $\tilde{\tau}$ such that all $x \in proj [U]$ obeys (3), (5), and (ii) for each $n$ there is an $s_n \in H$ such that $s_n \models \tau(\tilde{n}) = x(n)$, and (iii) there is an $s' \in H$ such that $s' \models \varphi[\tau, \tilde{a}]$. Since $X \triangleleft V_\lambda$, these elements of $H$ force the same sentences in $X$. Thus $H = \pi_X''H$ is $\pi_X(R)$-generic over $M_X$ and is such that

$$M_X[H] \models \varphi[x, \pi_X(a)].$$

Since $X$ is $(< \kappa)$-generically correct for $(\varphi, a)$, $\varphi[x, a]$ holds.

Now let $R$ be a partial ordering belonging to $V_\lambda$ and let $H$ be $R$-generic over $V$. We show that

$$(c) (\forall x \in \omega)\omega)(V[H] \models \varphi[x, a] \to x \in (proj [U])^V[H]);$$

$$(d) (\forall x \in \omega)\omega)(V[H] \models \neg \varphi[x, a] \to x \in (proj [U'])^V[H]).$$

We prove only (c). Let $x \in \omega \cap V[H]$ be such that $V[H] \models \varphi[x, a]$. Let $\tau$ be such that $\tau_H = x$. Let $X \triangleleft V_\lambda$ be closed under $f$ and be such that $R \in X$. Let $\tilde{r} : \omega \to X$ and $\tilde{s} : \omega \to H \cap X$ be surjections such that $\tilde{r}(0) = \tilde{\tau}$, such that $\tilde{r}(1) = \tau$, and such that all $\{x \upharpoonright k, \{\tilde{r} \upharpoonright k, \tilde{s} \upharpoonright k\}\}$ obey (3), (5), and (7)–(10). Then $\tilde{p}, \{\tilde{r}, \tilde{s}\}$ belongs to $[U]$ and witnesses that $x \in proj [U]$. 

(5) for each $m$ such that $2m + 3 < k$, $r(2m + 3) = s(m)$;

(6) $\{s(m) \mid m < k\}$ is a subset of $R$ having a common lower bound with respect to $\leq$;

(7) for each $m$ such that $2m + 2 < k$, if $r(m)$ is a dense subset of $R$ then $s(2m + 2) \in r(m)$;

(8) if $k > 1$ then $r(1)$ is an $R$-name and $s(0) \models r(1) : \tilde{\omega} \to \tilde{\omega}$;

(9) for each $m$ such that $2m + 3 < k$, $s(2m + 3) \models r(1)(m) = (p(m))$;

(10) if $k > 1$ then $s(1) \models \varphi[r(1), \tilde{a}]$.

Similarly define a tree $U'$, with $\neg \varphi[r(1), \tilde{a}]$ replacing $\varphi[r(1), \tilde{a}]$.
Let $R$ and $H$ be as in the last paragraph. By (c) and (d), the proof will be complete if we can show that
\[(\text{proj } [U])^V[H] \cap (\text{proj } [U'])^V[H] = \emptyset.\]

Let
\[S = \{\langle p, \langle r, s \rangle, \langle r', s' \rangle \rangle | \langle p, \langle r, s \rangle \rangle \in U \land \langle p, \langle r', s' \rangle \rangle \in U'\}.\]
The tree $S$ is wellfounded in $V$. Thus it is wellfounded in $V[H]$. This in turn means that proj $[U]$ and proj $[U']$ are disjoint in $V[H]$. □

Our next step is to get a useful sufficient condition for the hypotheses of Lemma 9.6.1 to hold. This sufficient condition will be formulated in terms of the notion of generic absoluteness, introduced in Section 5, and one further notion.

Let $\rho$ be a limit ordinal, let $a$ be a set, and let $\varphi$ be a formula. Say that $P_\rho$ is \textit{correct} for $\langle \varphi, a \rangle$ if, for every $G$ that is $P_\rho$-generic over $V$,
\begin{enumerate}
  \item [(a)] $\langle \omega, \omega \rangle^V[G] \subseteq \text{range (} \pi_G \text{)}$;
  \item [(b)] $(\forall x \in (\omega, \omega)^V[G]) (V[G] \models \varphi[x, a] \leftrightarrow \bigwedge \pi_G (V; e) \models \varphi[\pi_G^{-1}(x), i_G(a))]$.
\end{enumerate}

Similarly, say that $Q_\rho$ is \textit{correct} for $\langle \varphi, a \rangle$ if (a) and (b) hold for every $G$ that is $Q_\rho$-generic over $V$.

\textbf{Remark.} In our applications of these definitions, Ult$(V; G)$ will exist. When this is the case, clause (a) above asserts that $(\omega, \omega)^V[G] = (\omega, \omega)^{\text{Ult}(V; G)}$, and (b) says that
\[(\forall x \in (\omega, \omega)^V[G]) (V[G] \models \varphi[x, a] \leftrightarrow \text{Ult}(V; G) \models \varphi[x, i_G(a)]).\]

The following lemma follows immediately from Theorem 9.3.8.

\textbf{Lemma 9.6.2.} Let $\kappa$ be Woodin, let $\varphi(v_1, v_2)$ be a formula, and let $a \in \omega$. Then both $P_\kappa$ and $Q_\kappa$ are correct for $\langle \varphi, a \rangle$.

We now derive the sufficient condition mentioned above.

\textbf{Lemma 9.6.3.} Let $\kappa$ be inaccessible and let $a \in V_\kappa$. Let $\varphi(v_1, v_2)$ be a formula. Assume that $\langle \varphi, a \rangle$ is $\kappa$-generically absolute and that either $P_\kappa$ or $Q_\kappa$ is correct for $\langle \varphi, a \rangle$. Let $\lambda > \kappa$ be such that $V_\lambda \prec \Sigma_n V$, where $\varphi$ is, say, $\Sigma_n-10$.

Then the set $\tilde{S}$ of all countable $X \prec V_\lambda$ such that $X$ is $(<\kappa)$-generically correct for $\langle \varphi, a \rangle$ belongs to $\mathcal{F}_{P_\kappa}^{\text{club}}(V_\lambda)$.
Proof. Assume that $\bar{S} \notin \mathcal{F}^{\text{club}}_{\mathcal{P}_{\aleph_1}(V_\gamma)}$. For each countable $X \prec V_\lambda$ such that ${a, \kappa} \subseteq X$ and $X \notin \bar{S}$, there is a $\delta \in X$ such that $X$ is not $(<\delta)$-generically correct for $\langle \varphi, a \rangle$. (Let $\bar{R} = \pi_{\mathcal{R}}(\mathcal{R})$ witness that $X \notin \bar{S}$, and let $\delta$ be the least cardinal $\gamma$ such that $R \in V_\gamma$.) Since $\mathcal{P}_{\aleph_1}(V_\lambda) \setminus \bar{S}$ is stationary, the normality of $\mathcal{F}^{\text{club}}_{\mathcal{P}_{\aleph_1}(V_\lambda)}$ gives us a $\delta < \kappa$ such that $S_\delta$ is stationary in $\mathcal{P}_{\aleph_1}(V_\lambda)$, where $S_\delta$ is the set of all countable $X \prec V_\lambda$ such that $X$ is not $(<\delta)$-generically correct for $\langle \varphi, a \rangle$. Hence $S_\delta$ is stationary in $\mathcal{P}_{\aleph_1}(V_\lambda)$ for all sufficiently large $\delta < \kappa$. Fix such a $\delta$ with $\delta$ a strong limit cardinal and $a \in V_\delta$.

Let $Y \prec V_\lambda$ with $|Y| < \kappa$ and $V_{\delta+1} \subseteq Y$. Let $S$ be the set of all countable $X \prec M_Y$ such that $X$ is not $(<\delta)$-generically correct for $\langle \varphi, a \rangle$. For all but a nonstationary set of $X \in S_\delta$, $X \cap Y \prec Y$. For such an $X$, any witnesses that $X$ is not $(<\delta)$-generically correct for $\langle \varphi, a \rangle$ will directly give witnesses that $X \cap Y ^\ominus$ is not $(<\delta)$-generically correct for $\langle \varphi, a \rangle$. Hence $S$ and $\pi_{\mathcal{Y}}(S_\delta \res Y)$ differ by a nonstationary set. Therefore $S$ is stationary in $\mathcal{P}_{\aleph_1}(M_Y)$.

Let $S^*$ be defined from $S$ as in the statement of Theorem 9.5.1. Depending on whether $P_\kappa$ or $Q_\kappa$ is correct for $\langle \varphi, a \rangle$, let $G$ be $P_\kappa$-generic or $Q_\kappa$-generic over $V$ with $S^* \in G$.

Since $S \in G$, we have that

$$[\text{id}_S]_G \in G \ i'_G(S).$$

Thus $\prod_G (V; \in)$ satisfies “$[\text{id}_S]_G$ is not $(<i'_G(\delta))$-generically correct for $i'_G(\langle \varphi, a \rangle)$.”

By Lemma 9.2.8,

$$i'_G | \bigcup S : (\bigcup S; \in) \cong ([\text{id}_S]_G; \in)_G.$$

Since $\bigcup S = \bar{M}_Y$, the conclusion of the preceding paragraph gives that there is a partial ordering $\bar{R} \in V_\delta ^{\bar{M}_Y}$, there is an $\bar{H} \in \text{range}(\pi_G)$, and there is an $x \in (\omega) ^{\bar{M}_Y \res \bar{H}}$ such that $\bar{H}$ is $\bar{R}$-generic over $\bar{M}_Y$ and

$$\bar{M}_Y[\bar{H}] \models \varphi[x, a] \iff \prod_G (V; \in) \not\models \varphi[\pi_G^{-1}(x), i'_G(a)].$$

Note that $\bar{H}$ is also $\bar{R}$-generic over $V$.

Assume for definiteness that $\bar{M}_Y[\bar{H}] \models \varphi[x, a]$. Let $\tau_{\bar{H}} = x$ (in both models $\bar{M}_Y[\bar{H}]$ and $V[\bar{H}]$). For some $p \in \bar{R}$,

$$\bar{M}_Y \models p \models \varphi(\tau, \bar{a}).$$
Since \( Y \prec V_\lambda \prec \Sigma_n V \),
\[
V \models p \Vdash \varphi(\tau, \bar{a}).
\]
Hence
\[
V[\bar{H}] \models \varphi[x, a].
\]
Since \( \bar{H} \in V[G] \) and \( \langle \varphi, a \rangle \) is \( \kappa \)-generically absolute,
\[
V[G] \models \varphi[x, a].
\]
Since the relevant one of \( P_\kappa \) and \( Q_\kappa \) is correct for \( \langle \varphi, a \rangle \),
\[
\prod_G (V; \in) \models \varphi[\pi_G^{-1}(x), \bar{i}_G(a)].
\]
This is a contradiction. \( \square \)

We now combine Lemmas 9.6.1 and 9.6.3, getting what Woodin calls the Tree Production Lemma.

**Lemma 9.6.4.** Let \( \kappa \) be inaccessible and let \( a \in V_\kappa \). Let \( \varphi(v_1, v_2) \) be a formula. Assume that \( \langle \varphi, a \rangle \) is \( \kappa \)-generically absolute and that either \( P_\kappa \) or \( Q_\kappa \) is correct for \( \langle \varphi, a \rangle \).

Then \( \{ x \in \omega : L(R) \models \psi(x, a) \} \) is \((<\kappa)\)-universally Baire. Indeed, there are trees \( U \) and \( U' \) such that, for any partial ordering \( R \) of size less than \( \kappa \) and for any \( H \) that is \( R \)-generic over \( V \),

\[
\begin{align*}
(i) \{ x \in (\omega^{\omega}) \mid V[H] \models \varphi[x, a] \} & = (\text{proj}[U])^{V[H]}; \\
(ii) \{ x \in (\omega^{\omega}) \mid V[H] \not\models \varphi[x, a] \} & = (\text{proj}[U'])^{V[H]}.
\end{align*}
\]

**Theorem 9.6.5.** Let \( \langle \kappa_n \mid n \in \omega \rangle \) be a strictly increasing sequence of Woodin cardinals and let \( \kappa = \sup_{n \in \omega} \kappa_n \). Assume that there is a measurable cardinal greater than \( \kappa \). Let \( \psi(v_1, v_2) \) be a formula and let \( \varphi \) be \( \psi_L(R) \). Let \( a \in \omega^{\omega} \). Then \( \{ x \in \omega : \varphi(x, a) \} \) is \((<\kappa)\)-universally Baire. Indeed, there are trees \( U \) and \( U' \) such that, for any partial ordering \( R \) of size less than \( \kappa \) and for any \( H \) that is \( R \)-generic over \( V \),

\[
\begin{align*}
(i) \{ x \in (\omega^{\omega}) \mid V[H] \models \varphi[x, a] \} & = (\text{proj}[U])^{V[H]}; \\
(ii) \{ x \in (\omega^{\omega}) \mid V[H] \not\models \varphi[x, a] \} & = (\text{proj}[U'])^{V[H]}.
\end{align*}
\]
Proof. Lemmas 9.5.18, 9.6.2, and 9.6.4 give us, for each \( n \in \omega \), trees \( U_n \) and \( U'_n \), such that clauses (i) and (ii) hold for \( U_n \) and \( U'_n \) for every \( H \) that is \( R \)-generic over \( V \) for an \( R \) of size \( < \kappa_n \).

We get \( U \) and \( U' \) by combining the \( U_n \) and the \( U'_n \) respectively as follows. For \( m \in \omega \), \( t : m \to V \), and \( n \in \omega \), define \( t_n : m \to V \) by setting \( t_n(k) = \langle n, t(k) \rangle \) for each \( k < m \). Let

\[ U = \{ \langle p, t_n \rangle \mid n \in \omega \land \langle p, t_n \rangle \in U_n \}. \]

Similarly define \( U'_n \) in \( V \).

\[ \{ x \in (\omega^\omega) \mid \varphi(x,a) \} = \text{proj}[U]; \]
\[ \{ x \in (\omega^\omega) \mid \neg \varphi(x,a) \} = \text{proj}[U']. \]

By the argument at the end of the proof of Lemma 9.6.1, \( \text{proj}[U] \) and \( \text{proj}[U'] \) are disjoint in all forcing extensions of \( V \). It is thus easy to see that \( U \) and \( U' \) are as in the statement of the lemma. \( \square \)

Theorem 9.6.5 gives, under its hypotheses, that all subsets of \( \omega^\omega \) definable in \( L(R) \) from an element of \( \omega^\omega \) are \((<\kappa)\)-universally Baire. Though this will suffice for our determinacy proof, it is nevertheless true that the hypotheses of Theorem 9.6.5 imply that every subset of \( \omega^\omega \) in \( L(R) \) is \((<\kappa)\)-universally Baire. (See Exercise 9.6.2.) We will prove a better result in Theorem 9.6.19.

The proof of the next theorem is like that of Theorem 9.6.5, and we omit it.

Theorem 9.6.6. Let \( \langle \kappa_n \mid n \in \omega \rangle \) be a strictly increasing sequence of Woodin cardinals and let \( \kappa = \sup_{n \in \omega} \kappa_n \). Let \( \psi \) and \( \psi' \) be as in the statement of Lemma 9.5.19. Let \( \varphi \) be \( \psi^{L(R)} \). Then \( \{ x \in (\omega^\omega) \mid \varphi(x,a) \} \) is \((<\kappa)\)-universally Baire. Indeed, there are trees \( U \) and \( U' \) such that, for any partial ordering \( R \) of size less than \( \kappa \) and for any \( H \) that is \( R \)-generic over \( V \),

(i) \( \{ x \in (\omega^\omega)^{V[H]} \mid V[H] \models \varphi[x,a] \} = (\text{proj}[U])^{V[H]}; \)
(ii) \( \{ x \in (\omega^\omega)^{V[H]} \mid V[H] \not\models \varphi[x,a] \} = (\text{proj}[U'])^{V[H]} \).

We now turn to the proof that, for \( \kappa \) Woodin, all \( \kappa^+ \)-universally Baire sets are weakly \( \gamma \)-homogeneously Souslin, for all \( \gamma < \kappa \).

Lemma 9.6.7. Let \( \kappa \) be a cardinal number. Let \( P \) be a partial ordering such that \( |P| \leq \kappa \) and such that, for all \( p \in P \), \( p \models 2^{\aleph_0} \leq \kappa \). Let \( U \) be a tree on \( \omega \times Y \), for some set \( Y \). There is a subtree \( U' \) of \( U \) such that
(a) $|\bar{U}| \leq \kappa$;

(b) for every $G$ that is $P$-generic over $V$, $V[G] \models \text{proj}[\bar{U}] = \text{proj}[U]$.

**Proof.** Let $\tau$ be a $P$-name such that, for every $p \in P$, $p \models \text{"$\tau$ is a surjection of $\kappa$ onto $\omega$.}$. Let $\sigma$ be a $P$-name such that, for every $p \in P$ and every $\alpha < \kappa$, $p \models \text{"if $\tau(\check{\alpha}) \in \text{proj}[\check{U}]$ then $\langle |\tau(\check{\alpha}), \sigma(\check{\alpha})| \rangle \in [\check{U}]$.}$. Let $Y$ be the set of all $y \in Y$ such that, for some $p \in P$, some $n \in \omega$, and some $\alpha < \kappa$, $p \models \text{"$(\sigma(\check{\alpha}))(\check{n}) = y$.}$

Let $\bar{U} = \{ \langle s, t \rangle \in U \mid \text{range}(t) \subseteq \bar{Y} \}$. It is easy to see that $\bar{U}$ has the required properties.

**Lemma 9.6.8.** Let $\kappa$ be Woodin. Let $U$ and $U'$ be trees on $\omega \times Y$ and $\omega \times Y'$ respectively, for some sets $Y$ and $Y'$. Assume that

(i) $\text{proj}[U] = \omega \setminus \text{proj}[U']$;

(ii) for every $G$ that is $Q_\kappa$-generic over $V$, $V[G] \models \text{proj}[U] = \omega \setminus \text{proj}[U']$.

Then for every $\gamma < \kappa$ there is a full, $\gamma$-complete cover of $U$ by ultrafilters. (See page 426 for the definition.)

**Proof.** Every element of $(\omega^\omega)_{\text{Ult}(V; G)}$ is $\pi_G([f]_G)$ for some $f \in V_\kappa$. Therefore $|(\omega^\omega)_{\text{Ult}(V; G)}| \leq \kappa$ in $V[G]$. Since $(\omega^\omega)_{\text{Ult}(V; G)} = (\omega^\omega)^{V[G]}$, $V[G] \models 2^{\aleph_0} \leq \kappa$. Since this is true not only for the given $G$, but for any object that is $Q_\kappa$-generic over $V$, Lemma 9.6.7 authorizes us to assume that $U$ is a tree on $\omega \times \kappa$.

Let $G$ be $Q_\kappa$-generic over $V$.

From (i) and (ii) we get the following facts about the $\leq \omega$ projections of the trees $U$, $U'$, $i_G(U)$, and $i_G(U')$:

1. $\text{Ult}(V; G) \models \text{proj}[i_G(U)] = \omega \setminus \text{proj}[i_G(U')]$;
2. $V[G] \models \text{proj}[i_G(U)] = \omega \setminus \text{proj}[i_G(U')]$;
3. $V[G] \models \text{proj}[U] \subseteq \text{proj}[i_G(U)]$;
4. $V[G] \models \text{proj}[U'] \subseteq \text{proj}[i_G(U')]$;
5. $V[G] \models \text{proj}[U] = \text{proj}[i_G(U)]$. 


Theorem 9.6.9. Let $\kappa$ be a Woodin cardinal and let $A \subseteq {}^\omega \omega$ be $\kappa$-universally Baire. Then $A$ is weakly $\gamma$-homogeneously Souslin for every $\gamma < \kappa$.

Proof. The theorem follows directly from Lemma 9.6.8 and Theorem 8.1.7. $\Box$
Theorem 9.6.10. If $A \subseteq \omega_1$ is $\kappa$-universally Baire and there are at least two Woodin cardinals $\leq \kappa$, then $G(A;^{<\omega_1})$ is determined.

Proof. The theorem follows from Theorem 9.6.9, Corollary 8.2.8, and Theorem 4.3.5. □

Theorem 9.6.11. If there is a measurable cardinal that is larger than infinitely many Woodin cardinals, then $G(A;^{<\omega_1})$ is determined for every $A$ in $L(R)$. Hence the Axiom of Determinacy holds in $L(R)$.

Proof. Assume that there is a non-determined $A \subseteq \omega_1$ such that $A \in L(R)$. Every element of $L(R)$ is definable in $L(R)$ from an ordinal and an element of $\omega_1$. By minimizing the ordinal, we get a non-determined $A$ that is definable in $L(R)$ from some element $a$ of $\omega_1$. We get a contradiction by applying Theorem 9.6.5 to $a$ and a defining formula $\psi$ and then applying Theorem 9.6.10. □

Since Theorems 9.6.10 and 9.6.11 are proved by combining the Woodin results of this chapter with those of Martin–Steel presented in Chapter 8, a word is in order about the order in which the results were proved. Woodin, in the wake of Foreman–Magidor–Shelah [1988], proved versions of Theorems 9.6.5 and 9.6.9, using the stronger hypothesis that a supercompact cardinal exists. (See Woodin [1988].) Woodin’s isolation of the notion of Woodin cardinals occurred at about the same time. Next Martin and Steel proved Theorem 8.2.7 and Corollary 8.2.8. Finally Woodin was able to prove his results from hypotheses about Woodin cardinals. In the course of doing so, Woodin also proved the following theorem.

Theorem 9.6.12. Let $\langle \kappa_n \mid n \in \omega \rangle$ be a strictly increasing sequence of Woodin cardinals and let $\kappa = \sup_{n \in \omega} \kappa_n$. Let $H$ be $\text{Coll}(\omega, < \kappa)$-generic over $V$ and let $K$ be the symmetric model defined from $H$. Then the Axiom of Determinacy holds in $(L(R))^K$.

Proof. Assume that AD does not hold in $(L(R))^K$. There is a least ordinal $\alpha$ such that some $A \subseteq (\omega_1)^K$ belongs to $(L_\alpha(R))^K$ and is not determined in $(L(R))^K$. Then $(L_\alpha(R))^K \models \text{“}A \text{ is not determined.”}$ There is some $a \in (\omega_1)^K$ such that $A$ is definable in $(L_\alpha(R))^K$ from $a$. Let $\varphi$ be a formula witnessing this.

Let $\chi(v_0, v_2, v_3)$ be the $\Sigma_1(R)$ formula that is the conjunction of the formulas paraphrased as follows:
(a) \( v_0 \) is the least ordinal \( \beta \) such that AD does not hold in \( L_\beta(\mathcal{R}) \);
(b) \( v_2 \in {}^\omega \omega \);
(c) \( v_3 \) is the subset of \( {}^\omega \omega \) defined by \( \varphi \) in \( L_{v_0} \) from \( v_2 \);
(d) \( G(v_3; {}^<\omega \omega) \) not determined.

Let \( \psi(v_1, v_2) \) be the formula
\[
(\exists v_0)(\exists v_3)(\chi(v_0, v_2, v_3) \land v_1 \in v_3)
\]
and let \( \psi'(v_1, v_2) \) be the formula
\[
(\exists v_0)(\exists v_3)(\chi(v_0, v_2, v_3) \land v_1 \notin v_3).
\]
These formulas are trivially equivalent to \( \Sigma_1(\mathcal{R}) \) formulas. Clearly
(i) \( ZF \vdash (\forall v_1)(\forall v_2)(\neg \psi(v_1, v_2) \lor \neg \psi'(v_1, v_2)) \);
(ii) \( (\forall y \in {}^\omega \omega)(L(\mathcal{R})|K = \psi[y, a] \lor (L(\mathcal{R}))^K = \psi'[y, a]) \).

Thus \( \psi \) and \( \psi' \) satisfy the hypotheses of Lemma 9.5.19. Note also that, if \( M \) is any transitive class model of \( ZF \) such that \( a \in M \) and (ii) holds with \( "M" \) in place of \( "K" \), then \( M \models \"\{ x \in {}^\omega \omega \mid (\psi(x, a))^{L(\mathcal{R})} \}\) is not determined.

For some \( \gamma < \kappa \), \( a \) belongs to \( V[H_\gamma] \), where \( H_{\gamma} = H \cap \text{Coll}(\omega, < \gamma) \). By Lemma 9.5.10, there is an \( H' \in V[H] \) such that \( H' \) is \( \text{Coll}(\omega, < \kappa) \)-generic over \( V[H_\gamma] \) and such that the symmetric model defined from \( H' \) is \( K \). This means that \( \psi \) and \( \psi' \) satisfy in \( V[H_\gamma] \) the hypotheses of Lemma 9.5.19 and so those of Theorem 9.6.6.

Applying Theorem 9.6.6 in \( V[H_\gamma] \), we get that \( V[H_\gamma] \models \"\{ x \in {}^\omega \omega \mid (\psi(x, a))^{L(\mathcal{R})} \}\) is \( (< \kappa) \)-universally Baire.” Applying Theorem 9.6.10 in \( V[H_\gamma] \), we then get that \( V[H_\gamma] \models \"\{ x \in {}^\omega \omega \mid (\psi(x, a))^{L(\mathcal{R})} \}\) is determined.”

Applying Lemma 9.5.14 in \( V[H_\gamma] \) to \( \psi \) and to \( \psi' \), we get that (ii) above holds with \( "V[H_\gamma]" \) in place of \( "K" \). But this yields the contradiction that \( V[H_\gamma] \models \"\{ x \in {}^\omega \omega \mid (\psi(x, a))^{L(\mathcal{R})} \}\) is not determined.”

\[ \square \]

**Corollary 9.6.13.** If \( \text{ZFC} + \"\text{There are infinitely many Woodin cardinals}\" \) is consistent, then so are \( \text{ZF} + \text{AD} \) and \( \text{ZFC} + \text{AD}^{L(\mathcal{R})} \).

**Proof.** Assume that \( \text{ZFC} + \"\text{There are infinitely many Woodin cardinals}\" \) is consistent. It follows fairly directly from Theorem 9.6.12 that \( \text{ZF} + \text{AD} \) is consistent.
Let $K$ be as in the statement of Theorem 9.6.12. We show that $(L(R))^K$ satisfies the Axiom of Dependent Choices (DC). This in fact follows from the result of Kechris [19??] that $AD + V = L(R)$ implies DC. But we can also prove directly that $(L(R))^K$ satisfies DC. Because every set in $(L(R))^K$ is ordinal definable from an element of $\omega\omega$, it is enough to prove that DC holds in $(L(R))^K$ for relations on $\omega\omega$. Assume that this fails. Its failure is expressed in $(L(R))^K$ by a $\Sigma_1(R)$ sentence. By Lemma 9.5.14, this $\Sigma_1(R)$ sentence also holds in $(L(R))^V$. But Choice holds in $V$, and Choice implies $DC^{L(R)}$.

Given a transitive $M$ satisfying $ZF + AD + DC + V = L(R)$, let $P$ be in $M$ the partial ordering of all countable functions $f : \omega_1 \rightarrow \omega\omega$. Let $H$ be $P$-generic over $M$. It is easy to see that $(\omega\omega)^M[H] = (\omega\omega)^M$, and so that $M[H]$ satisfies $ZFC + AD^{L(R)}$. □

Corollary 9.6.13 is half of an equiconsistency result of Woodin. (See Exercise 9.6.4.)

Our hypothesis for Theorem 9.6.11 is stronger than necessary. We are now going to present an argument, due to John Steel, that allows one to weaken the hypothesis. Woodin had earlier found a different proof from an even weaker (and more technical) hypothesis.

It will be convenient in presenting this argument to generalize a bit the concept of generic ultrapowers for subsets $G$ of stationary towers.

Let $M$ be a transitive class, and let $\gamma$ be a limit ordinal of $M$. Assume that $(M; \in, \gamma)$ is a premouse. (See page 368.) Suppose that $G \subseteq P^M_\gamma$ or $G \subseteq Q^M_\gamma$. If $G$ is $P^M_\gamma$-generic or $Q^M_\gamma$-generic over $M$, then in $M[G]$ the generic ultrapower $\prod_G(M; \in)$ is defined. But the definition of $\prod_G(M; \in)$ works under weaker assumptions than the genericity of $G$. For $X \in V^M_\gamma$, let $G_X = \{ A \in G \mid \bigcup A = X \}$. Assume that each $G_X$ is an $M$-ultrafilter on $P^M(X)$ and that the $G_X$, $X \in V_\gamma$, are compatible in the sense of Lemma 9.2.6. (Note that these assumptions hold if, for each $n \in \omega$, $G \cap V_{\kappa_n}$ is $P^M_{\kappa_n}$-generic or $Q^M_{\kappa_n}$-generic over $M$.) Then the definition of $\prod_G(M; \in)$ makes sense, and so does the definition of $i^*_G : M \rightarrow \prod_G(M; \in)$. The usual proof shows that $i^*_G : M \rightarrow \prod_G(M; \in)$. Define $\pi^M_G$ as usual. If $\prod_G(M; \in)$ is wellfounded, then let Ult($M; G$) be the transitive class isomorphic to it, and let $i^M_G : M \rightarrow Ult(M; G)$ be defined as usual.

Our first lemma is analogous to Lemma 9.5.4. Its proof is like that of the earlier lemma. Indeed, a similar proof yields a common generalization of the two lemmas, a generalization we do not bother to state.
Lemma 9.6.14. Let $M$ be such that $(M; \in, \kappa)$ be a premouse. Let $\langle \kappa_n \mid n \in \omega \rangle$ be a strictly increasing sequence of Woodin cardinals of $M$ such that $\kappa = \bigcup_{n \in \omega} \kappa_n$. Let $G \subseteq Q^M_{\kappa_n}$ be such that $G_n$ is $Q^M_{\kappa_n}$-generic over $M$ for each $n \in \omega$, where $G_n = G \cap Q^M_{\kappa_n}$. Assume that $\prod G(M; \in)$ is wellfounded, so that $\text{Ult}(M; G)$ exists. Then

(1) $i^M_G(\omega_1) = \kappa$;
(2) $(\omega^\omega)^{\text{Ult}(M; G)} = \bigcup_{n \in \omega} (\omega^\omega)^{M[G \cap Q^M_{\kappa_n}]}$.

The next lemma is an analogue of Lemma 9.5.13. Its proof is like that of the earlier lemma. (The two lemmas have a common generalization.)

Lemma 9.6.15. Let $M$, $\kappa$, $\langle \kappa_n \mid n \in \omega \rangle$, and $G$, be as in the hypotheses of Lemma 9.6.14. In some forcing extension of $V$ (or in $V$ if $M$ is countable), there is an $H$ that is $\text{Coll}(\omega, < \kappa)$-generic over $M$ and is such that

$$(\omega^\omega)^{\text{Ult}(M; G)} = (\omega^\omega)^K,$$

where $K$ is the symmetric model defined from $H$,

Steel’s proof of $\text{AD}^{L(R)}$ from weaker hypotheses than those of Theorem 9.6.11 was inspired by a result of Woodin. (See Exercise 9.6.3.) The following lemma, which was also known to Woodin, is central to Steel’s proof.

Lemma 9.6.16. Let $\langle \kappa_n \mid n \in \omega \rangle$ be a strictly increasing sequence of Woodin cardinals and let $\kappa = \sup_{n \in \omega} \kappa_n$. Let $\lambda$ be a limit ordinal of cofinality greater than $\kappa$. Let $X \prec V_\lambda$ be such that $\{\kappa\} \cup \{\kappa_n \mid n \in \omega\} \subseteq X$ and such that $X \cap V_{\kappa_n} \subseteq \hat{S}$, where $\hat{S}$ is as in Corollary 9.5.2. Define $G \subseteq Q^{M_X}_{\pi_X(\kappa)} = \pi_X(Q_n)$ by

$$\tilde{S} \in G \leftrightarrow \pi_X^{-1} \pi_X^{-1}(\tilde{S}) \in \text{Ult}(\pi_X) \cap \hat{S} \in (\pi_X)^{-1}.$$ 

For $n \in \omega$, let $G_n = G \cap Q^{M_X}_{\pi_X(\kappa_n)} = G \cap \pi_X(Q_{\kappa_n})$.

(1) For each $n \in \omega$, $G_n$ is $Q^{M_X}_{\pi_X(\kappa_n)}$-generic over $M_X$.
(2) $\text{Ult}(M_X; G)$ exists.
(3) There is a $k : \text{Ult}(M_X; G) \prec V_\lambda$ with $k \circ i^{M_X}_G = \pi_X^{-1}$. 
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Proof. If \bar{S} \in \pi_X(Q_\kappa), then there is a unique \bar{S} \in Q_\kappa \cap X such that \bar{S} = \pi_X(S). Since \pi_X^{-1}^\prime \cup \bar{S} = X \cap \bigcup S for such \bar{S} and \bar{S}, we see that

\[ \bar{G} = \{ \pi_X(S) \mid S \in Q_\kappa \cap X \land X \cap \bigcup S \in S \}. \]

For (1) the only non-routine thing to show is that \bar{G} meets every dense subset of \( Q_{\pi_X(\kappa_n)} = \pi_X(Q_{\kappa_n}) \) that belongs to \( \bar{M}_X \). Let \( \bar{D} \in \bar{M}_X \) be a dense subset of \( \pi_X(Q_{\kappa_n}) \). Let \( \bar{D} \in X \) be such that \( \bar{D} = \pi_X(\bar{D}) \). The facts that \( X \in \bar{S} \), \( \bar{D} \in X \), and \( \bar{D} \) is dense in \( Q_{\kappa_n} \) imply that \( X \) captures \( \bar{D} \). Let then \( S \in \bar{D} \cap X \) be such that \( X \cap \bigcup S \in S \). We have that \( \pi_X(S) \in \bar{G} \) and \( \pi_X(S) \in \bar{D} \).

For each \( \bar{f} : \bar{S} \to \bar{M}_X \) with \( \bar{S} \in \bar{G} \) and \( \bar{f} \in \bar{M}_X \), set

\[ k'(\bar{f}|_\bar{G}) = f(X \cap \bigcup S), \]

where \( \bar{f} = \pi_X(f) \) and \( \bar{S} = \pi_X(S) \). It is easy to check that \( k' \) is well-defined and that \( k' : \prod_G(\bar{M}_X; \bar{\in}) \prec (\bar{V}_\lambda; \bar{\in}) \). Thus \( \prod_G(\bar{M}_X; \bar{\in}) \) is wellfounded, and so clause (2) of the lemma holds. Moreover \( k : \text{Ult}(\bar{M}_X; \bar{G}) \prec V_\lambda \), where \( k = k' \circ (\pi_G^{\bar{M}_X})^{-1} \).

To verify (3), let \( \bar{a} \in \bar{M}_X \) and let \( \bar{a} = \pi_X(a) \); let \( \bar{S} \in \bar{G} \) and let \( \bar{S} = \pi_X(S) \). Then

\[ k \circ i^{\bar{M}_X}(\bar{a}) = k \circ \pi_G^{\bar{M}_X}(i_\bar{G}(\bar{S})) = k'(i_{\bar{G}}(\bar{S})) = c_{\bar{a}}^\bar{S}(X \cap \bigcup S) = a. \]

Remark. If \( Y = \text{range}(k) \), then \( X \prec Y \prec V_\rho \), \( k = \pi_Y^{-1} \), and \( \pi_\bar{M}_X = \pi_Y \circ \pi_X^{-1} \). It can be shown that the members of \( Y \) are exactly those elements of \( V_\rho \) of the form \( f(X \cap a_1, \ldots, X \cap a_n) \) for \( a_1, \ldots, a_n \in V_\kappa \cap X \) and \( f \in X \) with \( f : V_\kappa \to V_\rho \). In other words \( X \) is a Skolem hull in \( V_\lambda \) of \( X \cup \{ a \cap X \mid a \in V_\kappa \cap X \} \).

We now use Lemma 9.6.16 to prove Lemma 9.5.15 from weaker hypothesis.

Lemma 9.6.17. Let \( \langle \kappa_n \mid n \in \omega \rangle \) be a strictly increasing sequence of Woodin cardinals and let \( \kappa = \sup_{n \in \omega} \kappa_n \). Assume that \( (V_\kappa)^{\#} \) exists. If \( H' \) is \( \text{Coll}(\omega, \kappa) \), then
If $K'$ is the symmetric model defined from $H'$, if $\psi(v_1,\ldots,v_n)$ is a formula of set theory, and if $a_1,\ldots,a_m$ belong to $(\omega\omega)^V$, then

$$(L(R))^{K'} \models \psi[a_1,\ldots,a_m] \iff L(R) \models \psi[a_1,\ldots,a_m].$$

\textbf{Proof.} Assume that the lemma is false for $\langle\kappa_n \mid n \in \omega\rangle$ and $\kappa$. Let $\lambda$ be such that $\text{cf}(\lambda) > \kappa$ and such that $V_\lambda \not\prec_{\Sigma_k} V$, where the statement of the lemma is $\Sigma_k$. We will obtain the contradiction that the lemma holds in $V_\lambda$ for $\langle\kappa_n \mid n \in \omega\rangle$ and $\kappa$.

Let $\hat{S}$ be as in the statement of Corollary 9.5.2. Since $\hat{S}$ is stationary in $P_{\aleph_1}(V_\kappa)$, let $X \prec V_\lambda$ be countable with $X \cap V_{\kappa+1} \in \hat{S}$ and with $\{\kappa,\langle\kappa_n \mid n \in \omega\rangle\} \subseteq X$. It suffices to show that the lemma holds in $\hat{M}_X$ for $\langle\pi_X(\kappa_n) \mid n \in \omega\rangle$ and $\pi_X(\kappa)$.

The hypotheses of Lemma 9.6.16 are satisfied by $X$. Let $G$ be as in the statement of that lemma, and let $k$ be given by clause (3) of that lemma.

Clauses (1) and (2) of Lemma 9.6.16 imply that the hypotheses of Lemma 9.6.15 are satisfied by $G$ and $M = \hat{M}_X$. Let $\bar{H}$ and $\bar{K}$ be the $H$ and $K$ given by Lemma 9.6.15. Let $\bar{R} = (\omega\omega)^{\text{Ult}(\hat{M}_X;G)} = (\omega\omega)^{\bar{K}}$. By Lemma 9.5.11 applied in $\hat{M}_X$ and by the elementarity of $G_{\bar{M}_X}$, the proof will be complete if we show that, for every formula $\psi(v_1,\ldots,v_m)$ and all $a_1,\ldots,a_n \in \bar{R}$,

$$(L(R))^\bar{K} \models \psi[a_1,\ldots,a_m] \iff (L(R))^{\text{Ult}(\hat{M}_X;G)} \models \psi[a_1,\ldots,a_m].$$

(As far as we know, $\bar{K}$ might have fewer ordinals than $\text{Ult}(\hat{M}_X;G)$, so we are unable to assert that $\bar{K} = \text{Ult}(\hat{M}_X;G)$.) We will prove that both $(L(R))^{\text{Ult}(\hat{M}_X;G)}$ and $(L(R))^\bar{K}$ are elementary submodels of $L(R)$.

Since $R^\#$ exists, “$R^\#$ exists” holds in $V_\lambda$ and thus in $\hat{M}_X$ and in $\text{Ult}(\hat{M}_X;G)$.

Let $Y = \text{range}(k)$. Observe that $R \cap Y = \bar{R}$. We have that

$$\omega\omega \cap H(L(R);C^R \cup \bar{R}) = \omega\omega \cap H(L(R);(C^R \cap Y) \cup \bar{R}) = \omega\omega \cap H((L(R))^{\text{Ult}(\hat{M}_X;G)}; (C^R)^{\text{Ult}(\hat{M}_X;G)} \cup \bar{R}) = R^{\text{Ult}(\hat{M}_X;G)} = \bar{R},$$

where the first equality is a consequence of the indiscernibility of the elements of $C^R$. It follows that

$$\text{trcoll}(H(L(R);C^R \cup \bar{R})) = L(\bar{R}).$$
and that
\[(L(R))^\text{Ult}(M_X; G) = \text{trcoll}(H(L(R); (C' \cap Y) \cup R)) \prec L(R).\]

Since \((V_\kappa)^#\) exists, \(\langle V_{\pi_X(\kappa)} \rangle^#\) exists holds in \(\bar{M}_X\). Thus
\[
V_\kappa \cap H(L(V_\kappa); C^{V_\kappa} \cup (V_\kappa \cap X)) = V_\kappa \cap H(L(V_\kappa); (C^{V_\kappa} \cup X) \cup (V_\kappa \cap X)) = \pi_X^{-1} u (V^{\bar{M}_X}_{\pi_X(\kappa)} \cap H(L(V^{\bar{M}_X}_{\pi_X(\kappa)}); (C^{V^{\bar{M}_X}_{\pi_X(\kappa)}(\bar{M}_X \cup V^{\bar{M}_X}_{\pi_X(\kappa)})) = \pi_X^{-1} u V^{\bar{M}_X}_{\pi_X(\kappa)}.
\]

It follows that
\[
\text{trcoll}(H(L(V_\kappa); C^{V_\kappa} \cup (V_\kappa \cap X))) = L(V^{\bar{M}_X}_{\pi_X(\kappa)})
\]
and that
\[
(L(V^{\bar{M}_X}_{\pi_X(\kappa)}))^{\bar{M}_X} = \text{trcoll}(H(L(V_\kappa); (C^{V_\kappa} \cap X) \cup (V_\kappa \cap X))) \prec L(V^{\bar{M}_X}_{\pi_X(\kappa)}).
\]

Trivially \(\bar{H}\) is \(\text{Coll}(\omega, < \kappa)\)-generic over \((L(V^{\bar{M}_X}_{\pi_X(\kappa)}))^{\bar{M}_X}\). It is not hard to see that \(R^{\bar{K}} = \bar{R}\) is definable in \((L(V^{\bar{M}_X}_{\pi_X(\kappa)}))^{\bar{M}_X}[\bar{H}]\) from \(\bar{H}\). Let \(\sigma\) be a \(\text{Coll}(\omega, < \pi_X(\kappa))\)-name belonging to \((L(V^{\bar{M}_X}_{\pi_X(\kappa)}))^{\bar{M}_X}\) and such that \(\sigma_{\bar{H}} = \bar{R}\). Let \(\varphi(v_1, \ldots, v_m)\) be a formula and let \(b_1, \ldots, b_m\) be elements of \((L(R))^{\bar{K}}\). There are \(\text{Coll}(\omega, < \pi_X(\kappa))\)-names \(\tau_1, \ldots, \tau_m\) belonging to \((L(V^{\bar{M}_X}_{\pi_X(\kappa)}))^{\bar{M}_X}\) and such that \(b_i = (\tau_i)_{\bar{H}}\) for each \(i\). For \(p \in \text{Coll}(\omega, < \kappa)\), the assertion that \(p \models \varphi^{L(\sigma)}(\tau_1, \ldots, \tau_m)\) is expressed in \((L(V^{\bar{M}_X}_{\pi_X(\kappa)}))^{\bar{M}_X}\) by some formula \(\chi(p, \tau_1, \ldots, \tau_m)\). Now \(\bar{H}\) is also \(\text{Coll}(\omega, < \kappa)\)-generic over \(L(V^{\bar{M}_X}_{\pi_X(\kappa)})\). Using the fact that \((L(V^{\bar{M}_X}_{\pi_X(\kappa)}))^{\bar{M}_X} \prec L(V^{\bar{M}_X}_{\pi_X(\kappa)})\), we get that \((L(R))^{\bar{K}} \models \varphi(b_1, \ldots, b_m) \leftrightarrow L(\bar{R}) \models \varphi(b_1, \ldots, b_m)\).

We can now weaken the measurable cardinal hypothesis of the earlier results of the present section.

**Lemma 9.6.18.** (a) Lemma 9.5.18 holds if the hypothesis that there is a measurable cardinal greater than \(\kappa\) is replaced by the hypothesis that \((V_\kappa)^#\) exists.

(b) Theorem 9.6.5 holds if the hypothesis that there is a measurable cardinal greater than \(\kappa\) is replaced by the hypothesis that \((V_\kappa)^#\) exists.
Theorem 9.6.19. If there is a $\kappa$ that is a limit of Woodin cardinals and such that $(V_\kappa)^\#$ exists, then every game $G(A;^{<\omega}\omega)$ with $A \in L(\mathcal{R})$ is determined, and so the Axiom of Determinacy holds in $L(\mathcal{R})$.

Theorem 9.6.19 is due to Woodin, but the proof we have given is due to Steel. Woodin’s proof uses the forcing machinery we introduced in the exercises for Chapters 6 and 7, and so uses also the full machinery of iteration trees. (Woodin also uses the stationary tower, but Steel has shown how to eliminate completely the stationary tower in favor of the other machinery.) Woodin’s proof has the advantage that it uses an even weaker hypothesis than the existence of $(V_\kappa)^\#$. Steel’s proof of Theorem 9.6.19 has its own advantage: it gives more determinacy. (See Exercise 9.6.2.)

Exercise 9.6.1. Assume that there is a completely Jonsson cardinal that is greater than infinitely many Woodin cardinals. Use Theorem 9.6.12 and Lemma 9.5.15 to prove that the Axiom of Determinacy holds in $L(\mathcal{R})$.

Exercise 9.6.2. Let $\langle \kappa_n \mid n \in \omega \rangle$ be a strictly increasing sequence of Woodin cardinals and let $\kappa = \sup_{n \in \omega} \kappa_n$.

(a) Use Exercise 9.5.4 to show that $\mathcal{R}^\#$ is $(<\kappa)$-universally Baire if there is a measurable cardinal greater than $\kappa$.

(b) Assume that $(V_\kappa)^\#$ exists and prove that $\mathcal{R}^\#$ is $(<\kappa)$-universally Baire.

(b) is a result of Steel.

Hint. For (b) note that, in the proof of Lemma 9.6.17, we could with little more effort have deduced that $(\mathcal{R}^\#)^K = (\mathcal{R}^\#)^{\text{Ult}(\mathcal{M}_X;G)}$.

Exercise 9.6.3. Assume that the ideal of nonstationary subsets of $\omega_1$ is $\omega_2$-saturated. (See Exercise 9.2.3.) Let $X \prec V_{\omega_2}$. Let

$$Y = \{ g(\omega_1 \cap X) \mid g \in X \land g : \omega_1 \to \mathcal{V}_{\omega_2} \}. $$

Let $\pi_X : X \cong M$ and let $\pi_Y : Y \cong N$ with $M$ and $N$ transitive. Let $j = \pi_Y \circ \pi_X^{-1} : M \prec N$. Show that there is a $G$ that is $\text{P}(\mathcal{I}_{\omega_1}^*)^M$-generic over $M$ and such that $N = \text{Ult}(M;G)$ and $j = \pi^M_G$.

Exercise 9.6.4. Work in ZF and assume $\text{AD} + V = L(\mathcal{R})$. Let $\delta_1^2$ be the least ordinal $\alpha$ such that $L_\alpha(\mathcal{R}) \prec_{\Sigma_1} L(\mathcal{R})$. Let HOD be the class of
CHAPTER 9. GAMES IN $L(\mathcal{R})$

hereditarily ordinal definable sets. Prove that, for every $\eta < (\delta^2_1)^+$, the class model $\text{HOD} \models \text{“} \delta^2_1 \text{ is } \eta\text{-strong}.”$ This is a theorem of John Steel.

**Hint.** (The hint assumes some familiarity with the consequences of AD.) Let $\eta < (\delta^2_1)^+$. Let $E \subseteq \delta^2_1 \times \delta^2_1$ be such that $(\delta^2_1; E) \cong (V^{\text{HOD}}_\eta; \in)$. For each $\alpha < \delta^2_1$ for which $(\alpha; E) \models \text{Extensionality}$, let $N_\alpha$ be transitive and such that $(\alpha; E) \cong (N_\alpha; \in)$. Let

$$S = \{\alpha < \delta^2_1 \mid N_\alpha \in \text{HOD}\}.$$ 

First use the fact that $L_{\delta^2_1}(\mathcal{R}) \prec_{\Sigma_1} L(\mathcal{R})$ to prove that $S$ is a stationary subset of $\delta^2_1$. By results of Kechris–Kleinberg–Moschovakis–Woodin [1981], $S$'s being stationary implies that there is a a uniform normal ultrafilter $\mathcal{U}$ on $\delta^2_1$ such that $S \in \mathcal{U}$. By a result of Kunen (see Solovay [1978]), every ultrafilter on an ordinal is ordinal definable; thus $\mathcal{U}$ is ordinal definable. Let $M$ be transitive and such that $(M; \in) \cong \prod_{\mathcal{U}} (\text{HOD}; \in)$. (The class $M$ is not the same as $\text{Ult}(\text{HOD}; \mathcal{U})$.) Let $j : \text{HOD} \prec M$ be the canonical embedding. Both $j$ and $M$ are classes in $\text{HOD}$. (In $\text{HOD}$, $j$ is $i_E$ with $E$ an extender rather than an ultrafilter.) Show that $j$ witnesses that $\delta^2_1$ is $\eta$-strong in $\text{HOD}$.

Woodin’s proof of the converse of Corollary 9.6.13 makes use of Steel’s idea. We now indicate very briefly the main steps in the lengthy journey from Steel’s theorem to Woodin’s. We do this by giving a sequence of assertions (in italics), with each assertion followed by a sketch of the steps in its proof. We continue to work in ZF and to assume AD + $V = L(\mathcal{R})$.

(1) Let $\delta^* \text{ be the least ordinal } \alpha > \delta^2_1 \text{ such that } L_\alpha(\mathcal{R}) \prec_{\Sigma_1} L(\mathcal{R})$. Then $\delta^2_1$ is $\delta^*$-strong in $\text{HOD}$.

If $\eta \geq (\delta^2_1)^+$, then the structure $V^{\text{HOD}}_\eta$ cannot be coded by a relation on $\delta^2_1$. Instead it can be coded with the aid of a prewellordering of $\omega^\omega$ of length $|V^{\text{HOD}}_\eta|$. If $\eta < \delta^*$, then there is such a prewellordering $\mathcal{P}$ that is is defined both by $\Sigma_1(\mathcal{R})$ formula and $\Pi_1(\mathcal{R})$ formulas from $\delta^2_1$ and parameters belonging to $\omega^\omega$. The Moschovakis Coding Lemma (see Chapter 7 of Moschovakis [1980]) then gives a coding of $V^{\text{HOD}}_\eta$ by a subset of $\omega^\omega$ that is similarly definable.

To go along with this coding, one needs an appropriate ultrafilter on $\delta^2_1$. If $\varphi$ is a $\Sigma_1$ formula and $y \in \omega^\omega$ is such that $\varphi(\mathcal{R}, y, \delta^2_1)$ holds, then the set of all $\delta < \delta^2_1$ such that $\varphi(\mathcal{R}, y, \delta)$ holds is unbounded, indeed stationary, in $\delta^2_1$. Coding pairs $\langle \varphi, y \rangle$ by elements $x$ of $\omega^\omega$, let

$$U = \{x \in \omega^\omega \mid \varphi_y(\mathcal{R}, y_x, \delta^2_1)\}.$$
and, for $x \in U$, let

$$Z_x = \{ \delta < \delta_1^2 \mid \varphi_x(\mathcal{R}, y_x, \delta) \}.$$ 

The sets $Z_x$, $x \in U$, generate a filter $\mathcal{F}$ on $\delta_1^2$. This filter can be extended to an ultrafilter as follows. Let $X \subseteq \delta_1^2$. Consider a game $G_X$ in $\omega^\omega \omega$ given as follows. Think of players $I$ and $II$ as choosing $\{x_{2i} \mid i \in \omega\}$ and $\{x_{2i+1} \mid i \in \omega\}$ respectively, with each $x_i$ belonging to $\omega^\omega$. If for some $i$, $x_i \notin U$, then the player responsible for least such $i$ loses. Otherwise let $\delta < \delta_1^2$ be the least member of $\bigcap_{x \in X} Z_x$. Then $I$ wins if and only if $\delta \in X$. Define an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$ by letting $X \in \mathcal{U}$ if and only if $I$ has a winning strategy for $G_X$.

With the aid of the Uniform Coding Lemma (see Kechris–Kleinberg–Moschovakis–Woodin [1981]), one can prove the following normality theorem for $\mathcal{U}$. Let $\eta < \delta^*$ and let $P$ be a prewellordering as above. It is easily seen that there is a $Z \in \mathcal{F}$ such that for each $\delta \in Z$ one gets a prewellordering $P_\delta$ when on replaces $\delta_1^2$ by $\delta$ in the definition(s) of $P$. Suppose that $f : \delta_1^2 \to \delta_1^2$ and that $\{ \delta < \delta_1^2 \mid f(\delta) < \text{length} (P_\delta) \} \in \mathcal{U}$. Then there is a $z \in \omega^\omega$ such that $\{ \delta \mid f(\delta) = \|z\|^P \} \in \mathcal{U}$, where $\|z\|^P$ is the length of the initial segment of $z$ in $P_\delta$.

Define $j : HOD \prec M$ from $\mathcal{U}$. Using the normality theorem, one can show as in the Steel proof that $j$ witnesses the $\eta$-strength of $\delta_1^2$ for every $\eta < \delta^*$, and so that $j$ witnesses (2).

(2) Let $\theta$ be the least ordinal that is not the surjective image of $\omega^\omega$. Then $\delta_1^2$ is $\eta$-strong in HOD for every $\eta < \theta$.

One can define a function $F$ with domain $\delta_1^2$ so that the following modification of the proof of (2) works for every $\eta < \theta$. It is easy to see that there is a ordinal definable prewellordering of $\omega^\omega$ of length $|V^\text{HOD}_\eta|$. Let $P$ be such a prewellordering. With an appropriate coding, let

$$U_P = \{ x \mid \varphi_y(\mathcal{R}, y_x, \delta_1^2, P) \}$$

and, for $x \in U$, let

$$Z_x = \{ \delta < \delta_1^2 \mid \varphi_x(\mathcal{R}, y_x, \delta, F(\delta)) \}.$$ 

Now proceed as in the proof of (1). (Roughly speaking $F(\delta)$ is the least ordinal definable prewellordering not having the reflection property with respect to $F \upharpoonright \delta$ that one wants $P$ to have with respect to $F$.)

(3) The ordinal $\theta$ is a Woodin cardinal in HOD.
Let $A \subseteq \theta$. Let $\delta_A$ be the least ordinal $\alpha$ such that $(L_\alpha(\mathcal{R}); \in, A \cap \alpha) \prec_{\Sigma_1} (L_\alpha(\mathcal{R}); \in, A)$. Proceed as in the proofs of (1) and (2), using $\delta_A$ in place of $\delta_1^2$.

(4) For every $x \in {}^\omega \omega$ of sufficiently large Turing degree, $\omega^L_2[x] = \theta^L[x]$ is Woodin in $\text{HOD}^L[x]$.

Our hypotheses imply that, for every $x$ of sufficiently large Turing degree, $\mathcal{L}[x] \models \text{"all $\Delta^1_1$ games in $<_\omega \omega$ are determined."}$ A result of Kechris–Solovay [19??] implies that, for all such $x$, $\mathcal{L}[x] \models \text{"all ordinal definable games in $<_\omega \omega$ are determined."}$ This makes it possible—though not by any means easy—to find an adaptation of the proof of (3) that works.

(5) Let $S$ be a class of ordinals. Then, for every $x \in {}^\omega \omega$ of sufficiently large Turing degree, $\omega^L_2[S,x] = \theta^L[S,x]$ is Woodin in $\text{HOD}^L[S,x]$, i.e., in the class of sets hereditarily ordinal definable in $\mathcal{L}[S,x]$ from $S$.

The proof is like that of (4), the special case $S = \emptyset$.

(6) There is a transitive class model of ZFC with $\omega$ Woodin cardinals.

Under our hypothesis, it can be shown that the class of sets hereditarily ordinal definable from an element $x$ of ${}^\omega \omega$ is the same as $\text{HOD}[x]$. For $x$ and $y \in {}^\omega \omega$, let $x \leq y$ if and only if $x$ ordinal definable from $y$. This gives us a notion of degree analogous to that of Turing degree. Let $D$ be the set of all degrees in this new sense. As with Turing degrees, the set of cones generates a countable complete ultrafilter $\mathcal{V}$ on $D$. Let $M_1$ be the transitive class isomorphic to the ultrapower of HOD by $\mathcal{V}$. Let $j_{0,1} : \text{HOD} \prec M_1$. Let $I$ be the set of all subsets $A$ of ${}^\omega \omega$ such that the set of degrees of elements of $A$ does not belong to $\mathcal{V}$. Let $\mathcal{G}$ be $\mathcal{P}(I)$-generic over $\mathcal{V}$. Then $M_1$ is the same as the transitive model isomorphic to $\prod_{\mathcal{G}}(\text{HOD}; \in)$. In $\mathcal{V}[\mathcal{G}]$ one can force over $M_1$ to get a set $\mathcal{R}_1$ such that $M_1 = \text{HOD}^L(\mathcal{R}_1)$ and such that the element $y$ of $\mathcal{V}[\mathcal{G}]$ corresponding to the equivalence class with respect to $\mathcal{G}$ of the identity function belongs to $\mathcal{R}_1$. This enables one to extend the embedding $j_{0,1}$ to a $j_{0,1} : L(\mathcal{R}) \prec L(\mathcal{R}_1)$. Continuing in this way, one gets

$$
\text{HOD} = M_0 \prec^{j_{0,1}} M_1 \prec^{j_{1,2}} M_2 \prec^{j_{2,3}} \cdots
$$

$$
L(\mathcal{R}) = L(\mathcal{R}_0) \prec^{j_{0,1}} L(\mathcal{R}_1) \prec^{j_{2,3}} L(\mathcal{R}_2) \prec^{j_{3,4}} \cdots
$$

Here each $M_i = \text{HOD}^L(\mathcal{R}_i)$. Let $M_\omega$ be the transitive class isomorphic to the direct limit model of the system $(\{M_n \mid n \in \omega\}, \{j_{m,n} \mid m \leq n \in \omega\})$. Let $\mathcal{R}_\omega = \bigcup_{n \in \omega} \mathcal{R}_n$. Then $\text{HOD}^L(\mathcal{R}_\omega) = M_\omega$. 
Let $S$ be a set of ordinals such that $\text{HOD} = L[S]$ for $x \in {}^{<\omega} \omega$, $\text{HOD}[x] = L[S,x]$. For each $x \in {}^{<\omega} \omega$, let $C(x)$ be the set of all $\omega$-Borel sets in $\text{HOD}[x]$ that have $\omega$-Borel codes ordinal definable from $S$ in $\text{HOD}[x]$. (See Exercise 6.3.7.) Let $C_1$ be element of $L(R_1)$ corresponding to the function $x \mapsto C(x)$. In $L(R_1)$, $C_1$ is countable. It can be shown that $\text{HOD}[C(x)] = \text{HOD}^{HOD}[x] = \text{HOD}_{\{S\}}^{HOD}[x]$, which is the same as $\text{HOD}^{L[S,x]}$. Thus (5) gives that $\omega_2^{\text{HOD}[x]}$ is Woodin in $\text{HOD}[C(x)]$ for a cone of $x$. This in turn implies that the ordinal $\theta_1$ corresponding to $x \mapsto \omega_2^{\text{HOD}[x]}$ is Woodin in $M_1[C_1]$.

Working in $L(R_1)$, define $C(x,C_1)$ as $C(x)$ was defined in $L(R)$, but replacing $\text{HOD}[x]$ by $\text{HOD}[x,C_1]$ (i.e., by $M_1[x,C_1]$), and replacing $\{S\}$ by $\{j_{0,1}(S),C_1\}$. Now let $C_2 \in L(R_2)$ correspond to the function $x \mapsto C(x,C_1)$. Since $\theta_1$ and $C_1$ are countable in $L(R_1)$, $j_{1,2}$ does not move either of these objects (or anything in their transitive closures). Hence the ordinal $\theta_2$ corresponding to $x \mapsto \omega_2^{M_1[x,C_1]}$ is Woodin in $M_2[C_1,C_2]$, and $\theta_1$ is Woodin in this model as well. Continuing in this manner one defines $\langle C_n \mid n \in \omega \rangle$ and ordinals $\theta_n$, $n \in \omega$, such that $\theta_i$ is Woodin in $M_n[C_1,\ldots,C_n]$ for $1 \leq i \leq n$. Finally, it can be shown that all the $\theta_i$ are Woodin in $M_\omega[\langle C_n \mid n \in \omega \rangle]$.

*Remark.* What the proof of (3) actually shows is that if $V = L[x]$ for some $x \in {}^{<\omega} \omega$ and if all games in $<^{<\omega} \omega$ are determined, then $\theta$ (i.e., $\omega_2$) is Woodin in HOD. This, together with Theorem ?? gives Woodin’s result that the ZFC + “there is a Woodin cardinal” is equiconsistent with ZFC + “all games in $<^{<\omega} \omega$ are determined.”
Bibliography


