1. The function to be averaged is \( f(x, y) = x^2 + y^2 \). Here is one way to parameterize the curve.

\[
c(\theta) = (\cos \theta, 1 - \sin \theta);
0 \leq \theta \leq \pi.
\]

The length unit \( ds = d\theta \). To see this, observe that \( c'(\theta) = (-\sin \theta, -\cos \theta) \) and so

\[
||c'(\theta)|| = \sin^2 \theta + \cos^2 \theta = 1.
\]

Computing, we get that

\[
f(c(\theta)) = f(\cos \theta, 1 - \sin \theta)
= \cos^2 \theta + (1 - \sin \theta)^2
= \cos^2 \theta + 1 - 2\sin \theta + \sin^2 \theta
= 2(1 - \sin \theta).
\]

The length of the curve is \( \pi \), so the average value of \( f \) is

\[
\frac{2}{\pi} \int_0^\pi (1 - \sin \theta) d\theta = \frac{2}{\pi} \left[ \theta + \cos \theta \right]_0^\pi = 2 - \frac{4}{\pi}.
\]

2. Since \( c'(t) = (-3 \sin t, 2 \cos t, 1) \), we get that

\[
F \cdot c' = -9 \sin t \cos t + 2 \cos t + 6 \sin t \cos t = -3 \sin t \cos t + 2 \cos t.
\]

Hence

\[
\int_c F \cdot ds = \int_0^\pi (-3 \sin t \cos t + 2 \cos t) dt
= \left[ \frac{3}{2} \cos^2 t + 2 \sin t \right]_0^\pi
= 0.
\]

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3. Let $g(x, y) = x^2 + y^2$. The area is given by

$$\int \int_D 1 \, dS = \int \int_D \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1} \, dx \, dy,$$

where we still have to determine $D$. Since $\frac{\partial g}{\partial x} = 2x$ and $\frac{\partial g}{\partial y} = 2y$, the area is

$$\int \int_D \sqrt{4x^2 + 4y^2 + 1} \, dx \, dy.$$

To determine $D$, we need to compute the values of $x$ and $y$ for which the two surfaces intersect. (Note that this is the exact same problem as computing the $x$ and $y$ limits in Problem 3 on the first midterm, so the computation that follows is just a repetition of that part of the posted solution to the first midterm problem.) Obviously where the intersection occurs depends only on $x^2 + y^2$, so let $u = x^2 + y^2$. The intersection occurs when

$$u = \sqrt{1 - u},$$

i.e., when

$$u^2 = 1 - u.$$

Solving this equation and taking the positive value of $u$, we find that

$$u = \frac{\sqrt{5} - 1}{2}.$$

Hence the intersection occurs at points on the circle

$$x^2 + y^2 = \frac{\sqrt{5} - 1}{2}.$$

Converting to polar coordinates, we get that the area is

$$\int_0^\sqrt{\frac{\sqrt{5} - 1}{2}} \int_0^{2\pi} r \sqrt{4r^2 + 1} \, d\theta \, dr = 2\pi \int_0^\sqrt{\frac{\sqrt{5} - 1}{2}} \sqrt{4r^2 + 1} \, dr = \frac{\pi}{6} \left[ (4r^2 + 1)^{\frac{3}{2}} \right]_0^\sqrt{\frac{\sqrt{5} - 1}{2}} = \frac{\pi}{6} \left( (2(\sqrt{5} - 1) + 1)^{\frac{3}{2}} - 1 \right).$$
4. For points \((x, y, z)\) on the unit sphere, the unit normal is just the vector \((x, y, z)\). Thus
\[
\mathbf{F} \cdot \mathbf{n} = xyz + z^4.
\]
In spherical coordinates,
\[
\mathbf{F} \cdot \mathbf{n} = \cos \theta \sin \theta \sin^2 \phi \cos \phi + \cos^4 \phi.
\]
Hence
\[
\int \int_S \mathbf{F} \cdot dS = \int \int_S \mathbf{F} \cdot \mathbf{n} \, dS
\]
\[
= \int_0^\pi \int_0^{2\pi} \sin \phi \left( \cos \theta \sin \theta \sin^2 \phi \cos \phi + \cos^4 \phi \right) \, d\theta \, d\phi
\]
\[
= \int_0^\pi \int_0^{2\pi} \left( \cos \theta \sin \theta \sin^3 \phi \cos \phi + \cos^4 \phi \sin \phi \right) \, d\theta \, d\phi
\]
\[
= 0 + 2\pi \int_0^\pi \cos^4 \phi \sin \phi \, d\phi
\]
\[
= \left[ \frac{2\pi}{5} \cos^5 \phi \right]_0^\pi
\]
\[
= \frac{4}{5} \pi.
\]