Math 223B Winter 2005 D.A. Martin

Set Theory

This course will be an introduction to independence proofs by forcing. Our basic treatment will be close to that in Kenneth Kunen's *Set Theory: an Introduction to Independence Proofs*, North-Holland, 1980. In particular, we will use Kunen's notation almost always.

1 Forcing

For the purposes of forcing, a *partially ordered set* (*poset*) is a triple $\langle \mathbf{P}, \leq, \mathbf{1} \rangle$ such that

(a) $\mathbf{P}$ is a nonempty set;

(b) $\leq$ is a *partial ordering* of $\mathbf{P}$; i.e., $\leq$ is transitive, reflexive relation in $\mathbf{P}$ (and so $p \leq q \wedge q \leq p \wedge p \neq q$ is not forbidden);

(c) for all $p \in \mathbf{P}$, $p \leq \mathbf{1}$. ($\mathbf{1}$ need not be the only such maximum element.)

We shall often write $\mathbf{P}$ for $\langle \mathbf{P}, \leq, \mathbf{1} \rangle$.

Let $\mathbf{P}$ be a poset. If p is an element of $\mathbf{P}$, then an *extension* of p is a $q \in \mathbf{P}$ such that $q \leq p$. Two elements of $\mathbf{P}$ are *compatible* if they have a common extension. We write $p \perp q$ to mean that p and q are incompatible. A subset D of $\mathbf{P}$ is *dense* in $\mathbf{P}$ if every element of $\mathbf{P}$ has an extension that belongs to D.

A *filter* on a poset $\mathbf{P}$ is a non-empty subset F of $\mathbf{P}$ satisfying:

(i) $(\forall p \in F)(\forall q \in P)(p \leq q \rightarrow q \in F)$;

(ii) any two elements of F have a common extension that belongs to F.

If M is a class and $\mathbf{P}$ is a poset, then a subset G of $\mathbf{P}$ is $\mathbf{P}$-*generic* over M if

(1) G is a filter on $\mathbf{P}$;

(2) $G \cap D \neq \emptyset$ for every dense subset D of $\mathbf{P}$ such that $D \in M$.

Remark. One often sees "M-generic on $\mathbf{P}$" used to mean what we mean by "$\mathbf{P}$-generic over M."

Example. Let M be a transitive class in which ZFC holds. Let $\mathbf{P}$ be the set of all finite functions f such that domain $(f) \subseteq \omega$ and range $(f) \subseteq \{0, 1\}$. Set $p \leq q \leftrightarrow q \subseteq p$, and let $\mathbf{1} = \emptyset$. Note that $\langle \mathbf{P}, \leq, \mathbf{1} \rangle \in M$.

Suppose that G is $\mathbf{P}$-generic over M. It follows from property (ii) of filters that any two elements of G are compatible functions. Thus $\bigcup G$ is a function. For $n \in \omega$, let

$$D_n = \{p \in \mathbf{P} \mid n \in \text{domain}\,(p)\}\,.$$

It is easy to see that each D_n is dense. By property (2) of G, it follows that domain $(\bigcup G) = \omega$. Thus $\bigcup G : \omega \to \{0, 1\}$.

Lemma 1.1. *Let M be a countable set and let $\mathbf{P}$ be a poset. For every $p \in \mathbf{P}$, there is a G that is $\mathbf{P}$-generic over M with $p \in G$.*

Proof. Let D_i, $i \in \omega$, be such that each D_i is dense in $\mathbf{P}$ and such that every $D \in M$ that is dense in $\mathbf{P}$ is among the D_i. Let $p \in \mathbf{P}$.

We construct a sequence

$$p = q_0 \geq q_1 \geq \cdots$$

of elements of $\mathbf{P}$. Given q_i, we let q_{i+1} be an extension of q_i that belongs to D_i.

Let $G = \{r \in \mathbf{P} \mid (\exists i \in \omega)\, q_i \leq r\}$. It is easy to see that G is $\mathbf{P}$-generic over M. □

Let $\mathbf{P}$ be a poset. An *atom* of $\mathbf{P}$ is an element p of $\mathbf{P}$ such that any two extensions of p are compatible. $\mathbf{P}$ is *atomless* if it has no atoms.

Lemma 1.2. *Let $\mathbf{P}$ be an atomless poset. Let M be a class such that the set $\mathbf{P}$ belongs to M and such that M is closed under relative complements. Let G be $\mathbf{P}$-generic over M. Then $G \notin M$.*

Proof. Assume that $G \in M$. Since M is closed under relative complements, $\mathbf{P} \setminus G \in M$. Since $G \cap (\mathbf{P} \setminus G) = \emptyset$, we shall have a contradiction if we can show that $\mathbf{P} \setminus G$ is dense. Let $p \in \mathbf{P}$. The atomlessness of $\mathbf{P}$ gives us incompatible extensions q and r of p. Since G is a filter on $\mathbf{P}$, at least one of q and r does not belong to G. □

Exercise 1.1. *Let M be a transitive class in which, say, ZF − Power Set holds. Let $\mathbf{P}$ be a poset with $\mathbf{P} \in M$ (i.e., with $\langle \mathbf{P}, \leq, \mathbf{1}\rangle \in M$) and let $p \in \mathbf{P}$ be an atom. Show that there is a filter $G \in M$ such that $p \in G$ and such that G is $\mathbf{P}$-generic over V.*

Let $\mathbf{P}$ be a poset By transfinite recursion on rank, we define the class $V^{\mathbf{P}}$ of $\mathbf{P}$-*names*. If $\operatorname{rank}(\tau) = \alpha$, then let

$$\tau \in V^{\mathbf{P}} \;\leftrightarrow\; (\tau \text{ is a relation } \wedge\; (\forall \langle \sigma, p\rangle \in \tau)(\sigma \in V^{\mathbf{P}} \cap V_\alpha \wedge p \in \mathbf{P}))\,.$$

Thus, for any set τ,

$$\tau \in V^{\mathbf{P}} \;\leftrightarrow\; (\tau \text{ is a relation } \wedge\; (\forall \langle \sigma, p\rangle \in \tau)(\sigma \in V^{\mathbf{P}} \wedge p \in \mathbf{P}))\,.$$

For classes M with $\mathbf{P} \in M$, let

$$M^{\mathbf{P}} = \{\tau \mid (\tau \text{ is a } \mathbf{P}\text{-name})^M\}\,.$$

If M is a transitive class in which ZFC holds, then absoluteness implies that $M^{\mathbf{P}} = V^{\mathbf{P}} \cap M$.

Fix $\mathbf{P}$. For any set G, we define $\operatorname{val}(\tau, G)$ for all sets τ, by transfinite recursion on $\operatorname{rank}(\tau)$:

$$\operatorname{val}(\tau, G) = \{\operatorname{val}(\sigma, G) \mid (\exists p \in G)\, \langle \sigma, p\rangle \in \tau\}\,.$$

Note that the operation val is absolute for transitive classes in which ZFC holds. We usually write τ_G for $\operatorname{val}(\tau, G)$.

If M is a transitive class in which ZFC holds, if $\mathbf{P} \in M$ is a poset, and if $G \subseteq \mathbf{P}$, then set

$$M[G] = \{\tau_G \mid \tau \in M^{\mathbf{P}}\}\,.$$

The absoluteness of val and of the property of being a $\mathbf{P}$-name gives us the following lemma.

Lemma 1.3. *Let M and N be transitive classes in which ZFC holds. Assume that $M \subseteq N$. Let $\mathbf{P} \in M$ be a poset and let $G \subseteq \mathbf{P}$ with $G \in N$. Then $M[G] \subseteq N$.*

Let $\mathbf{P}$ be a poset. We define, by transfinite recursion on $\operatorname{rank}(x)$, a $\mathbf{P}$-name $\check{x}^{\mathbf{P}}$ for each set x. Set

$$\check{x}^{\mathbf{P}} = \{\langle \check{y}^{\mathbf{P}}, \mathbf{1}\rangle \mid y \in x\}\,.$$

Note that the two-argument function $\check{x}^{\mathbf{P}}$ is absolute for transitive classes in which ZFC holds. We shall usually write $\check{x}$ for $\check{x}^{\mathbf{P}}$.

Lemma 1.4. *Let M be a transitive class in which ZFC holds, let $\mathbf{P} \in M$ be a poset, and let G be a filter on $\mathbf{P}$. Then*

(a) $(\forall x \in M)\, \check{x}_G = x$ *;*

(b) $M \subseteq M[G]$ *.*

Proof. We prove (a) by transfinite induction on rank.

$$\check{x}_G = \{\sigma_G \mid (\exists p \in G)\, \langle \sigma, p\rangle \in \check{x}\} = \{\check{y}_G \mid \mathbf{1} \in G \,\wedge\, y \in x\} = \{y \mid y \in x\} = x\,.$$

(b) follows from (a). $\square$

If $\mathbf{P}$ is a poset, let $\Gamma^{\mathbf{P}} = \{\langle \check{p}, p\rangle \mid p \in \mathbf{P}\}$. (We usually omit the superscript $\mathbf{P}$.)

Lemma 1.5. *Let M, $\mathbf{P}$, and G, be as in Lemma 1.4. Then $\Gamma_G = G$, and so $G \in M[G]$.*

Proof. $\Gamma_G = \{\check{p}_G \mid p \in G\} = \{p \mid p \in G\} = G$. $\square$

Lemma 1.6. *Let M, $\mathbf{P}$, and G be as in Lemma 1.4. Then $M[G]$ is transitive.*

Lemma 1.7. *Let M, $\mathbf{P}$, and G be as in Lemma 1.4. Then*

(a) for every $\tau \in M^{\mathbf{P}}$, $\mathrm{rank}(\tau_G) \le \mathrm{rank}(\tau)$ *;*

(b) $\mathrm{ON} \cap M[G] = \mathrm{ON} \cap M$ *.*

Proof. (a) is easily proved by transfinite induction, and (a) implies that $\mathrm{ON} \cap M[G] \subseteq \mathrm{ON} \cap M$. Since $M \subseteq M[G]$, the reverse inclusion also holds. $\square$

For any poset $\mathbf{P}$, we let

$$\begin{aligned} \mathrm{up}^{\mathbf{P}}(\sigma,\tau) &= \{\langle \sigma, \mathbf{1}\rangle, \langle \tau, \mathbf{1}\rangle\}\,; \\ \mathrm{op}^{\mathbf{P}}(\sigma,\tau) &= \mathrm{up}(\mathrm{up}(\sigma,\sigma), \mathrm{up}(\sigma,\tau))\,. \end{aligned}$$

Lemma 1.8. *Let M, $\mathbf{P}$, and G be as in Lemma 1.4 and let σ and τ belong to $M^{\mathbf{P}}$. Then*

(a) $\mathrm{up}(\sigma,\tau) \in M^{\mathbf{P}}$ *and* $(\mathrm{up}(\sigma,\tau))_G = \{\sigma_G, \tau_G\}$ *;*

(b) $\mathrm{op}(\sigma,\tau) \in M^{\mathbf{P}}$ *and* $(\mathrm{op}(\sigma,\tau))_G = \langle \sigma_G, \tau_G\rangle$ *.*

Lemma 1.9. *Let M, $\mathbf{P}$, and G be as in Lemma 1.4. Then Extensionality, Foundation, Pairing, and Union hold in $M[G]$.*

Proof. For Union, let $\tau \in M^{\mathbf{P}}$. Let $\pi = \mathcal{U}(\text{domain}(\tau))$, where $\mathcal{U}$ is the union operation. We show that $\mathcal{U}(\tau_G) \subseteq \pi_G$. Let $x \in \tau_G$. There is a $\sigma \in \text{domain}(\tau)$ such that $x = \sigma_G \in \tau_G$. By the definition of π, we have that $\sigma \subseteq \pi$. This implies that $\sigma_G \subseteq \pi_G$. Hence every element of τ_G is a subset of π_G. $\square$

Exercise 1.2. *Let M and $\mathbf{P}$ be as in Lemma 1.4 and let G be a filter on $\mathbf{P}$. Prove that $M[G]$ is closed under the operation $\mathcal{U}$.*

The next exercise requires the following definitions. Let $\mathbf{P}$ be a poset. A subset D of $\mathbf{P}$ is *predense* if every element of $\mathbf{P}$ is compatible with some element of D. An *antichain* in $\mathbf{P}$ is a pairwise incompatible subset of $\mathbf{P}$. Thus a maximal antichain is just a predense antichain.

Exercise 1.3. *Let M and $\mathbf{P}$ be as in Lemma 1.4 and let G be a filter on $\mathbf{P}$. Show that the following are equivalent:*

(i) G is $\mathbf{P}$-generic over M;

(ii) G meets every $D \in M$ that is predense in $\mathbf{P}$;

(iii) G meets every $A \in M$ that is a maximal antichain in $\mathbf{P}$.

Exercise 1.4. *Let M and $\mathbf{P}$ be as in Lemma 1.4 and let G satisfy the conditions (1) and (2) for being $\mathbf{P}$-generic over M but with condition (ii) in the definition of a filter on $\mathbf{P}$ replaced by the weaker condition that any two elements of G have a common extension in $\mathbf{P}$. Prove that G is $\mathbf{P}$-generic over M.*

Let M be a transitive class in which ZFC holds and let $\mathbf{P} \in M$ be a poset. Let $\mathcal{L}(\mathbf{P}, M)$ be the result of adjoining to the language of set theory each element of $M^{\mathbf{P}}$ as a new constant. We define a relation $\Vdash_{\mathbf{P},M} = \Vdash$ in $\mathbf{P}\times$ the class of all sentences of $\mathcal{L}(\mathbf{P}, M)$. Restricted to sentences of any fixed complexity, $\Vdash$ will be a proper class of M, definable in M from $\mathbf{P}$.

We first define by transfinite recursion the forcing relation restricted to $\mathbf{P}$ times the set of all atomic identity sentences of $\mathcal{L}(\mathbf{P}, M)$. We let $p \Vdash \tau_1 = \tau_2$ if and only if both (i) and (ii) below hold:

(i) For all $\langle \sigma_1, s_1 \rangle \in \tau_1$ the set

$$\{q \leq p \mid q \leq s_1 \;\rightarrow\; (\exists \langle \sigma_2, s_2 \rangle \in \tau_2)(q \leq s_2 \;\wedge\; q \Vdash \sigma_1 = \sigma_2)\}$$

is dense below p (i.e., is dense in $\{q \in \mathbf{P} \mid q \leq p\}$).

(ii) For all $\langle \sigma_2, s_2 \rangle \in \tau_2$ the set

$$\{q \leq p \mid q \leq s_2 \;\rightarrow\; (\exists \langle \sigma_1, s_1 \rangle \in \tau_1)(q \leq s_1 \;\wedge\; q \Vdash \sigma_1 = \sigma_2)\}$$

is dense below p.

Next we let $p \Vdash \tau_1 \in \tau_2$ iff the set

$$\{q \leq p \mid (\exists \langle \sigma, s \rangle \in \tau_2)(q \leq s \;\wedge\; q \Vdash \sigma = \tau_1)\}$$

is dense below p.

We finish the definition as follows.

(a) $p \Vdash (\varphi \wedge \psi) \;\leftrightarrow\; (p \Vdash \varphi \;\wedge\; p \Vdash \psi)$;

(b) $p \Vdash \neg\varphi \;\leftrightarrow\; (\forall q \leq p)\, q \nVdash \varphi$;

(c) $p \Vdash (\exists x)\, \varphi(x)$ if and only if $\{q \leq p \mid (\exists \sigma \in M^{\mathbf{P}})\, q \Vdash \varphi(\sigma)\}$ is dense below p.

Theorem 1.10. *Let M be a transitive class in which ZFC holds. Let $\mathbf{P} \in M$ be a poset. Let G be $\mathbf{P}$-generic over M. Then, for any formula $\varphi(v_1, \ldots, v_n)$ of the language of set theory and for any elements $\tau_1, \ldots, \tau_n$ of $M^{\mathbf{P}}$,*

(1) $(\forall p \in G)(p \Vdash \varphi(\tau_1, \ldots, \tau_n) \;\rightarrow\; (\varphi({\tau_1}_G, \ldots, {\tau_n}_G))^{M[G]})$;

(2) $(\varphi({\tau_1}_G, \ldots, {\tau_n}_G))^{M[G]} \;\rightarrow\; (\exists p \in G)(p \Vdash \varphi(\tau_1, \ldots, \tau_n))$.

Proof. Note before we start the proof that if $p \Vdash \varphi$ and $r \leq p$ then $r \Vdash \varphi$.

We begin by proving the theorem for the special case that φ is an atomic identity formula, which we may assume is $v_1 = v_2$. We prove both (1) and (2) by transfinite induction on the maximum of the ranks of τ_1 and τ_2.

For (1), assume that $p \in G$ and that $p \Vdash \tau_1 = \tau_2$. We use clause (i) of the definition to prove that ${\tau_1}_G \subseteq {\tau_2}_G$. A similar argument using (ii) gives that ${\tau_2}_G \subseteq {\tau_1}_G$. Let $a \in {\tau_1}_G$. Then $a = {\sigma_1}_G$ for some $\langle \sigma_1, s_1 \rangle \in \tau_1$ such that $s_1 \in G$. Fix such a $\langle \sigma_1, s_1 \rangle$ and let $r \in G$ be a common extension of p and s_1. Since $r \Vdash \tau_1 = \tau_2$ and since $r \leq s_1$, (i) gives that the set

$$E = \{q \leq r \mid (\exists \langle \sigma_2, s_2 \rangle \in \tau_2)(q \leq s_2 \;\wedge\; q \Vdash \sigma_1 = \sigma_2)\}$$

is dense below r. Let $D = E \cup \{q \in \mathbf{P} \mid q \perp r\}$. Then $D \in M$ and D is dense in $\mathbf{P}$. Let $q \in G \cap D$. Then q must belong to E; thus $q \leq r$ and there is a $\langle \sigma_2, s_2 \rangle \in \tau_2$ such that

$$q \leq s_2 \wedge q \Vdash \sigma_1 = \sigma_2 .$$

For such a $\langle \sigma_2, s_2 \rangle$, $s_2 \in G$ and so ${\sigma_2}_G \in {\tau_2}_G$. Furthermore, we have by induction that ${\sigma_1}_G = {\sigma_2}_G$. Thus we have shown that $a \in {\tau_2}_G$.

For (2), assume that ${\tau_1}_G = {\tau_2}_G$. Let D be the set of all $r \in \mathbf{P}$ such that one of the following holds:

(i′) For some $\langle \sigma_1, s_1 \rangle \in \tau_1$,

$$\{q \leq r \mid q \leq s_1 \rightarrow (\exists \langle \sigma_2, s_2 \rangle \in \tau_2)(q \leq s_2 \wedge q \Vdash \sigma_1 = \sigma_2)\} = \emptyset;$$

(ii′) For some $\langle \sigma_2, s_2 \rangle \in \tau_2$,

$$\{q \leq r \mid q \leq s_2 \rightarrow (\exists \langle \sigma_1, s_1 \rangle \in \tau_1)(q \leq s_1 \wedge q \Vdash \sigma_1 = \sigma_2)\} = \emptyset;$$

(iii′) $r \Vdash \tau_1 = \tau_2$.

We first show that D is dense. For this let $p \in \mathbf{P}$. We may assume that $p \not\Vdash \tau_1 = \tau_2$. Thus either clause (i) or clause (ii) of the definition fails. Assume that (i) fails. (The other case is similar.) This gives us $\langle \sigma_1, s_1 \rangle \in \tau_1$ such that the set

$$\{q \leq p \mid q \leq s_1 \rightarrow (\exists \langle \sigma_2, s_2 \rangle \in \tau_2)(q \leq s_2 \wedge q \Vdash \sigma_1 = \sigma_2)\}$$

is not dense below p. Therefore this set is empty below some $r \leq p$. Such an r satisifies (i′).

Since $D \in M$, let $r \in G \cap D$. It suffices to show that (i′) and (ii′) fail. Suppose that (i′) holds. Let $\langle \sigma_1, s_1 \rangle$ witness this. Note first that $r \leq s_1$, which implies that $s_1 \in G$. Thus ${\sigma_1}_G \in {\tau_1}_G$. But ${\tau_1}_G = {\tau_2}_G$ by hypothesis. Hence ${\sigma_1}_G \in {\tau_2}_G$. This means that there is a $\langle \sigma_2, s_2 \rangle \in \tau_2$ such that $s_2 \in G$ and ${\sigma_1}_G = {\sigma_2}_G$. It follows by induction that there is a $q \in G$ such that $q \Vdash \sigma_1 = \sigma_2$. Let $q' \in G$ be a common extension of q, r, and s_2. Since $q' \Vdash \sigma_1 = \sigma_2$, we have a contradiction. The assumption that (ii′) holds yields a similar contradiction.

Next we prove (1) and (2) for the case that φ is an atomic membership formula, which we may take to be $v_1 \in v_2$.

For (1), assume that $p \in G$ and that $p \Vdash \tau_1 \in \tau_2$. Thus the set

$$E = \{q \leq p \mid (\exists \langle \sigma, s \rangle \in \tau_2)(q \leq s \wedge q \Vdash \sigma = \tau_1)\}$$

is dense below p. Since $p \in G$, an argument like one in the proof of the $v_1 = v_2$ case gives us a $q \in G \cap E$. For this q we have a $\langle \sigma, s \rangle \in \tau_2$ such that $q \leq s$ and $q \Vdash \sigma = \tau_1$. But then $\sigma_G = \tau_{1_G}$ and $\sigma_G \in \tau_{2_G}$.

For (2), assume that $\tau_{1_G} \in \tau_{2_G}$. Let $\langle \sigma, s \rangle \in \tau_2$ be such that $s \in G$ and $\sigma_G = \tau_{1_G}$. Thus there is an $r \in G$ such that $r \Vdash \sigma = \tau_1$. Let $p \in G$ be a common extension of r and s. Then $p \Vdash \tau_1 \in \tau_2$. (Indeed, the set in question is not merely dense below p; the single object $\langle \sigma, s \rangle$ witnesses that it is the whole $\{q \in \mathbf{P} \mid q \leq p\}$.)

We now prove the theorem by induction on the complexity of the formula φ. In the interests of brevity, we shall shall omit "$\tau_1, \ldots, \tau_n$" and "$\tau_{1_G}, \ldots, \tau_{n_G}$."

Suppose that φ is $\neg\psi$.

For (1), assume that $p \in G$ and $p \Vdash \varphi$. Then there is no $q \leq p$ such that $q \Vdash \psi$. Hence there is no $q \in G$ such that $q \Vdash \psi$. By (2) for ψ, we get that ψ does not hold in $M[G]$ and so that $\varphi^{M[G]}$.

For (2), assume that $\varphi^{M[G]}$. It is obvious from the definition than $\{p \in \mathbf{P} \mid p \Vdash \psi \vee p \Vdash \neg\psi\}$ is dense. Hence some $p \in G$ belongs to this set. By (1) for ψ, it is impossible that $p \Vdash \psi$. Therefore $p \Vdash \varphi$.

Next suppose that φ is $\psi \wedge \chi$. If $p \in G$ is such that $p \Vdash \varphi$, then $p \Vdash \psi$ and $p \Vdash \chi$. By (1) for ψ and for χ, it follows that $\varphi^{M[G]}$. If $\varphi^{M[G]}$, then (2) for ψ and for χ gives us elements q and r of G such that $q \Vdash \psi$ and $r \Vdash \chi$. If $p \in G$ is a common extension of q and r, then $p \Vdash \varphi$.

Finally suppose that φ is $(\exists x)\, \psi(x)$.

For (1), assume that $p \in G$ and $p \Vdash \varphi$. Then some member of the set $\{q \leq p \mid (\exists \sigma \in M^{\mathbf{P}})\, q \Vdash \psi(\sigma)\}$ belongs to G. Choose such a q and choose a witness σ. By (1) for ψ, we have that $(\psi(\sigma_G))^{M[G]}$. Hence $\varphi^{M[G]}$.

For (2), assume that $\varphi^{M[G]}$. Let $\sigma_G \in M[G]$ be such that $(\psi(\sigma_G))^{M[G]}$. By (2) for ψ, there is a $p \in G$ such that $p \Vdash \psi(\sigma)$. For any such p, $p \Vdash \varphi$. □

Corollary 1.11. *Let M and $\mathbf{P}$ be as in Theorem 1.10. Assume that M is countable. Then for all sentences $\varphi(\tau_1, \ldots, \tau_n)$ of $\mathcal{L}(\mathbf{P}, M)$ (with only the indicated constants) and for all $p \in \mathbf{P}$, $p \Vdash \varphi(\tau_1, \ldots, \tau_n)$ if and only if, for every G with $p \in G$ such that G is $\mathbf{P}$-generic over M, $(\varphi(\tau_{1_G}, \ldots, \tau_{n_G}))^{M[G]}$*

Proof. We first prove by induction that, for all sentences φ of $\mathcal{L}(\mathbf{P}, M)$,

$$(*) \qquad p \Vdash \varphi \;\leftrightarrow\; \{q \leq p \mid q \Vdash \varphi\} \text{ is dense below } p\,.$$

If $p \Vdash \varphi$, then $q \Vdash \varphi$ for every $q \leq p$. For the other direction, note that if the set of $q \leq p$ such that D is dense below q is dense below p then D

is dense below p. This directly implies the $\leftarrow$ direction of $(*)$ for all cases except those where φ is a negation or a conjuction. For the case of negation, assume that $p \not\Vdash \neq \psi$. This means that there is an $r \leq p$ such that $r \Vdash \psi$. Nothing below r belongs to $\{q \leq p \mid q \Vdash \neg\psi\}$, and so this set is not dense below p. The the case of a conjunction follows by induction.

The $\rightarrow$ direction of the Corollary is just part (1) of Theorem 1.10. For the other direction, suppose that $p \not\Vdash \varphi(\tau_1, \ldots, \tau_n)$. We show that there is a G that $\mathbf{P}$-generic over M such that $p \in G$ but $\varphi({\tau_1}_G, \ldots, {\tau_n}_G)$ does not hold in $M[G]$. By $(*)$ there is a $q \leq p$ such that there is no $r \leq q$ satisfying $r \Vdash \varphi(\tau_1, \ldots, \tau_n)$. Fix such a q. By definition, $q \Vdash \neg\varphi(\tau_1, \ldots, \tau_n)$. By Lemma 1.1, let G be $\mathbf{P}$-generic over M with $q \in G$. By (1) of Theorem 1.10, $(\neg\varphi({\tau_1}_G, \ldots, {\tau_n}_G))^{M[G]}$. Since $p \in G$, our proof is complete. $\square$

Theorem 1.12. *Let M be a transitive class in which ZFC holds. Let $\mathbf{P} \in M$ be a poset. Let G be $\mathbf{P}$-generic over M. Then ZFC holds in $M[G]$.*

Proof. By Lemma 1.9, we know that Extensionality, Foundation, Pairing, and Union hold in $M[G]$.

To prove Comprehension, let $\varphi(x, z, w_1, \ldots, w_n)$ be a formula and let σ and $\tau_1, \ldots, \tau_n$ be elements of $M^{\mathbf{P}}$. We want to prove that $X \in M[G]$, where

$$X = \{a \in \sigma_G \mid (\varphi(a, \sigma_G, {\tau_1}_G, \ldots, {\tau_n}_G))^{M[G]}\}\,.$$

Let

$$\pi = \{\langle \mu, p\rangle \mid \mu \in \operatorname{domain}(\sigma) \ \wedge\ p \Vdash (\mu \in \sigma \ \wedge\ \varphi(\mu, \sigma, \tau_1, \ldots, \tau_n))\}\,.$$

Assume that $a \in X$. Then $a = \mu_G$ for some $\mu \in \operatorname{domain}(\sigma)$. By (2) of Theorem 1.10, there is a $p \in G$ such that $p \Vdash (\mu \in \sigma \ \wedge\ \varphi(\mu, \sigma, \tau_1, \ldots, \tau_n))$. For such a μ and p, $\langle \mu, p\rangle \in \pi$ and so $\mu_G \in \pi_G$.

If $\langle \mu, p\rangle \in \pi$ and $p \in G$ then, by (1) of Theorem 1.10, $\mu_G \in X$.

For Replacement, assume that

$$(\forall x \in \sigma_G)(\exists! y \in M[G])(\varphi(x, y, \sigma_G, {\tau_1}_G, \ldots, {\tau_n}_G))^{M[G]}\,.$$

For each $\mu \in \operatorname{domain}(\sigma)$ and each $p \in \mathbf{P}$, let $f(\mu, p)$ be the least ordinal α such that there is some $\rho \in M^{\mathbf{P}}$ with $\operatorname{rank}(\rho) = \alpha$ and such that $p \Vdash \varphi(\mu, \rho, \sigma, \tau_1, \ldots, \tau_n)$, if such an α exists. Otherwise let $f(\mu, p) = 0$. By Replacement in M, $f \in M$. Let $\beta > f(\mu, p)$ for all $\langle \mu, p\rangle \in \operatorname{domain}(f)$. Let $\pi = (M^{\mathbf{P}} \cap V_\beta) \times \{\mathbf{1}\}$. It is easy to see that π_G witnesses that the given instance of Replacement holds in $M[G]$.

We now know that ZF $-$ Power Set $-$ Infinity holds in $M[G]$. Since $\omega \in M \subseteq M[G]$, it follows that Infinity holds in $M[G]$.

For Power Set, let $\tau \in M^{\mathbf{P}}$. Let

$$S = \{\sigma \in M^{\mathbf{P}} \mid \operatorname{domain}(\sigma) \subseteq \operatorname{domain}(\tau)\}.$$

Let $\pi = S \times \{\mathbf{1}\}$. It is fairly easy to show that $\mathcal{P}(\tau_G) \cap M[G] \subseteq \pi_G$.

For Choice, it is enough to show that for $x \in M[G]$ there exist an ordinal α and a function $f : \alpha \to M[G]$ such that $f \in M[G]$ and $x \subseteq \operatorname{range}(f)$. (This implies that "Every set can be wellordered" holds in $M[G]$, and Choice in $M[G]$ readily follows.)

Let $\tau \in M^{\mathbf{P}}$. By Choice in M, let $g : \alpha \to M^{\mathbf{P}}$ be such that $\alpha \in \mathrm{ON} \cap M$, $g \in M$, and $\operatorname{range}(g) = \operatorname{domain}(\tau)$. Let

$$\pi = \{\langle \operatorname{op}(\check{\beta}, g(\beta)), \mathbf{1}\rangle \mid \beta < \alpha\}.$$

Then π_G is a function with domain α and, for each $\beta < \alpha$, $\pi_G(\beta) = (g(\beta))_G$. Thus $\tau_G \subseteq \operatorname{range}(\pi_G)$.

$\square$

Exercise 1.5. Let M be a countable transitive model of ZFC. Let $\mathbf{P}$ be the poset of the example on page 2. Show that there is a filter G on $\mathbf{P}$ such that $\bigcup G : \omega \to 2$ but $M[G]$ is not a model of ZFC.

2 Forcing with Partial Functions

Theorem 2.1. *Let M be a transitive class such that ZFC holds in M. Let $\mathbf{P}$ be the poset of the example on page 2. Let G be $\mathbf{P}$-generic over M. Then $V \neq L$ holds in $M[G]$. Indeed $\mathcal{P}(\omega) \nsubseteq L$ holds in $M[G]$. Thus if there is a countable transitive model of ZFC then there is a countable transitive model of ZFC + $\mathcal{P}(\omega) \nsubseteq L$.*

Proof. $\mathbf{P}$ is atomless, and so $G \notin M$. As we showed on page 2, $\bigcup G : \omega \to \{0, 1\}$. Now

$$G = \{p \in \mathbf{P} \mid p \subseteq \bigcup G\},$$

so it follows that $\bigcup G \notin M$. By absoluteness, $L^{M[G]} = L^M = L_{\mathrm{ON} \cap M}$. Hence $(\bigcup G \notin L)^M$. If x is the subset of ω whose characteristic function is $\bigcup G$, then $(x \notin L)^M$. $\square$

For sets I and J with $J \neq \emptyset$, let $\mathbf{Fn}(I, J)$ be the set of all functions $f : x \to J$ with x a finite subset of I. Partially order $\mathbf{Fn}(I, J)$ by reverse inclusion. (Hence $\mathbf{1} = \emptyset$.)

Lemma 2.2. *Let M be a transitive class in which ZFC holds and let I and J belong to M, with $J \neq \emptyset$. Let G be $\mathbf{Fn}(I, J)$-generic over M. Then $\bigcup G : I \to J$, and $M[G]$ is the smallest transitive class N such that ZFC holds in N, $\bigcup G \in N$, and $M \subseteq N$.*

Proof. The proof of the first assertion is like that for the special case $I = \omega$ and $J = 2$. The second assertion follows from the corresponding assertion with "G" replacing "$\bigcup G$," which follows from Lemmas 1.3, 1.4, 1.5, and 1.6, together with Theorem 1.12. $\square$

Lemma 2.3. *Let M be a transitive class in which ZFC holds. Let α be an ordinal of M. Let G be $\mathbf{Fn}(\alpha \times \omega, 2)$-generic over M. Then $(2^{\aleph_0} \geq |\alpha|)^{M[G]}$.*

Proof. For $\beta < \alpha$, let $g_\beta : \omega \to 2$ be given by $g_\beta(n) = (\bigcup G)(\beta, n)$. For β and β' less than α, let $D_{\beta,\beta'}$ be the set of all $p \in \mathbf{Fn}(\alpha \times \omega, 2)$ such that, for some $n \in \omega$, $p(\beta, n)$ and $p(\beta', n)$ are defined and different. For distinct β and β', $D_{\beta,\beta'}$ is dense and so meets G. $\square$

A Δ-*system* is a set A such that, for some set r (the *root* of the Δ-system), $a \cap a' = r$ for all distinct elements a and a' of A.

Lemma 2.4 (Δ-System Lemma). *Let A be an uncountable set of finite sets. Then there is an uncountable $B \subseteq A$ such that B is a Δ-system.*

Proof. Shrinking A if necessary, we may assume that $|A| = \aleph_1$. Thus $|\bigcup A| \leq \omega_1$ and we may assume that $\bigcup A \subseteq \omega_1$. By further shrinking, we may assume that, for some $n \in \omega$, $|a| = n$ for all $a \in A$. For $a \in A$ and for $1 \leq m \leq n$, let a_m be the mth element of a in order of magnitude. By shrinking A still further, we may assume that there is an m, $1 \leq m \leq n$, and there is an $r = \{r_1, \ldots, r_{m-1}\}$ such that

(i) $(\forall a \in A)(\forall k)(1 \leq k < m \to a_k = r_k)$;

(ii) $(\forall a \in A)(\forall a' \in A)(a \neq a' \to a_m \neq a'_m)$.

Define $\langle b_\alpha \mid \alpha < \omega_1 \rangle$ by transfinite recursion as follows. Let b_α be some element b of A such that

$$(\forall \beta < \alpha)\,(b_\beta)_n < b_m\,.$$

The set $B = \{b_\alpha \mid \alpha < \omega_1\}$ is a Δ-system. □

A poset $\mathbf{P}$ has the *countable chain condition* (ccc) if every antichain in $\mathbf{P}$ is countable.

Lemma 2.5. *Let I and J be sets with J non-empty and countable. Then $\mathbf{Fn}(I, J)$ has the* ccc.

Proof. Let $\{p_z \mid z \in Z\}$ be an uncountable antichain in $\mathbf{Fn}(I, J)$. Let $a_z = \text{domain}\,(p_z)$ for $z \in Z$. Since J is countable, $\{p_z \mid a_z = a\}$ is countable for each a. Thus $\{a_z \mid z \in Z\}$ is uncountable. Shrinking Z if necessary, we may assume that the function $z \mapsto a_z$ is one-one. By the Δ-System Lemma, let $X \subseteq Z$ be uncountable and such that $\{a_z \mid z \in X\}$ is a Δ-system. Let r be the root of this Δ-system There is an uncountable $Y \subseteq X$ such that, for some fixed p, $p_z \restriction r = p$ for all $z \in Y$. If z and z' belong to Y, then $p_z \cup p_{z'}$ is a common extension of p_z and $p_{z'}$. This is a contradiction. □

Lemma 2.6. *Let M be a countable transitive model of ZFC. Let $\mathbf{P} \in M$ be a poset such that "$\mathbf{P}$ has the* ccc*" holds in M. Let G be $\mathbf{P}$-generic over M. Let X and Y belong to M and let $f \in M[G]$ with $f : X \to Y$. Then there is a g such that*

(a) $g \in M$;

(b) $g : X \to \mathcal{P}(Y)$;

(c) $(\forall x \in X)\, f(x) \in g(x)$ *;*

(d) $(\forall x \in X)\, (|g(x)| \leq \aleph_0)^M$.

Proof. Let $f = \tau_G$. Let $\varphi(v_1, v_2, v_3)$ be the formula "v_1 is a function from v_2 to v_3." Then some $p \in G$ is such that $p \Vdash \varphi(\tau, \check{X}, \check{Y})$. For $x \in X$ let

$$g(x) = \{y \in Y \mid (\exists q \leq p)\, q \Vdash \tau(\check{x}) = \check{y}\}\,.$$

Clauses (a), (b), and (c) are clear. For (d) let $x \in X$. For each $y \in g(x)$, let $q_y \leq p$ be such that $q \Vdash \tau(\check{x}) = \check{y}$. By the fact that $\mathbf{P}$ has the ccc, it suffices to show that q_y and $q_{y'}$ are incompatible when y and y' are distinct elements of $g(x)$. If this fails for some y and y', then let r be a common extension of q_y and $q_{y'}$ and let H be $\mathbf{P}$-generic over M with $r \in H$. Then p, q_y, and $q_{y'}$ all belong to H, and so we have the contradiction that the function τ_H has two distinct values on the argument x. □

Remark. As we shall see later, the assumption that M is countable (or even a set) is unnecessary.

Exercise 2.1. Which of the following have the ccc?

(a) $\mathbf{Fn}(\omega, \omega_1)$;

(b) The set of all subsets A of $[0, 1]$ such that A is Lebesgue measurable and has positive Lebesgue measure, ordered by inclusion and with $\mathbf{1} = [0, 1]$.

Exercise 2.2. Let M be a countable transitive model of ZFC. Let $\mathbf{P}$ be the partial ordering defined in M by the definiton of the partial ordering (b) of Exercise 2.1. Let G be $\mathbf{P}$-generic over M. Show that $M[G] \models \mathcal{P}(\omega) \not\subseteq L$.

Let M be a transitive class in which ZFC holds. A poset $\mathbf{P} \in M$ *preserves cardinals* (with respect to M) if $\mathbf{1} \Vdash$ "$\check{\kappa}$ is a cardinal" for every cardinal κ of M. $\mathbf{P}$ *preserves cofinalities* (with respect to M) if $\mathbf{1} \Vdash$ "$\mathrm{cf}(\check{\alpha}) = (\mathrm{cf}(\alpha))\check{}$" for every limit ordinal α of M.

Lemma 2.7. *Let M be countable transitive model of ZFC. Let $\mathbf{P} \in M$ be a poset.*

(1) If $\mathbf{P}$ preserves cofinalities, then $\mathbf{P}$ preserves cardinals.

(2) If $\mathbf{1} \Vdash$ "$\mathrm{cf}(\check{\kappa}) = \check{\kappa}$" for every uncountable regular cardinal κ of M, then $\mathbf{P}$ preserves cofinalities.

Proof. (1) Suppose $\mathbf{P}$ preserves cofinalities. Let κ be a cardinal of M. Since M is countable, it is enough to prove that, for every G that is $\mathbf{P}$-generic over M, every cardinal of M is a cardinal of $M[G]$. Let G be $\mathbf{P}$-generic over M. If some cardinal κ of M is not a cardinal of $M[G]$, then the least such κ is a successor cardinal of M. Hence $M[G] \models (\mathrm{cf}(\kappa) \neq \kappa)$, and so $\mathbf{P}$ does not preserve cofinalities.

(2) Suppose that the antecedent of (2) holds. Since M is countable, it is enough to prove that, for every G that is $\mathbf{P}$-generic over M, the function cf is the same in M and $M[G]$. Let G be $\mathbf{P}$-generic over M. Let α be a limit ordinal of M and let $\kappa = \mathrm{cf}^M(\alpha)$. Since κ is regular in M, the antecedent of (2) implies that κ is regular in $M[G]$. Let $\kappa^* = \mathrm{cf}^{M[G]}(\alpha)$. We have $f : \kappa \to \alpha$ and $g : \kappa^* \to \alpha$ such that $f \in M$, $g \in M[G]$, both range(f) and range(g) are unbounded in α, and range$(f \restriction \gamma)$ is bounded in α for each $\gamma < \kappa$. Define $h : \kappa^* \to \kappa$ by letting $h(\beta)$ be the least ordinal $\gamma < \kappa$ such that $f(\gamma) \geq g(\beta)$. Then $h \in M[G]$ and range(h) is unbounded in κ. Hence $\kappa^* \geq \mathrm{cf}^{M[G]}(\kappa) = \kappa$. $\square$

Remark. As was the case with Lemma 2.6, the lemma holds without the assumption that M is countable or even a set. This is also true of the results that follow. This will be explained in the next section.

Lemma 2.8. *Let M be a countable transitive model of ZFC. Let $\mathbf{P} \in M$ be a poset such that "$\mathbf{P}$ has the* ccc*" holds in M. Then $\mathbf{P}$ preserves cofinalities and cardinals.*

Proof. Let G be $\mathbf{P}$-generic over M. Let κ be an uncountable regular cardinal of M. Let $\lambda < \kappa$ and let $f : \lambda \to \kappa$ with $f \in M[G]$. Let $g : \lambda \to \mathcal{P}(\kappa)$ be given by Lemma 2.6. Since $\mathrm{cf}^M(\kappa) > \omega$, we have that $h(\beta) = \bigcup g(\beta) < \kappa$ for every $\beta < \lambda$. Since κ is regular in M, $\bigcup \mathrm{range}(h) < \kappa$. But $f(\beta) \leq h(\beta)$ for every $\beta < \lambda$, and so the range of f is bounded in κ. $\square$

If $\mathbf{P}$ is a poset and $\sigma \in V^{\mathbf{P}}$, then a *nice name for a subset of* σ is a $\tau \in V^{\mathbf{P}}$ such that, for some function $\pi \mapsto A_\pi$ defined on all $\pi \in \mathrm{domain}(\sigma)$ and such that each A_π is an antichain in $\mathbf{P}$,

$$\tau = \bigcup \{\{\pi\} \times A_\pi \mid \pi \in \mathrm{domain}(\sigma)\},$$

Remark. Note that being a nice name for a subset of σ depends on $\mathbf{P}$ as well as on σ. Note also that "$x \in V^{\mathbf{P}}$ and y is a nice name for a subset of x" is absolute for transitive class models of ZFC to which $\mathbf{P}$ belongs.

Lemma 2.9. *Let M be a countable transitive model of ZFC and let $\mathbf{P} \in M$ be a poset. Let σ and μ belong to $M^{\mathbf{P}}$. Then there is a nice name τ for a subset of σ such that $\tau \in M$ and*

$$\mathbf{1} \Vdash (\mu \subseteq \sigma \to \mu = \tau)\,.$$

Proof. For $\pi \in \operatorname{domain}(\sigma)$, let A_π be an antichain in $\mathbf{P}$ such that

(1) $(\forall r \in A_\pi)\, r \Vdash \pi \in \mu$;

(2) $(\forall r \in \mathrm{P})(r \Vdash \pi \in \mu \to$ r is compatible with a member of A_π).

Do this so that the function $\pi \mapsto A_\pi$ belongs to M. Let τ be the nice name for a subset of σ given by $\pi \mapsto A_\pi$.

Let G be $\mathbf{P}$-generic over M. (1) implies that $\tau_G \subseteq \mu_G$. Assume that $\mu_G \subseteq \sigma_G$. We must show that $\mu_G \subseteq \tau_G$. Let $a \in \mu_G$. Since $a \in \sigma_G$, there is a $\pi \in \operatorname{domain}(\sigma)$ such that $a = \pi_G$. Let D be the set all $q \in \mathbf{P}$ such that either $q \leq p$ for some $p \in A_\pi$ or $q \perp p$ for all $p \in A_\pi$. Then $D \in M$ and D is dense in $\mathbf{P}$. Let then $q \in G \cap D$. Since $\pi_G \in \mu_G$, there must be an $r \in G$ such that $r \Vdash \pi \in \mu$, and hence there is such an r that is $\leq q$. By (2), r (and hence q) must be compatible with some element of A_π. By the definition of D, there is a $p \in A_\pi$ such that $q \leq p$. But then $\langle \pi, p\rangle \in \tau$ and $p \in G$, and so $\pi_G \in \tau_G$. $\square$

Lemma 2.10. *Let M be a countable transitive model of ZFC and let $\mathbf{P} \in M$ be a poset such that "$\mathbf{P}$ has the ccc" holds in M. Let κ be an infinite cardinal of M such that $(|\mathbf{P}| \leq \kappa)^M$. Let λ be an infinite cardinal of M and let θ be such that $(\kappa^\lambda = \theta)^M$. Let G be $\mathbf{P}$-generic over M. Then $2^\lambda \leq \theta$ holds in $M[G]$.*

Proof. Work in M. (That is, construe our assertions as relativized to M.) The number of antichains of $\mathbf{P}$ is $\leq \kappa^{\aleph_0}$. Since $\operatorname{domain}(\check{\lambda}) = \{\check{\alpha} \mid \alpha < \lambda\}$, we have that $|\operatorname{domain}(\check{\lambda})| = \lambda$. The number of nice names for subsets of $\check{\lambda}$ is thus $\leq (\kappa^{\aleph_0})^\lambda = \kappa^\lambda = \theta$. Let $\alpha \mapsto \tau_\alpha$ have domain θ and range the set of all nice names for subsets of $\check{\lambda}$. Now in $M[G]$ let $f(\alpha) = (\tau_\alpha)_G$ for $\alpha < \theta$. By Lemma 2.9, $\operatorname{range}(f) \supseteq \mathcal{P}(\lambda) \cap M[G]$. $\square$

Theorem 2.11. *Let M be a countable transitive model of ZFC and let κ be any infinite cardinal of M such that $\kappa^{\aleph_0} = \kappa$ holds in M. Let G be* $\mathrm{Fn}(\kappa, 2)$*-generic over M. Then all cardinals of M are cardinals of $M[G]$ and $(2^{\aleph_0} = \kappa)^{M[G]}$.*

Proof. The first assertion holds by Lemma 2.8. By Lemmas 2.5 and 2.10, $(2^{\aleph_0} \le \kappa)^{M[G]}$. By Lemma 2.3, $(2^{\aleph_0} \ge \kappa)^{M[G]}$. $\square$

Remarks:

(a) If M is a countable transitive model of ZFC then so is L^M, and the GCH holds in L^M. The GCH implies that any cardinal κ such that $\mathrm{cf}(\kappa) > \omega$ satisfies the condition $\kappa^{\aleph_0} = \kappa$.

(b) By a theorem of König, $\mathrm{cf}(2^{\aleph_0}) > \omega$, so, for M as in the statement of the theorem, the conclusion $(2^{\aleph_0} = \kappa)^{M[G]}$ must fail for κ such that $\mathrm{cf}(\kappa) = \omega$ holds in M. In fact, it is easy to see that the conclusion fails for all κ such that $\kappa^{\aleph_0} = \kappa$ does not hold in M.

For sets I, non-empty sets J, and infinite cardinals λ, let $\mathbf{Fn}(I, J, \lambda)$ be the set of all functions $f : x \to J$ with $x \subseteq I$ and $|x| < \lambda$. Partially order $\mathbf{Fn}(I, J, \lambda)$ by reverse inclusion. For $\lambda > \omega$, it turns out that $\mathbf{Fn}(I, J, \lambda)$ is not absolute for transitive M in which ZFC holds.

Lemma 2.12. *Let M be a transitive class in which ZFC holds, let I and J be members of M with $J \neq \emptyset$, let λ be an infinite cardinal of M, and let G be $\mathbf{Fn}^M(I, J, \lambda)$-generic over M. Then $\bigcup G : I \to J$, and $M[G]$ is the smallest class N such that ZFC holds in N, $\bigcup G \in N$, and $M \subseteq N$.*

Lemma 2.13. *Let M be a transitive class in which ZFC holds. Let λ and $\lambda' \le \lambda$ be infinite cardinals of M and let α be an ordinal of M. Let G be $\mathbf{Fn}^M(\alpha \times \lambda, 2, \lambda')$-generic over M. Then $(2^\lambda \ge |\alpha|)^{M[G]}$.*

Proof. The proof is like that of Lemma 2.3. $\square$

Lemma 2.14 (General Δ-System Lemma). *Let κ be an infinite cardinal. Let $\theta > \kappa$ be regular and satisfy $(\forall \gamma < \theta)\, |{}^{<\kappa}\gamma| < \theta$. Let A be set of size $\ge \theta$ of sets of size $< \kappa$. Then there is an $B \subseteq A$ such that $|B| = \theta$ and B is a Δ-system.*

Proof. Shrinking A if necessary, we may assume that $|A| = \theta$. Thus $|\bigcup A| \le \theta$ and we may assume that $\bigcup A \subseteq \theta$. By further shrinking, we may assume that, for some $\rho < \kappa$, $\mathrm{ot}(a) = \rho$ for all $a \in A$. For $a \in A$ let $\langle a_\alpha \mid \alpha < \rho \rangle$ enumerate the elements of a in order of magnitude.

By shrinking A still further, we may assume that there is an $\alpha < \rho$ and there is an $r = \{ r_\beta \mid \beta < \alpha \}$ such that

(i) $(\forall a \in A)(\forall \beta < \alpha)\, a_\beta = r_\beta$;

(ii) $(\forall a \in A)(\forall a' \in A)(a \neq a' \to a_\alpha \neq a'_\alpha)$.

To see this, let α be the least ordinal such that either $\alpha = \rho$ or else $|\{a_\alpha \mid a \in A\}| = \theta$. The regularity of θ implies that there is a $\gamma < \theta$ such that $\{\langle a_\beta \mid \beta < \alpha\rangle \mid a \in A\}$ is a subset of ${}^\alpha\gamma$. By the hypothesis that $(\forall\gamma < \theta)\,|{}^{<\kappa}\gamma| < \theta$, it follows that $|\{\langle a_\beta \mid \beta < \alpha\rangle \mid a \in A\}| < \theta$. From this we get both that $\alpha < \rho$ and that $\langle a_\beta \mid \beta < \alpha\rangle$ is constant on a subset of A of size θ.

Define $\langle b(\xi) \mid \xi < \theta\rangle$ by transfinite recursion as follows. Let $b(\xi)$ be some element b of A such that

$$(\forall\eta < \xi) \bigcup b(\eta) < b_\alpha .$$

The set $B = \{b(\xi) \mid \xi < \theta\}$ is a Δ-system. □

For cardinals θ, a poset $\mathbf{P}$ has the *θ-chain condition* (θ-cc) if every antichain in $\mathbf{P}$ has size $< \theta$.

Lemma 2.15. *Let I and $J \neq \emptyset$ be sets and let λ be an infinite cardinal. Then $\mathbf{Fn}(I, J, \lambda)$ has the $(|J|^{<\lambda})^+$-cc.*

Proof. We may assume that $|J| \geq 2$. Suppose that $\{p_z \mid z \in Z\}$ is antichain in $\mathbf{Fn}(I, J, \lambda)$ with $|\{p_z \mid z \in Z\}| \geq (|J|^{<\lambda})^+ = \theta$.

Assume first that λ is regular. Let $a_z = \text{domain}\,(p_z)$ for $z \in Z$. Let $A = \{a_z \mid z \in Z\}$. For each $a \in A$, $|\{z \mid a_z = a\}| \leq |J|^{<\lambda}$. Thus $|\{a_z \mid z \in Z\}| \geq \theta$. Shrinking Z if necessary, we may assume that the function $z \mapsto a_z$ is one-one. Since λ is regular, $(|J|^{<\lambda})^{<\lambda} = |J|^{<\lambda} < \theta$. Therefore the General Δ-System Lemma applies with $\kappa = \lambda$. By that application, let $X \subseteq Z$ be such that $|X| = \theta$ and $\{a_z \mid z \in X\}$ is a Δ-system. Let r be the root of this Δ-system. There is a $Y \subseteq X$ such that $|Y| = \theta$ and such that, for some fixed p, $p_z \restriction r = p$ for all $z \in Y$. If z and z' belong to Y, then $p_z \cup p_{z'}$ is a common extension of p_z and $p_{z'}$. This is a contradiction.

Now assume that λ is singular. Since θ is regular and λ is not of the form δ^+, there is a regular $\lambda' < \lambda$ such that $|\{z \in Z \mid |p_z| < \lambda'\}| \geq \theta$. The argument of the preceding paragraph shows that this is impossible. □

Lemma 2.16. *Let M be a countable transitive model for ZFC. Let θ be a cardinal number of M. Let $\mathbf{P} \in M$ be a poset such that "$\mathbf{P}$ has the θ-cc" holds in M. Let G be $\mathbf{P}$-generic over M. Let X and Y belong to M and let $f \in M[G]$ with $f : X \to Y$. Then there is a g such that*

(a) $g \in M$;

(b) $g : X \to \mathcal{P}(Y)$ *;*

(c) $(\forall x \in X)\, f(x) \in g(x)$ *;*

(d) $(\forall x \in X)\, (|g(x)| < \theta)^M$*.*

Proof. The proof is just like that of Lemma 2.6. $\square$

Let M be a transitive class in which ZFC holds. Let θ be a cardinal of M. A poset $\mathbf{P} \in M$ *preserves cardinals* $\geq \theta$ $[\leq \theta]$ (with respect to M) if $\mathbf{1} \Vdash$ "$\check{\kappa}$ is cardinal" for every cardinal κ of M such that $\kappa \geq \theta$ $[\kappa \leq \theta]$. $\mathbf{P}$ *preserves cofinalities* $\geq \theta$ $[\leq \theta]$ (with respect to M) if $\mathbf{1} \Vdash$ "$\mathrm{cf}(\check{\alpha}) = (\mathrm{cf}(\alpha))\check{}$" for every limit ordinal α of M such that $\mathrm{cf}^M(\alpha) \geq \theta$ $[\mathrm{cf}^M(\alpha) \leq \theta]$.

Lemma 2.17. *Let M be a countable transitive model of ZFC. Let θ be an infinite cardinal of M. Let $\mathbf{P} \in M$ be a poset such that "$\mathbf{P}$ has the θ-cc" holds in M. Then $\mathbf{P}$ preserves cofinalities $\geq \theta$, and if θ is regular in M then $\mathbf{P}$ preserves cardinals $\geq \theta$.*

Proof. The proof is like that of Lemma 2.8. $\square$

If λ is a cardinal number, then a poset $\mathbf{P}$ is λ-*closed* if, whenever $\gamma < \lambda$ and $\langle p_\beta \mid \beta < \gamma \rangle$ is a decreasing sequence of elements of $\mathbf{P}$, then there is a $q \in \mathbf{P}$ such that $q \leq p_\beta$ for all $\beta < \gamma$.

Lemma 2.18. *If λ is regular, then $\mathbf{Fn}(I, J, \lambda)$ is λ-closed.*

Proof. If λ is regular and $\gamma < \lambda$ and $\langle p_\beta \mid \beta < \gamma \rangle$ is a decreasing sequence of elements of $\mathbf{Fn}(I, J, \lambda)$, then $\bigcup \{p_\beta \mid \beta < \gamma\} \in \mathbf{Fn}(I, J, \lambda)$. $\square$

Lemma 2.19. *Let M be a countable transitive model of ZFC. Let λ be a cardinal of M. Let $\mathbf{P} \in M$ be a poset such that "$\mathbf{P}$ is λ-closed" holds in M. Let X and Y belong to M with $|X|^M < \lambda$. Let G be $\mathbf{P}$-generic over M. Let $f : X \to Y$ with $f \in M[G]$. Then $f \in M$.*

Proof. Let $\tau_G = f$. Let $q \in G$ be such that $q \Vdash \tau : \check{X} \to \check{Y}$. Work in M. Let $\alpha \mapsto x_\alpha$ be a one-one function from some ordinal $\beta < \lambda$ onto X.

For $p \leq q$, we use transfinite recursion to associate with p a decreasing sequence $\langle p_\alpha \mid \alpha \leq \beta \rangle$. Let $p_0 = p$. Given $\langle p_\alpha \mid \alpha < \gamma \rangle$ with $\gamma \leq \beta$, let $r_\gamma \leq p_\alpha$ for all $\alpha < \gamma$. If $\gamma < \beta$, let $p_\gamma \leq r_\gamma$ be such that $p_\gamma \Vdash \tau(\check{x_\gamma}) = \check{y_\gamma}$ for some $y_\gamma \in Y$. If $\gamma = \beta$, let $p_\gamma = r_\gamma$. Then $p_\beta \Vdash \tau(\check{x_\gamma}) = \check{y_\gamma}$ for all $\gamma < \beta$. Thus $p_\beta \Vdash \tau = \check{g}$," where g is $\gamma \mapsto y_\gamma$. Since the set of all p_β, $p \leq q$, is dense below q and belongs to M, some p_β belongs to G.

Corollary 2.20. *If M, λ, and $\mathbf{P}$ are as in Lemma 2.19, then $\mathbf{P}$ preserves cofinalities and cardinals $\leq \lambda$.*

Lemma 2.21. *Let M be a countable transitive model of ZFC and let I and $J \neq \emptyset$ belong to M. Let λ be an infinite regular cardinal of M such that $(|J|^{<\lambda} = \lambda)^M$. Then $\mathbf{Fn}^M(I, J, \lambda)$ preserves cofinalities and cardinals.*

Proof. Work in M. By Lemma 2.15, $\mathbf{Fn}(I, J, \lambda)$ has the λ^+-cc. By Lemma 2.17, $\mathbf{Fn}(I, J, \lambda)$ preserves cofinalities and cardinals $\geq \lambda^+$. By Corollary 2.20, $\mathbf{Fn}(I, J, \lambda)$ preserves cofinalities and cardinals $\leq \lambda$. $\square$

Theorem 2.22. *Let M be a countable transitive model of ZFC. Let λ and κ be infinite cardinals of M such that, in M, λ is regular, $\lambda < \kappa$, $2^{<\lambda} = \lambda$, and $\kappa^\lambda = \kappa$. Let G be $\mathbf{P}$-generic over M, where $\mathbf{P} = \mathbf{Fn}(\kappa \times \lambda, 2, \lambda)^M$. Then cardinals and cofinalities are the same in M and $M[G]$, $(2^{\lambda'})^{M[G]} = (2^{\lambda'})^M$ for $\lambda' < \lambda$, and $(2^\lambda = \kappa)^{M[G]}$.*

Proof. The first assertion follows from Lemma 2.21. The second assertion then follows from Lemma 2.19.

Work in M. The cardinal of P is $\leq \kappa^{<\lambda} \cdot 2^{<\lambda} = \kappa \cdot \lambda = \kappa$. Lemma 2.15 implies that $\mathbf{P}$ has the λ^+-cc. Hence the set of all antichains in $\mathbf{P}$ has cardinal $\leq \kappa^\lambda = \kappa$. Thus there are no more than $\kappa^\lambda = \kappa$ nice names for subsets of $\check{\lambda}$.

The argument of just given implies that $(2^\lambda \leq \kappa)^{M[G]}$. Lemma 2.13 implies that $(2^\lambda \geq \kappa)^{M[G]}$. $\square$

Exercise 2.3. Let M be a countable transitive model of ZFC + GCH. Let λ and $\kappa > \lambda$ be infinite cardinals of M with λ regular in M and $\mathrm{cf}^M(\kappa) \geq \lambda$. Let G be $\mathbf{Fn}^M(\lambda, \kappa, \lambda)$-generic over M. Show the following:

(1) If δ is a cardinal of M and $\delta \leq \lambda$ or $\delta > \kappa$, then δ is a cardinal of $M[G]$.

(2) $(|\kappa| = \lambda)^{M[G]}$.

Exercise 2.4. Let M be a countable transitive model of ZFC. Let $(\kappa \geq 2^{\aleph_0})^M$, and let G be $\mathbf{Fn}^M(\omega_1, \kappa, \omega_1)$-generic over M. Prove that the continuum hypothesis holds in $M[G]$.

3 Relative Consistency and Boolean Valued Models

Weakening hypotheses.

Most of the results of §2 have the hypothesis that M is a countable transitive model of ZFC. Except for results whose conclusions assert the existence of generic objects, we can always weaken this hypothesis, requiring only that M is a transitive class in which ZFC holds.

To indicate why this is so, we discuss the case of Lemma 2.6. Suppose we change the statement of Lemma 2.6 as follows:

(a) Replace the hypotheses that G is $\mathbf{P}$-generic over M, $f \in M[G]$, and $f : X \to Y$ by the hypothesis that $p \in \mathbf{P}$, $\tau \in M^{\mathbf{P}}$, and $p \Vdash \tau : \check{X} \to \check{Y}$.

(b) Replace clause (c) of the conclusion by $(\forall x \in X)\, p \Vdash \tau(\check{x}) \in \check{g}(\check{x})$.

It is easy to see that

(i) for M a countable transitive model of ZFC, the modified Lemma 2.6 for M is equivalent to the original Lemma 2.6 for M;

(ii) for M a transitive class such that ZFC holds in M, the modified Lemma 2.6 for M implies the original Lemma 2.6 for M.

It is also easy to see that there is a sentence σ such that, for each transitive class M such that ZFC holds in M,

(iii) the modified Lemma 2.6 for M is equivalent to σ^M.

It can be verified that there is some $m \in \omega$ such that our proof of Lemma 2.6 for a countable transitive M goes through when we require only that ZFC_m holds in M, where ZFC_m is the set of the first m axioms of ZFC. (Fix some reasonable enumeration of the axioms of ZFC.)

Suppose that M is a transitive class in which ZFC holds. By the Reflection schema (applied to the V_α^M), the Löwenheim–Skolem Theorem, and Mostowski Collapse, let N be a countable transitive model of ZFC_m such that $\sigma^N \leftrightarrow \sigma^M$. Then Lemma 2.6 holds for N. Hence σ^N holds. Hence σ^M holds. Hence Lemma 2.6 holds for M.

Some of the results of §2 are like Lemma 2.6 in that they can be reformulated in the form "for all countable transitive models M of ZFC, σ^M holds." Others are already essentially in this form.

Relative Consistency.

To see how the results of §2 give relative consistency results, let us as an example indicate how Theorem 2.1 yields a proof that ZFC + $V \neq L$ is consistent if ZFC is consistent.

Let $\mathbf{P}$ be the partial ordering of Theorem 2.1. One can see that our proofs give a (computable) function $f : \omega \to \omega$ and, for each $n \in \omega$, a proof in ZFC that if M is a countable transitive model of $\mathrm{ZFC}_{f(n)}$ and G is $\mathbf{P}$-generic over M, then $M[G]$ is a countable transitive model of ZFC_n + $V \neq L$.

All instances of The Reflection schema are provable in ZFC, and the Löwenheim–Skolem and Mostowski's Lemma are provable in ZFC. Thus we know that for each $n \in \omega$ there is a proof in ZFC (of a sentence expressing) that there exists a countable transitive model M of $\mathrm{ZFC}_{f(n)}$. Combining this with the fact mentioned in the preceding paragraph, we get that for each n there is a proof in ZFC that there exists a countable transitive model N of $\mathrm{ZFC}_n + V \neq L$.

Suppose that ZFC + $V \neq L$ is inconsistent. Then there is an $n \in \omega$ such that from $\mathrm{ZFC}_n + V \neq L$ a contradiction is provable. For this n, it is provable in ZFC that $\mathrm{ZFC}_n + V \neq L$ is inconsistent. Since the Soundness Theorem is provable in ZFC, there is a proof in ZFC that there does not exist any model of $\mathrm{ZFC}_n + V \neq L$. Thus in ZFC both a sentence and its negation are provable, i.e., ZFC is inconsistent.

Complete embeddings and dense embeddings.

Let $\langle \mathbf{P}, \leq_{\mathbf{P}}, \mathbf{1}_{\mathbf{P}} \rangle$ and $\langle \mathbf{Q}, \leq_{\mathbf{Q}}, \mathbf{1}_{\mathbf{Q}} \rangle$ be posets. A function $i : \mathbf{P} \to \mathbf{Q}$ is a complete embedding (of $\langle \mathbf{P}, \leq_{\mathbf{P}}, \mathbf{1}_{\mathbf{P}} \rangle$ into $\langle \mathbf{Q}, \leq_{\mathbf{Q}}, \mathbf{1}_{\mathbf{Q}} \rangle$) if

(1) $(\forall p \in \mathbf{P})(\forall p' \in \mathbf{P})(p \leq p' \to i(p) \leq i(p'))$;

(2) $(\forall p \in \mathbf{P})(\forall p' \in \mathbf{P})(p \perp p' \leftrightarrow i(p) \perp i(p'))$;

(3) $(\forall q \in \mathbf{Q})(\exists p \in \mathbf{P})(\forall p' \leq p)\, i(p')$ is compatible with q .

(Note that we suppress subscripts on "$\leq$" when there is no danger of confusion.) For q and p as in (3), p is called a *reduction of* q *to* $\mathbf{P}$.

Say that $\langle \mathbf{P}, \leq_{\mathbf{P}}, \mathbf{1}_{\mathbf{P}} \rangle \subseteq_c \langle \mathbf{Q}, \leq_{\mathbf{Q}}, \mathbf{1}_{\mathbf{Q}} \rangle$ if $\mathbf{P} \subseteq \mathbf{Q}$, $\leq_{\mathbf{P}} = \leq_{\mathbf{Q}} \cap (\mathbf{P} \times \mathbf{P})$ and $\mathrm{id} : \mathbf{P} \to \mathbf{Q}$ is a complete embedding.

Lemma 3.1. *If $I \subseteq I'$ then $\mathbf{Fn}(I, J, \lambda) \subseteq_c \mathbf{Fn}(I', J, \lambda)$.*

Proof. (1) and (2) are easily verified. For (3), note that if $q \in \mathbf{Fn}(I', J, \lambda)$ then $q \restriction I$ is a reduction of q to $\mathbf{P}$. □

Lemma 3.2. *Let M be transitive and such that ZFC holds in M. Let $\mathbf{P}$ and $\mathbf{Q}$ be posets and let $i : \mathbf{P} \to \mathbf{Q}$ be a complete embedding. Assume that $\mathbf{P}$, $\mathbf{Q}$, and i all belong to M. Let H be $\mathbf{Q}$-generic over M. Then*

(a) $i^{-1}(H)$ *is* $\mathbf{P}$*-generic over* M*;*

(b) $M[i^{-1}(H)] \subseteq M[H]$.

Proof. Let $G = i^{-1}(H)$. To prove (a), we use Exercise 1.3.

Assume $p \leq p'$ and $p \in G$. Then (1) implies that $i(p) \leq i(p')$ and so that $i(p') \in H$. Thus $p' \in G$.

If p and p' belong to G and $p \perp p'$, then (2) gives the contradiction that $i(p) \perp i(p')$.

Let $D \in M$ with D dense in $\mathbf{P}$. Let

$$E = \{q' \in \mathbf{Q} \mid (\exists p \in D)\, q' \leq i(p)\}\,.$$

We show that E is dense in $\mathbf{Q}$. Let $q \in \mathbf{Q}$. By (3), let p be a reduction of q to $\mathbf{P}$. Let $p' \leq p$ with $p' \in D$. Then $i(p')$ and q are compatible. Let q' be a common extension of $i(p')$ and q. Since $p' \in D$ and $q' \leq i(p')$, we get that $q' \in E$.

Now let $q \in E \cap H$. Let $p \in D$ with $q \leq i(p)$. Then $i(p) \in H$, and so $p \in G$.

For (b), note that $G \in M[H]$, and so $M[G] \subseteq M[H]$ by Lemma 1.3. □

For posets $\mathbf{P}$ and $\mathbf{Q}$, a function $i : \mathbf{P} \to \mathbf{Q}$ is a dense embedding if

(1) $(\forall p \in \mathbf{P})(\forall p' \in \mathbf{P})(p \leq p' \to i(p) \leq i(p'))\,;$

(2) $(\forall p \in \mathbf{P})(\forall p' \in \mathbf{P})(p \perp p' \leftrightarrow i(p) \perp i(p'))\,;$

(3) $i``\mathbf{P}$ is dense in $\mathbf{Q}$.

Obviously every isomorphism is a dense embedding.

Lemma 3.3. *Every dense embedding is a complete embedding.*

Proof. If $i : \mathbf{P} \to \mathbf{Q}$ is a dense embedding and $q \in \mathbf{Q}$, then any $p \in \mathbf{P}$ with $i(p) \leq q$ is a reduction of q to $\mathbf{P}$. □

Theorem 3.4. *Let M be transitive and such that ZFC holds in M. Let $\mathbf{P}$ and $\mathbf{Q}$ be posets and let $i : \mathbf{P} \to \mathbf{Q}$ be a dense embedding. Assume that $\mathbf{P}$, $\mathbf{Q}$, and i all belong to M. For $G \subseteq \mathbf{P}$, let*

$$\tilde{\imath}(G) = \{q \in \mathbf{Q} \mid (\exists p \in G)\, i(p) \leq q\}\,.$$

Then

(a) If H is $\mathbf{Q}$-generic over M, then $i^{-1}(H)$ is $\mathbf{P}$-generic over M and $\tilde{\imath}(i^{-1}(H)) = H$.

(b) If G is $\mathbf{P}$-generic over M, then $\tilde{\imath}(G)$ is $\mathbf{Q}$-generic over M and $i^{-1}(\tilde{\imath}(G)) = G$.

(c) If $G = i^{-1}(H)$ in (a) or if $H = \tilde{\imath}(G)$ in (b), then $M[G] = M[H]$.

Proof. (a) By Lemmas 3.2 and 3.3, $i^{-1}(H)$ is $\mathbf{P}$-generic over M. Since $i``i^{-1}(H) \subseteq H$ and since every member of $\tilde{\imath}(i^{-1}(H))$ is $\geq$ some member of $i``i^{-1}(H)$,

$$\tilde{\imath}(i^{-1}(H)) \subseteq H\,.$$

To see that $H \subseteq \tilde{\imath}(i^{-1}(H)) \subseteq H$, let $q \in H$. The set $\{q' \leq q \mid q' \in \text{range}(i)\}$ is dense below q, so some q' belongs both to this set and to H.

(b) Let $H = \tilde{\imath}(G)$. We use Exercise 1.3. If $q \leq q'$ and $q \in H$, then it follows directly that $q' \in H$. Let q and q' be any members of H. There are members p and p' of G such that $i(p) \leq q$ and $i(p') \leq q'$. Since p and p' are compatible in $\mathbf{P}$, property (2) of dense embeddings implies that q and q' are compatible in $\mathbf{Q}$. Let $E \in M$ with E dense in $\mathbf{Q}$. Let

$$D = \{p \in \mathbf{P} \mid (\exists q \in E)\, i(p) \leq q\}\,.$$

To see that D is dense in $\mathbf{P}$, let $p \in \mathbf{P}$. Let $q \in E$ be such that $q \leq i(p)$. By property (3) of dense embeddings, let $p' \in \mathbf{P}$ with $i(p') \leq q$. By property (2), p and p' are compatible. Let p'' be a common extension of p and p'. By property (1), $i(p'') \leq i(p') \leq q$, and so $p'' \in D$. By the density of D, let $p \in D \cap G$. Let $q \in E$ with $i(p) \leq q$. Then $q \in H$. If $p \in G$ then $i(p) \in \tilde{\imath}(G)$ and so $p \in i^{-1}(\tilde{\imath}(G))$. For the reverse inclusion, let $p \in i^{-1}(\tilde{\imath}(G))$. Since $\{p' \in \mathbf{P} \mid p' \leq p \vee p' \perp p\}$ is dense in $\mathbf{P}$, some $p' \in G$ belongs to this set. Both p and p' belong to $i^{-1}(\tilde{\imath}(G))$, and so $p' \not\perp p$, It follows that $p \in G$.

(c) That $G \in M[H]$ and $H \in M[G]$ is clear. By Lemma 1.3, it follows that $M[G] = M[H]$. □

If $\mathbf{P}$ and $\mathbf{Q}$ are posets and $i : \mathbf{P} \to \mathbf{Q}$ is a complete embedding, define inductively, for $\tau \in V^{\mathbf{P}}$,

$$i_*(\tau) = \{\langle i_*(\sigma), i(p)\rangle \mid \langle \sigma, p\rangle \in \tau\}\,.$$

Lemma 3.5. *Let M be a transitive class such that ZFC holds in M. Let $\mathbf{P}$ and $\mathbf{Q}$ be posets and let $i : \mathbf{P} \to \mathbf{Q}$ be a complete embedding. Assume that $\mathbf{P}$, $\mathbf{Q}$, and i all belong to M. Let H be $\mathbf{Q}$-generic over M.*

(a) For all $\tau \in M^{\mathbf{P}}$, $\tau_{i^{-1}(H)} = (i_*(\tau))_H$.

(b) *If $\varphi(x_1, \ldots, x_n)$ is asolute for transitive models of ZFC, then*

$$(\forall p \in \mathbf{P})(p \Vdash \varphi(\tau_1, \ldots, \tau_n) \leftrightarrow i(p) \Vdash \varphi(i_*(\tau_1), \ldots, i_*(\tau_n)))\,.$$

(c) *If i is a dense embedding, then the conclusion of (b) holds for all formulas $\varphi(x_1, \ldots, x_n)$.*

Proof. We prove (a) by induction on rank(τ). To show that $\tau_{i^{-1}(H)} \subseteq i_*(\tau)_H$, let $\langle \sigma, p\rangle \in \tau$ and $p \in i^{-1}(H)$. Then $\langle i_*(\sigma), i(p)\rangle \in i_*(\tau)$ and $i(p) \in H$. Therefore $i_*(\sigma)_H \in i_*(\tau)_H$. By induction, it follows that $\sigma_{i^{-1}(H)} \in i_*(\tau)_H$. The proof of the reverse inclusion is similar.

For M countable, (b) and (c) follow from part (b) of Lemma 3.2 and part (c) of Theorem 3.4. For general M they can be proved using the method sketched at the beginning of this section. □

A poset $\mathbf{P}$ is *separative* if

(a) $(\forall p \in \mathbf{P})(\forall q \in \mathbf{P})((p \leq q \wedge q \leq p) \to p = q)$;

(b) $(\forall p \in \mathbf{P})(\forall q \in \mathbf{P})(p \not\leq q \to (\exists r \leq p)\, r \perp q)$.

Exercise 3.1. Let $\mathbf{P}$ and $\mathbf{Q}$ be separative posets and let $i : \mathbf{P} \to \mathbf{Q}$ be a complete embedding. Show that i is one-one, that $i(\mathbf{1_P}) = \mathbf{1_Q}$, and that

$$(\forall p \in \mathbf{P})(\forall p' \in \mathbf{P})(p \leq p' \leftrightarrow i(p) \leq i(p'))\,.$$

Lemma 3.6. *Let $\mathbf{P}$ be a poset. There is a separative poset $\mathbf{Q}$ such that $|\mathbf{Q}| \leq |\mathbf{P}|$ and such that there is a dense embedding $i : \mathbf{P} \to \mathbf{Q}$.*

Proof. Let $\leq\, = \,\leq_{\mathbf{P}}$ and let $\mathbf{1} = \mathbf{1_P}$. For elements p_1 and p_2 of $\mathbf{P}$, set

$$p_1 \leq' p_2 \;\leftrightarrow\; (\forall r \leq p_1)\, r \not\perp p_2\,.$$

It is easy to check that $\leq'$ is a partial ordering of $\mathbf{P}$, that $\langle \mathbf{P}, \leq', \mathbf{1}\rangle$ satisfies clause (b) in the definition of "separative," and that the identity is a dense embedding of $\langle \mathbf{P}, \leq, \mathbf{1}\rangle$ into $\langle \mathbf{P}, \leq', \mathbf{1}\rangle$.

For elements p_1 and p_2 of $\mathbf{P}$, set

$$p_1 \sim p_2 \;\leftrightarrow\; (p_1 \leq' p_2 \wedge p_2 \leq' p_1)\,.$$

Let $\mathbf{Q}$ be the set of all equivalence classes with respect to the equivalence relation $\sim$, let $\leq_{\mathbf{Q}}$ be the induced partial ordering of the equivalence classes, and let $\mathbf{1_Q}$ be the equivalence class of $\mathbf{1}$. One readily verifies that $\langle \mathbf{Q}, \leq_{\mathbf{Q}}$

$, \mathbf{1_Q}\rangle$ is a separative poset and that the function $i : \mathbf{P} \to \mathbf{Q}$ that sends each $p \in \mathbf{P}$ to its equivalence class is a dense embedding of $\langle \mathbf{P}, \leq', \mathbf{1}\rangle$ into $\langle \mathbf{Q}, \leq_{\mathbf{Q}}, \mathbf{1_Q}\rangle$ and hence is a dense embedding of $\langle \mathbf{P}, \leq, \mathbf{1}\rangle$ into $\langle \mathbf{Q}, \leq_{\mathbf{Q}}, \mathbf{1_Q}\rangle$. $\square$

Boolean-Valued Models

A *Boolean algebra* is a separative poset $\langle \mathbf{B}, \leq, \mathbf{1}\rangle$ with the following properties.

(a) Any two elements b and c of $\mathbf{B}$ have a lub $b \vee c$ and a glb $b \wedge c$.

(b) $\vee$ and $\wedge$ distribute over one another.

(c) There is an operation $b \mapsto b'$ such that $b \vee b' = \mathbf{1}$, $(b')' = b$, $(b \vee c)' = b' \wedge c'$, and $(b \wedge c)' = b' \vee c'$ for any elements b and c of $\mathbf{B}$.

It is not hard to see that the operation $'$ is uniquely determined by (c).

Remark. The definition just given has redundancies. For example, clause (b) from the definition of "separative" is redundant.

A Boolean algebra $\mathbf{B}$ is *complete* if every subset S of $\mathbf{B}$ has a lub $\bigvee S$ and a glb $\bigwedge S$.

Example. If A and B are Lebesgue measurable subsets of the unit interval, set

$$A \sim B \;\leftrightarrow\; \mu(A \triangle B) = 0$$

where μ is Lebesgue measure and $A \triangle B$ is the symmetric difference of A and B. Let $\mathbf{M}$ be the set of all equivalence classes with respect to $\sim$. Partially order $\mathbf{M}$ by letting the equivalence class of A be $\leq$ that of B if $\mu(A \setminus B) = 0$. Then $\mathbf{M}$ is a complete Boolean algebra. Indeed, this follows from the fact that $\mathbf{M}$ is a σ-algebra ($\bigvee S$ and $\bigwedge S$ exist for all countable $S \subseteq \mathbf{M}$) together with the fact (Exercise 2.1) that $\mathbf{M} \setminus \{\mathbf{0}\}$ has the ccc.

Lemma 3.7. *Let* $\mathbf{P}$ *be a poset. There is a complete Boolean algebra* $\mathbf{B}$ *such that there is a dense embedding* $i : \mathbf{P} \to \mathbf{B} \setminus \{\mathbf{0}\}$*, where* $\mathbf{0} = \mathbf{1}'$*. Moreover* $\mathbf{B}$ *is unique up to isomorphism.*

Proof. If X is a topological space, a subset Y of X is *regular open* if $Y = \mathrm{int}(\mathrm{cl}(Y))$, where $\mathrm{int}(Z)$ is the interior of Z and $\mathrm{cl}(Z)$ is the closure of Z. The *regular open algebra* of X, $\mathrm{ro}(X)$, is the poset of regular open subsets of X, ordered by inclusion (and with $\mathbf{1} = X$). It is fairly easy to check that $\mathrm{ro}(X)$ is a complete Boolean algebra and that

$$\begin{aligned} b \wedge c &= b \cap c\,; \\ b \vee c &= \mathrm{int}(\mathrm{cl}(b \cup c))\,; \\ b' &= \mathrm{int}(X \setminus b)\,; \\ \bigwedge S &= \mathrm{int}(\bigcap S)\,; \\ \bigvee S &= \mathrm{int}(\mathrm{cl}(\bigcup S))\,. \end{aligned}$$

We make $\mathbf{P}$ into a topological space by taking as a base all sets of the form $N_p,\ p \in P$, where

$$N_p = \{q \in \mathbf{P} \mid q \leq p\}\,.$$

Let $\mathbf{B} = \mathrm{ro}(\mathbf{P})$. Define $i : \mathbf{P} \to \mathbf{B} \setminus \{\mathbf{0}\}$ by

$$i(p) = \mathrm{int}(\mathrm{cl}(N_p))\,.$$

We have that $p \leq q \Rightarrow N_p \subseteq N_q \Rightarrow \mathrm{int}(\mathrm{cl}(N_p)) \subseteq \mathrm{int}(\mathrm{cl}(N_q)) \Rightarrow i(p) \subseteq i(q)$. Moreover, $p \perp q \Rightarrow N_p \cap N_q = \emptyset \Rightarrow \mathrm{cl}(N_p) \cap N_q = \emptyset \Rightarrow \mathrm{int}(\mathrm{cl}(N_p)) \cap N_q = \emptyset \Rightarrow \mathrm{int}(\mathrm{cl}(N_p)) \cap \mathrm{cl}(N_q) = \emptyset \Rightarrow \mathrm{int}(\mathrm{cl}(N_p)) \cap \mathrm{int}(\mathrm{cl}(N_q)) = \emptyset \Rightarrow i(p) \cap i(q) = \emptyset \Rightarrow i(p) \perp i(q)$. For the converse, note that $i(p) \perp i(q) \Rightarrow \mathrm{int}(\mathrm{cl}(N_p)) \cap \mathrm{int}(\mathrm{cl}(N_q)) = \emptyset \Rightarrow N_p \cap N_q = \emptyset$. To see that the range of i is dense in $\mathbf{B} \setminus \{\mathbf{0}\}$, let $b \in \mathbf{B} \setminus \{\mathbf{0}\}$. Since b is open, there is a $p \in \mathbf{P}$ such that $N_p \subseteq b$. But then $\mathrm{int}(\mathrm{cl}(N_p)) \subseteq \mathrm{int}(\mathrm{cl}(b)) = b$, since b is regular open.

For the uniqueness of $\mathbf{B}$, note that if $i : \mathbf{P} \to \mathbf{B} \setminus \{0\}$ is a dense embedding and $b \in \mathbf{B}$, then

$$\bigvee \{i(p) \mid p \in \mathbf{P} \,\wedge\, i(p) \leq b\} = b\,;$$

for otherwise $b \wedge (\bigvee \{i(p) \mid p \in \mathbf{P} \,\wedge\, i(p) \leq b\})' \neq \mathbf{0}$, and so range (i) is not dense in $\mathbf{B} \setminus \{\mathbf{0}\}$. This fact tells us how to define an isomorphism beteween $\mathbf{B}$ and $\mathbf{B}'$, given dense embeddings $i : \mathbf{P} \to \mathbf{B} \setminus \{\mathbf{0}\}$ and $i' : \mathbf{P}$ to $\mathbf{B}' \setminus \{\mathbf{0}\}$.

$\square$

If $\mathbf{P}$ is a poset, then the complete Boolean algebra given by Lemma 3.7 is called the *completion* of $\mathbf{P}$.

If $\mathbf{B}$ is a complete Boolean algebra and φ is a sentence of $\mathcal{L}(\mathbf{B} \setminus \{\mathbf{0}\}, V)$, then let us make the convention that $\mathbf{0} \Vdash \varphi$.

Lemma 3.8. *Let $\mathbf{B}$ be a complete Boolean algebra. Let φ be a sentence of $\mathcal{L}(\mathbf{B} \setminus \{\mathbf{0}\}, V)$. There is a greatest element b of $\mathbf{B}$ such that $b \Vdash \varphi$.*

Proof. Let $b = \bigvee\{c \mid c \Vdash \varphi\}$. It is easy to check that b is as required. $\square$

For each φ, let us call the b given by Lemma 3.8 the *truth-value* of φ and denote it by $[\![\varphi]\!]$.

If $\mathbf{B}$ is a complete Boolean algebra, and $\sigma \in V^{\mathbf{B}\setminus\{\mathbf{0}\}}$, a *very nice name for a subset of* σ is an element of $V^{\mathbf{B}\setminus\{\mathbf{0}\}}$ of the form

$$\{\langle \pi, f(\pi)\rangle \mid \pi \in \mathrm{domain}(\sigma) \;\wedge\; f(\pi) \neq \mathbf{0}\}\,,$$

where $f : \mathrm{domain}\,(\sigma) \to \mathbf{B}$. For complete Boolean algebras, Lemma 2.9 holds with "very nice name" replacing "nice name": given μ, let τ be the very nice name gotten by setting $f(\pi) = [\![\pi \in \mu]\!]$.

Using only very nice names, we one can construct an alternative version of $V^{\mathbf{B}\setminus\{\mathbf{0}\}}$. For any ordinal α, $V_{\alpha+1}$ is the collection of all sets of the form $\{x \in V_\alpha \mid f(x) = 1\}$, for $f : V_\alpha \to \{0,1\}\}$. If we think of $\{0,1\}$ as the two-element complete Boolean algebra and if we ignore the difference between a set and its characteristic function, then we can regard the following definition as a generalization of that of the V_α hierarchy.

$$\begin{aligned}
V_0^{\mathbf{B}} &= \emptyset\,;\\
V_{\alpha+1}^{\mathbf{B}} &= \{\tau \mid \tau : V_\alpha \to \mathbf{B}\}\,;\\
V_\lambda^{\mathbf{B}} &= \bigcup\{V_\beta^{\mathbf{B}} \mid \beta < \lambda\} \text{ for limit } \lambda\,;\\
V^{\mathbf{B}} &= \bigcup\{V_\alpha^{\mathbf{B}} \mid \alpha \in \mathrm{ON}\}\,.
\end{aligned}$$

An important difference between the general case and that of $\mathbf{B} = \{\mathbf{0}, \mathbf{1}\}$ is that we can have $\tau \in V_{\alpha+1}^{\mathbf{B}}$ and $\sigma \in V_\alpha^{\mathbf{B}}$ such that $[\![\sigma \in \tau]\!] > \tau(\sigma)$.

Instead of using $V^{\mathbf{B}}$ as an alternative version of the class of $\mathbf{B} \setminus \{\mathbf{0}\}$-names, one can simply construe it a a *Boolean-valued model.* If $\mathbf{B}$ is a complete Boolean algebra, then a $\mathbf{B}$-valued model $\mathfrak{A}$ (for a relational language) is a set A together with an assignment to each n-ary relation symbol P of a function $P_{\mathfrak{A}} : {}^n A \to \mathbf{B}$. Satisfaction (truth-value relative to an assignment of variables to elements of A) is defined using the Boolean operations, e.g., $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$. One can show that if a sentence φ follows logically from a set Σ of sentences, then $\bigwedge\{[\![\psi]\!] \mid \psi \in \Sigma\} \leq [\![\varphi]\!]$. Thus using $V^{\mathbf{B}}$ as a Boolean-valued (class) model, we can prove our relative consistency results by an *inner model* method, as one can prove relative consistency results using L, except now the inner model is Boolean-valued.

A *complete homomorphism* between complete Boolean algebras $\mathbf{B}$ and $\mathbf{B}'$ is an $i : \mathbf{B} \to \mathbf{B}'$ such that i preserves all the finite and infinite Boolean operations. We omit the routine proof of the following lemma.

Lemma 3.9. *Let* $\mathbf{B}$ *and* $\mathbf{B}'$ *be complete Boolean algebras and let* $i : \mathbf{B} \to \mathbf{B}'$. *Then* $i \restriction \mathbf{B} \setminus \{\mathbf{0}\}$ *is a complete embedding (into* $\mathbf{B}' \setminus \{\mathbf{0}\}$*) if and only if* i *is a one-one complete homomorphism, and* $i \restriction \mathbf{B} \setminus \{\mathbf{0}\}$ *is a dense embedding if and only if* i *is an isomorphism.*

Lemma 3.10. *Let* M *be a transitive class in which ZFC holds. In* M*, let* $\mathbf{B}$ *be a complete Boolean algebra. Let* G *be* $\mathbf{B} \setminus \{\mathbf{0}\}$*-generic over* M*. Let* $X \in M$ *and let* $Y \in M[G]$ *with* $Y \subseteq X$*.*

In M *there is a complete subalgebra* $\mathbf{C}$ *of* $\mathbf{B}$ *with the following properties. Let* $H = G \cap \mathbf{C}$*. Then* H *is* $\mathbf{C} \setminus \{\mathbf{0}\}$*-generic over* M*, and* $M[H]$ *is the smallest transitive class* N *such that* $M \subseteq N$*,* $Y \in N$*, and ZFC holds in* N*.*

Proof. Let $\tau_G = Y$. In M, let $\mathbf{C}$ be the complete subalgebra of $\mathbf{B}$ generated by

$$\{[\![\check{x} \in \tau]\!] \mid x \in X\}\,.$$

That H is $\mathbf{C} \setminus \{\mathbf{0}\}$-generic over M follows from Lemma 3.9 and Lemma 3.2.

Since $x \in Y$ if and only if $[\![\check{x} \in \tau]\!] \in H$, we have that $Y \in M[H]$.

Let N be transitive and such that $M \subseteq N$, $Y \in N$, and ZFC holds in N. In N, define $h : \mathbf{C} \to \{\mathbf{0}, \mathbf{1}\}$ by

$$\begin{aligned} h([\![\check{x} \in \tau]\!]) &= \begin{cases} \mathbf{1} & \text{if } x \in Y; \\ \mathbf{0} & \text{otherwise;} \end{cases} \\ h(c') &= (h(c))' \\ h(\bigvee S) &= \bigvee h(S). \end{aligned}$$

It is easy to show by transfinite induction that, for all $c \in \mathbf{C}$, $c \in H$ if and only if $h(c) = \mathbf{1}$. $\square$

4 Products, Iterations, and Measurability

Products.

Let us fix, for the first part of this section, posets $\langle \mathbf{P}_0, \leq_0, \mathbf{1}_0 \rangle$ and $\langle \mathbf{P}_1, \leq_1, \mathbf{1}_1 \rangle$. Define the *product*

$$\langle \mathbf{P}_0 \ \leq_0, \mathbf{1}_0 \rangle \times \langle \mathbf{P}_1 \ \leq_1, \mathbf{1}_1 \rangle$$

of these two posets to be

$$\langle \mathbf{P}_0 \times \mathbf{P}_1, \leq, \mathbf{1} \rangle\,,$$

where $\mathbf{1} = \langle \mathbf{1}_0, \mathbf{1}_1 \rangle$ and

$$\langle p_0, p_1 \rangle \leq \langle q_0, q_1 \rangle \ \leftrightarrow \ (p_0 \leq_0 q_0 \ \wedge \ p_1 \leq_1 q_1)\,.$$

Define $i_0 : \mathbf{P}_0 \to \mathbf{P}_0 \times \mathbf{P}_1$ and $i_1 : \mathbf{P}_1 \to \mathbf{P}_0 \times \mathbf{P}_1$ by

$$\begin{aligned} i_0(p_0) &= \langle p_0, \mathbf{1}_1 \rangle\,; \\ i_1(p_1) &= \langle \mathbf{1}_0, p_1 \rangle\,. \end{aligned}$$

We omit the easy proof of the following lemma.

Lemma 4.1. *i_0 and i_1 are complete embeddings.*

Corollary 4.2. *Let M be transitive and such that ZFC holds in M. Assume that $\mathbf{P}_0$ and $\mathbf{P}_1$ belong to M. Let G be $\mathbf{P}_0 \times \mathbf{P}_1$-generic over M. Then*

(i) ${i_0}^{-1}(G)$ is $\mathbf{P}_0$-generic over M ;

(ii) ${i_1}^{-1}(G)$ is $\mathbf{P}_1$-generic over M ;

(iii) $G = {i_0}^{-1}(G) \times {i_1}^{-1}(G)$.

Proof. (i) and (ii) follow from Lemma 4.1 and Lemma 3.2. For (iii), observe that

$$\begin{aligned} \langle p_0, p_1 \rangle \in {i_0}^{-1}(G) \times {i_1}^{-1}(G) &\leftrightarrow \langle p_0, \mathbf{1}_1 \rangle \in G \wedge \langle \mathbf{1}_0, p_1 \rangle \in G \\ &\leftrightarrow \langle p_0, p_1 \rangle \in G\,. \end{aligned}$$ $\square$

Lemma 4.3. *Let M be transitive and such that ZFC holds in M. Assume that $\mathbf{P}_0$ and $\mathbf{P}_1$ belong to M. Let $G_0 \subseteq \mathbf{P}_0$ and $G_1 \subseteq \mathbf{P}_1$. Then the following are equivalent:*

(1) *$G_0 \times G_1$ is $\mathbf{P}_0 \times \mathbf{P}_1$-generic over M.*

(2) *G_0 is $\mathbf{P}_0$-generic over M, and G_1 is $\mathbf{P}_1$-generic over $M[G_0]$.*

(3) *G_1 is $\mathbf{P}_1$-generic over M, and G_0 is $\mathbf{P}_0$-generic over $M[G_1]$.*

If (1)–(3) hold, then

$$M[G_0 \times G_1] = M[G_0][G_1] = M[G_1][G_0]\,.$$

Proof. We first show that (1) implies (2). Assume that (1) holds. That G_0 and G_1 are filters and that G_0 is $\mathbf{P}_0$-generic over M follow from Corollary 4.2. Thus we need only show that G_1 is $\mathbf{P}_1$-generic over $M[G_0]$.

Let $D_1 \in M[G_0]$ be dense in $\mathbf{P}_1$. Let $\tau \in M^{\mathbf{P}_0}$ be such that $\tau_{G_0} = D_1$. We may assume that domain $(\tau) \subseteq$ domain $(\check{\mathbf{P}}_1)$. Let $p_0 \in G_0$ be such that $p_0 \Vdash$ "τ is dense in $\check{\mathbf{P}}_1$." Let

$$D = \{\langle q_0, q_1\rangle \mid q_0 \leq_0 p_0 \wedge q_0 \Vdash \check{q}_1 \in \tau\}\,.$$

We argue as follows that D is dense below $\langle p_0, \mathbf{1}_1\rangle$ in $\mathbf{P}_0 \times \mathbf{P}_1$: Let $\langle r_0, r_1\rangle \leq \langle p_0, \mathbf{1}_1\rangle$. Since $r_0 \leq_0 p_0$,

$$r_0 \Vdash (\exists x \in \check{\mathbf{P}}_1)(x \in \tau \wedge x(\leq_1)\check{\ } \check{r}_1)\,.$$

By the definition of forcing, there are $q_0 \leq_0 r_0$ and $q_1 \in \mathbf{P}_1$ such that

$$q_0 \Vdash \check{q}_1 \in \tau \wedge \check{q}_1(\leq_1)\check{\ } \check{r}_1\,.$$

Hence $q_1 \leq_1 r_1$ and $\langle q_0, q_1\rangle \in D$. Moreover $\langle q_0, q_1\rangle \leq \langle r_0, r_1\rangle$.

Since $\langle p_0, \mathbf{1}_1\rangle \in G_0 \times G_1$, there is a $\langle q_0, q_1\rangle \in (G_0 \times G_1) \cap D$. Since $q_0 \in G_0$ and $q_0 \Vdash \check{q}_1 \in \tau$, we have that $q_1 \in D_1$ and so that $q_1 \in G_1 \cap D_1$.

The proof that (1) implies (3) is like the proof that (1) implies (2).

The proof that (2) implies (1) is left as Exercise 4.1. The proof that (3) implies (1) is similar.

The last assertion of the lemma is easily verified. □

Exercise 4.1. Do the (2) ⇒ (1) case of the proof of Lemma 4.3.

Two-stage iterations.

Let $\mathbf{P}$ be a poset. A $\mathbf{P}$-*name for a poset* is a triple

$$\langle \pi, \pi', \pi''\rangle\,,$$

where π, π', and π'' are all $\mathbf{P}$-names such that $\mathbf{1_P} \Vdash$ "π' is a partial ordering of π with π'' a greatest element." We often write π for the triple, and we write $\leq_\pi$ for π' and $\mathbf{1}_\pi$ for π''.

If $\mathbf{P}$ is a poset and π is a $\mathbf{P}$-name for a poset, then let

$$\mathbf{P} * \pi = \{\langle p, \tau \rangle \mid p \in \mathbf{P} \wedge \tau \in \text{domain}(\pi) \wedge p \Vdash \tau \in \pi\}.$$

Partially order $\mathbf{P} * \pi$ by

$$\langle p, \tau \rangle \leq \langle q, \sigma \rangle \leftrightarrow (p \leq_{\mathbf{P}} q \wedge p \Vdash \tau \leq_\pi \sigma).$$

Let $\mathbf{1}_{\mathbf{P}*\pi} = \langle \mathbf{1_P}, \mathbf{1}_\pi \rangle$. Define the *canonical embedding* $i : \mathbf{P} \to \mathbf{P} * \pi$ by

$$i(p) = \langle p, \mathbf{1}_\pi \rangle.$$

Lemma 4.4. *Let $\mathbf{P}$ be a poset and let π be a $\mathbf{P}$-name for a poset. Then the canonical embedding i is a complete embedding.*

Proof. We have the following facts:

(a) $p \leq p' \leftrightarrow \langle p, \mathbf{1}_\pi \rangle \leq \langle p', \mathbf{1}_\pi \rangle$;

(b) $i(\mathbf{1_P}) = \mathbf{1}_{\mathbf{P}*\pi}$;

(c) $p \perp p' \to \langle p, \tau \rangle \perp \langle p', \tau' \rangle$;

(d) $p \perp p' \leftrightarrow \langle p, \tau \rangle \perp \langle p', \mathbf{1}_\pi \rangle$;

(e) $p \perp p' \leftrightarrow i(p) \perp i(p')$.

(a), (b), and (c) follow readily from the definitions. The $\to$ part of (d) follows from (c). For the $\leftarrow$ part of (d), note that

$$(q \leq p \wedge q \leq p') \to (\langle q, \tau \rangle \leq \langle p, \tau \rangle \wedge \langle q, \tau \rangle \leq \langle p', \mathbf{1}_\pi \rangle).$$

(e) follows from (d). The lemma follows from (a), (e), and (d), since (d) implies that p is a reduction to $\mathbf{P}$ of $\langle p, \tau \rangle$. □

Let M be transitive and such that ZFC holds in M. Suppose that in $\mathbf{P} \in M$ is a poset and that, in M, π is a $\mathbf{P}$-name for a poset. Let G be $\mathbf{P}$-generic over M and let $H \subseteq \pi_G$. Define

$$G * H = \{\langle p, \tau \rangle \in \mathbf{P} * \pi \mid p \in G \wedge \tau_G \in H\}.$$

Exercise 4.2. Let M be transitive and such that ZFC holds in M. Let $\mathbf{P} \in M$ be a poset. In M let π be a $\mathbf{P}$-name for a poset. Let K be $\mathbf{P} * \pi$-generic over M. Let i be the canonical embedding and let $G = i^{-1}(K)$. Let

$$H = \{\tau_G \mid \tau \in \operatorname{domain}(\pi) \wedge (\exists q)\, \langle q, \tau\rangle \in K\}\,.$$

Let $\tau \in \operatorname{domain}(\pi)$ be such that $\tau_G \in H$. Prove that there is a $q \in \mathbf{P}$ such that $\langle q, \tau\rangle \in K$.

Lemma 4.5. *Let M, $\mathbf{P}$, π, K, i, G, and H be as in Exercise 4.2. Then G is $\mathbf{P}$-generic over M, H is π_G-generic over $M[G]$, $K = G * H$, and $M[K] = M[G][H]$.*

Proof. We shall write $\leq$ for $\leq_{\mathbf{P}}$, for $\leq_{\mathbf{P}*\pi}$, and for the name $\leq_\pi$ (i.e., π').

Since i is a complete embedding, G is $\mathbf{P}$-generic over M.

To see that $H \subseteq \pi_G$, assume that $a \in H$. There are q and τ such that $a = \tau_G$, $\tau \in \operatorname{domain}(\pi)$, and $\langle q, \tau\rangle \in K$. Thus $q \in G$ and $q \Vdash \tau \in \pi$. It follows that $\tau_G \in \pi_G$.

Next let us show that H is a filter.

Let $a \leq_{\pi_G} b$ with $a \in H$. There are q, τ, and σ such that $a = \tau_G$, $b = \sigma_G$, and $\langle q, \tau\rangle \in K$. There is a $p \in G$ such that $p \Vdash \tau \leq \sigma$. Since $\langle p, \mathbf{1}_\pi\rangle \in K$, there is an $\langle r, \mu\rangle \in K$ such that $\langle r, \mu\rangle$ is a common extension of $\langle p, \mathbf{1}_\pi\rangle$ and $\langle q, \tau\rangle$. Since $r \leq p$, $r \Vdash \tau \leq \sigma$. By the definition of $\leq$, $r \Vdash \mu \leq \tau$. Hence $r \Vdash \mu \leq \sigma$. But then $\langle r, \mu\rangle \leq \langle r, \sigma\rangle$, and so $\langle r, \sigma\rangle \in K$. This implies that $\sigma_G \in H$.

For the other filter property, let a and b belong to H. Let p, q, σ, and τ be such that $a = \sigma_G$, $b = \tau_G$, $\langle p, \sigma\rangle \in K$, and $\langle q, \tau\rangle \in K$. Let $\langle r, \mu\rangle \in K$ be a common extension of $\langle p, \sigma\rangle$ and $\langle q, \tau\rangle$. Then $\mu_G \in H$ and μ_G is a common extension of σ_G and τ_G.

The following proof that H meets every dense subset of π_G that belongs to $M[G]$ is similar to the analogous part of the argument that (1) $\Rightarrow$ (2) in the proof of Lemma 4.3.

Let $D_1 \in M[G]$ be dense in π_G. Let $\tau_G = D_1$. We may assume that $\operatorname{domain}(\tau) \subseteq \operatorname{domain}(\pi)$. Let $p \in G$ be such that $p \Vdash$ "τ is dense in π." Let

$$D = \{\langle q, \sigma\rangle \in \mathbf{P} * \pi \mid q \leq p \wedge q \Vdash \sigma \in \tau\}\,.$$

To see that D is dense below $\langle p, \mathbf{1}_\pi\rangle$ in $\mathbf{P} * \pi$, let $\langle r, \mu\rangle \leq \langle p, \mathbf{1}_\pi\rangle$. Since $r \leq p$,

$$r \Vdash (\exists x \in \pi)(x \in \tau \wedge x \leq \mu)\,.$$

By the definition of forcing, there are $q \leq r$ and $\sigma \in \operatorname{domain}(\pi)$ such that

$$q \Vdash (\sigma \in \tau \wedge \sigma \leq \mu) .$$

Hence $\langle q, \sigma\rangle \leq \langle r, \mu\rangle$ and $\langle q, \sigma\rangle \in D$.

Since $\langle p, \mathbf{1}_\pi\rangle \in K$, there is a $\langle q, \sigma\rangle \in K \cap D$. Since $q \in G$ and $q \Vdash \sigma \in \tau$, we have that $\sigma_G \in D_1$ and so that $\sigma_G \in H \cap D_1$.

To see that $K = G * H$, first observe that

$$\begin{aligned} \langle p, \tau\rangle \in K \quad &\rightarrow \quad \langle p, \tau\rangle \in \mathbf{P} * \pi \wedge p \in G \wedge \tau_G \in H \\ &\rightarrow \quad \langle p, \tau\rangle \in G * H . \end{aligned}$$

Next assume that $\langle p, \tau\rangle \in G * H$. Then

$$\tau \in \operatorname{domain}(\pi) \wedge \langle p, \tau\rangle \in \mathbf{P} * \pi \wedge p \in G \wedge \tau_G \in H .$$

By Exercise 4.2, let q be such that $\langle q, \tau\rangle \in K$. Let $\langle r, \sigma\rangle \in K$ be a common extension of $\langle p, \mathbf{1}_\pi\rangle$ and $\langle q, \tau\rangle$. Then $r \Vdash \sigma \leq \tau$, so $\langle r, \sigma\rangle \leq \langle p, \tau\rangle$. Hence $\langle p, \tau\rangle \in K$.

Since G and H belong to $M[K]$, $M[G][H] \subseteq M[K]$. Since $K = G * H \in M[G][H]$, $M[K] \subseteq M[G][H]$. $\square$

Lemma 4.6. *Let M be transitive and such that ZFC holds in M. In M, let $\mathbf{B}$ and $\mathbf{C}$ be complete Boolean algebras such that $\mathbf{C}$ is a complete subalgebra of $\mathbf{B}$. For $b \in \mathbf{B} \setminus \{\mathbf{0}\}$, let $h(b) \in \mathbf{C} \setminus \{\mathbf{0}\}$ be given by*

$$h(b) = \bigwedge\{c \in \mathbf{C} \mid b \leq c\} .$$

Let K be $(\mathbf{B} \setminus \{\mathbf{0}\})$-generic over M. Let $G = K \cap \mathbf{C}$. In $M[G]$ let

$$\mathbf{Q} = \{b \in \mathbf{B} \setminus \{\mathbf{0}\} \mid h(b) \in G\} .$$

Then G is $(\mathbf{C} \setminus \{\mathbf{0}\})$-generic over M, K is $\mathbf{Q}$-generic over $M[G]$, and $M[G][K] = M[K]$.

Proof. Lemmas 3.2 and 3.9 imply that G is $(\mathbf{C} \setminus \{\mathbf{0}\})$-generic over M.

Note that $K \subseteq \mathbf{Q}$, since $b \in K \Rightarrow h(b) \in K \Rightarrow h(b) \in G$. Since K is a filter on $\mathbf{B} \setminus \{\mathbf{0}\}$, it follows that K is a filter on $\mathbf{Q}$.

Let $\pi \in M^{\mathbf{C} \setminus \{\mathbf{0}\}}$ be given by

$$\pi = \{\langle \check{b}, h(b)\rangle \mid b \in \mathbf{B} \setminus \{\mathbf{0}\}\} .$$

For $b \in \mathbf{B} \setminus \{\mathbf{0}\}$, $[\![\check{b} \in \pi]\!] = h(b)$, and thus $\pi_G = \mathbf{Q}$.

Let σ_G be dense in π_G. Changing σ if necessary, we may assume (a) that domain $(\sigma) \subseteq$ domain $(\pi) = \{\check{b} \mid b \in \mathbf{B} \setminus \{\mathbf{0}\}\}$ and (b) that $\mathbf{1}_{\mathbf{C}\setminus\{\mathbf{0}\}} \Vdash$ "$\sigma \subseteq \pi$ and σ is dense in π." (Clearly we may assume (a). Suppose (a) holds and (b) fails. For some $c \in \mathbf{C} \setminus \{\mathbf{0}\}$, (b) holds with c replacing $\mathbf{1}_{\mathbf{C}\setminus\{\mathbf{0}\}}$. Replace σ by $\{\langle \check{b}, c \wedge c_1\rangle \mid \langle b, c_1\rangle \in \sigma \,\wedge\, c \wedge c_1 \neq \mathbf{0}\} \cup \{\langle \check{b}, c' \cap h(b)\rangle \mid b \in \mathbf{B} \setminus \{\mathbf{0}\}\}$.)

We show that $K \cap \sigma_G \neq \emptyset$. Let

$$D = \{b \in \mathbf{B} \setminus \{\mathbf{0}\} \mid (\exists b^* \in \mathbf{B} \setminus \{\mathbf{0}\})(b \leq b^* \,\wedge\, h(b) \Vdash \check{b}^* \in \sigma)\}\,.$$

To prove that D is dense in $\mathbf{B} \setminus \{\mathbf{0}\}$, let $b_0 \in \mathbf{B} \setminus \{\mathbf{0}\}$. Since $h(b_0) \Vdash \check{b}_0 \in \pi$, we get from (b) that

$$h(b_0) \Vdash (\exists x \in \pi)(x \leq \check{b}_0 \,\wedge\, x \in \sigma)\,.$$

The definition of forcing and (a) give us a $c \in \mathbf{C} \setminus \{\mathbf{0}\}$ and a $b^* \in \mathbf{B} \setminus \{\mathbf{0}\}$ such that

$$c \leq h(b_0) \,\wedge\, c \Vdash (\check{b}^* \leq \check{b}_0 \,\wedge\, \check{b}^* \in \sigma)\,.$$

Hence $b^* \leq b_0$ and $c \Vdash \check{b}^* \in \sigma$. The latter fact and (b) imply that

$$c \leq [\![\check{b}^* \in \pi]\!] = h(b^*)\,.$$

If $c \perp b^*$, then $b^* \leq c'$, and so $h(b^*) \leq c'$. Hence c and b^* are compatible. Let $b \in \mathbf{B} \setminus \{\mathbf{0}\}$ be a common extension of b^* and c. Since $h(b) \leq c$, $h(b) \Vdash \check{b}^* \in \sigma$. Because $b \leq b_0$, our proof that D is dense is complete.

By the density of D, let $b \in D \cap K$. Thus $h(b) \in G$. By the definition of D, there is a $b^* \geq b$ such that $h(b) \Vdash \check{b}^* \in \sigma$. For such a b^*, $b^* \in K$ and so $b^* \in K \cap \sigma_G$.

The last assertion of the lemma obviously holds. $\square$

Corollary 4.7. *Let M, $\mathbf{B}$, $\mathbf{C}$, K, and G be as in Lemma 4.6. Let $\mathbf{P} \subseteq \mathbf{B} \setminus \{0\}$ with $\mathrm{id} : \mathbf{P} \to \mathbf{B} \setminus \{0\}$ a complete embedding. There is a subset $\mathbf{Q}$ of P such that $\mathbf{Q} \in M[G]$, $K \cap \mathbf{Q}$ is $\mathbf{Q}$-generic over $M[G]$, and $M[G][K \cap \mathbf{Q}] = M[K]$.*

Lemma 4.8. *Let M be a transitive model of ZFC and let $\mathbf{P} \in M$ be a poset. Let $\varphi(x, y_1, \ldots, y_n)$ be a formula of the language of set theory and let $\sigma_1, \ldots, \sigma_n$ be elements of $M^{\mathbf{P}}$. Let $p \in \mathbf{P}$ be such that*

$$p \Vdash (\exists x)\, \varphi(x, \sigma_1, \ldots, \sigma_n)\,.$$

Then there is a $\tau \in M^{\mathbf{P}}$ such that

$$p \Vdash \varphi(\tau, \sigma_1, \ldots, \sigma_n)\,.$$

Proof. Let $A \subseteq \mathbf{P}$ be a maximal antichain below p such that, for each $q \in A$,

$$(\exists \mu)(q \Vdash \varphi(\mu, \sigma_1, \ldots, \sigma_n)\,.$$

For each $q \in A$, let μ_q be such that $q \Vdash \varphi(\mu_q, \sigma_1, \ldots, \sigma_n)$. Now let

$$\tau = \{\langle \rho, r\rangle \mid (\exists q \in A)(\exists s \in \mathbf{P})(r \leq q \,\wedge\, r \leq s \,\wedge\, \langle \rho, s\rangle \in \mu_q)\}\,.$$

For $q \in A,\ q \Vdash \tau = \mu_q$ and so $q \Vdash \varphi(\tau, \sigma_1, \ldots, \sigma_n)$. $\square$

Theorem 4.9. *Let M be transitive and such that ZFC holds in M. Let λ be an uncountable cardinal number of M. Let $\mathbf{P}_\lambda$ be the dense subposet of $\mathbf{Fn}(\omega, \lambda)$ consisting of those $p \in \mathbf{Fn}(\omega, \lambda)$ whose domain belongs to ω. Let $\mathbf{Q} \in M$ be a separative poset such that $(|\mathbf{Q}| = \lambda)^M$ and such that, in M, $\mathbf{Q}$ collapses λ to ω (i.e., such that, in M, $\mathbf{1_Q} \Vdash |\check{\lambda}| = \check{\omega}$). Then there is a dense embedding $i : \mathbf{P}_\lambda \to \mathbf{Q}$ such that $i \in M$.*

Proof. Work in M. For $q \in \mathbf{Q}$, let $\mathbf{Q}_q = \{r \mid r \leq q\}$. Note that $\langle \mathbf{Q}_q, \leq, q\rangle$ is a poset. We first show:

(†) $(\forall q \in \mathbf{Q})(\exists A)(A$ is an antichain in $\mathbf{Q}_q \,\wedge\, |A| = \lambda)$.

Proof of (†)**.** Assume that (†) fails for q. This means that $\mathbf{Q}_q$ has the λ-cc. If λ is regular, then Lemma 2.17 implies that $\mathbf{Q}_q$ preserves cardinals $\geq \lambda$, and this contradicts the hypothesis that $\mathbf{Q}$ collapses λ to ω. Assume then that λ is singular. If $\mathbf{Q}_q$ has the λ'-cc for some cardinal $\lambda' < \lambda$, then Lemma 2.17 again yields a contradiction. Thus $\mathbf{Q}_q$ has antichains of every size $< \lambda$. Let A be an antichain in $\mathbf{Q}_q$ of size $\mathrm{cf}(\lambda)$. Let $f : A \to \lambda$ with $\mathrm{range}(f)$ unbounded in λ. For each $a \in A$, let B_a be an antichain in $\mathbf{Q}_a$ with $|B_a| = f(a)$. Then $\bigcup\{B_a \mid a \in A\}$ is antichain in $\mathbf{Q}_q$ of size λ. $\square$

The hypotheses of the theorem imply that

$$\mathbf{1_Q} \ \Vdash \ (\exists f)\, f : \check{\omega} \xrightarrow{\text{onto}} \Gamma\,.$$

(Recall that Γ is the canonical name for the generic of object.) By Lemma 4.8, let $\tau \in M^{\mathbf{P}}$ be such that

$$\mathbf{1_Q} \ \Vdash \ \tau : \check{\omega} \xrightarrow{\text{onto}} \Gamma\,.$$

We define $i(p)$ by induction on $\ell\mathrm{h}(p)$. Let $i(\emptyset) = \mathbf{1_Q}$. Suppose that $i(p)$ is defined and that $\ell\mathrm{h}(p) = n$. By (†) let A be a maximal antichain in $\mathbf{Q}_{i(p)}$ such that $|A| = \lambda$. For each $a \in A$, let B_a be a maximal antichain in $\mathbf{Q}_a$

such that every $r \in B_a$ *decides* $\tau(\check{n})$, i.e., such that for all $r \in B_a$ there is an $s \in \mathbf{Q}$ such that $r \Vdash \tau(\check{n}) = \check{s}$. Such an antichain exists because the set of all $r \leq a$ that decide $\tau(\check{n})$ is dense below a. Let $B = \bigcup\{B_q \mid q \in A\}$. Then B is a maximal antichain in $\mathbf{Q}_{i(p)}$, $|B| = \lambda$, and each member of B decides $\tau(\check{n})$. Let $\alpha \mapsto q_\alpha$ be a bijection between λ and B. Define

$$i(p \cup \{\langle n, \alpha \rangle\}) = q_\alpha$$

for each $\alpha < \lambda$.

Note that induction on n yields that $\{i(p) \mid p \in \mathbf{P} \wedge \ell\mathrm{h}(p) = n\}$ is a maximal antichain in $\mathbf{Q}$ for each $n \in \omega$. Note also that $(\forall p)(\forall p')(p \leq p' \leftrightarrow i(p) \leq i(p')$.

To check that i is a dense embedding it is thus enough to check that the range of i is dense in $\mathbf{Q}$. Let $q \in \mathbf{Q}$. Clearly $q \Vdash \check{q} \in \Gamma$. Hence $q \Vdash (\exists y \in \check{\omega})\, \tau(y) = \check{q}$. Thus there are $n \in \omega$ and $r \leq q$ such that $r \Vdash \tau(\check{n}) = \check{q}$. Since $\{i(p) \mid p \in \mathbf{P} \wedge \ell\mathrm{h}(p) = n+1\}$ is a maximal antichain in $\mathbf{Q}$, there is a $p \in \mathbf{P}$ such that $\ell\mathrm{h}(p) = n+1$ and $i(p)$ is compatible with r. Since $i(p)$ decides $\tau(\check{n})$, it must be that $i(p) \Vdash \tau(\check{n}) = \check{q}$. Thus $i(p) \Vdash \check{q} \in \Gamma$. Since $\mathbf{Q}$ is separative, it follows that $i(p) \leq q$. □

Corollary 4.10. *Let M be transitive and such that ZFC holds in M. Let α be an ordinal of M that is uncountable in M. Let G^* be $\mathbf{Fn}(\omega,\alpha)$-generic over M. Let $\mathbf{R} \in M$ be a poset and suppose that $G \in M[G^*]$ is $\mathbf{R}$-generic over M. Assume that α is uncountable in $M[G]$. Then there is an $H \in M[G^*]$ such that H is $\mathbf{Fn}(\omega,\alpha)$-generic over $M[G]$ and such that $M[G][H] = M[G^*]$.*

Proof. Let $\mathbf{B}$ be the completion in M of $\mathbf{Fn}(\omega,\alpha)$. We may assume that $\mathbf{Fn}(\omega,\alpha) \subseteq \mathbf{B} \setminus \{\mathbf{0}\}$ and that the identity is a dense embedding. The filter K on $\mathbf{B} \setminus \{\mathbf{0}\}$ that is generated by G^* is $(\mathbf{B} \setminus \{0\})$-generic over M and $M[K] = M[G^*]$.

Apply Lemma 3.10 with K, $\mathbf{R}$, and G as the G, X, and Y respectively of Lemma 3.10. This gives us a $\mathbf{C}$ that is in M a complete subalgebra of $\mathbf{B}$ and is such that $M[K \cap \mathbf{C}] = M[G]$.

Now apply Corollary 4.7 with with $\mathbf{P} = \mathbf{Fn}(\omega,\alpha)$. We get a poset $\mathbf{Q}$ such that

(i) $\mathbf{Q} \in M[G]$;

(ii) $\mathbf{Q} \subseteq \mathbf{Fn}(\omega,\alpha)$;

(iii) $K \cap \mathbf{Q}$ is $\mathbf{Q}$-generic over $M[G]$;

(iv) $M[G][K \cap \mathbf{Q}] = M[K]$.

Since α is countable in $M[G^*] = M[K]$, we may assume, replacing $\mathbf{Q}$ by a subordering if necessary, that $\mathbf{1_Q} \Vdash |\check{\alpha}| = \check{\omega}$. This, together with the fact that α is uncountable in $M[G] = M[K \cap \mathbf{C}]$, implies that $|\mathbf{Q}|^{M[G]} = |\alpha|^{M[G]}$. Clause (ii) implies that $\mathbf{Q}$ is separative. We may thus apply Theorem 4.9 to conclude that $\mathbf{Q}$ has the same completion in $M[G]$ as $\mathbf{Fn}(\omega, |\alpha|^{M[G]})$. The latter is isomorphic in $M[G]$ to $\mathbf{Fn}(\omega, \alpha)$, so the results of §3 give us our H. $\square$

For any ordinal α, let $\mathbf{Lv}(\alpha)$ be the set of all finite functions f such that

(i) $\mathrm{domain}\,(f) \subseteq \alpha \times \omega$;

(ii) $(\forall \langle \beta, n\rangle \in \mathrm{domain}\,(f))\, f(\beta, n) < \beta$.

Exercise 4.3. Let M be transitive and such that ZFC holds in M. Let κ be an uncountable regular cardinal of M. Let G be $\mathbf{Lv}(\kappa)$-generic over M.

(a) Prove that κ is the ω_1 of $M[G]$.

(b) Let $X \subseteq \alpha < \kappa$ with $x \in M[G]$. Show that there is an ordinal $\kappa' < \kappa$ such that $x \in M[G \cap \mathbf{Lv}(\kappa')]$.

Hint. Show that $\mathbf{Lv}(\kappa)$ has the κ-cc.

Lemma 4.11. *Let M be transitive and such that ZFC holds in M. Let κ be inaccessible in M. Let $\tilde{G}$ be $\mathbf{Lv}(\kappa)$-generic over M. Let $\mathbf{R} \in M$ be a poset such that $|R|^M < \kappa$. Suppose that $G \in M[\tilde{G}]$ is $\mathbf{R}$-generic over M. Then there is an $\tilde{H} \in M[\tilde{G}]$ such that $\tilde{H}$ is $\mathbf{Lv}(\kappa)$-generic over $M[G]$ and such that $M[G][\tilde{H}] = M[\tilde{G}]$.*

Proof. Observe that, for $\alpha < \kappa$, $\mathbf{Lv}(\kappa)$ is isomorphic to the product of $\mathbf{Lv}(\alpha)$ and $\mathbf{Lv}_\alpha(\kappa)$, where $\mathbf{Lv}_\alpha(\kappa) = \{p \in \mathbf{Lv}(\kappa) \mid \mathrm{domain}\,(p) \cap (\alpha \times \omega) = \emptyset\}$. By part (b) of Exercise 4.3, there is a $\beta < \kappa$ such that $G \in M[\tilde{G} \cap \mathbf{Lv}(\beta)]$. We shall show that there is an α and there is an $H^* \in M[\tilde{G}]$ such that $\beta \leq \alpha < \kappa$, such that H^* is $\mathbf{Lv}(\alpha)$-generic over $M[G]$, and such that $M[G][H^*] = M[\tilde{G} \cap \mathbf{Lv}(\alpha)]$. This will suffice, as we see as follows. By Lemma 4.3, $\tilde{G} \cap \mathbf{Lv}_\alpha(\kappa)$ is $\mathbf{Lv}_\alpha(\kappa)$-generic over $M[\tilde{G} \cap \mathbf{Lv}(\alpha)]$, and so over $M[G][H^*]$. By Lemma 4.3 again, $H^* \times (\tilde{G} \cap \mathbf{Lv}_\alpha(\kappa))$ is $(\mathbf{Lv}(\alpha) \times \mathbf{Lv}_\alpha(\kappa))$-generic over $M[G]$.

Let α be such that $\beta \leq \alpha < \kappa$ and such that α is uncountable in $M[G]$ but α is not a cardinal in M. By Theorem 4.9, $\mathbf{Lv}(\alpha)$ and $\mathbf{Fn}(\omega, \alpha)$ have the same completion in M. Hence there is a $G^* \in M[\tilde{G} \cap \mathbf{Lv}(\alpha)]$ such that

G^* is $\mathbf{Fn}(\omega,\alpha)$-generic over M and such that $M[G^*] = M[\tilde{G} \cap \mathbf{Lv}(\alpha)]$. By Corollary 4.10, there is an $H \in M[\tilde{G} \cap \mathbf{Lv}(\alpha)]$ such that H is $\mathbf{Fn}(\omega,\alpha)$-generic over $M[G]$ and such that $M[G][H] = M[G^*]$. Since $\mathbf{Lv}(\alpha)$ and $\mathbf{Fn}(\omega,\alpha)$ have the same completion in $M[G]$, we get our H^*. □

Lemma 4.12. *Let M be transitive and such that ZFC holds in M. Let κ be any infinite ordinal of M. Let φ be a sentence of $\mathcal{L}(\mathbf{Lv}(\kappa), M)$ all of whose constants are of the form $\check{x}$ for elements x of M. Then*

$$\mathbf{1} \Vdash \varphi \ \vee \ \mathbf{1} \Vdash \neg\varphi\,.$$

Proof. We may assume that M is countable. Suppose that $p \Vdash \varphi$. Let G be $\mathbf{Lv}(\kappa)$-generic over M. The proof will be complete if we show that $\varphi^{M[G]}$. Let $q \in G$ with domain $(q) =$ domain (p). Let $F : \kappa \setminus \{0\} \times \omega \to \kappa$ be given by

$$F(\alpha, n) = \begin{cases} p(\alpha, n) & \text{if } \langle \alpha, n\rangle \in \text{domain}\,(p)\,; \\ (\bigcup G)(\alpha, n) & \text{otherwise.} \end{cases}$$

Let $G' = \{r \in \mathbf{Lv}(\kappa) \mid r \subseteq F\}$. It is clear that G' is a filter on $\mathbf{Lv}(\kappa)$, that $p \in G'$, and that $M[G'] = M[G]$. If we show that G' is $\mathbf{Lv}(\kappa)$-generic over M then we will know that $\varphi^{M[G']}$ and so that $\varphi^{M[G]}$.

Let D' be dense in $\mathbf{Lv}(\kappa)$. Let

$$D = \{(r \setminus p) \cup q \mid r \in D' \ \wedge\ p \subseteq r\}\,.$$

Then D is dense below q and so $G \cap D \neq \emptyset$. But then $G' \cap D' \neq \emptyset$. □

Let μ be Lebesgue measure on $[0,1]$ until further notice. Let us say that a set A *satisfies condition* $\mathcal{C}$ if A is a set of pairwise disjoint closed subsets of $[0,1]$ of positive Lebesgue measure and $\mu(\bigcup A) = 1$. Note that any such A must be countable.

For transitive M such that ZFC holds in M, a member x of $[0,1]$ is *random over* M if, for every $A \in M$ such that "A satisfies condition $\mathcal{C}$" holds in M, $x \in \bigcup\{\text{cl}(p) \mid p \in A\}$. (Here cl$(p)$ is the closure of p.)

Lemma 4.13. *Let M be transitive and such that ZFC holds in M. Assume that $(2^{\aleph_0})^M$ is countable. Then the set of $x \in [0,1]$ such that x is random over M has Lebesgue measure 1.*

Proof. We first prove three absoluteness results, the last of which will not be needed until the proof of Lemma 4.14. Let p and q be subsets of $[0,1]^M$ that are closed in M. Then

(a) $\mu(\mathrm{cl}(p)) = \mu^M(p)$;

(b) $p \cap q = \emptyset \rightarrow \mathrm{cl}(p) \cap \mathrm{cl}(q) = \emptyset$;

(c) $\mathrm{cl}(p \cap q) = \mathrm{cl}(p) \cap \mathrm{cl}(q)$.

If $r \subseteq [0, 1]$ is open, then r is representable in a unique way as a disjoint union of open intervals. The Lebesgue measure of r is the sum of the lengths of the associated open intervals. The intervals of M associated in M with the complement of p in M are the same as those associated with the complement of $\mathrm{cl}(p)$ in V, in the sense that they have the same endpoints. From this (a) follows. If $p \cap q = \emptyset$, then there are disjoint open sets of M, p' and q', such that $p \subseteq p'$ and $q \subseteq q'$. The open sets in V whose associated intervals are the same as (have the same enpoints as) those for p' and q' in M are disjoint and cover $\mathrm{cl}(p)$ and $\mathrm{cl}(q)$ respectively. This gives (b). Finally, suppose $x \notin \mathrm{cl}(p \cap q)$. Then x belongs to interval I with endpoints in M that is disjoint from $p \cap q$. Assuming $x \in \mathrm{cl}(p) \cap \mathrm{cl}(q)$, one can get the contradiction that some $y \in M$ belongs to $I \cap p \cap q$.

If $A \in M$ and "A satisfies condition $\mathcal{C}$" holds in M, then (a) and (b) give that $\mu(\bigcup\{\mathrm{cl}(p) \mid p \in A\}) = \sum_{p \in A} \mu(\mathrm{cl}(p)) = \sum_{p \in A} \mu^M(p) = \mu^M(\bigcup A) = 1$. Since $(2^{\aleph_0})^M$ is countable, there are only countably many such A. Hence the conclusion of the lemma follows by the countable additivity of μ. $\square$

Lemma 4.14. *Let M be transitive and such that ZFC holds in M. In M let $\mathbf{P}$ be the set of all Lebesgue measurable subsets of $[0, 1]$ of positive measure, ordered by inclusion. An element x of $[0, 1]$ is random over M if and only if there is a G that is $\mathbf{P}$-generic over M and such that $\{x\} = \bigcap\{\mathrm{cl}(p) \mid p \in G\}$.*

Proof. Assume first that G is $\mathbf{P}$-generic over M and $\{x\} = \bigcap\{\mathrm{cl}(p) \mid p \in G\}$. If $A \in M$ and "A satisfies condition $\mathcal{C}$" holds in M, then A is a maximal antichain in $\mathbf{P}$. Thus there is a $p \in A \cap G$. For such p, $x \in \mathrm{cl}(p)$.

Before proving the converse, we note that a filter G on $\mathbf{P}$ is $\mathbf{P}$-generic over M just in case G meets every $A \in M$ that satisfies condition $\mathcal{C}$ in M. This follows from the fact that for every $B \in M$ that is a maximal antichain in $\mathbf{P}$ there is an $A \in M$ such that "A satisfies condition $\mathcal{C}$" holds in M and every element of A is $\leq$ some element of B. This fact can in turn be proved from the facts that the set of elements of $\mathbf{P}$ that are closed in M is dense in $\mathbf{P}$ and that $\mathbf{P}$ has the ccc in M.

Assume that x is random over M. Let $G = \{p \in \mathbf{P} \mid x \in \mathrm{cl}(p)\}$. If $p \leq q$ and $p \in G$, then $q \in G$. Suppose that p and q belong to G. By (c) from the proof of Lemma 4.13, $\mathrm{cl}(p \cap q) = \mathrm{cl}(p) \cap \mathrm{cl}(q)$, and so $x \in \mathrm{cl}(p \cap q)$. If $\mu^M(p \cap q) > 0$, then $p \cap q$ belongs to G and is a common extension of p

and q. Assume that $\mu^M(p \cap q) = 0$. Then there is an $A \in M$ that satisfies condition $\mathcal{C}$ in M and whose members are disjoint from $p \cap q$. Since x is random over M, there is an $r \in A$ such that $x \in \mathrm{cl}(r)$. By (b) from the proof of Lemma 4.13, we get the contradiction that $\mathrm{cl}(r)$ and $\mathrm{cl}(p \cap q)$ are disjoint. We have now shown that G is a filter on $\mathbf{P}$. By the remark in the preceding paragraph, G is $\mathbf{P}$-generic over M. To show that $\{x\} = \bigcap\{\mathrm{cl}(p) \mid p \in G\}$, assume for definiteness that $y > x$. If a is any rational number such that $x < a < y$, then $[0, a]^M \in G$. Thus $y \notin \bigcap\{\mathrm{cl}(p) \mid p \in G\}$. □

Lemma 4.15. *Let M be transitive and such that ZFC holds in M. Let κ be inaccessible in M. Let $\tilde{G}$ be $\mathbf{Lv}(\kappa)$-generic over M. Let $\varphi(v_0, \ldots, v_n)$ be a formula and let $\alpha_1, \ldots, \alpha_n$ be ordinals of M.*

Then "$\{x \in [0,1] \mid \varphi(x, \alpha_1, \ldots, \alpha_n)\}$ is Lebesgue measurable" holds in $M[\tilde{G}]$.

Proof. Work in $M[\tilde{G}]$.

By Lemma 4.13, the set of $x \in [0,1]$ that are random over M has Lebesgue measure one. Thus we need only find a measurable set X such that, for every x random over M,

$$\varphi(x, \alpha_1, \ldots, \alpha_n) \;\leftrightarrow\; x \in X\,.$$

Le $\mathbf{P}$ be as in Lemma 4.14.

For G that is $\mathbf{P}$-generic over M, the set $\bigcap\{\mathrm{cl}(p) \mid p \in G\}$ is a singleton. The argument of the last step of the proof of Lemma 4.14 shows that this set has at most one member. That it is non-empty follows from the compactness of $[0,1]$ and the fact that any finite intersection of closures of members of G is non-empty. Let $x(G)$ be the unique member of $\bigcap\{\mathrm{cl}(p) \mid p \in G\}$ and let $\tau \in M^{\mathbf{P}}$ be such that $\tau_G = x(G)$ for all $\mathbf{P}$-generic G. (The existence of τ follows from Lemma 4.8.)

Using Lemma 4.12 and using once more the density in M of the sets closed in M and the fact that $\mathbf{P}$ has the ccc in M, we get an $A \in M$ satisfying condition $\mathcal{C}$ in M and be such that, for every $p \in A$,

$$p \Vdash (\mathbf{1}_{\mathbf{Lv}(\check{\kappa})} \Vdash \varphi(\tau, \check{\alpha}_1, \ldots, \check{\alpha}_n)) \;\vee\; p \Vdash (\mathbf{1}_{\mathbf{Lv}(\check{\kappa})} \Vdash \neg\varphi(\tau, \check{\alpha}_1, \ldots, \check{\alpha}_n))\,.$$

(Here we are being careless with notation: we should have written $\check{\tau}$ instead of τ, and we should have put two checks on the α_i.) Let

$$A_1 = \{p \in A \mid p \Vdash (\mathbf{1}_{\mathbf{Lv}(\check{\kappa})} \Vdash \varphi(\tau, \check{\alpha}_1, \ldots, \check{\alpha}_n))\}$$

and let

$$A_2 = \{p \in A \mid p \Vdash (\mathbf{1}_{\mathbf{Lv}(\check{\kappa})} \Vdash \neg\varphi(\tau, \check{\alpha}_1, \ldots, \check{\alpha}_n))\}\,.$$

Let

$$X = \bigcup\{\mathrm{cl}(p) \mid p \in A_1\}.$$

To show that X has the required property, let x be random over M. Let p_x be the unique $p \in A$ such that $x \in \mathrm{cl}(p)$. We must prove that

$$\varphi(x, \alpha_1, \ldots, \alpha_n) \;\leftrightarrow\; p \in A_1 \,.$$

By Lemma 4.14, let G be $\mathbf{P}$-generic over M and such that $x(G) = x$. Then $p \in G$.

Let us quit working in $M[\tilde{G}]$, in order to talk about it. Assume that $p \in A_1$. (The case that $p \in A_2$ is similar.) Then in $M[G]$ it is true that

$$\mathbf{1}_{\mathbf{Lv}(\kappa)} \Vdash \varphi(\tau, \check{\alpha}_1, \ldots, \check{\alpha}_n)\,.$$

By Lemma 4.11, there is $\tilde{H} \in M[\tilde{G}]$ such that $\tilde{H}$ is $\mathbf{Lv}(\kappa)$-generic over $M[G]$ and such that $M[G][\tilde{H}] = M[\tilde{G}]$. Thus $\varphi^{M[\tilde{G}]}(x, \alpha_1, \ldots, \alpha_n)$ holds. □

Lemma 4.16. *Let M, κ, and $\tilde{G}$ be as in the statement of Lemma 4.15. Let $\varphi(v_0, \ldots, v_{m+n})$ be a formula, let $y_1, \ldots y_m$ be elements of $[0,1] \cap M[\tilde{G}]$, and let $\alpha_1, \ldots, \alpha_n$ be ordinals of M. Then "$\{x \in [0,1] \mid \varphi(x, y_1, \ldots, y_m, \alpha_1, \ldots, \alpha_n)\}$ is Lebesgue measurable" holds in $M[\tilde{G}]$.*

Proof. By Exercise 4.3, we get that there is an $\alpha < \kappa$ such that every y_i belongs to $M[G \cap \mathbf{Lv}(\alpha)]$. By Lemma 4.11, there is an $\tilde{H} \in M[\tilde{G}]$ such that $\tilde{H}$ is $\mathbf{Lv}(\kappa)$-generic over $M[G \cap \mathbf{Lv}(\alpha)]$ and such that $M[G \cap \mathbf{Lv}(\alpha)][\tilde{H}] = M[\tilde{G}]$. Thus it suffices to prove the lemma in the special case that the y_i belong to M. But then the proof of Lemma 4.15 works, since that proof did not need that the α_i were ordinals but only that they belonged to M. □

A set X is called *ordinal definable* if there is an ordinal α such that X is definable in V_α from ordinal parameters. (The parameters aren't really needed.) If X is a set definable in V from ordinal parameters, then Reflection implies that X is ordinal definable. The converse is obvious.

The *hereditarily ordinal definable* sets are those sets x such that x and all members of its transitive closure are ordinal definable. HOD is the class of all hereditarily ordinal definable sets.

Lemma 4.17. HOD *is transitive and ZFC holds in* HOD.

We omit the proof, which is not difficult. (A proof is in Kunen's book.)

The class of sets *ordinal definable from reals* and the class HOD($\mathbb{R}$) are defined as were the class of ordinal definable sets and the class HOD, except that real as well as ordinal parameters are allowed.

Lemma 4.18. HOD($\mathbb{R}$) *is transitive and ZF holds in* HOD($\mathbb{R}$).

Theorem 4.19. *Let M be transitive and such that ZFC holds in M. Let κ be inaccessible in M. Let $\tilde{G}$ be $\mathbf{Lv}(\kappa)$-generic over M.*

(1) "All subsets of $[0, 1]$ *ordinal definable from reals are Lebesgue measurable" holds in $M[\tilde{G}]$.*

(2) "ZF + all subsets of $[0, 1]$ *are Lebesgue measurable" holds in the inner model* $(\mathrm{HOD}(\mathbb{R}))^{M[\tilde{G}]}$.

Proof. (1) is a restatement of Lemma 4.16. (2) follows from (1), Lemma 4.18, and the fact that measurability is absolute for HOD($\mathbb{R}$). $\square$

Theorem 4.20. *(1) If "ZFC + there is an inaccessible cardinal" is consistent, then so is "ZFC + all subsets of* $[0, 1]$ *ordinal definable from reals are Lebesgue measurable."*

(2) If "ZFC + there is an inaccessible cardinal" is consistent, then so is "ZF + all subsets of $[0, 1]$ *are are Lebesgue measurable."*