Homework Problem 18

H11. Let $E = \langle E_a \mid a \in [\lambda]^{<\omega} \rangle$ be such that each $E_a$ is a $\kappa$-complete ultrafilter on $[\kappa]^{|a|}$ and such that, for all $a$, $b$, and $X \subseteq [\kappa]^{|a|}$,

$$a \subseteq b \rightarrow (X \in E_a \leftrightarrow \{ s \mid \operatorname{proj}_{b,a}(s) \in X \} \in E_b).$$

Define the directed system $\langle \langle M_a \mid a \in [\lambda]^{<\omega}\rangle, \langle j_{a,b} \mid a \subseteq b \in [\lambda]^{<\omega}\rangle \rangle$ in the usual way. Prove that the direct limit model of this system is well-founded only if the Well-Foundedness property of extenders hold for $E$.

*Hint.* Let $\langle a_i \mid i \in \omega \rangle$ and $\langle X_i \mid i \in \omega \rangle$ be a counterexample to Well-Foundedness. First show that you may assume that $i \leq j$ implies both

1. $a_i \subseteq a_j$;
2. $(\forall s \in X_j) \operatorname{proj}_{a_j,a_i}(s) \in X_i$.

Now let

$$U = \{ u \mid (\exists i)(u : a_i \rightarrow \kappa \land \operatorname{range}(u) \in X_i) \}.$$ 

Note that the relation $\supseteq$ is well-founded on $U$. Let $f : U \rightarrow \text{ON}$ witness this. For $n \in \omega$, let $g_n : [\kappa]^{|a_n|} \rightarrow \text{ON}$ be such that $g_n(s) = f(u)$ for the unique order preserving $u \in U$ with range $s$, if there is such a $u$. Show that the $g_n$ witness ill-foundedness of the direct limit model.