Determinacy Consequences of the Existence of $0^\#$

Let α be a countable ordinal. A set $A \subseteq {}^\omega\omega$ is α-$\mathbf{\Pi}^1_1$ if there exists $\langle A_\beta \mid \beta < \alpha\rangle$ such that each A_β is $\mathbf{\Pi}^1_1$ and

$$(\forall x)(x \in A \leftrightarrow \mu\beta[\beta = \alpha \vee x \notin A_\beta] \text{ is odd}).$$

For small enough α, say for $\alpha < \omega_1^{\mathrm{CK}}$, we can define a lightface notion, that of being α-Π^1_1, by requiring that $\{(\beta, x) \mid x \in A_\beta\}$ is Π^1_1.

The existence of $0^\#$ implies that α-Π^1_1 determinacy holds for every $\alpha < \omega^2$. Below we will give the proof for the special case α=2, and we will describe the auxiliary game used in the proof for $\alpha = \omega n$.

Theorem 1. *If $0^\#$ exists then 2-Π^1_1 determinacy holds.*

Let $\langle A_0, A_1\rangle$ witness that A is 2-Π^1_1. For $i < 2$ there there is for each $p \in {}^{<\omega}\omega$ a linear ordering $<^i_p$ of $\ell\mathrm{h}(p)$ such that:

(i) 0 is $<^i_p$-maximal if $\ell\mathrm{h}(p) > 0$;

(ii) $p \subseteq p' \rightarrow <^i_p = <^i_{p'} \restriction \ell\mathrm{h}(p)$;

(iii) $(\forall x \in {}^\omega\omega)(x \in A_i \leftrightarrow <^i_x$ is a wellordering);

(iv) The function $p \mapsto <^i_p$ is recursive.

Here $<_x = \bigcup_n <_{x\restriction n}$.

Let G be the game in ${}^{<\omega}\omega$ with A as I's winning set. Consider the game G^* played as follows.

$$\begin{array}{lccccccc}
\mathrm{I}: & \langle x(0), \alpha_0\rangle & & \langle x(2), \alpha_2\rangle & & \dots & \\
\mathrm{II}: & & \langle x(1), \alpha_1\rangle & & \langle x(3), \alpha_3\rangle & & \dots
\end{array}$$

Each α_n must be a countable ordinal. Player I is trying to make $n \mapsto \alpha_{2n}$ an embedding of $(\omega, <^0_x)$ into $(\omega_1, <)$, and II is trying to make $n \mapsto \alpha_{2n+1}$ an embedding of $(\omega, <^1_x)$ into $(\omega_1, <)$. If either player fails, the first player to fail (to have the wrong order of the ordinals played) loses. Otherwise II wins.

Lemma 1. *One of the players has a winning strategy that is definable in L from ω_1 (the ω_1 of V).*

The proof of the Lemma, which we omit, is similar to the proof of Lemma 4.4.1 in the course text.

Assume first that II has a winning strategy τ^* for G^* that is definable in L from ω_1.

We define a strategy τ for II for G. Given a position p of length $2k+1$ in G, we define a set of positions p^* in G^*, all extending p and all having length $2k+1$. Each of these positions is gotten as follows. Let $n \mapsto \alpha_{2n}$ embed $(k+1, <_p^0)$ into $(C_{\omega_1}, <)$, where C_{ω_1} is the set of all countable indiscernibles. Let the ordinals α_{2n+1} be given by τ^*. For each of our positions p^*, there is a formula φ such that $\tau^*(p^*) = f_\varphi(p, c_{\gamma_0}, \ldots, c_{\gamma_k}, c_{\omega_1})$, where the c_{γ_i} form an increasing sequence of countable indiscernibles. Since the first component of $\tau^*(p^*)$ has only countably many possible values, indiscernibility implies that it has only one possible value. We set $\tau(p)$ equal to the first component of $\tau^*(p^*)$.

Assume that $x \in A$ is a play of G consistent with τ. Then $x \in A_0$, so $<_x^0$ is a wellordering. Extend x to a play x^* of G^* consistent with τ^* by letting $n \mapsto \alpha_{2n}$ embed $(\omega, <_x^0)$ into $(C_{\omega_1}, <)$ and letting the α_{2n+1} be given by τ^*. Since τ^* is a winning strategy, x^* is a win for II. Hence $<_x^1$ is a wellordering, and so we have the contradiction that $x \notin A$.

Now assume that I has a winning strategy σ^* for G^* that is definable in L from ω_1.

Note that the ordinal α_0 is played by I before II plays any ordinals. Since 0 is maximal in every $<_x^0$, all of I's remaining ordinals have to be $< \alpha_0$ for I to win.

We define a strategy σ for I for G. Given a position p of length $2k$ in G, we define a set of positions in G^*, all extending p and all having length $2k$. Each of these positions is gotten as follows. Let $n \mapsto \alpha_{2n+1}$ embed $(k, <_p^0)$ into $(C_{\omega_1}, <)$, with all the $\alpha_{2n+1} > \alpha_0$. Let the ordinals α_{2n} be given by σ^*. Since σ^* is a winning strategy, all these ordinals are $< \alpha_0$. Since $\sigma^*(p^*)$ has only countably many possible values, indiscernibility implies that it has only one possible value. Set $\sigma(p)$ equal to the first component of $\sigma^*(p^*)$.

Assume that $x \notin A$ is a play of G consistent with σ. We will extend x to a play x^* of G^* consistent with σ^*. By the argument of the last paragraph, if II's ordinals are indiscernibles $> \alpha_0$ and are in the right order, then $\sigma^*(p^*)$ is independent of which ordinals II plays. Let then I play the α_{2n} that σ^* would call for if II had played indiscernibles $> \alpha_0$ in the right order. Since σ^* is a winning strategy, I's ordinals are in the right order. Thus $x \in A_0$. Since $x \notin A$, $x \in A_1$. Get x^* by having $n \mapsto \alpha_{2n+1}$ embed $(\omega, <_x^1)$ into $(C_{\omega_1}, <)$. This play is a win for II, contradicting the fact that σ^* is a winning strategy. $\square$

Let A be ωn-Π^1_1 with n a positive integer, and let $\langle A_\beta \mid \beta < \omega n\rangle$ witness this. Let G be the game in ${}^{<\omega}\omega$ with A as I's winning set. For $\beta < \omega n$, let $p \mapsto <^\beta_p$ associate a linear ordering of $\ell\mathrm{h}(p)$ with each position p in G in such a way that conditions (i)-(iv), with "i" replaced by "β," are satisfied.

Let $\langle \beta, i\rangle \mapsto k(\beta, i)$ be a recursive bijection between $\omega n \times \omega$ and ω such that

(a) β even $\leftrightarrow k(\beta, i)$ even;

(b) $i < i' \rightarrow k(\beta, i) < k(\beta, i')$

(c) $j < j' \rightarrow (k(\omega m + j, 0) < k(\omega m + j', i))$.

Let G^* be the game played as follows.

$$\begin{array}{rlllllll} \mathrm{I}: & \langle x(0), \alpha_0\rangle & & \langle x(2), \alpha_2\rangle & & \ldots & \\ \mathrm{II}: & & \langle x(1), \alpha_1\rangle & & \langle x(3), \alpha_3\rangle & & \ldots \end{array}$$

For $\beta_k < \omega m$, α_k must be an ordinal $< \omega(m+1)$. For each even β, I is trying to make $i \mapsto \alpha_{k(\beta,i)}$ an embedding of $(\omega, <^\beta_x)$ into $(\omega_n, <)$. For odd β, II is trying to make $i \mapsto \alpha_{k(\beta,i)}$ an embedding of $(\omega, <^\beta_x)$ into $(\omega_n, <)$. If either player fails at one of these tasks, then the first player to fail loses. Otherwise II wins.

The first stated requirement on the α_k makes sure that if β is in a lower ω-block than β' then there are more ordinals that are available choices for $\alpha_{k(\beta',i')}$ than for any $\alpha_{k(\beta,i)}$. Condition (c) guarantees that if β and β' are in the same ω-block, then the ordinal $\alpha_{k(\beta,0)}$ is chosen before any $\alpha_{k(\beta',j)}$ is chosen.

The game G^* is open. One can prove that one of the players has a winning strategy that is definable in L from $\{\omega_1, \ldots, \omega_n\}$.

Suppose, e.g., that I has a winning strategy σ^* for G^* that is definable in L from $\{\omega_1, \ldots, \omega_n\}$.

We define a strategy σ^* for I for G^* by assuming that II's ordinal moves are all indiscernibles and are in the right order. We assume also that $\alpha_{k(\beta',i')} > \alpha_{k(\beta,0)}$ for $\beta < \beta'$. By an argument like that in the proof of the theorem, the natural number moves and the ordinal moves $\alpha_{k(\beta,i)}$ given by σ^* are independent of II's ordinal moves $\alpha_{k(\beta',i')}$ for $\beta' > \beta$. This fact allows us to define a strategy σ for I for G and to prove that it is a winning strategy.