Appendix 2

The following theorem is a special case of the theorem that Problem x6.19 asks for a proof of. What is special about the theorem below is that the relation $E(u, v)$ of Problem x6.18 is assumed to be $\in\restriction z$ for some set z. Our proof of the theorem will use ordinal recursion and induction, rather than more general forms of transfinite recursion and induction.

Theorem. *Let z be a set such that $(z, \in\restriction z)$ satisfies Extensionality and such that $\in\restriction z$ is a well-founded relation. There is a unique transitive set w such that $(z, \in\restriction z) \cong (w, \in\restriction w)$. Moreover the isomorphism $\pi : (z, \in\restriction z) \cong (w, \in\restriction w)$ is unique.*

Proof. Define $G : V \to V$ by $G(u) = \{x \in z \mid x \cap z \subseteq \bigcup \text{Range}(u)\}$. By ordinal recursion, let $F : \text{ON} \to V$ be such that $F(\alpha) = G(F \restriction \alpha)$ for all $\alpha \in \text{ON}$. Thus

$$(\forall \alpha \in \text{ON})\, F(\alpha) = \{x \in z \mid x \cap z \subseteq \bigcup F[\alpha]\}.$$

For $x \in \bigcup F[\text{ON}]$, let $\text{rank}(x) =$ the least α such that $x \in F(\alpha)$. Note that if $\text{rank}(x)$ exists then $\text{rank}(y)$ exists for each $y \in x \cap z$ and

$$\text{rank}(x) = \bigcup \{\text{rank}(y) + 1 \mid y \in x \cap z\}.$$

Lemma 1. $(\exists \alpha \in \text{ON})(\forall \gamma \geq_{\text{ON}} \alpha)\, F(\gamma) = F(\alpha)$.

Proof. Assume that no such α exists. Then rank is a surjection of $\bigcup F[\text{ON}]$ onto ON. Since $\bigcup F[\text{ON}] \subseteq z$, this contradicts Replacement. □

Let ρ be the least α such that $F(\gamma) = F(\alpha)$ for all $\gamma \geq \alpha$.

Lemma 2. $F(\rho) = z$.

Proof. Otherwise let $y = \{x \in z \mid x \notin F(\rho)\}$. Let x be an $\in$-minimal element of y. Since $x \subseteq F(\rho)$, $x \in F(\rho')$, contradicting the definition of ρ. □

We define $\pi(x)$ for $x \in z$ by ordinal recursion on $\text{rank}(x)$ by setting

$$\pi(x) = \{\pi(y) \mid y \in x \cap z\}.$$

(More formally stated, here is what we do. We first define by ordinal recursion $h : \rho \to V$ such that each $h(\alpha)$ is a function with domain $\{x \in z \mid \operatorname{rank}(x) = \alpha\}$ and such that $(h(\alpha))(x) = \{(h_{\operatorname{rank}}(y))(y) \mid y \in x \cap z\}$. Then we define $\pi = \bigcup h[\rho + 1]$.)

Let $w = \operatorname{Range}(\pi)$. By ordinal induction, one can easily see that, for any transitive w' and any isomorphism $\pi' : (z, \in\restriction z) \cong (w \in\restriction w')$, π must agree with π and so must satisfy $w' = w$. Hence we need only show that π is an isomorphism and that w is transitive.

It is immediate from the definition of π that

$$x \in y \Rightarrow \pi(x) \in \pi(y)$$

for all x and y in z.. The definition also implies that w is transitive, since

$$t \in u \in w \Rightarrow (\exists x) u = \pi(x) \Rightarrow (\exists y) t = \pi(y) \Rightarrow t \in w.$$

To prove that $\pi(y) \in \pi(x) \Rightarrow y \in x$ for all x and y in z, it is enough to show that π is injective, and so this will complete the proof that $\pi : (z \in\restriction z) \cong (w, \in\restriction w)$.

By ordinal induction on the maximum of $\operatorname{rank}(x_1)$ and $\operatorname{rank}(x_2)\}$, we show that $\pi(x_1) = \pi(x_2) \Rightarrow x_1 = x_2$. We have

$$\begin{aligned} \pi(x_1) = \pi(x_2) &\Rightarrow \{\pi(y) \mid y \in x_1 \cap z\} = \{\pi(y) \mid y \in x_2 \cap z\} \\ &\Rightarrow \text{(by induction) } \{y \mid y \in x_1 \cap z\} = \{y \mid y \in x_2 \cap z\} \\ &\Rightarrow \text{(by Extensionality) } x_1 = x_2 \,. \end{aligned}$$

□