CHAPTER 6

INTRODUCTION TO FORMAL SET THEORY

We summarize here briefly the basic facts about sets which can be proved in the standard axiomatic set theories, primarily to prepare the ground for the introduction to the metamathematics of these theories in the next chapter.

6A. The intended universe of sets

It may be useful to review at this point our intuitive conception of the standard model for set theory, the universe V of sets. This *does not* contain all "arbitrary collections of objects" in Cantor's eloquent phrase: it is well known that this naive approach leads to contradictions. Instead, we admit as "sets" only those collections which occur in the *complete (transfinite) cumulative sequence of types*—the hierarchy obtained by starting with the empty set and iterating "indefinitely" the "power operation."

To be just a little more precise—and using "intuitive set theory", as we have been doing all along—suppose we are given an operation P on sets which assigns to each set x a collection $P(x)$ of subsets of x

(6A-1) $$y \in P(x) \Longrightarrow y \subseteq x.$$

Suppose we are also given a collection $\mathcal{S}$ of *stages*, wellordered by a relation $\leq_{\mathcal{S}}$, i.e., for ζ, η, ξ in $\mathcal{S}$,

(6A-2) $$\zeta \leq_{\mathcal{S}} \zeta, \quad (\zeta \leq_{\mathcal{S}} \eta \,\&\, \eta \leq_{\mathcal{S}} \xi) \Longrightarrow \zeta \leq_{\mathcal{S}} \xi,$$
$$(\zeta \leq_{\mathcal{S}} \eta \,\&\, \eta \leq_{\mathcal{S}} \zeta) \Longrightarrow \zeta = \eta, \quad \zeta \leq_{\mathcal{S}} \eta \quad \text{or} \quad \eta \leq_{\mathcal{S}} \zeta$$

(6A-3) if $A \subseteq \mathcal{S}$ is any collection of stages, $A \neq \emptyset$, then
there is some $\xi \in A$ such that for every $\eta \in A$, $\xi \leq_{\mathcal{S}} \eta$.

Call *the least stage* 0 and for $\xi \in \mathcal{S}$, let $\xi + 1$ be *the next stage*—the least stage which is greater than ξ. If λ is a stage $\neq 0$ and $\neq \xi + 1$ for every ξ, we call it a *limit stage*.

For fixed P, $\mathcal{S}$, $\leq_{\mathcal{S}}$ satisfying (6A-1) – (6A-3) we define the hierarchy

$$V_\xi = V_\xi(P, \mathcal{S}, \leq_{\mathcal{S}}) \qquad (\xi \in \mathcal{S})$$

by recursion on $\xi \in \mathcal{S}$:

$$\begin{aligned} V_0 &= \emptyset \\ V_{\xi+1} &= V_\xi \cup P(V_\xi), \\ V_\lambda &= \bigcup_{\xi<\lambda} V_\xi \quad \text{if } \lambda \text{ is a limit stage.} \end{aligned}$$

The collection of sets

$$V = V(P, \mathcal{S}, \leq_{\mathcal{S}}) = \bigcup_{\xi \in \mathcal{S}} V_\xi$$

is *the universe generated with P as the power operation, on the stages $\mathcal{S}$*. It is very easy to check that

$$\xi \leq_{\mathcal{S}} \eta \Longrightarrow V_\xi \subseteq V_\eta$$

and that each V_ξ is a *transitive set*, i.e.,

$$(x \in V_\xi \,\&\, y \in x) \Longrightarrow y \in V_\xi.$$

For example, suppose we take

$$P(x) = \mathcal{P}(x) = \{y : y \subseteq x\}$$

and

$$\mathcal{S} = \omega 2 = \{0, 1, 2, \ldots, \omega, \omega+1, \omega+2, \ldots\},$$

where the stages $0, 1, 2, \ldots, \omega, \omega+1, \omega+2, \ldots$ are all assumed distinct and ordered as we have enumerated them. In this case we obtain the universe

$$V^Z = V(\mathcal{P}, \omega 2, \leq_{\omega 2}) = V_0 \cup V_1 \cup V_2 \cup \cdots \cup V_\omega \cup V_{\omega+1} \cup \cdots,$$

often called the *universe of Zermelo*. It is well known that all the familiar structures of classical mathematics have *isomorphic copies* within V^Z—we can locate in V^Z (faithful representations of) the natural and real numbers, all functions on the reals to the reals, etc.

For a very different universe of sets, we might choose a small power operation, e.g.,

$$\mathbf{Def}(x) = \{y \subseteq x : y \text{ is elementary in the structure } (x, \in\restriction x, \{t\}_{t \in x})\}.$$

We may want to take $\mathcal{S}$ quite long this time, say

$$\mathcal{S} = \omega^\omega = \{0, 1, 2, , \ldots, \omega, \omega+1, \ldots, \omega 2, \omega 2+1, \ldots, \omega n, \omega n+1, \ldots, \ldots\},$$

so that ω^ω is the union of infinitely many disjoint copies of $\mathbb{N}$ put side-by-side. Using notation we will justify later, set

$$L_{\omega^\omega} = V(\mathbf{Def}, \omega^\omega, \leq_{\omega^\omega}).$$

It is easy to see that $V^Z \not\subseteq L_{\omega^\omega}$, because V^Z is uncountable while L_{ω^ω} is a countable set. It is a little more difficult to show also that $L_{\omega^\omega} \not\subseteq V^Z$, so that these two constructions yield two incomparable *set universes*, in which we can interpret the axioms of axiomatic set theory and check out which are true for each of them.

It is clear that the universe $V(P, \mathcal{S}, \leq_{\mathcal{S}})$ does not depend on the particular objects that we have chosen to call *stages* but only on the *length* (the *order type*) of the ordering $\leq_{\mathcal{S}}$; i.e., if the structures $(\mathcal{S}, \leq_{\mathcal{S}})$ and $(\mathcal{S}', \leq'_{\mathcal{S}})$ are isomorphic, then

$$V(P, \mathcal{S}, \leq_{\mathcal{S}}) = V(P, \mathcal{S}', \leq'_{\mathcal{S}}).$$

This definition of the universes $V(P, \mathcal{S}, \leq_{\mathcal{S}})$ is admittedly vague, and the results about them that we have claimed are grounded on intuitive ideas about sets and wellorderings which we have not justified. It is clear that we cannot expect to give a precise, mathematical definition of the basic notions of set theory, unless we use notions of some richer theory which in turn would require interpretation. At this point, we claim only that the intuitive description of $V(P, \mathcal{S}, \leq_{\mathcal{S}})$ is sufficiently clear so we can formulate meaningful propositions about these set universes and argue rationally about their truth or falsity.

Most mathematicians accept that there is *a largest meaningful* operation P satisfying (6A-1) above, *the true power operation* which takes x to the collection $\mathcal{P}(x)$ of *all* subsets of x. This is one of the cardinal assumptions of *realistic* (meaningful, not formal) set theory. Similarly, it is not unreasonable to assume that there is a *longest* collection of stages ON along which we can *meaningfully* iterate the power operation.

Our intended standard universe of sets is then

$$V = V(\mathcal{P}, \mathrm{ON}, \leq_{\mathrm{ON}}),$$

where $(\mathrm{ON}, \leq_{\mathrm{ON}})$ *is the longest meaningful collection of stages*—the wellordered class of *ordinal numbers*, as we will call it later. The axioms of the standard axiomatic set theory ZFC (Zermelo-Fraenkel set theory with Choice and Foundation) are justified by appealing to this intuitive understanding of what sets are. We will formulate it (again) carefully in the following sections and then derive its most basic consequences.

What is less obvious is that if we take $(\mathrm{ON}, \leq_{\mathrm{ON}})$ to be the same "largest, meaningful collection of stages", then the set universe

$$L = (\mathbf{Def}, \mathrm{ON}, \leq_{\mathrm{ON}})$$

is another plausible understanding of the notion of set, Gödel's *universe of constructible sets*: the central theorem of this and the next Chapter is that L *also satisfies all the axioms of* ZFC. Moreover, Gödel's proof of this surprising result does not depend on the Axiom of Choice, and so it also shows the consistency of ZFC relative to its choiceless fragment.

6B. ZFC and its subsystems

To simplify the formulation of the formal axioms of set theory, we state here a simple result of logic which could have been included in Chapter 1, right after Definition 1H.11:

Proposition 6B.1 (Eliminability of descriptions). *Fix a signature τ, and suppose $\phi(\vec{v}, w) \equiv \phi(v_1, \dots, v_n, w)$ is a full extended τ-formula and F is an n-ary function symbol not in τ.*

(1) *With each full, extended (τ, F)-formula $\theta'(\vec{u})$ we can associate a full, extended τ-formula $\theta(\vec{u})$ such that*

$$(\forall \vec{v})(\exists! w)\phi(\vec{v}, w) \ \& \ (\forall \vec{v})\phi(\vec{v}, F(\vec{v})) \vdash \theta'(\vec{u}) \leftrightarrow \theta(\vec{u}).$$

(2) *Suppose T is a τ-theory axiomatized by schemes such that*

$$T \vdash (\forall \vec{v})(\exists! w)\phi(\vec{v}, w),$$

and let T' be the (τ, F)-theory whose axioms are those of T, the sentence $(\forall \vec{v})\phi(\vec{v}, F(\vec{v}))$, and all instances with (τ, F) formulas of the axiom schemes of T. Then T' is a conservative extension of T, i.e., for all τ-sentences θ,

$$T' \vdash \theta \iff T \vdash \theta.$$

There is also an analogous result where we add to the language a new n-ary relation symbol C and the axiom

$$(\forall \vec{v})[R(\vec{v}) \leftrightarrow \phi(\vec{v})] \quad (\phi(\vec{v}) \text{ full extended}), \tag{6B-4}$$

but it is simpler, and it can be avoided by treating (6B-4) as an abbreviation. In applying these constructions we will refer to T' as an *extension of T by definitions.*

We leave the precise definition of "axiomatization by schemes" and the proof for Problem x6.1*. The thing to notice here is that all the set theories we will consider are axiomatized by schemes, and so the proposition allows us to introduce—and use with no restriction—names for constants and operations defined in them. If, for example,

$$T \vdash (\exists! z)(\forall t)[t \notin z],$$

as all the theories we are considering do, we can then extend T with a constant $\emptyset$ and the axiom

$$(\forall t)[t \notin \emptyset]$$

and we can use this constant in producing instances of the axiom schemes of T without adding any new theorems which do not involve $\emptyset$.

We now restate for easy reference (from Definitions 1A.5, 1G.12) the axioms of set theory and their formal versions in the language $\mathbb{FOL}(\in)$.

We will be using the common abbreviations for *restricted quantification*

$$
\begin{aligned}
(\exists x \in z)\phi &:\equiv (\exists x)[x \in z \,\&\, \phi],\\
(\forall x \in z)\phi &:\equiv (\forall x)[x \in z \to \phi],\\
(\exists ! x \in z)\phi &:\equiv (\exists y \in z)(\forall x \in z)[\phi \leftrightarrow y = x]
\end{aligned}
$$

(1) *Extensionality Axiom*: two sets are equal exactly when they have the same members:

$$(\forall x, y)[x = y \leftrightarrow [(\forall u \in x)(u \in y) \ \& \ (\forall u \in y)(u \in x)]].$$

(2) *Emptyset and Pairing Axioms*: there exists a set with no members, and for any two sets x, y, there is a set z whose members are exactly x and y:

$$(\exists z)(\forall u)[u \notin z], \quad (\forall x, y)(\exists z)(\forall u)[u \in z \leftrightarrow (u = x \vee u = y)]$$

It follows by the Extensionality Axiom that there is exactly one empty set and one pairing operation, and we name them $\emptyset$ and $\{x, y\}$, as usual. (And in the sequel we will omit this ceremony of stating separately the unique existence condition before baptizing the relevant operation with its customary name.)

(3) *Unionset Axiom*: for each set x, there is exactly one set $z = \bigcup x$ whose members are the members of members of x, i.e.,

$$(\forall u)[u \in \bigcup x \leftrightarrow (\exists y \in x)[u \in y]].$$

(4) *Infinity Axiom*: there exists a set z such that $\emptyset \in z$ and z is closed under the *set successor operation* x',

$$(\exists z)(\forall x \in z)[x' \in z],$$

where $u \cup v = \bigcup\{u, v\}$ and $x' = x \cup \{x\}$.

(5) *Replacement Axiom Scheme*: For each extended formula $\phi(u, v)$ in which the variable z does not occur and $x \not\equiv u, v$, the universal closure of the following formula is an axiom:

$$(\forall u)(\exists ! v)\phi(u, v) \to (\exists z)\Big[(\forall u \in x)(\exists v \in z)\phi(u, v) \\ \& \ (\forall v \in z)(\exists u \in x)\phi(u, v)\Big].$$

The instance of the Replacement Axiom for a full extended formula $\phi(\vec{y}, u, v)$ says that if for some tuple $\vec{y}$ the formula defines an operation

$$F_{\vec{y}}(u) = (\text{the unique } v)[\phi(\vec{y}, u, v)],$$

then the image

$$F_{\vec{y}}[x] = \{F_{\vec{y}}(u) : u \in x\}$$

of any set x by this operation is also a set. This is most commonly used to justify definitions of operations, in the form

$$G(\vec{y}, x) = \{F(\vec{y}, u) : u \in x\}. \tag{6B-5}$$

(6) *Powerset Axiom*: for each set x there is exactly one set $\mathcal{P}(x)$ whose members are all the subsets of x, i.e.,

$$(\forall u)[u \in \mathcal{P}(x) \leftrightarrow (\forall v \in u)[v \in x]].$$

(7) *Axiom of Choice,* **AC**: for every set x whose members are all non-empty and pairwise disjoint, there exists a set z which intersects each member of x in exactly one point:

$$\begin{aligned}(\forall x)\Big(\Big[&(\forall u \in x)(u \neq \emptyset)\\ &\& \ (\forall u, v \in x)[u \neq v \rightarrow [(\forall t \in u)(t \notin v) \ \& \ (\forall t \in v)(t \notin u)]]\Big]\\ &\qquad\qquad \rightarrow (\exists z)(\forall u \in x)(\exists! t \in z)(t \in u)\Big).\end{aligned}$$

(8) *Foundation Axiom*: Every non-empty set x has a member z from which it is disjoint:

$$(\forall x)[x \neq \emptyset \rightarrow (\exists z \in x)(\forall t \in z)[t \notin x]]$$

The most important of the theories we will consider are

- $\mathsf{ZF}^- = (1) - (5)$, i.e., the axioms of extensionality, emptyset and pairing, unionset, infinity and the Axiom Scheme of Replacement,
- $\mathsf{ZF}^-_g = (1) - (5) + (8) = \mathsf{ZF}^- +$ Foundation,
- $\mathsf{ZF} = (1) - (6) = \mathsf{ZF}^- +$ Powerset $= \mathsf{ZFC} -$ Foundation $- \mathbf{AC}$,
- $\mathsf{ZF}_g = (1) - (6) + (8) = \mathsf{ZF} +$ Foundation $= \mathsf{ZFC} - \mathbf{AC}$.
- $\mathsf{ZFC} = (1) - (8) = \mathsf{ZF}_g + \mathbf{AC}$.

We have included the alternative, more commonly used names of ZF and ZF_g which specify them as subtheories of ZFC.

The *Zermelo-Fraenkel set theory with choice* ZFC is the most widely accepted standard in mathematical practice: if a mathematician claims to have proved some proposition P about sets, then she is expected to be able to supply (in principle) a proof of its formal version θ_P from the axioms of ZFC. (This, in fact, applies to propositions in any part of mathematics, as they can all be interpreted faithfully by set-theoretic statements using familiar methods—which we will not discuss in any detail here.)

The weaker theories will also be very important to us, however, primarily as technical tools: to show the consistency of ZFC relative to ZF, for example, we will need to verify that a great number of theorems can be established in ZF—without appealing to the Axiom of Foundation or **AC**.

Convention: *All results in this Chapter will be derived from the axioms of* ZF$^-$ (or extensions of ZF$^-$ by definitions) *unless otherwise specified*—most often by a discreet notation (ZF) or (ZFC) added to the statement.

We will assume that the theorems we prove are interpreted in a structure $(\mathcal{V}, \in)$, which may be very different from the intended interpretation $(V, \in)$ of ZFC we discussed in Section 6A, especially as $(\mathcal{V}, \in)$ need not satisfy the powerset, choice and foundation axioms.

Finally, there is the matter of *mathematical propositions and proofs* versus *formal sentences of* $\mathbb{FOL}(\in)$ *and formal proofs* in one of the theories above—which are, in practice, impossible to write down in full and not very informative. We will choose the former over the latter for statements (and certainly for proofs), although in some cases we will put down the formal version of the conclusion, or a reasonable misspelling of it, cf. 1B.7. The following terminology and conventions help.

A full extended formula

$$\varphi(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y}_1, \dots, \mathbf{y}_m) \equiv \varphi(\vec{\mathbf{x}}, \vec{\mathbf{y}})$$

together with an m-tuple $\vec{y} = y_1, \dots, y_m \in \mathcal{V}$ determines an n-ary (class) *condition* on the universe $\mathcal{V}$,

$$P(\vec{x}) \iff (\mathcal{V}, \in) \models \varphi[\vec{x}, \vec{y}],$$

and it is a *definable condition* if there are no parameters, i.e., $m = 0$. For example, $t \in y$ is a condition for each y, and $x \in y$, $x = y$ are definable conditions.

A *collection* of sets $M \subseteq \mathcal{V}$ is a *class* if membership in M is a unary condition, i.e., if there is some full extended formula $\varphi(\mathbf{s}, \vec{\mathbf{y}})$ of $\mathbb{FOL}(\in)$ and sets $\vec{y}$ such that

$$M = \{s : (\mathcal{V}, \in) \models \varphi[s, \vec{y}]\}, \text{ i.e., } s \in M \iff (\mathcal{V}, \in) \models \varphi[s, \vec{y}].$$

It is a *definable class*

$$M = \{s : (\mathcal{V}, \in) \models \varphi[s]\}$$

if no parameters are used in its definition.

If a class M has the same members as a set x, we then identify it with x, so that, in particular, every set x is a class; and x is a *definable set* if it is definable as a class, i.e., if the condition $t \in x$ is definable.

A class is *proper* if it is not a set.

Finally, of $M_1, \dots, M_n$ are classes, then a *class operation*

$$F : M_1 \times \cdots \times M_n \to \mathcal{V}$$

is any $F : \mathcal{V}^n \to \mathcal{V}$ such that

$$s_1 \notin M_1 \vee \cdots \vee s_n \notin M_n \Longrightarrow F(s_1, \dots, s_n) = \emptyset,$$

and for some full extended formula $\varphi(\mathbf{s}_1, \dots, \mathbf{s}_n, \mathbf{w}, \vec{\mathbf{y}})$ and suitable $\vec{y}$,

$$F(s_1, \dots, s_n) = w \iff (\mathcal{V}, \in) \models \varphi[s_1, \dots, s_n, w, \vec{y}].$$

Such a class operation is then determined by its values $F(s_1, \dots, s_n)$ for arguments $s_1 \in M_1, \dots, s_n \in M_n$. A class operation is *definable* if it can be defined by a formula without parameters.

When there is any possibility of confusion, we will use capital letters for classes, conditions and operations to distinguish them from sets, relations and functions (sets of ordered pairs) which are members of our interpretation.

It is important to remember that theorems about classes, conditions and operations are expressed formally by theorem schemes.

It helps to do this explicitly for a while, as in the following

Proposition 6B.2 (The Comprehension Scheme)**.** *If A is a class and z is a set, then the intersection*

$$A \cap z = \{t \in z : t \in A\} \tag{6B-6}$$

is a set, i.e., for every full extended formula $\phi(s, \vec{x})$,

$$\mathsf{ZF}^- \vdash (\forall \vec{x})(\exists w)(\forall s)\Big[s \in w \leftrightarrow (\phi(s, \vec{x}) \ \&\ s \in z)\Big].$$

Proof. If $(\forall t \in z)[t \notin A]$, then $A \cap z = \emptyset$ and $\emptyset$ is a set. If there is some $t_0 \in A \cap z$, let

$$F(t) = \begin{cases} t, & \text{if } t \in z \ \&\ t \in A, \\ t_0, & \text{otherwise,} \end{cases}$$

and check easily that $F[z] = A \cap z$. ⊣

The Comprehension Scheme is also called the *Subset* or *Separation Property* and it is one of the basic axioms in Zermelo's first axiomatization of set theory,

- ZC = (1) − (4) + (6) + (7) + Comprehension.

It is most useful in showing that simple sets exist and defining class operations by setting

$$F(z, \vec{x}) = \{s \in z : P(z, \vec{x})\}$$

where $P(z, \vec{x})$ is a definable condition, e.g.,

$$x \cap y = \{t \in x : t \in y\}, \quad x \setminus y = \{t \in x : t \notin y\}.$$

In fact, almost all of classical mathematics can be developed in ZC, without using replacement, but it is not a strong enough theory for our purposes here and so we will not return to it.

6B.3. Note. Zermelo's formulation of the Axiom of Infinity (given in Definition 1A.5) was different from (4), and so the universe of sets that can be constructed by his axioms is not exactly the collection V^Z defined in Section 6A. Zermelo's Axiom of Infinity is equivalent (in ZF$^-$ − Infinity) to (4), but the proof requires establishing first some basic facts in Zermelo's theory.

6C. Set theory without powersets, AC or foundation, ZF$^-$

Set theory is mostly about the size (cardinality) of sets, and not much about size can be established without the Powerset Axiom and the Axiom of Choice. It is perhaps rather surprising that all the basic results about wellfounded relations, wellorderings and ordinal numbers can be developed in this fairly weak system.

We start with a list of basic and useful definable sets, classes and operations, some of which we have already introduced and some new ones which will not be motivated until later. In verifying the parts of the next theorem, we will often appeal (without explicit mention) to the following lemma, whose proof we leave for Problem x6.4:

Lemma 6C.1. *If $H, G_1, \dots, G_m$ are definable class operations, then their* (generalized) *composition*

$$F(\vec{x}) = H(G_1(\vec{x}), \dots, G_m(\vec{x}))$$

is also definable.

Theorem 6C.2. *The following classes, conditions, operations and sets are definable, and the claims made about them hold:*

#1. $x \in y \iff x$ is a member of y.

#2. $x \subseteq y \iff (\forall t \in x)[t \in y]$.

#3. $x = y \iff x$ is equal to y.

#4. $\{x, y\} =$ the unordered pair of x and y;
$\{x, y\} = w \iff x \in w \,\&\, y \in w \,\&\, (\forall t \in w)[t = x \vee t = y]$.

#5. $\emptyset = 0 =$ the empty set; $1 = \{\emptyset\}$;
$w = \emptyset \iff (\forall t \in w)[t \notin w]$.

#6. $\bigcup x = \{t : (\exists s \in x)[t \in s]\}$;
$\bigcup x = w \iff (\forall s \in x)(\forall t \in s)[t \in w] \,\&\, (\forall t \in w)(\exists s \in x)[t \in s]$.

#7. $x \cup y = \bigcup\{x, y\}, \quad x \cap y = \{t \in x : t \in y\}, \quad x \setminus y = \{t \in x : t \notin y\}$.

#8. $x' = x \cup \{x\}$.

#9. $\omega =$ the $\subseteq$-least set satisfying the Axiom of Infinity;
$t \in \omega \iff (\forall z)\Big([\emptyset \in z \;\&\; (\forall x \in z)(x' \in z)] \rightarrow t \in z\Big).$

#10. $\langle x, y\rangle = \big\{\{x\}, \{x, y\}\big\},$
$\langle x_1, \dots, x_{n+1}\rangle = \big\langle \langle x_1, \dots, x_n\rangle, x_{n+1}\big\rangle.$

Notice that for any x, y,

$$x, y \in \bigcup \langle x, y\rangle, \quad \langle x, y\rangle \in r \Longrightarrow x, y \in \bigcup\bigcup r.$$

#11. $u \times v = \{\langle x, y\rangle : x \in u \,\&\, y \in v\},$
$u_1 \times \cdots \times u_{n+1} = (u_1 \times \cdots \times u_n) \times u_{n+1},$
$u \uplus v = (\{0\} \times u) \cup (\{1\} \times v)$ *(disjoint union)*

#12. $\text{OrdPair}(w) \iff w$ is an ordered pair
$\iff (\exists x \in \bigcup w)(\exists y \in \bigcup w)[w = \langle x, y\rangle].$

#13. $\text{Relation}(r) \iff r$ is a set of ordered pairs
$\iff (\forall w \in r)\text{OrdPair}(w).$

#14. $\text{Domain}(r) = \{x \in \bigcup\bigcup r : (\exists y \in \bigcup\bigcup r)[\langle x, y\rangle \in r]\},$
$\text{Domain}(r) = w \iff (\forall x \in \bigcup\bigcup r)(\forall y \in \bigcup\bigcup r)$
$[\langle x, y\rangle \in r \Longrightarrow x \in w] \,\&\, (\forall x \in w)(\exists y \in \bigcup\bigcup r)[\langle x, y\rangle \in r].$

#15. $\text{Image}(r) = \{y \in \bigcup\bigcup r : (\exists x \in \bigcup\bigcup r)[\langle x, y\rangle \in r]\},$
$\text{Image}(r) = w \iff (\forall x \in \bigcup\bigcup r)(\forall y \in \bigcup\bigcup r)$
$[\langle x, y\rangle \in r \Longrightarrow y \in w] \,\&\, (\forall y \in w)(\exists x \in \bigcup\bigcup r)[\langle x, y\rangle \in r].$

#16. $\text{Field}(r) = \text{Domain}(r) \cup \text{Image}(r).$

#17. $\text{Function}(f) \iff f$ is a function (as a set of ordered pairs)
$\iff \text{Relation}(f)$
$\& \big(\forall x \in \text{Domain}(f)\big)\big(\forall y \in \text{Image}(f)\big)$
$\big(\forall y' \in \text{Image}(f)\big)$
$\big[[\langle x, y\rangle \in f \,\&\, \langle x, y'\rangle \in f] \rightarrow y = y'\big].$

If f is a function, we put

$$f(x) = y \iff \langle x, y\rangle \in f.$$

#18. $f : a \to b \iff \text{Function}(f) \;\&\; \text{Domain}(f) = a \;\&\; \text{Image}(f) \subseteq b,$
$f : a \rightarrowtail b \iff f$ is an injection from a to b,
$f : a \twoheadrightarrow b \iff f$ is a surjection from a to b,
$f : a \rightarrowtail\!\!\!\!\twoheadrightarrow b \iff f$ is a bijection from a to b
$f : a \to \mathcal{V} \iff (\exists b)[f : a \to b]$
(and similarly with all the other arrows).

#19. $F \restriction a =$ the restriction of the operation F to a
$= \{\langle x, w\rangle : x \in a \;\&\; F(x) = w\}$

#20. $r \restriction u = \{w \in r : (\exists x \in u)(\exists y \in \mathrm{Image}(r))[w = \langle x, y\rangle\}$.

$$\begin{aligned} r \restriction u = w \iff & w \subseteq r \,\&\, \mathrm{Relation}(w) \\ & \& \big(\forall x \in \mathrm{Domain}(r)\big)\big(\forall y \in \mathrm{Image}(r)\big) \\ & \qquad [\langle x, y\rangle \in w \leftrightarrow x \in u]. \end{aligned}$$

#21. $$\begin{aligned} \mathrm{Iso}(f, r_1, r_2) & \iff f \text{ is an isomorphism of } r_1 \text{ and } r_2 \\ & \iff f : \mathrm{Field}(r_1) \rightarrowtail\!\!\!\!\twoheadrightarrow \mathrm{Field}(r_2) \\ & \quad \& \ (\forall s, t \in \mathrm{Field}(r_1))[\langle s, t\rangle \in r_1 \leftrightarrow \langle f(s), f(t)\rangle \in r_2]. \end{aligned}$$

#22. $$\begin{aligned} \mathrm{WF}(r) & \iff r \text{ is a (strict) wellfounded relation} \\ & \iff \mathrm{Relation}(r) \ \& \ (\forall x \neq \emptyset)(\exists y \in x)(\forall t \in x)[\langle t, y\rangle \notin r]. \end{aligned}$$

A point y is *r-minimal in x* if $y \in x$ & $(\forall t \in x)[\langle t, y\rangle \notin r]$

#23. $$\begin{aligned} x \leq_r y & \iff \langle x, y\rangle \in r, \\ x <_r y & \iff \langle x, y\rangle \in r \ \& \ \langle y, x\rangle \notin r. \end{aligned}$$

These are notation conventions, to facilitate dealing with partial orderings and wellfounded relations. The second defines the *strict part* of the relation r, and $<_r = r$ if r is already strict, i.e., if we never have $\langle x, y\rangle$ & $\langle y, x\rangle$; this is true, in particular for wellfounded r, since

$$\langle s, t\rangle, \langle t, s\rangle \in r \Longrightarrow \{s, t\} \text{ has no } r\text{-minimal member.}$$

Notice that

$$\{x : x <_r y\} = \{x \in \bigcup\bigcup r : x <_r y\}$$

is a set, as is $\{x : x \leq_r y\}$.

#24. $$\begin{aligned} \mathrm{PO}(r) & \iff r \text{ is a partial ordering (or } \textit{poset}) \\ & \iff \mathrm{Relation}(r) \\ & \quad \& \ (\forall x \in \mathrm{Field}(r)[x \leq_r x] \\ & \quad \& \ (\forall x, y, z \in \mathrm{Field}(r))[[x \leq_r y \ \& \ y \leq_r z] \rightarrow x \leq_r z] \\ & \quad \& \ (\forall x, y \in \mathrm{Field}(r)[[x \leq_r y \ \& \ y \leq_r x] \rightarrow x = y] \end{aligned}$$

In the terminology introduced by Definition 1A.2 and used in the preceding chapters, a partial ordering is a pair $(x, \leq_x)$ where $\mathrm{PO}(\leq_x)$ and $x = \mathrm{Field}(\leq_x)$ by this notation. We will sometimes revert to the old notation when it helps clarify the discussion.

#25. $$\begin{aligned} \mathrm{LUB}(c, r, w) & \iff w \text{ is a least upper bound of } c \subseteq \mathrm{Field}(r) \\ & \iff \mathrm{PO}(r) \ \& \ (\forall x \in c)(x \leq_r w) \\ & \qquad \& \ (\forall v \in \mathrm{Field}(r))\Big((\forall x \in c)(x \leq_r v) \rightarrow w \leq_r v\Big). \end{aligned}$$

#26. $\sup_r(c) =$ the least upper bound of c in r, if it exists, otherwise $\emptyset$

$$\sup_r(c) = w \iff \mathrm{LUB}(c, r, w) \vee [(\forall v \in \mathrm{Field}(r))\neg\mathrm{LUB}(c, r, w) \ \& \ w = \emptyset]$$

#27. $$\begin{aligned} \mathrm{Chain}(c, r) & \iff c \text{ is a chain in the relation } r \\ & \iff (\forall x, y \in c)[x \leq_r y \vee y \leq_r x]. \end{aligned}$$

#28. $$\begin{aligned} \mathrm{LO}(r) & \iff r \text{ is a linear ordering} \\ & \iff \mathrm{PO}(r) \ \& \ \mathrm{Chain}(\mathrm{Field}(r), r). \end{aligned}$$

#29. $\mathrm{WO}(r) \iff r$ is a wellordering
$\iff \mathrm{LO}(r)\ \&\ \mathrm{WF}(<_r)$.

We will appeal repeatedly (and silently) to the easy fact that

$$\mathrm{WO}(r) \Longrightarrow (\forall x)\mathrm{WO}(r \cap (x \times x)). \tag{6C-7}$$

#30. $r_1 =_o r_2 \iff r_1$ and r_2 are similar (isomorphic) wellorderings
$\iff \mathrm{WO}(r_1)\ \&\ \mathrm{WO}(r_2)\ \&\ (\exists f)[\mathrm{Iso}(f, r_1, r_2)]$

#31. $\mathrm{Transitive}(x) \iff x$ is a transitive set
$\iff \bigcup x \subseteq x$
$\iff (\forall s \in x)[s \subseteq x]$
$\iff (\forall s \in x)(\forall t \in s)[t \in x]$.

#32. A is a transitive class $\iff (\forall s \in A)(\forall t \in s)[t \in A]$.

#33. $\mathrm{Ordinal}(\xi) \iff \xi$ is an ordinal (number)
$\iff \mathrm{Transitive}(\xi)$
$\&\ \mathrm{WO}(\{\langle x, y\rangle : x, y \in \xi\ \&\ [x = y \vee x \in y]\})$.

#34. $\mathrm{ON} = \{\xi : \mathrm{Ordinal}(\xi)\}$ = the class of ordinals.

#35. $x \leq_\xi y \iff x, y \in \xi \in \mathrm{ON}\ \&\ (x = y \vee x \in y)$.

#36. $\eta \leq_{\mathrm{ON}} \xi \iff \eta, \xi \in \mathrm{ON}\ \&\ [\eta = \xi \vee \eta \in \xi]$,
$\eta <_{\mathrm{ON}} \xi \iff \eta \leq_{\mathrm{ON}} \xi\ \&\ \eta \neq \xi$.

Proof. All the parts of the theorem follow very easily from the axioms and the properties of elementary definability, except perhaps for the following three.

(#9) The Axiom of Infinity guarantees that there is a set z^* which satisfies it, and we set

$$\omega = \Big\{x \in z^* : (\forall z \subseteq z^*)\Big[[\emptyset \in z\ \&\ (\forall x \in z)[x' \in z]] \to x \in z\Big]\Big\}.$$

It is easy to verify that ω satisfies the Axiom of Infinity and is the least such.

(#11) The existence of cartesian products is proved by two applications of replacement in the form (6B-5):

$$u \times v = \bigcup \Big\{\{\langle x, y\rangle : x \in u\} : y \in v\Big\}.$$

(#19) Let $G(x) = \langle x, F(x)\rangle$ and using replacement, set

$$F \restriction a = G[a] = \{\langle x, F(x)\rangle : x \in a\}. \qquad \dashv$$

Next we establish the basic properties of ω, which models the natural numbers. The first—and most fundamental—is immediate from its definition:

Proposition 6C.3 (The Induction Principle)**.** *For every set x,*

$$\Big(0 \in x \subseteq \omega\ \&\ (\forall n)[n \in x \Longrightarrow n' = n \cup \{n\} \in x]\Big) \Longrightarrow x = \omega.$$

This justifies in ZF^- the usual method of *proof by induction* of claims of the form

$$(\forall n \in \omega) P(n, \vec{y})$$

for any condition $P(n, \vec{y})$, taking $x = \{n \in \omega : P(n, \vec{y})\}$.

For a first, trivial application of the induction principle, we observe that:

Proposition 6C.4. (1) *If $x \in \omega$, then either $x = 0$ or $x = k \cup \{k\}$ for some $k \in \omega$.*

(2) Transitive(ω).

PROOF. (1) is immediate from the definition of ω, since the set

$$\{x \in \omega : x = 0 \vee (\exists k \in \omega)[x = k \cup \{k\}]\}$$

contains 0 and is closed under the successor operation $k \mapsto k \cup \{k\}$.

(2) We prove by induction that $(\forall n \in \omega)[n \subseteq \omega]$. The basis is trivial since $0 = \emptyset \subseteq \omega$. In the induction step, assuming that $n \subseteq \omega$, we get immediately that $n' = n \cup \{n\} \subseteq \omega$. $\dashv$

Anticipating the next result, we set

$$m \leq_\omega n \iff m = n \vee m \in n, \quad (m, n \in \omega).$$

The proof of the next theorem is quite simple, but it depends essentially on the identification of $<_\omega$ with $\in$,

$$m <_\omega n \iff m \in n \quad (m, n \in \omega)$$

which is not a very natural (and so confusing) definition of a strict ordering condition and takes some getting used-to. It implies that for any set x,

$$\begin{aligned} y \text{ is } \leq_\omega\text{-minimal in } x &\iff y \text{ is } \in\text{-minimal in } x \\ &\iff y \in x \;\&\; (\forall t \in y)(t \notin x) \iff y \in x \;\&\; y \cap x = \emptyset. \end{aligned}$$

Theorem 6C.5 (Basic properties of ω)**.** *The relation $\leq_\omega$ on ω is a wellordering.*

It follows that ω is an ordinal, and every $n \in \omega$ is an ordinal.

PROOF. We verify successively a sequence of properties of ω and $\leq_\omega$ which then together imply the statements in the theorem.

(a) *$\leq_\omega$ is wellfounded.*

Suppose that $x \subseteq \omega$ has no $\in$-minimal member. It is enough to show that for all $n \in \omega$, $n \cap x = \emptyset$, since this implies that $(n \cup \{n\}) \cap x = \emptyset$ for every $n \in \omega$ and so $x = \emptyset$.

The claim is trivial for $n = 0$, which has no members. In the inductive step, suppose $n \cap x = \emptyset$ but $(n \cup \{n\}) \cap x \neq \emptyset$; this means that $n \in x$, and n then is $\in$-minimal in x since none of its members are in x—contradicting the hypothesis.

(b) $\leq_\omega$ *is transitive*, i.e., $\Big(k \leq_\omega n \;\&\; n \leq_\omega m\Big) \Longrightarrow k \leq_\omega m$.

The claim here is that *each* $m \in \omega$ *is a transitive set* and we prove it by contradiction, using (a): if m is $\in$-minimal among the assumed non-transitive members of ω, it can't be 0 (which is transitive), and so $m = k \cup \{k\}$ for some k. Now $k \subseteq m$, and by the choice of m, $t \in k \Longrightarrow t \subseteq k \subseteq m$; hence $t \in m \Longrightarrow t \subseteq m$, which contradicts the assumption that m is not transitive.

(c) $\leq_\omega$ *is a partial ordering.*

We only need to show antisymmetry, so suppose that $m \leq_\omega n \leq_\omega m$. If $m \neq n$, this gives $m \in n \in m$ which contradicts (a), since it implies that the set $\{m, n\}$ has no $\in$-minimal element.

(d) $\leq_\omega$ *is a linear ordering.*

Notice first that by (a),

$$0 \neq m \in \omega \Longrightarrow 0 \in m;$$

because if m is not 0 and $\in$-least so that $0 \notin m$, then $m = k \cup \{k\}$ for some k by (a) of Proposition 6C.4, and then the choice of m yields an immediate contradiction.

Suppose now that the trichotomy law fails, and

(i) choose an $\in$-minimal n such that for some m

$$m \notin n \;\&\; m \neq n \;\&\; n \notin m; \qquad (*)$$

(ii) for this n, choose an $\in$-minimal m so that (*) holds.

By the first observation, $m, n \neq 0$, so for suitable k, l,

$$n = k \cup \{k\}, \quad m = l \cup \{l\}.$$

By (*), $m \notin k \cup \{k\}$, and so $m \notin k, m \neq k$; but then the choice of n means that

$$k \in m = l \cup \{l\}.$$

By (*) again, $n \notin m = l \cup \{l\}$, so $n \notin l, n \neq l$; but then the choice of m means that

$$l \in n = k \cup \{k\}.$$

Since $k \neq l$ (otherwise $n = m$), the last two displayed formulas imply that

$$k \in l \;\&\; l \in k,$$

which in turn implies that the set $\{k, l\} \subseteq \omega$ has no $\in$-minimal element, contradicting (a).

Now (a) and (d) together with (2) of Proposition 6C.4 mean exactly that ω is an ordinal. Moreover, since each $n \in \omega$ is a subset of ω, the restriction of $\in$ to n is a wellordering; and n is a transitive set by the transitivity of $\leq_\omega$, since

$$k \in m \in n \Longrightarrow k \leq_\omega m \leq_\omega n \Longrightarrow k \leq_\omega n \Longrightarrow k \in n,$$

the last because the alternative by (d) would produce a subset of ω with no $\in$-minimal element, as above. $\dashv$

Theorem 6C.6 (Definition by recursion on ω)**.** *From any two, given operations $G(\vec{x}), H(s, n, \vec{x})$, we can define an operation $F(n, \vec{x})$ such that*

$$F(0, \vec{x}) = G(\vec{x}),$$
$$F(n', \vec{x}) = H(F(n, \vec{x}), n, \vec{x}) \quad (n \in \omega).$$

In particular, with no parameters, from any a and $H(s, n)$, we can define a function $\overline{f} : \omega \to \mathcal{V}$ such that

$$\overline{f}(0) = a, \quad \overline{f}(n') = H(F(n), n).$$

Proof. Set

$$\begin{aligned} P(n, \vec{x}, f) \iff\ & n \in \omega \ \&\ \text{Function}(f) \\ & \&\ \text{Domain}(f) = n' \ \&\ f(0) = G(\vec{x}) \\ & \&\ (\forall m \in \text{Domain}(f))[m' \in \text{Domain}(f) \Longrightarrow f(m') = H(f(m), m, \vec{x})]. \end{aligned}$$

Immediately from the definition

$$P(0, \vec{x}, f) \iff f = \{\langle 0, G(\vec{x})\rangle\}, \quad P(n', \vec{x}, f) \Longrightarrow P(n, \vec{x}, f \setminus \{\langle n, f(n)\rangle\}),$$

and using these we can show easily by induction that

$$(\forall n \in \omega)(\exists! f) P(n, \vec{x}, f).$$

The required operation is

$$F(n, \vec{x}) = w \iff (\exists f)[P(n', \vec{x}, f) \ \&\ f(n) = w].$$

For the second claim, we apply the first with no parameters $\vec{x}$ to get $F(n)$ such that

$$F(0) = a, \quad F(n') = H(F(n), n),$$

and then appeal to the Replacement Axiom to set

$$\overline{f} = \{\langle n, F(n)\rangle : n \in \omega\}. \qquad \dashv$$

Corollary 6C.7. *Every set x is a member of some transitive set.*

Proof. By Theorem 6C.6, for each x there is a function $\text{TC}_x : \omega \to \mathcal{V}$ satisfying the equations

$$\text{TC}_x(0) = \{x\}, \quad \text{TC}_x(n') = \bigcup \text{TC}_x(n).$$

Let $y = \bigcup \text{TC}_x[\omega]$. Clearly $x \in y$ and y is transitive—because if $t \in u \in y$, then there is some n such that $t \in u \in \text{TC}_x(n)$ and so $t \in \text{TC}_x(n') \subseteq y$. $\dashv$

The *transitive closure of* x is the $\subseteq$-least transitive set which contains x as a member,

$$\text{(6C-8)}\quad \mathrm{TC}(x) = \bigcup \mathrm{TC}_x[\omega] = \bigcup_{n\in\omega} \mathrm{TC}_x(n) = \bigcap\{z : \mathrm{Transitive}(z) \ \&\ x \in z\}.$$

It is easy to check that if x is transitive, then $\mathrm{TC}(x) = x \cup \{x\}$, cf. Problem x6.8.

Note. Sometimes the transitive closure of x is defined as the least transitive set which contains x as a subset,

$$\text{(6C-9)}\qquad \mathrm{TC}'(x) = \bigcap\{y \in \mathrm{TC}(x) : \mathrm{Transitive}(y) \ \&\ x \subseteq y\}.$$

Normally, $\mathrm{TC}'(x) = \mathrm{TC}(x) \setminus \{x\}$, but it could be that $\mathrm{TC}'(x) = \mathrm{TC}(x)$ if $x \in x$—which is not ruled out without assuming the Foundation Axiom!

We collect in one definition some basic and familiar conditions on sets whose definition refers to ω and the transitive closure operation.

Definition 6C.8. (1) Two sets are *equinumerous* if their members can be put into a one-to-one correspondence, i.e.,

$$x =_c y \iff (\exists f)[f : x \rightarrowtail\!\!\!\!\twoheadrightarrow y];$$

x is *no larger than* y *in size* if x can be embedded in y,

$$x \le_c y \iff (\exists f)[f : \rightarrowtail y];$$

and x is *smaller than* y *in size* if the converse does not hold,

$$x <_c y \iff x \le_c y \ \&\ x \ne_c y.$$

(2) A set x is *finite* if $x =_c n$ for some $n \in \omega$; and it is *hereditarily finite* if $\mathrm{TC}(x)$ is finite.

(3) A set x is *countable* (or *denumerable*, or *enumerable*) if either it is finite or equinumerous with ω; and it is *hereditarily countable* if $\mathrm{TC}(x)$ is countable.

(4) A set x is *grounded* (wellfounded) if the restriction of $\in$ to $\mathrm{TC}(x)$ is a wellfounded relation, in symbols

$$x \text{ is grounded} \iff \mathrm{WF}(\{\langle s,t\rangle \in \mathrm{TC}(x)\times\mathrm{TC}(x) : s \in t\}).$$

Next we establish the basic properties of the class ON of ordinal numbers, which suggest that it is a (very long) *number system*, a proper-class-size extension of ω. As with the results about ω, the (similar) proofs about ON are simple, but they depend essentially on the somewhat perverse identification of $<_\xi$ on each $\xi \in \mathrm{ON}$ and $<_{\mathrm{ON}}$ on ON with $\in$,

$$x <_\xi y \iff x \in y, \quad \eta <_{\mathrm{ON}} \xi \iff \eta \in \xi.$$

Theorem 6C.9 (Basic properties of ON). (1) ON *is a transitive class wellordered by* $\leq_{\mathrm{ON}}$, *i.e., for all* $\eta, \zeta, \xi \in \mathrm{ON}$,

$$\xi \in \mathrm{ON} \Longrightarrow \xi \subseteq \mathrm{ON},$$
$$\eta \leq_{\mathrm{ON}} \zeta \leq_{\mathrm{ON}} \xi \Longrightarrow \eta \leq_{\mathrm{ON}} \xi, \quad (\eta \leq_{\mathrm{ON}} \xi \ \&\ \xi \leq_{\mathrm{ON}} \eta) \Longrightarrow \eta = \xi,$$
$$\eta \leq_{\mathrm{ON}} \xi \vee \eta = \xi \vee \xi \leq_{\mathrm{ON}} \eta,$$

and every non-empty class $A \subseteq \mathrm{ON}$ *has an* $\leq_{\mathrm{ON}}$*-least member.*
In other words:

$$\eta \in \zeta \in \xi \Longrightarrow \eta \in \xi, \qquad \eta \in \xi \vee \eta = \xi \vee \xi \in \eta,$$
$$\exists \eta \in A \subseteq \mathrm{ON} \Longrightarrow (\exists \xi \in A)(\forall \eta \in \xi)[\eta \notin A].$$

(2) *For each* $\xi \in \mathrm{ON}$, $\xi' = \xi \cup \{\xi\}$ *is the successor of* ξ *in* $\leq_{\mathrm{ON}}$, *i.e.,*

$$\xi <_{\mathrm{ON}} \xi' \ \&\ (\forall \eta)[\xi <_{\mathrm{ON}} \eta \Longrightarrow \xi \leq_{\mathrm{ON}} \eta].$$

(3) *Every ordinal is grounded.*

(4) *For every* $x \subseteq \mathrm{ON}$,

$$\bigcup x = \sup\{\xi : \xi \in x\} = \text{the least ordinal } \eta \text{ such that } (\forall \xi \in x)[\xi \leq_{\mathrm{ON}} \eta].$$

(5) *For every ordinal* ξ, *exactly one of the following three conditions holds:*

(i) $\xi = 0$.
(ii) ξ *is a successor ordinal, i.e.,* $\xi = \eta' = \eta \cup \{\eta\}$ *for a unique* $\eta < \xi$.
(iii) ξ *is a limit ordinal, i.e.,* $(\forall \eta <_{\mathrm{ON}} \xi)[\eta' <_{\mathrm{ON}} \xi]$ *and* $\xi = \bigcup \xi$.

It follows, in particular, that ON *is a proper class.*

Proof. We first show three properties of ON and $\leq_{\mathrm{ON}}$ which together imply (1).

(a) ON *is transitive*, i.e., every member of an ordinal is an ordinal.

Suppose, towards a contradiction, that $\xi \in \mathrm{ON}$ but $\xi \not\subseteq \mathrm{ON}$. Since $\leq_\xi$ wellorders ξ, there is a $\leq_\xi$-least $x \in \xi$ which is not an ordinal. Since ξ is transitive, $x \subseteq \xi$, and x is also transitive, because

$$s \in t \in x \Longrightarrow s <_\xi t <_\xi x \Longrightarrow s <_\xi x \Longrightarrow s \in x.$$

Moreover, x is wellorderd by the relation $\leq_x$ because $\leq_x = \leq_\xi \cap (x \times x)$. It follows that $x \in \mathrm{ON}$, which is a contradiction.

(b) *The condition* $\leq_{\mathrm{ON}}$ *is wellfounded*, i.e., for all classes A,

$$\emptyset \neq A \subseteq \mathrm{ON} \Longrightarrow (\exists \xi \in A)(\forall \eta \in A)[\eta \not<_{\mathrm{ON}} \xi].$$

Supposed $\emptyset \neq A \subseteq \mathrm{ON}$ and choose some $\xi \in A$. If ξ is $\in$-minimal in A, there is nothing to prove. If not, then there is some $\eta \in (\xi \cap A)$ and ξ is wellordered by $\leq_\xi$, so there is an $\in$-least η in $\xi \cap A$. We claim that this η is $\in$-minimal in A; if not, then there is some $\zeta \in A$ such that $\zeta <_{\mathrm{ON}} \eta$, which means that $\zeta \in \eta$—but then $\zeta \in \xi$, since ξ is transitive, and this contradicts the choice of η.

(c) *For any two ordinals* η, ξ,

$$(*) \qquad \eta \in \xi \vee \eta = \xi \vee \xi \in \eta.$$

Assume not, and choose by (a) an $\in$-minimal ξ so that $(*)$ fails for some η, and then choose an $\in$-minimal η for which $(*)$ fails with this ξ. In particular, $\xi \neq \eta$.

If $x \in \eta$, then $\xi \in x \vee x = \xi \vee x \in \xi$ by the choice of η, and the first two of these alternatives are not possible, because they both imply $\xi \in \eta$ which implies $(*)$; it follows that $x \in \xi$, and since x was an arbitrary member of η, $\eta \subseteq \xi$.

If $x \in \xi$, then $\eta \in x \vee \eta = x \vee x \in \eta$ by the choice of ξ, and the first two of these alternatives are not possible because they both imply $\eta \in \xi$ which again implies $(*)$; it follows that $x \in \eta$, so that $\xi \subseteq \eta$—which together with together with the conclusion of the preceding paragraph gives $\xi = \eta$, and that contradicts our hypothesis.

Now (a), (b) and (c) complete the proof of (1) in the theorem.

(2) – (5) and the claim that ON is a proper class follow from (1) and simple or similar arguments and we leave them for problems. $\dashv$

We will not cover *ordinal arithmetic* in this class (except for a few problems), but it is convenient to introduce the notation

$$\xi + 1 = \xi' = \xi \cup \{\xi\}$$

which is part of the definition of *ordinal addition*. We will also use a limit notation for increasing sequences of ordinals,

$$\lim_{n\to\infty} \xi_n = \sup\{\xi_n : n \in \omega\} \qquad (\xi_0 < \xi_1 < \cdots).$$

Theorem 6C.10 (Wellfounded recursion)**.** *For each operation* $G(f,t)$ *and each wellfounded relation* r*, there is exactly one function* $\overline{f} : \mathrm{Field}(r) \to \mathcal{V}$ *such that*

$$\text{(6C-10)} \qquad \overline{f}(t) = G(\overline{f} \restriction \{s : s <_r t\}, t) \qquad (t \in \mathrm{Field}(r)).$$

Moreover, if $G(f,t) = H(f,t,\vec{x})$ *with a definable operation* $H(f,t,\vec{x})$*, then there is a definable operation* $H^*(t,r,\vec{x})$ *such that for every wellordering* $\leq$*,* $\overline{f}(t) = H^*(t, \leq, \vec{x})$*.*

Proof. Define "f is a piece of the function we want" by

$$\begin{aligned} P(f) \iff & \mathrm{Function}(f) \ \&\ \mathrm{Domain}(f) \subseteq \mathrm{Field}(r) \\ & \&\ (\forall t \in \mathrm{Domain}(f))(\forall s \in \mathrm{Field}(r))[s <_r t \Longrightarrow s \in \mathrm{Domain}(f)] \\ & \qquad \&\ (\forall t \in \mathrm{Domain}(f))[f(t) = G(f \restriction \{s : s <_r t\}, t)]. \end{aligned}$$

Lemma. If $P(f), P(g)$ *and* $t \in \mathrm{Domain}(f) \cap \mathrm{Domain}(g)$*, then* $f(t) = g(t)$*.*

Proof. Suppose not, let f, g witness the failure of the Lemma, and let $t \in \mathrm{Field}(r)$ be $\leq_r$-minimal such that $f(t) \neq g(t)$. We know that

$$\{s : s <_r t\} \subseteq \mathrm{Domain}(f) \cap \mathrm{Domain}(g) \ \& \ f \restriction \{s : s <_r t\} = g \restriction \{s : s <_r t\}$$

by the definition of the condition P and the choice of t, and so by the definition of P, again,

$$f(t) = G(f \restriction \{s : s <_r t\}, t) = G(g \restriction \{s : s <_r t\}, t) = g(t),$$

contradicting the choice of t. (Lemma) ⊣

Set now

$$y = \Big\{ t \in \mathrm{Field}(r) : (\exists f)[P(f) \ \& \ t \in \mathrm{Domain}(f)] \Big\},$$
$$Q(t, w) \iff t \in y \ \& \ (\exists f)[P(f) \ \& \ f(t) = w].$$

The Lemma insures that

$$(\forall t \in y)(\exists! w) Q(t, w),$$

and so the Replacement Scheme guarantees a function $\overline{f}$ with $\mathrm{Domain}(\overline{f}) = y$ such that

$$\overline{f}(t) = G(\overline{f} \restriction \{s : s <_r t\}, t) \quad (t \in y),$$

so to conclude the proof, we only need verify that $y = \mathrm{Field}(r)$. Suppose this fails, choose an r-minimal $t \in \mathrm{Field}(r) \setminus y$ and set

$$f^* = \overline{f} \cup \{\langle t, G(\overline{f}, t) \rangle\}.$$

This is a function and it is easy to verify (directly from the definition) that $P(f^*)$, so $f^* \subseteq \overline{f}$, contradicting the assumption. ⊣

The next three, basic theorems are among the numerous applications of wellfounded recursion. We verify first a simple lemma about wellorderings which deserves separate billing:

Lemma 6C.11. *Suppose* $\mathrm{WO}(\leq)$ *and* $\pi : \mathrm{Field}(\leq) \rightarrowtail \mathrm{Field}(\leq)$ *is an injection which preserves the strict ordering, i.e.,*

$$x < y \Longrightarrow \pi(x) < \pi(y);$$

it follows that for every $x \in \mathrm{Field}(\leq)$, $x \leq \pi(x)$.

Proof. Assume the opposite and let x be $\leq$-least in $\mathrm{Field}(\leq)$ such that $\pi(x) < x$; by the hypothesis then, $\pi(\pi(x)) < \pi(x)$, which contradicts the choice of x. ⊣

In the next theorem we confuse—as is common—an ordinal ξ with the wellordering $\leq_\xi$ which is determined by ξ.

Theorem 6C.12. *Every wellordering $\leq$ is similar with exactly one ordinal*

$$\text{(6C-11)} \qquad \mathrm{ot}(\leq) = \text{the unique } \xi \in \mathrm{ON} \text{ such that } \leq =_o \leq_\xi .$$

The ordinal $\mathrm{ot}(\leq)$ is the *order type* or *length* of $\leq$.

Proof. Let

$$G(f,t) = f[\{s \in \mathrm{Field}(\leq) : s < t\}] = \{f(s) : s < t\}$$

when $t \in \mathrm{Field}(\leq)$ & $\mathrm{Function}(f)$ & $\{s : s < t\} \subseteq \mathrm{Domain}(f)$, and set $G(f,t) = 0$ (or any other, irrelevant value) otherwise. By Theorem 6C.10, there exists a function $\pi : \mathrm{Field}(\leq) \to \mathcal{V}$ such that

$$\text{(6C-12)} \qquad \pi(t) = \Big\{\pi(s) : s < t\Big\} = G(\pi \restriction \{s : s < t\}, t).$$

We verify that the image

$$\xi = \pi[\mathrm{Field}(\leq)]$$

is the required ordinal and π is the required similarity. This is trivial if $\mathrm{Field}(\leq) = \emptyset$, so we assume that we are dealing with a non-trivial wellordering.

For any $\emptyset \neq x \subseteq \mathrm{Field}(\leq)$, let

$$\min(x) = \text{the } \leq\text{-least } t \in x.$$

(1) ξ *is transitive.*

Because if $x \in \pi(t) \in \xi$, then $x \in \{\pi(t') : t' < t\}$, so $x = \pi(t')$ for some t' and $x \in \xi$.

(2) $\pi : \mathrm{Field}(\leq) \rightarrowtail\!\!\!\!\rightarrow \xi$ *is a bijection.*

It is a surjection by the definition, so assume that it is not injective, let

$$t = \min\{t' : \text{for some } s > t', \pi(t') = \pi(s)\},$$

and choose some s which witnesses the characteristic property of t, i.e.,

$$t < s \;\&\; \{\pi(t') : t' < t\} = \pi(t) = \pi(s) = \{\pi(s') : s' < s\}.$$

Since $t < s$, $\pi(t) \in \pi(s) = \pi(t)$ and so there is some $t' < t$ such that $\pi(t) = \pi(t')$, which contradicts the choice of t.

(3) $s < t \iff \pi(s) \in \pi(t)$.

Immediately from the definition, $s < t \Longrightarrow \pi(s) \in \{\pi(t') : t' < t\} = \pi(t)$. For the converse, assume that $\pi(s) \in \pi(t)$ but $s \not< t$ and consider the two possibilities.

(i) $s = t$, so that $\pi(s) = \pi(t)$ and $\pi(t) \in \pi(t) = \{\pi(t') : t' < t\}$; so $\pi(t) = \pi(t')$ for some $t' < t$ contradicting (2).

(ii) $t < s$, so that by the forward direction

$$\{\pi(t') : t' < t\} = \pi(t) \in \pi(s) \in \pi(t);$$

so $\pi(s) = \pi(t')$ for some $t' < t < s$, which also contradicts (2).

(2) and (3) together give us that

$$s \leq t \iff \pi(s) = \pi(t) \vee \pi(s) \in \pi(t) \iff \pi(s) \leq_\xi \pi(t),$$

and so $\pi : \mathrm{Field}(\leq) \rightarrowtail\!\!\!\!\rightarrow \xi$ carries the wellordering $\leq$ to the relation $\leq_\xi$, which is then a wellordering. And since ξ is also transitive by (1), it is an ordinal and π is a similarity.

Finally, to prove that $\leq$ cannot be similar to two, distinct ordinals, assume the opposite, i.e.,

$$\leq_\xi =_o \leq =_o \leq_\eta \text{ for some } \xi < \eta.$$

It follows that $\xi =_o \eta$, and so we have a similarity $\pi : \eta \rightarrowtail\!\!\!\!\rightarrow \xi$ such that $\pi(\xi) <_\eta \xi$. But $\pi : \eta \rightarrowtail \eta$ is an injection which preserves the strict ordering, and so $\xi \leq_\eta \pi(\xi)$ by Lemma 6C.11, which is a contradiction. ⊣

Definition 6C.13. A *decoration* or *Mostowski surjection* of a relation r is any function $d : \mathrm{Field}(r) \to \mathcal{V}$ such that

(6C-13) $$d(u) = \{d(v) : \langle v, u \rangle \in r\} \qquad (u \in \mathrm{Field}(r)).$$

A set x is *wellorderable* if it admits a wellordering,

(6C-14) $$\mathrm{WOable}(x) \iff (\exists r)[\mathrm{WO}(r) \;\&\; x = \mathrm{Field}(r)].$$

It is easy to check that the class WOable is closed under (binary) unions and cartesian products, cf. Problem x6.30.

Theorem 6C.14 (Mostowski Collapsing Lemma)**.** (1) *Every grounded relation r admits a unique decoration, d_r.*

(2) *A set x is grounded if and only if there exists a grounded relation r such that $x \in d_r[\mathrm{Field}(r)]$. Moreover, is $\mathrm{TC}(x)$ is wellorderable, then we can choose r so that $\mathrm{Field}(r)$ is an ordinal.*

PROOF. (1) is immediate by wellfounded recursion—in fact the required decoration which satisfies (6C-13) is defined exactly like the similarity π in the proof of Theorem 6C.12, only we do not assume that r is a wellordering.

(2) Suppose x is grounded, let

$$r = \{\langle u, v \rangle \in \mathrm{TC}(x) \times \mathrm{TC}(x) : u \in v\},$$

and let $d_r : \mathrm{TC}(x) \to \mathcal{V}$ be the unique decoration of r. Notice that d_r is the identity on its domain,

$$d_r(u) = u \quad (u \in \mathrm{TC}(x));$$

because if u is an $\in$-minimal counterexample to this, then

$$\begin{aligned} d_r(u) = \{d_r(v) : v \in \mathrm{TC}(x) \;\&\; v \in u\} \\ = \{v : v \in \mathrm{TC}(x) \;\&\; v \in u\} = \{v : v \in u\} = u, \end{aligned}$$

by the choice of u and the fact that $\mathrm{TC}(x)$ is transitive, which insures that $u \subseteq \mathrm{TC}(x)$. In particular, $d_r(x) = x$, as required.

If $\mathrm{TC}(x)$ is wellorderable, then there is a bijection $\pi : \lambda \rightarrowtail\!\!\!\!\twoheadrightarrow \mathrm{TC}(x)$ of an ordinal λ with it, and we can use this bijection to carry r to λ,

$$r' = \{\langle \xi, \eta \rangle \in \lambda \times \lambda : \pi(\xi) \in \pi(\eta)\}.$$

Easily

$$d_{r'}(\xi) = d_r(\pi(\xi)),$$

directly from the definitions of these two decorations, and so if $\pi(\xi) = x$, then $d_{r'}(\xi) = d_r(x) = x$. ⊣

There is an immediate, "foundational" consequence of the Mostowski collapsing lemma: if we know all the sets of ordinals, then we know all grounded sets. The theorem also has important mathematical implications, especially in its "class form", cf. Problems x6.17*, x6.18*.

Finally, we extend to the class of ordinals the principles of *proof by induction* and *definition by recursion*:

Theorem 6C.15 (Ordinal induction)**.** *If $A \subseteq \mathrm{ON}$ and*

$$(\forall \xi \in \mathrm{ON})\Big((\forall \eta \in \xi)(\eta \in A) \Longrightarrow \xi \in A\Big),$$

then $A = \mathrm{ON}$.

Proof. Assume the hypothesis on A and (toward a contradiction) that $\xi \notin A$ for some ξ. The hypothesis implies that $\eta \notin A$ for some $\eta \in \xi$; so let $\eta^* = \min\{\eta \in \xi : \eta \notin A\}$ and infer

$$(\forall \eta < \eta^*)(\eta \in A), \quad \eta^* \notin A$$

from the choice of η^*, which contradicts the hypothesis. ⊣

Theorem 6C.16 (Ordinal recursion)**.** *For any operation $G : \mathcal{V}^2 \to \mathcal{V}$, there is an operation $F : \mathrm{ON} \to \mathcal{V}$ such that*

$$F(\xi) = G(F \restriction \xi, \xi) \quad (\xi \in \mathrm{ON}). \tag{6C-15}$$

More generally, for any operation $G : \mathcal{V}^{m+2} \to \mathcal{V}$ there is an operation $F : \mathcal{V}^{m+1} \to \mathcal{V}$ such that

$$F(\xi, \vec{x}) = G(\{F(\eta, \vec{x}) : \eta \in \xi\}, \xi, \vec{x}) \quad (\xi \in \mathrm{ON}).$$

Moreover, in both cases, if G is definable, then so is F.

Proof. For the first claim, we apply Theorem 6C.10 to obtain for each ξ a unique function $\overline{f}_\xi : \xi \to \mathcal{V}$ such that

$$\overline{f}_\xi(\eta) = G(\overline{f}_\xi \restriction \eta, \eta) \quad (\xi \in \mathrm{ON})$$

and verify easily that these functions *cohere*, i.e.,

$$\eta < \zeta < \xi \Longrightarrow \overline{f}_\zeta(\eta) = \overline{f}_\xi(\eta).$$

We then set

$$F(\xi) = \overline{f}_{\xi+1}(\xi) \ (= \overline{f}_\zeta(\xi) \text{ (for any } \zeta > \xi).$$

The case with parameters is proved similarly, and the last claim follows from the uniformity of the argument. ⊣

Ordinal recursion is (perhaps) the most basic tool that we will use in this chapter. Many of its applications are theorems of ZF, because they require the Powerset Axiom, but it is worth including here a few, simple corollaries of it which can be established in ZF^-.

A partially ordering $\leq$ is *chain-complete* if every chain has a least upper bound in $\leq$. One needs the Powerset Axiom to construct interesting chain-complete posets, but the basic fact about them can be proved in ZF^-:

Proposition 6C.17 (The Fixed Point Theorem)**.** *If $\leq$ is a chain-complete partial ordering, $\pi : \mathrm{Field}(\leq) \to \mathrm{Field}(\leq)$ and for every $x \in \mathrm{Field}(\leq)$, $x \leq \pi(x)$, then $\pi(x^*) = x^*$ for some x^*.*

Proof. Notice first that every chain-complete poset has a least element,

$$\bot_\leq = \sup_\leq(\emptyset).$$

Assume, towards a contradiction that $x < \pi(x)$ for all $x \in \mathrm{Field}(\leq)$, and define $F : \mathrm{ON} \to \mathrm{Field}(\leq)$ by

$$F(\xi) = \begin{cases} \bot_\leq, & \text{if } \xi = 0, \\ \pi(F(\eta)), & \text{if } \xi = \eta + 1, \\ \sup_\leq(\{F(\eta) : \eta < \xi\}), & \text{otherwise.} \end{cases}$$

It is easy to check (by transfinite induction on ξ) that

$$\eta \leq \xi \Longrightarrow F(\eta) \leq F(\xi);$$

but then there must be some ξ such that

$$x = F(\xi) = F(\xi + 1) = \pi(x),$$

otherwise F injects the class of ordinals into the set $\mathrm{Field}(\leq)$, so that $\mathrm{ON} = F^{-1}(\mathrm{Field}(\leq))$ is a set. ⊣

Definition 6C.18. A class K of ordinals is *unbounded* if

$$(\forall \xi)(\exists \eta > \xi)[\eta \in K];$$

and K is *closed* if for every limit ordinal λ,

$$(\forall \eta < \lambda)(\exists \zeta)[\eta < \zeta < \lambda \,\&\, \zeta \in K] \Longrightarrow \lambda \in K,$$

i.e., if K is closed in the natural order topology on ON.

Proposition 6C.19. (1) *If K_1 and K_2 are closed, unbounded classes of ordinals, then $K_1 \cap K_2$ is also closed and unbounded.*

(2) *If $F : \mathrm{ON} \to \mathrm{ON}$ is a class operation on ordinals, then the class*

$$K^* = \{\xi : (\forall \eta < \xi)[F(\eta) < \xi]\}$$

is closed and unbounded.

Proof. (1) $K_1 \cap K_2$ is obviously closed. To see that it is unbounded, given ξ, define (by recursion on ω) $\xi_0, \xi_1, \xi_2, \ldots$ so that

$$\begin{array}{rcl} \xi < \xi_0 & \text{and} & \xi_0 \in K_1, \\ \xi_0 < \xi_1 & \text{and} & \xi_1 \in K_2, \\ \xi_1 < \xi_2 & \text{and} & \xi_2 \in K_1, \\ & \text{etc.} & \end{array}$$

and check that $\xi^* = \lim_n \xi_n \in K_1 \cap K_2$ because both K_1, K_2 are closed.

(2) Again, K^* is obviously closed. Given ξ, define ξ_n the recursion on ω,

$$\xi_0 = \xi$$
$$\xi_{n+1} = \text{the least } \xi \text{ such that supremum}\{f(\eta) : \eta < \xi_n\} + 1 < \xi$$

where the supremum exists by replacement and verify that $\eta = \xi_0 < \xi_1 < \cdots$ and $\lim_{n\to\infty} \xi_n \in K^*$. ⊣

Next we collect the few, basic results about *equinumerocity* which can be proved in ZF^-.

Theorem 6C.20. (1) *For any sets x, y, z, $x =_c y \Longrightarrow x \leq_c y$ and*

$$x =_c x, \quad x =_c y \Longrightarrow y =_c x, \quad (x =_c y =_c z) \Longrightarrow x =_c z,$$
$$(x \leq_c y \leq_c z) \Longrightarrow x \leq_c z.$$

(2) (The Schröder-Bernstein Theorem). *For any two sets x, y,*

$$(x \leq_c y \ \&\ y \leq_c x) \Longrightarrow x =_c y.$$

(1) is trivial, but the Schröder-Bernstein Theorem is actually quite difficult, cf. Problem x6.28*.

Every wellorderable set is equinumerous with an ordinal number by the basic Theorem 6C.12, and so we can measure their size—and compare them—using ordinals.

Definition 6C.21 (von Neumann cardinals). Set

$$\begin{array}{rcl} |x| & = & \text{the least } \xi \in \mathrm{ON} \text{ such that } x =_c \xi \quad (\mathrm{WOable}(x)), \\ \mathrm{Card}(\kappa) & \Longleftrightarrow & (\exists x)[\mathrm{WOable}(x) \ \&\ \kappa = |x|] \\ & \Longleftrightarrow & (\forall \xi \in \kappa)[\xi <_c \kappa], \end{array}$$

and on the class Card define

$$\kappa + \lambda = |\kappa \uplus \lambda| \quad (\kappa, \lambda \in \text{Card}),$$
$$\kappa \cdot \lambda = |\kappa \times \lambda| \quad (\kappa, \lambda \in \text{Card}).$$

Set also

$$\textstyle\sum_{\eta<\zeta} \kappa_\eta = \Big|\{\langle \eta, \xi \rangle : \xi \in \kappa_\eta\}\Big|,$$

where $\{\eta \mapsto \kappa_\eta\}_{\eta\in\zeta} : \zeta \to \text{Card}$ is any function from an ordinal ζ with cardinal values.

Theorem 6C.22. (1) *Each $n \in \omega$ and ω are cardinals.*

(2) $\kappa + 0 = \kappa;\ \kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu;\ \kappa + \lambda = \lambda + \kappa.$

(3) *The absorption law for addition:*

$$\omega \leq \max\{\kappa, \lambda\} \Longrightarrow \kappa + \lambda = \max\{\kappa, \lambda\}.$$

(4) $\kappa\cdot 0 = 0,\ \kappa\cdot 1 = \kappa;\ \kappa\cdot(\lambda\cdot\mu) = (\kappa\cdot\lambda)\cdot\mu;\ \kappa\cdot\lambda = \lambda\cdot\kappa,\ \kappa\cdot(\lambda+\mu) = \kappa\cdot\lambda+\kappa\cdot\mu.$

(5) *The absorption law for multiplication:*

$$\Big(\kappa, \lambda \neq 0 \ \&\ \omega \leq \max\{\kappa, \lambda\}\Big) \Longrightarrow \kappa \cdot \lambda = \max\{\kappa, \lambda\}.$$

(6) $\Big(|\zeta| \leq \kappa \ \&\ (\forall \eta \in \zeta)[\kappa_\eta \leq \kappa] \ \&\ \kappa \geq \omega\Big) \Longrightarrow \sum_{\eta<\zeta} \kappa_\eta \leq \kappa.$

We leave the proofs the problems; (1) – (4) are easy, if a bit fussy, and (6) follows immediately from (5), but the absorption law for multiplication is not trivial. Of course nothing in this theorem produces an infinite cardinal greater than ω—and we will show that, indeed, it is consistent with ZF^- that ω is the only infinite cardinal number.

6D. Set theory without AC or foundation, ZF

We now add the Powerset Axiom and start with two, basic results about cardinality which can be established without the Axiom of Choice.

Theorem 6D.1 (ZF, Cantor's Theorem)**.** *For every set x, $x <_c \mathcal{P}(x)$.*

Proof is left for Problem x6.38. ⊣

This gives an infinite sequence of ever increasing infinite size

$$\omega <_c \mathcal{P}(\omega) <_c \mathcal{P}(\mathcal{P}(\omega)) <_c \cdots,$$

perhaps Cantor's most important discovery. But we cannot prove in ZF that *every two sets are $\leq_c$-comparable* which, as we will see, is equivalent to the Axiom of Choice. The best we can do without **AC** in this direction is the following, simple but very useful fact:

Theorem 6D.2 (ZF, Hartogs' Theorem)**.** *For every set x, there is an ordinal ξ which cannot be injected into x,*

$$(\forall x)(\exists \xi \in \mathrm{ON})[\xi \not\leq_c x].$$

Proof. Assume towards a contradiction that every ordinal can be injected into x and set

$$y = \Big\{\mathrm{ot}(r) : r \subseteq x \times x \;\&\; \mathrm{WO}(r)\Big\}.$$

This is the image of a subset of $\mathcal{P}(x \times x)$ by a class operation, and so it is a set. The assumption on x implies that $y = \mathrm{ON}$, contradicting the fact that ON is not a set. ⊣

An immediate consequence of Hartogs' Theorem is that

$$(\forall \eta \in \mathrm{ON})(\exists \xi \in \mathrm{ON})[\eta <_c \xi]$$

and so we can define the *next cardinal operation*:

$$\kappa^+ = \text{the least } \lambda \in \text{Card such that } \kappa < \lambda; \tag{6D-16}$$

and we can iterate this operation:

Definition 6D.3 (ZF, the alephs)**.** We define for each ξ the *ξ'th infinite cardinal number* $\aleph_\xi$ by the ordinal recursion

$$\begin{aligned}\aleph_0 &= \omega,\\ \aleph_{\xi+1} &= \aleph_\xi^+\\ \aleph_\lambda &= \sup\{\aleph_\xi : \xi < \lambda\}, \text{ if } \lambda \text{ is a limit ordinal.}\end{aligned}$$

It is easy to check that every infinite cardinal κ is $\aleph_\xi$, for some ξ and that

$$\eta < \xi \Longrightarrow \aleph_\eta <_c \aleph_\xi,$$

cf. Problem x6.37.

We can iterate in the same way the powerset operation:

Definition 6D.4 (ZF, the cumulative hierarchy of grounded sets)**.** Define V_ξ for each $\xi \in \mathrm{ON}$ by the ordinal recursion

$$\begin{aligned}V_0 &= \emptyset\\ V_{\xi+1} &= \mathcal{P}(V_\xi),\\ V_\lambda &= \textstyle\bigcup_{\xi<\lambda} V_\xi, \text{ if } \lambda \text{ is a limit ordinal,}\end{aligned}$$

and set

$$\mathrm{rank}(x) = \text{the least } \xi \text{ such that } x \in V_{\xi+1} \quad (x \in \textstyle\bigcup_{\xi \in \mathrm{ON}} V_\xi).$$

Let also V be the class of all grounded sets,

$$V = \{x : \mathrm{WF}(\{\langle s, t\rangle \in \mathrm{TC}(x) \times \mathrm{TC}(x) : s \in t\})\}.$$

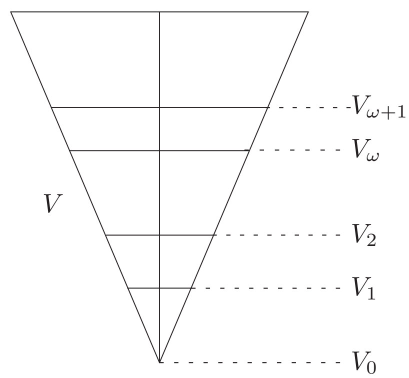

Figure 2

Theorem 6D.5 (ZF)**.** (1) *Each V_ξ is a transitive, grounded set,*

$$\eta \leq \xi \Longrightarrow V_\eta \subseteq V_\xi,$$

and $V = \bigcup_{\xi \in \mathrm{ON}} V_\xi$, i.e., every grounded set occurs in some V_ξ.

(2) *If $x \subseteq V$, then $x \in V$.*

(3) *The von Neumann universe V is a proper, transitive class.*

(4) *For each ordinal ξ,* $\mathrm{rank}(\xi) = \xi$*, so that, in particular, the operation $\xi \mapsto V_\xi$ is strictly increasing.*

Proof is left for the problems. ⊣

This hierarchy of partial universes gives a precise version of the intuitive construction for the universe of sets which we discussed in the introduction to this chapter, where *for stages we take the ordinals*. It suggests strongly that the Axiom of Foundation is true and, indeed, there is no competing intuitive idea of "what sets are" which justifies the axioms of ZF without also justifying foundation. We will not make it part of our "standard theory" yet, mostly because it is simply not needed for what we will do—and it is also not needed for developing classical mathematics in set theory.

Definition 6D.6 (Relativization)**.** For each definable class M and each $\mathbb{FOL}(\in)$-formula ϕ, we define recursively the *relativization* $(\phi)^M$ *of ϕ to M*:

$$(\mathbf{v}_i \in \mathbf{v}_j)^M :\equiv \mathbf{v}_i \in \mathbf{v}_j, (\mathbf{v}_i = \mathbf{v}_j)^M :\equiv \mathbf{v}_i = \mathbf{v}_j$$
$$(\neg\phi)^M :\equiv \neg\phi^M,\ (\phi \,\&\, \psi)^M :\equiv \phi^M \,\&\, \psi^M,$$
$$(\phi \vee \psi)^M :\equiv \phi^M \vee \psi^M,\ (\phi \to \psi)^M :\equiv \phi^M \to \psi^M,$$
$$(\exists \mathbf{v}_i \phi)^M :\equiv \exists \mathbf{v}_i(\mathbf{v}_i \in M \,\&\, \phi^M),\ (\forall \mathbf{v}_i \phi)^M :\equiv \forall \mathbf{v}_i(\mathbf{v}_i \in M \to \phi^M).$$

If $\phi(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is a full extended formula, we also set

$$M \models \phi[x_1, \dots, x_n] :\equiv x_1, \dots, x_n \in M \ \&\ (\phi(x_1, \dots, x_n))^M.$$

This definition and the accompanying notational convention extend easily to classes definable with parameters (cf. Problem x6.52) and they allow us to interpret $\mathbb{FOL}(\in)$ in any "class structure" $(M, \in\restriction M)$. Notice that $M \models \phi[x_1, \dots, x_n]$ is a formula which expresses the truth of $\phi(\mathbf{x}_1, \dots, \mathbf{x}_n)$ when we interpret each variable $\mathbf{x}_i$ by x_i, assume that each $x_i \in M$ and restrict all the quantifiers in the formula to M; and that the relativization ϕ^M depends on the formula which defines the class M.

We will prove the next, basic result in a general context because it has many applications, but in a first reading one may as well take $C_\xi = V_\xi$.

Theorem 6D.7 (The Reflection Theorem)**.** *Let $\xi \mapsto C_\xi$ be an operation on ordinals to sets which is definable in $\mathbb{FOL}(\in)$ and satisfies the following two conditions:*

(i) $\zeta \leq \xi \Longrightarrow C_\zeta \subseteq C_\xi$.

(ii) *If λ is a limit ordinal, then $C_\lambda = \bigcup_{\xi<\lambda} C_\xi$.*

Let $C = \bigcup_\xi C_\xi$.

It follows that for any full extended formula $\varphi(\mathbf{x}_1, \dots, \mathbf{x}_n)$ of $\mathbb{FOL}(\in)$, there is closed, unbounded class of ordinals K such that for $\xi \in K$ and $x_1, \dots, x_n \in C_\xi$,

$$C \models \varphi[x_1, \dots, x_n] \iff C_\xi \models \varphi[x_1, \dots, x_n].$$

In particular, if φ is any sentence of $\mathbb{FOL}(\in)$, then

$$C \models \varphi \Longrightarrow \text{for some } \xi,\ C_\xi \models \varphi.$$

Proof. We use induction on $\varphi(\mathbf{x}_1, \dots, \mathbf{x}_n)$, the result being trivial for prime formulas and following easily from the induction hypothesis for negations and conjunctions.

Suppose $(\exists\mathbf{y})\varphi(\mathbf{y}, \mathbf{x}_1, \dots, \mathbf{x}_n)$ is given and assume that K satisfies the result for $\varphi(\mathbf{y}, \mathbf{x}_1, \dots, \mathbf{x}_n)$. Let

$$G(x_1, \dots, x_n) = \begin{cases} \text{least } \xi \text{ such that } (\exists y \in C_\xi)[C \models \varphi[y, x_1, \dots, x_n]] \\ \quad \text{if one such } \xi \text{ exists,} \\ 0 \quad \text{otherwise} \end{cases}$$

and take

$$F(\xi) = \text{supremum}\{G(x_1, \dots, x_n) : x_1, \dots, x_n \in C_\xi\}$$

by replacement. By Proposition 6C.19, the class of ordinals

$$K \cap \{\xi : (\forall \eta < \xi)[F(\eta) < \xi]\} \cap \{\xi : \xi \text{ is limit}\}$$

is closed and unbounded and it is easy to verify that it satisfies the theorem for the formula $(\exists\mathbf{y})\varphi(\mathbf{y}, \mathbf{x}_1, \dots, \mathbf{x}_n)$. ⊣

Corollary 6D.8 (ZF). *$V \models \mathsf{ZF}_g$ and if* **AC** *holds, then $V \models \mathsf{ZFC}$.*

It follows that if ZF is consistent, then it remains consistent when we add the Axiom of Foundation; and if ZF + **AC** *is consistent, then so is ZFC.*

Proof is left for Problem x6.53. ⊣

Gödel's Theorem 7C.9 in the next Chapter is a much stronger relative consistency result, and it is proved by appealing to Theorem 7A.7, which in its turn is a much stronger version of the first claim here. This theorem, however, was proved by von Neumann considerably before Gödel's work, and it was the first non-trivial relative consistency proof in set theory. It provided the general plan for Gödel's work.

Finally, we include in this section the basic list of equivalents of the Axiom of Choice which can be formulated and proved in ZF.

Theorem 6D.9 (ZF). *The following statements are equivalent:*

(1) *The Axiom of Choice,* **AC**.

(2) (The logical form of **AC**). *For every binary condition $R(u, v)$ and any two sets a, b,*

$$(\forall u \in a)(\exists v \in b)R(u, v) \Longrightarrow (\exists f : a \to b)(\forall u \in a)R(u, f(u)).$$

(3) *For every set x, there is a function $\varepsilon : \mathcal{P}(x) \setminus \{\emptyset\} \to x$ such that*

$$(\forall y \subseteq x)[y \neq \emptyset \Longrightarrow \varepsilon(y) \in y]. \tag{6D-17}$$

We call any such ε a choice function for a.

(4) (Maximal Chain Principle). *In every very partial ordering $\leq$ there is a maximal chain.*

(5) (Zorn's Lemma). *If $\leq$ is a partial ordering on $x = \mathrm{Field}(\leq)$ in which every chain has an upper bound, then $\leq$ has a maximal element, some $a \in x$ such that $(\forall t \in x)(a \not< t)$.*

(6) (Cardinal Comparability Principle). *For any two sets $x, y,$, either $x \leq_c y$ or $y \leq_c x$.*

(7) (Zermelo's Wellordering Theorem). *Every set is equinumerous with an ordinal number.*

We have established all the ingredients needed for a simple round-robin proof $(1) \Longrightarrow (2) \Longrightarrow \cdots \Longrightarrow (7) \Longrightarrow (1)$, cf. Problem x6.41.

From the foundational point of view, the most interesting part of this theorem is the triple equivalence in ZF of the logical form of **AC** (2), which had been viewed as an obvious principle of logic, with the cardinal comparability principle (6), which looks like a technical result and with the wellordering principle (7), which had been considered false before Zermelo's proof—by many mathematicians, though not Cantor. From the point of view of its applications, all these "versions" of **AC** are useful in various parts of mathematics, but perhaps the most natural one is the existence of a choice functions (3): it makes it possible to say "choose a $y \in a$ such that … " after showing that "there exists a $y \in a$ such that … " in the course of a proof, with **AC** justifying in the end the validity of the argument.

6E. Cardinal arithmetic and ultraproducts, ZFC

We include in this Section a (very) few results about cardinal arithmetic and the ultraproduct construction, which need **AC**.

The most immediate effect of the Axiom of Choice is that it makes it possible to define *cardinal exponentiation*, which requires that the function space $(\lambda \to \kappa)$ is wellorderable,

$$\kappa^\lambda = |(\lambda \to \kappa)| \quad (\kappa, \lambda \in \text{Card}).$$

The definition gives (easily) "the laws of exponents":

Theorem 6E.1 (ZFC)**.** (1) *For every* $\kappa \in \text{Card}$, $2^\kappa = |\mathcal{P}(\kappa)|$.

(2) *For all cardinal numbers* κ, λ, μ,

$$\kappa^0 = 1, \kappa^1 = \kappa,\ \kappa^n = \underbrace{\kappa \cdots \kappa}_{n \text{ times}}\ (n \in \omega)$$

$$(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu,\ \kappa^{(\lambda+\mu)} = \kappa^\lambda \cdot \kappa^\mu,\ \left(\kappa^\lambda\right)^\mu = \kappa^{\lambda \cdot \mu}.$$

(3) *For all cardinal numbers* κ, λ, μ,

$$\begin{aligned} &\kappa \le \mu \implies \kappa + \lambda \le \mu + \lambda,\ \kappa \cdot \lambda \le \mu \cdot \lambda, \\ &\lambda \le \mu \implies \kappa^\lambda \le \kappa^\mu \qquad (\kappa \ne 0), \\ &\kappa \le \lambda \implies \kappa^\mu \le \lambda^\mu. \end{aligned}$$

These are proved by constructing the required bijections and injections, without, in fact, using **AC**. For example, (1) and (2) follow from the following theorems of ZF:

$$\begin{aligned} \mathcal{P}(x) &=_c (x \to \{0,1\}) \quad (y \mapsto \chi_y : x \to \{0,1\}, \\ &\qquad\qquad (\chi_y = \text{ the characteristic function of } y \subseteq x), \\ (z \to (x \times y)) &=_c (z \to x) \times (z \to y), \\ ((x \uplus y) \to z) &=_c (x \to z) \times (y \to z), \\ ((x \times y) \to z) &=_c (x \to (y \to z)). \end{aligned}$$

On the other hand, we must be careful with strict inequalities between infinite cardinal numbers because they are not always respected by the algebraic operations. For example,

$$\aleph_0 < \aleph_1 \text{ but } \aleph_0 + \aleph_1 = \aleph_1 + \aleph_1 \ (= \aleph_1).$$

A simple but basic consequence of **AC** to which we will appeal constantly (and silently) is

(6E-18) $$(\exists f)[f : a \twoheadrightarrow b] \Longrightarrow b \leq_c a,$$

which is proved by fixing a choice function $\varepsilon_a : \mathcal{P}(a) \setminus \{\emptyset\} \to a$ and defining the required injection $g : b \rightarrowtail a$ by

$$g(t) = \varepsilon_a(\{s \in a : f(s) = t\}).$$

It is known that (6E-18) cannot be proved in ZF, but its exact axiomatic strength is not clear—for all I know, it may imply **AC**.

One of the basic problems in set theory—perhaps its most basic problem—is the size of the powerset $\mathcal{P}(\omega)$ or, equivalently, the size of *Baire space* or the real numbers, since we can show in ZF that

$$\mathcal{P}(\omega) =_c \mathcal{N} =_c \mathbb{R},$$

cf. Problems x6.33, x6.34. Cantor's famous *Continuum Hypothesis* expresses the natural conjecture about this, that there are no sets intermediate in size between ω and its powerset:

(**CH**) $$(\forall x \subseteq \mathcal{P}(\omega))[x \leq_c \omega \vee x =_c \mathcal{P}(\omega)].$$

The corresponding hypothesis for arbitrary sets is the *Generalized Continuum Hypothesis*,

(**GCH**) $$(\forall y)(\forall x \subseteq \mathcal{P}(y))[x \leq_c y \vee x =_c \mathcal{P}(y)].$$

The Continuum Hypothesis is intimately related to the Cardinal Comparability Principle, because it could fail for some $x \subset \mathcal{P}(\omega)$ such that $x <_c \mathcal{P}(\omega)$ simply because x is not $\leq_c$-comparable to ω—i.e., x is uncountable, smaller than $2^{\aleph_0}$, but has no infinite, countable subsets. In ZFC, these two hypotheses take the simple "cardinal arithmetic" forms

$$2^{\aleph_0} = \aleph_1, \quad 2^{\aleph_\xi} = \aleph_{\xi+1}.$$

This does not help determine their truth value.

Many of the consequences of the Axiom of Choice can be formulated as theorems of ZF about wellorderable sets. We state here a few, very basic facts whose proofs use **AC** in such a fundamental way (often within an argument by contradiction), that there is no useful way to view them as theorems of ZF.

An *indexed set* (or *family*) *of sets* is a function $a : I \to \mathcal{V}$. We often write $a_i = a(i)$ for these indexed sets, and we use them to define indexed unions and products,

$$\bigcup_{i\in I} a_i = \bigcup\{a_i : i \in I\},$$
$$\prod_{i\in I} a_i = \{f : I \to \bigcup_{i\in I} a_i : (\forall i \in I)[f(i) \in a_i]\}.$$

The infinite product comprises all choice functions which pick just one member from each a_i, and the equivalence

$$(\forall (i \mapsto a_i))[(\forall i \in I)[a_i \neq \emptyset] \iff \prod_{i\in I} a_i \neq \emptyset]. \tag{6E-19}$$

is (easily) equivalent to **AC**, cf. Problem x6.44.

For indexed families of cardinal numbers, we also set

$$\sum_{i\in I} \kappa_i = |\{\langle i, t\rangle : i \in I \ \&\ t \in \kappa_i\}|,$$
$$\prod_{i\in I} \kappa_i = |\{f : I \to \bigcup_{i\in I} \kappa_i : (\forall i \in I)[f(i) \in \kappa_i]\}|,$$

so that $\prod_{\xi\in\lambda} \kappa = \kappa^\lambda$. (Use of the same notation for products and cardinal numbers of products is traditional and should not cause confusion.)

Theorem 6E.2 (ZFC, König's Theorem). *For any two families of sets $(i \mapsto a_i)$ and $(i \mapsto b_i)$ on the same index set $I \neq \emptyset$,*

$$\text{if } (\forall i \in I)[a_i <_c b_i], \text{then } \bigcup_{i\in I} a_i <_c \prod_{i\in I} b_i. \tag{6E-20}$$

In particular, for families of cardinals, $(i \mapsto \kappa_i)$ and $(i \mapsto \lambda_i)$,

$$\text{if } (\forall i \in I)[\kappa_i <_c \lambda_i], \text{then } \sum_{i\in I} \kappa_i <_c \prod_{i\in I} \lambda_i. \tag{6E-21}$$

PROOF. The hypothesis and **AC** yield for each i an injection $\pi_i : a_i \rightarrowtail b_i$; and since π_i cannot be a surjection, there is also a function $c : I \to \bigcup_{i\in I} b_i$ such that for each i, $c(i) \in b_i \setminus \pi_i[a_i]$. For any $x \in \bigcup_{i\in I} a_i$, we set

$$f(x, i) = \begin{cases} \pi_i(x), & \text{if } x \in a_i, \\ c(i), & \text{if } x \notin a_i, \end{cases}$$
$$g(x) = (i \mapsto f(x, i)) \in \prod_{i\in I} b_i.$$

If $x \neq y$ and x, y belong to the same a_i for some i, then

$$g(x)(i) = \pi_i(x) \neq \pi_i(y) = g(y)(i),$$

because π_i is an injection, and hence $g(x) \neq g(y)$. If no a_i contains both x and y, suppose $x \in a_i$, $y \notin a_i$; it follows that $g(x)(i) = \pi_i(x) \in \pi_i[a_i]$ and $g(y)(i) = c(i) \in b_i \setminus \pi_i[a_i]$ so that again $g(x) \neq g(y)$. We conclude that the mapping $g : \bigcup_{i\in I} a_i \rightarrowtail \prod_{i\in I} b_i$ is an injection, and hence

$$\bigcup_{i\in I} a_i \leq_c \prod_{i\in I} b_i.$$

Suppose, towards a contradiction that there existed a bijection

$$h : \bigcup_{i\in I} a_i \rightarrowtail\!\!\!\!\rightarrow \prod_{i\in I} b_i,$$

so that these two sets are equinumerous. For every i, the function

$$h_i(x) =_{\mathrm{df}} h(x)(i) \quad (x \in a_i)$$

is (easily) a function of a_i into b_i and by the hypothesis it cannot be a surjection; hence by **AC** there exists a function ε which selects in each b_i some element not in its image, i.e.,

$$\varepsilon(i) \in b_i \setminus h_i[a_i], \quad (i \in I).$$

By its definition, $\varepsilon \in \prod_{i \in I} b_i$, so there must exist some $x \in A_j$, for some j, such that $h(x) = \varepsilon$; this yields

$$\varepsilon(j) = h(x)(j) = h_j(x) \in h_j[A_j],$$

contrary to the characteristic property of ε.

The cardinal version (6E-21) follows by applying (6E-20) to $a_i = \{i\} \times \kappa_i$ and $b_i = \lambda_i$. ⊣

Definition 6E.3 (Cofinality, regularity)**.** A limit ordinal ξ is *cofinal* with a limit ordinal $\zeta \leq \xi$ if there exists a function $f : \zeta \to \xi$ which is unbounded, i.e., $\sup\{f(\eta) : \eta < \zeta\} = \xi$. (So each limit ξ is cofinal with itself.)

The *cofinality* of ξ is the least limit ordinal $\zeta \leq \xi$ which is cofinal with ξ,

$$\mathrm{cf}(\xi) = \min\{\zeta \leq \xi : (\exists f : \zeta \to \xi)[\sup\{f(\eta) \mid \eta < \zeta\} = \xi]\}.$$

A limit ordinal ξ is *regular* if $\mathrm{cf}(\xi) = \xi$, otherwise it is *singular.*

For example, ω is regular, since there is no limit ordinal less than it with which it could be cofinal, and $\aleph_\omega$ is singular, since (easily) $\mathrm{cf}(\aleph_\omega) = \omega$.

Proposition 6E.4. (1) *If ξ is cofinal with $\zeta \leq \xi$ and ζ is cofinal with $\mu \leq \zeta$, then ξ is cofinal with μ.*

(2) *For every limit ordinal ξ, $\mathrm{cf}(\xi)$ is a cardinal.*

(3) (ZF). *For every limit ordinal λ, $\mathrm{cf}(\aleph_\lambda) = \mathrm{cf}(\lambda)$.*

(4) *If $\lambda = \mathrm{cf}(\xi)$, then there is an injection $f : \lambda \rightarrowtail \xi$ which is cofinal and order preserving, i.e.,*

$$\eta_1 < \eta_2 < \lambda \Longrightarrow f(\eta_1) < f(\eta_2) < \xi, \qquad \sup\{f(\eta) : \eta < \lambda\} = \xi.$$

Proof is easy and left for Problem x6.46. ⊣

Theorem 6E.5 (ZFC)**.** *Every infinite, successor cardinal κ^+ is regular.*

Proof. Suppose towards a contradiction that some $f : \kappa \to \kappa^+$ is unbounded, so that

$$\kappa^+ = \sup\{f(\xi) : \xi < \kappa\}.$$

Now each $f(\xi) \leq_c \kappa$, since κ^+ is an initial ordinal; so choose surjections

$$\pi_\xi : \kappa \twoheadrightarrow \max(1, f(\xi)) \quad \text{(just in case } f(\xi) = 0),$$

and define $\pi : \kappa \times \kappa \to \kappa^+$ by

$$\pi(\xi, \eta) = \pi_\xi(\eta).$$

The assumptions imply that π is a surjection, because if $\zeta \in \kappa^+$, then $\zeta \in f(\xi)$ for some $\xi \in \kappa$, and so $\zeta = \pi_\xi(\eta) = \pi(\xi, \eta)$ for some $\eta \in \kappa$; but this is a contradiction, because $|\kappa \times \kappa| = \kappa < \kappa^+$, and so there cannot be a surjection of $\kappa \times \kappa$ onto κ^+. ⊣

So $\aleph_0, \aleph_1, \dots \aleph_n, \dots$, are all regular, $\aleph_\omega$ is singular, $\aleph_{\omega+1}, \aleph_{\omega+2}, \dots$ are regular, etc.

Theorem 6E.6 (ZFC, König's inequality). *For every infinite cardinal κ,*

$$\kappa < \kappa^{\mathrm{cf}(\kappa)}. \tag{6E-22}$$

PROOF. Let $\lambda = \mathrm{cf}(\kappa) \leq \kappa$ and fix an unbounded function $f : \lambda \to \kappa$, so that

$$f(\xi) <_c \kappa \quad (\xi < \lambda)$$

since $f(\xi) \in \kappa$ and κ is a cardinal. By König's Theorem 6E.2,

$$\kappa = \bigcup_{\xi\in\lambda} f(\xi) <_c \prod_{\xi\in\lambda} \kappa = \kappa^\lambda, \qquad ⊣$$

König's inequality was the strongest, known result about cardinal exponentiation in ZFC until the 1970s, when Silver proved that that if the **GCH** holds up to $\kappa = \aleph_{\aleph_1}$, then it holds at κ,

$$(\forall \xi < \aleph_1)[2^{\aleph_\xi} = \aleph_{\xi+1}] \Longrightarrow 2^{\aleph_{\aleph_1}} = \aleph_{\aleph_1+1}.$$

In fact Silver proved much stronger results in ZFC and others, after him extended them substantially, but none of these results affects the Continuum Hypothesis; and it can not, because it was already known from the work of Paul Cohen in 1963 that for any $n \geq 1$, the statement $2^{\aleph_0} = \aleph_n$ is consistent with ZFC.

Definition 6E.7 (ZF, Inaccessible cardinals). A limit cardinal κ is *weakly inaccessible* if it is regular and closed under the cardinal succession operation,

$$\lambda < \kappa \Longrightarrow \lambda^+ < \kappa; \tag{6E-23}$$

it is (strongly) *inaccessible* if it is regular and closed under exponentiation,

$$\lambda < \kappa \Longrightarrow 2^\lambda < \kappa. \tag{6E-24}$$

Notice that weakly inaccessible cardinals can be defined in ZF. We can also define strongly inaccessibles without **AC**, if we understand the definition to require that $\mathcal{P}(\lambda)$ is wellorderable for $\lambda < \kappa$, but nothing interesting about them can be proved without assuming **AC**. With **AC**, strongly inaccessible cardinals are weakly inaccessible, since

$$\lambda^+ \leq 2^\lambda.$$

We cannot prove in ZFC the existence of strongly inaccessible cardinals, cf. Problems x6.48*, x6.50*. In fact, ZFC does not prove the existence of weakly inaccessible cardinals either, as we will show in the next Chapter.

Finally, we include here the bare, minimum facts about ultrafilters and ultraproducts which have numerous applications in model theory.

Definition 6E.8. A (proper) *filter* on an infinite set I is a collection $F \subset \mathcal{P}(I)$ which satisfies the following conditions:

(1) If $X \in F$ and $X \subseteq Y$, then $Y \in F$.
(2) If $X_1, X_2 \in F$, then $X_1 \cap X_2 \in F$.
(3) F is neither empty nor the whole of $\mathcal{P}(I)$: i.e., $\emptyset \notin F$ and $I \in F$.

A filter on I is *maximal* or an *ultrafilter* if

$$X \in F \text{ or } X^c = (I \setminus X) \in F \qquad (X \subseteq I),$$

or F *decides* every $X \subseteq I$, as we will say.

For example, if $\emptyset \neq A \subseteq I$, then the set

$$F_A = \{X \subseteq I : A \subseteq X\}$$

of all supersets of A is a filter; and if $A = \{a\}$ is a singleton, then

$$F_{\{a\}} = U_a = \{X \subseteq I : a \in X\}$$

is an ultrafilter, the *principal ultrafilter* determined by a.

A more interesting example is the collection of *cofinite subsets* of I,

$$F_0(I) = \{X \subseteq I : X^c \text{ is finite}\}.$$

This is clearly not F_A for any $A \subseteq I$, and it is not an ultrafilter.

Intuitively, a filter F determines a notion of "largeness" for subsets of I, and its classical examples arise in this way: for example F might be the collection of sets of real numbers whose complement has (Lebesgue) measure 0 or whose complement is meager.

Theorem 6E.9 (ZFC). *Every filter F on an infinite set I can be extended to an ultrafilter $U \supseteq F$.*

PROOF. Consider the set of all filters which extend F,

$$\mathcal{F} = \{F' \subset \mathcal{P}(I) : F \subseteq F' \ \& \ F' \text{ is a filter}\},$$

and view it as a poset $(\mathcal{F}, \subseteq)$. Every chain $\mathcal{C} \subset \mathcal{F}$ (easily) has an upper bound, namely its union $\bigcup \mathcal{C}$; and so by Zorn's Lemma, $\mathcal{F}$ has a maximal member U. It suffices to prove that U decides every $X \subseteq I$, so suppose that for some X_0

$$X_0 \notin U \text{ and } X_0^c \notin U.$$

Let $G = \{Y : (\exists X \in U)[Y \supseteq (X \cap X_0)]\}$. Clearly $U \subsetneq G$, since G contains $X_0 = I \cap X_0$, and G is trivially closed under supersets. It is also closed under intersections, since if for some $X_1, X_2 \in U$,

$$Y_1 \supseteq (X_1 \cap X_0),\ Y_2 \supseteq (X_2 \cap X_0),$$

then $Y_1 \cap Y_2 \supseteq (X_1 \cap X_2 \cap X_0)$, and $X_1 \cap X_2 \in U$. Since G cannot be a (proper) filter because U is maximal, it must be that $\emptyset \supseteq (X \cap X_0)$ for some $X \in U$; which implies that $X \subseteq X_0^c$, and so $X_0^c \in U$, contrary to our assumption. ⊣

The most interesting immediate corollary is the existence of non-principal ultrafilters on every infinite set I: because if $U \supset F_0(I)$ extends the filter of cofinite subsets of I, then U is not principal. As far as the strength of these claims goes, it is known that the existence of non-principal ultrafilters cannot be proved in ZF_g, but even the stronger claim in the theorem does not imply **AC**.

Suppose U is an ultrafilter on I and $\{A_i\}_{i\in I}$ is a family of sets indexed by I, and let

$$f \sim_U g \iff \{i \in I : f(i) = g(i)\} \in U \quad (f, g \in \textstyle\prod_{i\in I} A_i).$$

It is easy to check that $\sim_U$ is an equivalence relation on $\prod_{i\in I} A_i$. We let

$$\overline{f} = \{g \in \textstyle\prod_{i\in I} A_i : f \sim_U g\} \qquad (f \in \prod_{i\in I} A_i)$$

be the equivalence class of f modulo $\sim_U$, so that

$$\overline{f} = \overline{g} \iff \{i \in I : f(i) = g(i)\} \in I. \tag{6E-25}$$

We will also let

$$\overline{A} = \Big(\textstyle\prod_{i\in I} A_i\Big)/U = \{\overline{f} : f \in \prod_{i\in I} A_i\} \tag{6E-26}$$

for the corresponding set of equivalence classes. The notation is compact (and in particular does not show explicitly the dependence on U) but it is useful.

Definition 6E.10 (ZFC, ultraproducts)**.** Suppose $\{\mathbf{A}_i\}_{i\in I}$ a family of τ-structures indexed by an infinite set I and U is an ultrafilter on I. The *ultraproduct*

$$\overline{\mathbf{A}} = \Big(\textstyle\prod_{i\in I} \mathbf{A}_i\Big)/U \tag{6E-27}$$

of the family $\{\mathbf{A}_I\}_{i\in I}$ modulo U is the τ-structure defined as follows:

(1) The universe $\overline{A}$ is the set of equivalence classes as in (6E-26).

(2) For each constant c in τ,

$$c^{\overline{\mathbf{A}}} = \overline{g},\ \text{where } g(i) = c^{\mathbf{A}_i}.$$

(3) For each relation symbol R in τ,

$$R^{\overline{\mathbf{A}}}(\overline{f}_1,\dots,\overline{f}_k) \iff \{i \in I : R^{\mathbf{A}_i}(f_1(i),\dots,f_k(i))\} \in I.$$

(4) For each function symbol f in τ,

$$f^{\overline{\mathbf{A}}}(\overline{f}_1,\dots,\overline{f}_k) = \overline{g} \text{ where } g(i) = f^{\mathbf{A}_i}(f_1(i),\dots,f_k(i)).$$

If $\mathbf{A}_i = \mathbf{A}$ for all $i \in I$, then $\overline{\mathbf{A}} = \Big(\prod_{i\in I}\mathbf{A}\Big)/I$ is the *ultrapower* of $\mathbf{A}$ module U.

To make sense of the last two clauses in this definition we need to check that if $f_1 \sim_U f_1', \dots f_k \sim_U f_k'$, then

$$\{i \in I : R^{\mathbf{A}_i}(f_1(i),\dots,f_k(i)) \iff R^{\mathbf{A}_i}(f_1'(i),\dots,f_k'(i))\} \in U,$$
$$\{i \in I : f^{\mathbf{A}_i}(f_1(i),\dots,f_k(i)) = f^{\mathbf{A}_i}(f_1'(i),\dots,f_k'(i))\} \in U.$$

These are true because the claimed equivalence and identity hold on

$$X = \bigcap_{j=1,\dots,k}\{i \in I : f_j(\vec{x}) = f_j'(\vec{x})\}$$

and $X \in U$ by the hypothesis.

Theorem 6E.11 (ZFC, Łós' Theorem). *Let $\{\mathbf{A}_i\}_{i\in I}$ be family of τ-structures indexed by an infinite set I and let $\overline{\mathbf{A}}$ be their ultraproduct modulo a ultrafilter U as in* (6E-27). *Then for each full extended formula $\phi(\mathbf{x}_1,\dots,\mathbf{x}_n)$ and all $\overline{f}_1,\dots,\overline{f}_n \in \overline{A}$,*

(6E-28) $$\overline{\mathbf{A}} \models \phi[\overline{f}_1,\dots,\overline{f}_n] \iff \{i \in I : \mathbf{A}_i \models \phi[f_1(i),\dots,f_n(i)]\} \in I.$$

In particular, for every sentence θ,

$$\overline{\mathbf{A}} \models \theta \iff \{i \in I : \mathbf{A}_i \models \theta\} \in I.$$

Proof. We first check by induction that for each term $t(\mathbf{x}_1,\dots,\mathbf{x}_n)$,

$$t^{\overline{\mathbf{A}}}[\overline{f}_1,\dots,\overline{f}_n] = \overline{g} \text{ where } g(i) = t^{\mathbf{A}_i}[f_i(i),\dots,f_n(i)],$$

and then we check (6E-28) by another, simple induction on ϕ. The only case where some thought is required is when

$$\phi(\vec{\mathbf{x}}) \equiv (\exists \mathbf{y})\psi(\vec{\mathbf{x}},\mathbf{y}),$$

and this is where **AC** comes in. We leave the detail for Problem x6.56. ⊣

We have put in the problems a few additional facts about ultrapowers, including a classical, purely semantic proof of the Compactness Theorem for arbitrary signatures. But it should be emphasized that the subject is large—especially rich in its applications to non-standard models—and we will not cover it here.

6F. Problems for Chapter 6

For each vocabulary τ, let

$$\tau' = \tau \cup \{\mathsf{P}_i^n : n, i \in \mathbb{N}\}$$

be the expansion of τ by infinitely many n-ary relation symbols

$$\mathsf{P}_0^n, \mathsf{P}_1^n, \mathsf{P}_1^n, \dots$$

for each n and no new function symbols or constants. A τ-*axiom scheme* is any τ'-sentence θ; and a τ-*instance* of θ is the τ-sentence constructed by associating with each P_i^n which occurs in θ a full, extended τ-formula $\phi_i^n(v_1, \dots, v_n)$ and replacing each prime formula $\mathsf{P}_i^n(t_1, \dots, t_n)$ in θ with the τ-formula $\phi_i^n(t_1, \dots, t_n)$, where the substitution $\{v_1 :\equiv t_1, \dots, v_n :\equiv t_n\}$ is assumed free.

For example, the sentence

$$\theta \equiv (\forall x)(\exists w)(\forall u)[u \in w \leftrightarrow [u \in x \ \& \ P(u)]]$$

is an $\in$-scheme, and the instances of it are all $\in$-sentences of the form

$$\theta\{P(v) :\equiv \phi(v)\} \equiv (\forall x)(\exists w)(\forall u)[u \in w \leftrightarrow [u \in x \ \& \ \phi(u)]],$$

where $\phi(u)$ is an arbitrary, full extended $\in$-formula.

A τ-theory T is *axiomatized by schemes* if its axioms (i.e., the members of T) comprise a set of τ-sentences and all τ-instances of a set of axiom schemes.

Problem x6.1. Prove that Peano arithmetic PA and ZF^- are axiomatized by schemes.

Problem x6.2* (Eliminability of descriptions, 6B.1)**.** Fix a signature τ, and suppose $\phi(\vec{v}, w) \equiv \phi(v_1, \dots, v_n, w)$ is a full extended τ-formula and F is an n-ary function symbol not in τ.

(1) With each full, extended (τ, F)-formula $\theta'(\vec{u})$ we can associate a full, extended τ-formula $\theta(\vec{u})$ such that

$$(\forall \vec{v})(\exists! w)\phi(\vec{v}, w) \ \& \ (\forall \vec{v})\phi(\vec{v}, F(\vec{v})) \vdash \theta'(\vec{u}) \leftrightarrow \theta(\vec{u}).$$

(2) Suppose T is a τ-theory axiomatized by schemes such that

$$T \vdash (\forall \vec{v})(\exists! w)\phi(\vec{v}, w),$$

and let T' be the (τ, F)-theory whose axioms are those of T, the sentence $(\forall \vec{v})\phi(\vec{v}, F(\vec{v}))$, and all instances with (τ, F) formulas of the axiom schemes of T. Then T' is a conservative extension of T, i.e., for all τ-sentences θ,

$$T' \vdash \theta \iff T \vdash \theta.$$

Problem x6.3. Prove that a set x is definable if and only if its singleton $\{x\}$ is a definable class.

Problem x6.4 (Lemma 6C.1)**.** Prove that if $H, G_1, \dots, G_m$ are definable class operations, then their (generalized) composition

$$F(\vec{x}) = H(G_1(\vec{x}), \dots, G_m(\vec{x}))$$

is also definable.

Problem x6.5. Prove that for every set x,

$$\text{Russel}(x) = \{t \in x : t \notin t\} \notin x.$$

Infer that the class $\mathcal{V}$ of all sets is not a set.

Problem x6.6. Determine which of the claims in Theorem 6C.2 is a formal *theorem scheme* (rather than a theorem) of ZF^- and write out these schemes.

Problem x6.7. Prove that if every member of x is transitive, then $\bigcup x$ is transitive.

Problem x6.8. Prove that if x is transitive, then $\mathrm{TC}(x) = x \cup \{x\}$.

Problem x6.9. Prove that the restriction $S = \{\langle n, n' \rangle : n \in \omega\}$ of the operation $x' = x \cup \{x\}$ to ω is a bijection of ω with $\omega \setminus \{0\}$. (This and the Induction Principle 6C.3 together comprise the *Peano axioms* for the structure $(\omega, 0, S)$.)

Problem x6.10* (Zermelo's Axiom of Infinity)**.** Prove that

$$(\mathsf{Z}\text{-infty}) \qquad (\exists z)[\emptyset \in z) \ \& \ (\forall t \in z)[\{t\} \in z]].$$

Outline a proof of the Axiom of Infinity in

$$\mathsf{ZF} - \text{Infinity} + (\mathsf{Z}\text{-infty}).$$

Problem x6.11. Prove that the following are equivalent for every x:

(1) x is finite, i.e., $x =_c n$ for some $n \in \omega$.
(2) There is exactly one $n \in \omega$ such that $x =_c n$.
(3) $x <_c \omega$.

Problem x6.12. Prove that a set x is countable exactly when $x \leq_c \omega$.

Problem x6.13. Prove that for each relation r, if $r' = <_r$, then $<_{r'} = <_r$.

Problem x6.14 ((2) and (3) of Theorem 6C.9)**.** Prove that for each ordinal ξ, $\xi' = \xi \cup \{\xi\}$ is the successor of ξ in $\leq_{\mathrm{ON}}$, i.e.,

$$\xi <_{\mathrm{ON}} \xi' \ \& \ (\forall \eta)[\xi <_{\mathrm{ON}} \eta \Longrightarrow \xi \leq_{\mathrm{ON}} \eta].$$

Infer that every ordinal is a grounded set.

Problem x6.15 ((4) of Theorem 6C.9)**.** Prove that for every $x \subseteq \mathrm{ON}$,

$$\bigcup x = \sup\{\xi : \xi \in x\} = \text{the least ordinal } \eta \text{ such that } (\forall \xi \in x)[\xi \leq_{\mathrm{ON}} \eta].$$

Problem x6.16. Prove that a set $x \subseteq \mathrm{ON}$ of ordinals is an ordinal if and only if x is transitive.

Problem x6.17 ((5) of Theorem 6C.9)**.** Prove that every ordinal number is (uniquely) 0, a successor or a limit, and also that ON is a proper class.

Problem x6.18* (Mostowski collapsing for classes)**.** Suppose $E(u, v)$ is a binary condition such that:

(1) For each v, $\{u : E(u, v)\}$ is a set.
(2) $(\forall x \neq \emptyset)(\exists t \in x)(\forall u \in x)\neg E(u, t)$, i.e., $E(x, y)$ is (strict and) grounded.

Prove that there is exactly one operation $D : \mathrm{Field}(E) \to \mathcal{V}$ such that

(6F-1) $$D(v) = \{D(u) : E(u, v)\} \qquad (v \in \mathrm{Field}(E)).$$

Verify that the hypotheses of the problem are satisfied if the Axiom of Foundation holds and for some class M,

$$E_M(u, v) \iff u, v \in M \ \&\ u \in v.$$

The operation D is the *decoration* or *Mostowski surjection* of the condition $E(u, v)$.

Problem x6.19*. Suppose $E(u, v)$ satisfies (1) and (2) of Problem x6.17* and it is also *extensional*, i.e.,

(6F-2) $$(\forall t)[E(t, u) \leftrightarrow E(t, v)] \to u = v \qquad (u, v \in \mathrm{Field}(E)).$$

Let $D : \mathrm{Field}(E) \to \mathcal{V}$ be the Mostowski surjection of $E(u, v)$.

Prove that the image $\overline{\mathrm{Field}(E)} = \{D(v) : v \in \mathrm{Field}(E)\}$ is a transitive, grounded class and D is an injection which carries E to the membership relation, i.e., for $u, v \in \mathrm{Field}(E)$,

(6F-3) $$u = v \iff D(u) = D(v), \quad D(u) \in D(v) \iff E(u, v).$$

Problem x6.20 (Ordinal addition)**.** Define a binary operation $\alpha + \beta$ on ordinals such that

$$\begin{aligned} \alpha + 0 &= \alpha, \\ \alpha + (\beta + 1) &= (\alpha + \beta) + 1, \\ \alpha + \lambda &= \sup\{\alpha + \beta : \beta \in \lambda\} \quad (\lambda \text{ limit}). \end{aligned}$$

Show that $\alpha + \beta = \mathrm{ot}(\leq_{\alpha \uplus \beta})$ where $\leq_{\alpha \uplus \beta}$ is the wellordering defined by adding $\leq_\beta$ at the end of $\leq_\alpha$:

$$\mathrm{Field}(\leq_{\alpha \uplus \beta}) = \alpha \uplus \beta,$$
$$\langle i, \xi \rangle <_{\alpha \uplus \beta} \langle j, \eta \rangle \iff i < j \vee [i = j \ \&\ [\xi \in \eta] \qquad (i = 0, 1).$$

Problem x6.21 (Ordinal addition inequalities)**.** Show that for all ordinals $\alpha, \beta, \gamma, \delta$:

$$\begin{gathered}0 + \alpha = \alpha, \text{ and } n \in \omega \leq \alpha \Longrightarrow n + \alpha = \alpha,\\ 0 < \beta \Longrightarrow \alpha < \alpha + \beta,\\ \alpha \leq \beta \;\&\; \gamma \leq \delta \Longrightarrow \alpha + \gamma \leq \beta + \delta,\\ \alpha \leq \beta \;\&\; \gamma < \delta \Longrightarrow \alpha + \gamma < \beta + \delta.\end{gathered}$$

Show also that, in general,

$$\alpha < \beta \text{ does not imply } \alpha + \gamma < \beta + \gamma.$$

Problem x6.22. Give examples of strictly increasing sequences of ordinals such that

$$\begin{gathered}\lim_n(\alpha_n + \beta) \neq \lim_n \alpha_n + \beta,\\ \lim_n(\alpha_n + \beta_n) \neq \lim_n \alpha_n + \lim_n \beta_n.\end{gathered}$$

Problem x6.23 (Ordinal multiplication)**.** Define a binary operation $\alpha \cdot \beta$ on ordinals such that

$$\begin{aligned}\alpha \cdot 0 &= 0,\\ \alpha \cdot (\beta + 1) &= (\alpha \cdot \beta) + \alpha,\\ \alpha \cdot \lambda &= \sup\{\alpha \cdot \beta : \beta \in \lambda\} \quad (\lambda \text{ limit}).\end{aligned}$$

Show that $\alpha \cdot \beta = \mathrm{ot}(\leq_{\alpha\times\beta})$ where $\leq_{\alpha\times\beta}$ is the inverse lexicographic wellordering on $\alpha \times \beta$,

$$\begin{gathered}\mathrm{Field}(\leq_{\alpha\times\beta}) = \alpha \times \beta,\\ \langle \xi_1, \eta_1 \rangle <_{\alpha,\beta} \langle \xi_2, \eta_2 \rangle \iff \eta_1 \in \eta_2 \vee [\eta_1 = \eta_2 \;\&\; \xi_1 \in \xi_2],\end{gathered}$$

so that $\alpha \cdot \beta$ is the rank of the wellordering constructed by laying out β copies of α one after the other. Verify that

$$\begin{gathered}\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma,\\ \alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma.\end{gathered}$$

Problem x6.24. Show that $2 \cdot \omega = \omega$ while $\omega < \omega \cdot 2$, so that ordinal multiplication is not in general commutative. Show also that for all $\alpha \geq \omega$,

$$\begin{aligned}&(\alpha + 1) \cdot n = \alpha \cdot n + 1 \quad (1 < n < \omega),\\ &(\alpha + 1) \cdot \omega = \alpha \cdot \omega,\end{aligned}$$

and infer that in general

$$(\alpha + \beta) \cdot \gamma \neq \alpha \cdot \gamma + \beta \cdot \gamma.$$

Problem x6.25 (Cancellation laws)**.** For all ordinals α, β, γ,

$$\begin{aligned} \alpha + \beta < \alpha + \gamma &\Longrightarrow \beta < \gamma, \\ \alpha + \beta = \alpha + \gamma &\Longrightarrow \beta = \gamma, \\ \alpha \cdot \beta < \alpha \cdot \gamma &\Longrightarrow \beta < \gamma, \\ 0 < \alpha \ \&\ \alpha \cdot \beta = \alpha \cdot \gamma &\Longrightarrow \beta = \gamma. \end{aligned}$$

Show also that, in general,

$$0 < \alpha \ \&\ \beta \cdot \alpha = \gamma \cdot \alpha \text{ does not imply } \beta = \gamma.$$

A *rank function* for a relation r is any

$$f : \text{Field}(r) \to \text{ON such that } x <_r y \Longrightarrow f(\xi) \in f(y).$$

A rank function is *tight* if its image $f[\text{Field}(r)]$ is an ordinal.

Problem x6.26. Prove that a relation r is wellfounded if and only if it admits a rank function. Show also that a wellfounded relation admits a unique tight rank function.

Problem x6.27. Prove that a set x is grounded if and only if every $y \in x$ is grounded.

Problem x6.28. Prove that the Axiom of Foundation holds if and only if every set is grounded.

Problem x6.29* ((2) of Theorem 6C.20)**.** Prove that for any two sets x, y,

$$(x \leq_c y \ \&\ y \leq_c x) \Longrightarrow x =_c y.$$

Problem x6.30. Prove that if x and y are wellorderable, then so are $x \cup y$ and $x \times y$.

Problem x6.31. Prove (1) – (4) of Theorem 6C.22.

Problem x6.32*. Prove the absorption law for cardinal multiplication, (5) of Theorem 6C.22.

Problem x6.33 (ZF)**.** Prove that $\mathcal{P}(\omega) =_c \mathcal{N}$, where $\mathcal{N} = (\omega \to \omega)$ is *Baire space*, the set of all functions on the natural numbers.

Problem x6.34 (ZF)**.** In one of the standard *arithmetizations of analysis*, the real numbers are identified with the set of *Dedekind cuts* of rationals,

$$\begin{aligned} \mathbb{R} = \{x \subseteq \mathbb{Q} : \emptyset \neq x \neq \mathbb{Q} \\ \&\ (\forall u \in x)(\forall v \in \mathbb{Q})[v < u \Longrightarrow v \in x] \ \&\ (\forall u \in x)(\exists v \in x)[u < v]. \end{aligned}$$

Outline a proof of $\mathbb{R} =_c \mathcal{P}(\omega)$ based on this definition of $\mathbb{R}$. (You will need to define $\mathbb{Q}$ in some natural way and check that $\mathbb{Q} =_c \omega$.)

Problem x6.35 (ZF)**.** Prove that for every set x, there is an ordinal ξ onto which x cannot be surjected,

$$(\forall x)(\exists \xi \in \mathrm{ON})(\forall f : x \to \xi)[f[x] \subsetneq \xi].$$

Problem x6.36 (ZF)**.** Prove that the class Card of cardinal numbers is proper, closed and unbounded.

Problem x6.37 (ZF)**.** Prove that for all ordinals η, ξ,

$$\eta < \xi \Longrightarrow \aleph_\eta < \aleph_\xi.$$

Problem x6.38 (ZF, Cantor's Theorem 6D.1)**.** Prove that for every set x, $x <_c \mathcal{P}(x)$.

Problem x6.39 (ZF)**.** Prove that a set x is grounded if and only if $\mathcal{P}(x)$ is grounded.

Problem x6.40 (ZF)**.** Prove Theorem 6D.5, the basic properties of the cumulative hierarchy of sets.

Problem x6.41 (ZF)**.** Prove the equivalence of the basic, elementary expressions of the Axiom of Choice, Theorem 6D.9.

Problem x6.42* (ZF)**.** Prove that if the powerset of every wellorderable set is wellorderable, then every grounded set is wellorderable.

Problem x6.43 (ZF)**.** Prove that $V \models \mathsf{ZF} +$ Foundation, specifying whether this is a theorem or a theorem scheme. Infer that ZF cannot prove the existence of an *illfounded* (not grounded) set.

Problem x6.44 (ZF)**.** Prove that the equivalence

$$(\forall (i \mapsto a_i))[(\forall i \in I)[a_i \neq \emptyset] \iff \textstyle\prod_{i\in I} a_i \neq \emptyset]$$

is equivalent to **AC**.

Problem x6.45 (ZFC)**.** Prove the cardinal equations and inequalities in Theorem 6E.1, and determine the values of λ, μ for which the implication

$$\lambda \leq \mu \Longrightarrow 0^\lambda \leq 0^\mu \qquad (\kappa \neq 0)$$

fails.

Problem x6.46. Prove Theorem 6E.4.

Problem x6.47. Write out the theorem scheme which is expressed by the Reflection Theorem 6D.7.

Problem x6.48***.** Prove that ZF, ZF_g and ZFC are not finitely axiomatizable (unless, of course, they are inconsistent). (Recall that by Definition 4A.6, a τ-theory T is finitely axiomatizable if there is a finite set T of τ-sentences which has the same theorems as T.)

Problem x6.49* (ZFC)**.** Prove that if κ is a strongly inaccessible cardinal, then $V_\kappa \models \mathsf{ZFC}$, specifying whether this is a theorem or a theorem scheme. Infer that

$$\mathsf{ZFC} \nvdash (\exists\kappa)[\kappa \text{ is strongly inaccessible}].$$

Problem x6.50 (ZFC)**.** Prove that if there exists a strongly inaccessible cardinal, then there exists a countable, transitive set M such that

$$M \models \mathsf{ZFC}.$$

Problem x6.51* (ZFC)**.** True or false: *if* $V_\kappa \models \mathsf{ZFC}$*, then* κ *is strongly inaccessible.* You must prove your answer.

Problem x6.52. Give a correct version of the construction $\phi \mapsto (\phi)^M$ in Definition 6D.6 when M is a class defined by a formula with parameters.

Problem x6.53 (ZF)**.** Prove Theorem 6D.8.

Problem x6.54 (ZFC)**.** What is $\Big(\prod_{i\in I} \mathbf{A}_i\Big)/U$ is U is a principal ultrafilter on I?

Problem x6.55 (ZFC)**.** Let $\overline{\mathbf{A}} = \Big(\prod_{i\in I} \mathbf{A}\Big)/U$ be the ultrapower of a structure $\mathbf{A}$ modulo an ultrafilter U. Prove that there exists an elementary embedding $\pi : \mathbf{A} \rightarrowtail \overline{\mathbf{A}}$, and that π is an isomorphism if and only if U is principal. (Elementary embeddings are defined in Definition 2A.1.)

Problem x6.56 (ZFC)**.** Finish the argument in the proof of Łós's Theorem 6E.11.

Problem x6.57. Let I be an infinite set and F_0 a set of non-empty subsets of I which has the (weak) *finite intersection property*, i.e.,

$$X_1, X_2 \in F_0 \Longrightarrow (\exists X \in F_0)[X_1 \cap X_2 \supseteq X].$$

Prove that the set

$$F = \{Z \subseteq I : (\exists X \in F_0)[Z \supseteq X]\}$$

is a filter which extends F_0.

Problem x6.58. Give a proof of the Compactness Theorem 1J.1 for languages of arbitrary cardinality following the hint below.

Compactness Theorem (ZFC). *For any signature τ, if T is a τ-theory and every finite subset of T has a model, then T has a model.*

Hint: Let I be the set of all finite conjunctions $\phi_1 \,\&\, \cdots \,\&\, \phi_n$ of sentences in T, and choose (by the hypothesis) for each $\phi \in I$ a τ-structure $\mathbf{A}_\phi$ such that $\mathbf{A}_\phi \models \phi$. Let

$$X_\phi = \{\psi \in I : \mathbf{A}_\phi \models \psi\}, \quad F_0 = \{X_\phi : \phi \in I\}.$$

Check that each $X_\phi \neq \emptyset$ and that $X_\phi \cap X_\psi \supseteq X_{\phi \,\&\, \psi}$, so that F_0 has the weak intersection property and can be extended to a filter F by Problem x6.57 and then to an ultrafilter U on I by Theorem 6E.9. Now apply Łós's Theorem.

CHAPTER 7

THE CONSTRUCTIBLE UNIVERSE

Our main aim in this Chapter is to define Gödel's class L of constructible sets and to prove (in ZF) that it satisfies all the axioms of ZFC, as well as the Generalized Continuum Hypothesis. One of many corollaries will be the consistency of $\mathsf{ZFC} + \mathbf{GCH}$ relative to ZF.

Convention: *Unless otherwise specified* (as in Chapter 6), *all results in this Chapter are proved from the axioms of* ZF^-_g, i.e., ZF^- + Foundation.

This, means, in effect, that we are working in von Neumann's universe V of grounded sets but do not appeal to the powerset axiom—except as specified.

In fact, most of the arguments we will give do not depend on the axiom of foundation, and in a few cases, where it is important, we will point this out. It simplifies the picture, however, to include it in the background theory.

7A. Preliminaries and the basic definition

Our main aim in this section is to define L and show (in ZF_g) that is it is a model of ZF_g. The method is robust and can be extended to define many interesting "inner models" of set theory.

We have often made the argument that all classical mathematics can be "developed" in set theory. This is certainly true of mathematical logic, as we covered the subject in the first five chapter of these lecture notes, and perhaps more naturally than it is true of (say) analysis or probability, since the basic notions of logic are inherently set theoretical.

To be just a bit more specific:

- We fix once and for all a specific sequence $\mathbf{v} : \omega \to V$ whose values $\mathbf{v}_0, \mathbf{v}_1, \dots$, are the *variables*, (perhaps setting $\mathbf{v}_i = 2i \in \omega$).
- We fix once and for all specific sets for the logical symbols $\neg$, $\&$, $\dots \exists, \forall$, the parentheses and the comma (perhaps $\neg = 1, \& = 3, \dots$).

- A vocabulary (or signature) is any finite tuple

$$\tau = \langle \mathrm{Const}, \mathrm{Rel}, \mathrm{Funct}, \mathrm{arity} \rangle,$$

  such that the sets Const, Rel, Funct are pairwise disjoint (and do not contain any variables, logical or punctuation symbols as we chose those), and arity : $\mathrm{Const} \cup \mathrm{Rel} \cup \mathrm{Funct} \to \omega$.

The syntactic objects of $\mathbb{FOL}(\tau)$ (terms, formulas, etc.) are now finite sequences from these basic sets and their formal definitions in $\mathbb{FOL}(\in)$ are obtained by formalizing their customary definitions. Structures of a specific signature τ are tuples of the form

$$\mathbf{A} = \langle A, \{c^{\mathbf{A}}\}_{c \in \mathrm{Const}}, \{R^{\mathbf{A}}\}_{R \in \mathrm{Rel}}, \{f^{\mathbf{A}}\}_{f \in \mathrm{Funct}} \rangle$$

which satisfy the obvious conditions, and the definitions of all the other semantic notions (homomorphisms, satisfaction, etc.) are also assumed to have been formalized in $\mathbb{FOL}(\in)$. Especially interesting is the *structure of arithmetic*

$$\mathbf{N} = \langle \omega, 0, S, +, \cdot \rangle \tag{7A-1}$$

which is definable in ZF^-, since ω is definable and addition and subtraction on ω can be defined by recursion, Theorem 6C.6. We will often use without explicit mention the fact that arithmetical relations on ω are definable in $\mathbb{FOL}(\in)$.

We do not need to get into the details of these formalizations of the basic notions of logic or the proofs in axiomatic set theory of the results in Chapters 1 – 5 any more than we need to do this in topology or probability theory. Except for one thing: for some of the metamathematical results with which we are concerned, it is sometimes very important to note that some theorems can be proved in a relatively weak set theory—ZF^- or ZF_g (without **AC**) for example—and so we will need to notice this. *As a general rule, most every result in Chapters 1 – 5 can be formalized and proved in* ZF^-, without using the Powerset, Foundation or Choice axioms. (The most notable exception is the Downward Skolem-Löwenheim Theorem 2B.1 which depends on **AC**.)

When we use variables m, n, k in the next theorem, it is understood that the conditions in question do not hold and the operations in question are set $= \emptyset$, unless $m, n, k \in \omega$. We continue with the numbering in Theorem 6C.2.

Theorem 7A.1. *The following conditions and operations on sets are definable:*

#37. $\mathrm{Formula}(m, n) \iff m$ is the code of a (full extended) formula $\varphi(\mathbf{v}_0, \dots, \mathbf{v}_{n-1})$ of the language $\mathbb{FOL}(\in)$

$$\#38.\ \mathrm{Sat}(m,n,x,A,e) \iff \mathrm{Formula}(m,n) \\ \&\, x : n \to A \,\&\, e \subseteq A \times A \\ \&\, \big[\text{ if } \varphi(\mathbf{v}_0,\dots,\mathbf{v}_{n-1}) \text{ is the formula with code } m\text{, then } (A,e) \models \varphi\big[x(0),\dots,x(n-1)\big]\big]$$

$$\#39.\ \mathbf{Def}_1(m,n,x,A,e) = \big\{s \in A : \mathrm{Sat}(m,n+1,x \cup \{\langle n,s\rangle\},A,e)\big\}$$

$$\#40.\ \mathbf{Def}(A) = \big\{\mathbf{Def}_1(m,n,x,A,\{\langle u,v\rangle : u \in v \,\&\, u \in A \,\&\, v \in A\}) : m \in \omega \,\&\, n \in \omega \,\&\, x : n \to A\big\}$$

Proof. #37 is immediate since $\mathrm{Formula}(m,n)$ is recursive.

#39 and #40 will follow immediately once we prove #38, that the satisfaction condition is definable.

To prove #38, let

$$F_1(m,n,x,A,e) = \begin{cases} 1 & \text{if } m \text{ is the code of some full extended formula} \\ & \varphi(\mathbf{v}_0,\dots,\mathbf{v}_{n-1}) \text{ and } x : n \to A \text{ and } e \subseteq A \times A \\ & \text{and } (A,e) \models \varphi\big[x(0),\dots,x(n-1)\big], \\ 0 & \text{otherwise} \end{cases}$$

and put

$$F(m,A,e) = \{\langle i,n,x,F_1(i,n,x,A,e)\rangle : n \in \omega \,\&\, i < m \in \omega \\ \&\, x : n \to A \,\&\, e \subseteq A \times A\};$$

it is enough to show that F is definable in $\mathbb{FOL}(\in)$, since

$$\mathrm{Sat}(m,n,x,A,e) \iff \langle m,n,x,1\rangle \in F(m+1,A,e).$$

To define F by recursion, applying Theorem 6C.6, we need definable operations G_1, G_2 such that

$$F(0,A,e) = G_1(A,e),$$
$$F(m+1,A,e) = G_2\big(F(m,A,e),m,A,e\big).$$

The first of these is trivial, since $F(0,A,e) = \emptyset$. On the other hand,

$$F(m+1,A,e) = F(m,A,e) \cup G_3(m,A,e)$$

where $G_3(m,A,e) = \emptyset$, unless m is the code of some full extended formula $\varphi(\mathbf{v}_0,\dots,\mathbf{v}_{n-1})$; and if m is the code of some such formula, then we can easily compute $G_3(m,A,e)$ from $F(m,A,e)$ because of the inductive nature of the definition of satisfaction—and the fact that formulas are assigned bigger codes than their proper subformulas. We will skip the details. ⊣

In (mathematical) English:

$$x \in \mathbf{Def}(A) \iff x \subseteq A \text{ and there is a full extrended formula } \varphi(\mathbf{v}_0,\dots,\mathbf{v}_{n-1},\mathbf{v}_n) \text{ in the language } \mathbb{FOL}(\in) \text{ and members } x_0,\dots,x_{n-1} \text{ of } A\text{, such that for all } s \in A,\ s \in x \iff (A,\in) \models \varphi[x_0,\dots,x_{n-1},s].$$

Definition 7A.2 (ZF$^-$)**.** We now define the constructible hierarchy by the ordinal recursion

$$\begin{aligned} L_0 &= \emptyset, \\ L_{\xi+1} &= \mathbf{Def}(L_\xi), \\ L_\lambda &= \textstyle\bigcup_{\xi<\lambda} L_\xi, \text{ if } \lambda \text{ is a limit ordinal} \end{aligned}$$

and we set $L = \bigcup_\xi L_\xi$. This is Gödel's class of *constructible sets*.

More generally, for any set A, put

$$\begin{aligned} L_0(A) &= \mathrm{TC}(A), \\ L_{\xi+1}(A) &= \mathbf{Def}\big(L_\xi(A)\big), \\ L_\lambda(A) &= \textstyle\bigcup_{\xi<\lambda} L_\xi(A), \text{ if } \lambda \text{ is a limit ordinal,} \end{aligned}$$

and set $L(A) = \bigcup_\xi L_\xi(A)$. This is the class of sets *constructible from* A.

Theorem 7A.3 (ZF$^-$)**.** (i) *The operation* $\xi \mapsto L_\xi$ *is definable and* L *is a definable class.*

(ii) $\eta \leq \xi \Longrightarrow L_\eta \subseteq L_\xi$.

(iii) *Each* L_ξ *is a transitive, grounded set,* L *is a transitive class and* $L \subseteq V$.

Similarly,

(ia) *The operation* $(\xi, A) \mapsto L_\xi(A)$ *is definable, and if* A *is a definable set, then* $L(A)$ *is a definable class.*

(iia) $\eta \leq \xi \Longrightarrow L_\eta(A) \subseteq L_\xi(A)$.

(iiia) *Each* $L_\xi(A)$ *is a transitive set and* $L(A)$ *is a transitive class. If, in addition,* A *is grounded, then every* $L_\xi(A)$ *is grounded and* $L(A) \subseteq V$.

PROOF. (i) follows immediately from Theorem 6C.16.

To prove (ii) and (iii) we show simultaneously by ordinal induction that for each ξ,

$$L_\xi \text{ is transitive, grounded and } \eta < \xi \Longrightarrow L_\eta \subseteq L_\xi.$$

This is trivial for $\xi = 0$ or limit ordinals ξ.

If $\xi = \zeta + 1$, suppose first that $\eta = \zeta$ and $x \in L_\zeta$. The induction hypothesis gives us that $x \subseteq L_\zeta$; and since x is clearly definable in L_ζ by the formula $\mathbf{v}_i \in x$ (with the parameter x), we have $x \in L_{\zeta+1}$. So $L_\zeta \subseteq L_{\zeta+1}$, and the transitivity of $L_{\zeta+1}$ follows immediately. If $\eta < \zeta$, then the induction hypothesis gives again $x \in L_\zeta$, and so $x \in L_{\zeta+1}$ by what we have just proved.

Now L is easily transitive as the union of transitive sets and (ia)–(iiia) are proved similarly. ⊣

To prove that L satisfies ZF^-_g, we need to look a little more carefully at its definition.

Definition 7A.4 (Σ_0 formulas)**.** Let Σ_0 be the smallest collection of formulas in the language $\mathbb{FOL}(\in)$ which contains all prime formulas

$$\mathbf{v}_i \in \mathbf{v}_j, \quad \mathbf{v}_i = \mathbf{v}_j$$

and is closed under the propositional operations and the bounded quantifiers, so that if φ and ψ are in Σ_0, then so are the formulas

$$\neg(\varphi),\ (\varphi)\,\&\,(\psi),\ (\varphi)\vee(\psi),\ (\varphi)\to(\psi),\ (\exists \mathbf{v}_i \in \mathbf{v}_j)\phi,\ (\forall \mathbf{v}_i \in \mathbf{v}_j)\phi.$$

Proposition 7A.5. *Prove that the conditions #1, #2, #8, #9, #14, #17 and #18 or 6C.2 are definable by Σ_0 formulas.*

One of the simplifying consequences of the Axiom of Foundation is that the class of ordinals becomes definable by a Σ_0 formula:

(7A-2) $\quad \xi \in \mathrm{ON}$

$$\iff (\forall x \in \xi)(\forall x \in y)[x \in \xi]\ \&\ (\forall x, y \in \xi)[x \in y \vee x = y \vee y \in x].$$

This is very useful, because of the following, simple but very basic fact about Σ_0:

Lemma 7A.6. *Let M be a transitive class.*

(i) *If $\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is a full extended Σ_0 formula and $x_1, \ldots, x_n \in M$, then*

$$V \models \varphi[x_1, \ldots, x_n] \iff M \models \varphi[x_1, \ldots, x_n].$$

(ii) *M satisfies the Axioms of Extensionality and Foundation.*

(iii) *If M is closed under pairing and union, then it satisfies the Pairing and Unionset axioms.*

(iv) *If some infinite ordinal $\lambda \in M$, then M satisfies the Axiom of Infinity.*

Proof. (i) Reverting to the notation of Theorem 1C.8 which is more appropriate here, we need to verify that if φ is any formula in Σ_0 and $\pi : \text{Variables} \to M$ is any assignment into M, then

$$V, \pi \models \varphi \iff M, \pi \models \varphi.$$

This is immediate for prime formulas, e.g.,

$$\begin{aligned} V, \pi \models \mathbf{v}_i \in \mathbf{v}_j &\iff \pi(\mathbf{v}_i) \in \pi(\mathbf{v}_j) \\ &\iff M, \pi \models \mathbf{v}_i \in \mathbf{v}_j \end{aligned}$$

(because π takes values in M) and if the required equivalence holds for φ and ψ, it obviously holds for $\neg(\varphi)$ and for $(\varphi)\,\&\,(\psi)$. By induction on the length of formulas then, in one of the non-trivial cases,

$$\begin{aligned} V, \pi \models (\exists \mathbf{v}_i)[\mathbf{v}_i \in \mathbf{v}_j \,\&\, \varphi] &\iff \text{for some } z \in \pi(\mathbf{v}_j),\ V, \pi\{\mathbf{v}_i := z\} \models \varphi \\ &\iff \text{for some } z \in \pi(\mathbf{v}_j),\ M, \pi\{\mathbf{v}_i := z\} \models \varphi \\ &\iff M, \pi \models (\exists \mathbf{v}_i)[\mathbf{v}_i \in \mathbf{v}_j \,\&\, \varphi], \end{aligned}$$

where we have used the transitivity of M and (again) the fact that π takes values in M in the main, middle equivalence.

(ii) Both of these axioms are expressed in $\mathbb{FOL}(\in)$ by formulas of the form $(\forall \mathbf{x}_1)\cdots(\forall \mathbf{x}_n)\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ where $\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is in Σ_0 and

$$\text{for all } x_1, \ldots, x_n \in M, V \models \phi[x_1, \ldots, x_n];$$

this implies with (i) that $M \models (\forall \mathbf{x}_1)\cdots(\forall \mathbf{x}_n)\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$.

(iii) Again, it is easy to find a formula $\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ in Σ_0 such that for x, y, z,

$$z = \{x, y\} \iff V \models \varphi[x, y, z].$$

To show that M satisfies the Pairing Axiom then, we must verify that for each $x \in M$, $y \in M$, there is some $z \in M$ such that $M \models \varphi[x, y, z]$; of course, we take $z = \{x, y\}$ and we use (i).

The argument for the Unionset Axiom is similar.

(v) If $\lambda \in M$ and λ is infinite, then either $\omega = \lambda$ or $\omega \in \lambda$ and in either case, by the transitivity of M, $\omega \in M$. Checking the definition of ω in Theorem 6C.2, we can construct a Σ_0 formula $\varphi(\mathbf{x})$ such that

$$x = \omega \iff V \models \varphi[x];$$

in part $\varphi(\mathbf{x})$ asserts that $\mathbf{x}$ is the z required to exist by the Axiom of Infinity. Clearly $V \models \varphi[\omega]$ and then by (i), $M \models \varphi[\omega]$ so that M satisfies the Axiom of Infinity. ⊣

The lemma implies immediately that L satisfies all the axioms of ZF_g except perhaps for the Power and Replacement Axioms. The key to deriving these for L is the Reflection Theorem 6D.7, but it is worth putting down a general result.

It is convenient to call a class M *grounded* if every set in it is grounded, i.e., if $M \subseteq V$ (cf. Problem x7.5). All classes are grounded in ZF^-_g, but it is instructive make an exception to the general convention of this Chapter and show the next theorem with the minimum hypotheses.

Theorem 7A.7 (ZF^-). *Let $\xi \mapsto C_\xi$ be an operation on ordinals to sets which satisfies the following four conditions, where $C = \bigcup_{\xi \in \mathrm{ON}} C_\xi$.*

(i) *Each C_ξ is a grounded, transitive set.*

(ii) $\zeta \leq \xi \Longrightarrow C_\zeta \subseteq C_\xi$.

(iii) *If λ is a limit ordinal, then* $C_\lambda = \bigcup_{\xi<\lambda} C_\xi$.

(iv) *For each ξ,* $\mathbf{Def}(C_\xi) \subseteq C$, *i.e., for each ξ, if $x \subseteq C_\xi$ is elementary in the structure*

$$\mathbf{C}_\xi = (C_\xi, \in\restriction C_\xi, \{s : s \in C_\xi\}),$$

then there is some ζ such that $x \in C_\zeta$.

It follows that C is a transitive subclass of V, it contains all the ordinals and $C \models \mathsf{ZF}_g^-$; and if, in addition, we assume the Powerset Axiom, then $C \models \mathsf{ZF}_g$.

In particular, $L \subseteq V$, it is a transitive model of ZF_g^- which contains all the ordinals, and if we assume the Powerset Axiom, then $L \models \mathsf{ZF}_g$.

Similarly for $L(A)$, if A is grounded.

PROOF. To begin with, we know from Lemma 7A.6 that C satisfies extensionality, pairing and unionset, since condition (iv) in the hypothesis implies easily that C is closed under pairing and union and these parts of Lemma 7A.6 where proved without the axiom of foundation. Also, $C_\xi \subseteq V$ by ordinal induction, and so $C \subseteq V$—and then it satisfies the Axiom of Foundation because V does.

We argue that C must contain all ordinals: if not, let λ be the least ordinal not in C and choose ξ large enough so that $\lambda \subseteq C_\xi$. Since V satisfies the Axiom of Foundation, for $x \in V$,

$$\text{Ordinal}(x) \iff V \models \phi_{\text{ON}}[x]$$

where

$$\phi_{\text{ON}}(\mathbf{x}) \equiv (\forall u \in \mathbf{x})(\forall v \in u)[v \in \mathbf{x}] \;\&\; (\forall u, v \in \mathbf{x})[u \in v \vee u = v \vee v \in u]$$

is a Σ_0-formula, as in (7A-2). Since no ordinal $\geq \lambda$ can be in C_ξ (by transitivity), we have

$$\{x \in C_\xi : \mathbf{C}_\xi \models \varphi_{\text{ON}}[x]\} = \lambda;$$

hence by condition (iv), $\lambda \in C$, which is a contradiction.

It follows in particular that $\omega \in C$, so that C also satisfies the Axiom of Infinity by 7A.6.

Verification of the Powerset Axiom (ZF). It is enough to show that for each $x \in C$, there is some $z \in C$ such that z has as members precisely all the members of C which are subsets of x—from this we can infer that C satisfies the Powerset Axiom as above. Let

$$\text{rank}_C(u) = \begin{cases} \text{least } \eta \text{ such that } u \in C_\eta, & \text{if } u \in C, \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$\lambda = \bigcup \text{rank}_C[\mathcal{P}(x)]$$

so that if $u \in C$ and $u \subseteq x$, then $u \in C_\lambda$. Thus

$$z = \{u \in C_\lambda : u \subseteq x\}$$

has as members precisely the subsets of x which are in C and since z is clearly definable in $\mathbf{C}_\lambda$, it is a member of C by (iv).

Verification of the Axiom Scheme of Replacement. Suppose $x \in C$ and $F : C \to C$ is an operation which is definable (with parameters) on C, i.e., for some formula $\psi(\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{s}, \mathbf{t})$ and fixed $y_1, \dots, y_n \in C$,

$$F(s) = t \iff C \models \psi[y_1, \dots, y_n, s, t] \qquad (s, t \in C);$$

as above, it is enough to show that the image

$$F[x] = \{F(s) : s \in x\}$$

is also a member of C.

Using the Reflection Theorem 6D.7, choose λ so that $x, y_1, \dots, y_n \in C_\lambda$ and for $s, t \in C_\lambda$,

$$C \models \psi[y_1, \dots, y_n, s, t] \iff C_\lambda \models \psi[y_1, \dots, y_n, s, t],$$

make sure as in the argument above that $F[x] \subseteq C_\lambda$, and set

$$\psi^*(\mathbf{y}_1, \dots, \mathbf{y}_n, \mathbf{x}, \mathbf{t}) \equiv (\exists \mathbf{s} \in \mathbf{x})\psi(\mathbf{y}_1, \dots, \mathbf{y}_n, \mathbf{x}, \mathbf{t}).$$

Clearly

$$F[x] = \{t \in C_\lambda : \mathbf{C}_\lambda \models \psi^*[y_1, \dots, y_n, \mathbf{s}, t]\},$$

and hence $F[x]$ is elementary in $\mathbf{C}_\lambda$ and must be in C by (iv).

This concludes the proof of the main part of the theorem and the fact that L and $L(A)$ satisfy the hypotheses follows easily from their definitions. $\dashv$

The recursive definition of the constructible hierarchy $\{L_\xi : \xi \in \mathrm{ON}\}$ makes it possible to define explicitly a wellordering of L. We prove this in some detail, as it is the key to our showing in the next section that the Axiom of Choice holds in L.

Theorem 7A.8 (The wellordering of L)**.** *There is a definable binary condition $x \leq_L y$ which wellorders L, and in such a way that*

$$x \leq_L y \,\&\, y \in L_\xi \Longrightarrow x \in L_\xi.$$

Proof. The idea is to define by ordinal recursion an operation

$$F : \mathrm{ON} \to V$$

so that for each ξ, $F(\xi) = \ \leq_\xi$ is a wellordering of L_ξ, i.e., $\leq_\xi \subseteq L_\xi \times L_\xi$ and $\leq_\xi$ wellorders L_ξ.

We will build up F step-by-step.

Step 1. There is a definable operation $F_1 : \omega \times V \times V \to V$ such that if w wellorders A, then $F_1(n, w, A)$ wellorders the set $(n \to A)$ of n-term sequences from A.

Proof. Order the n-tuples from A lexicographically, using w.

Step 2. There is a definable operation $F_2 : V^2 \to V$ such that if w wellorders A, then $F_2(w, A)$ wellorders $A^* = \bigcup_{n\in\omega}(n \to A)$.

Proof. For x, x' in A^*, put

$$\begin{aligned}\langle x, x'\rangle \in F(w, A) \iff\ & \mathrm{Domain}(x) < \mathrm{Domain}(x') \\ & \vee(\exists n)[\mathrm{Domain}(x) = \mathrm{Domain}(x') = n \\ & \qquad\qquad \& \langle x, x'\rangle \in F_1(n, w, A)].\end{aligned}$$

Step 3. There is a definable operation $F_3 : V^2 \to V$ such that if w wellorders A, then $F_3(w, A)$ wellorders $\mathbf{Def}(A)$.

Proof. Using the operation $\mathbf{Def}_1$ of Theorem 7A.1, put

$$G_1(m, n, x, A) = \mathbf{Def}_1(m, n, A, \{\langle u, v\rangle : u \in A \,\&\, v \in A \,\&\, u \in v\})$$

and for $y \in \mathbf{Def}(A)$ define successively:

$$G_2(y, w, A) = \text{least } m \text{ such that } (\exists n)(\exists x : n \to A)[y = G_1(m, n, x, A)],$$

$$G_3(y, w, A) = \text{least } n \text{ such that } (\exists x : n \to A)[y = G_1\big(G_2(y, w, A), n, x, A\big)],$$

$$G_4(y, w, A) = \text{least } x \text{ in the ordering } F_2(w, A) \text{ such that}$$
$$y = G_1\big(G_2(y, w, A), G_3(y, w, A), x, A\big).$$

Now each $y \in \mathbf{Def}(A)$ is completely determined by the triple

$$\big(G_2(y, w, A), G_3(y, w, A), G_4(y, w, A)\big)$$

and we can order these triples lexicographically, using the wellordering $F_2(w, A)$ in the last component.

Step 4. There is a definable operation $F : \mathrm{ON} \to V$ such that for each ξ, $F(\xi)$ is a wellordering of L_ξ.

We define $F(\xi)$ by ordinal recursion, taking cases on whether ξ is 0, a successor or limit.

Two of the cases are trivial: we set $F(0) = \emptyset$ and $F(\xi{+}1) = F_3\big(F(\xi), L_\xi\big)$.

If λ is limit, define first $G : L \to \mathrm{ON}$ by

$$G(x) = \text{least } \xi \text{ such that } x \in L_\xi$$

and put

$$\begin{aligned}F(\lambda) = \big\{\langle x, y\rangle \in\ & L_\lambda \times L_\lambda : G(x) < G(y) \\ & \vee\big[G(x) = G(y) \,\&\, \langle x, y\rangle \in F\big(G(x)\big)\big]\big\}.\end{aligned}$$

The theorem follows from this by setting again

$$x \leq_L y \iff G(x) < G(y) \vee \big[G(x) = G(y) \,\&\, \langle x, y\rangle \in F\big(G(x)\big)\big]. \qquad \dashv$$

7B. Absoluteness

At first blush, it seems like Theorem 7A.8 proves that L satisfies **AC**: we defined a condition $x \leq_L y$ on the constructible sets and we showed that it wellorders L, from which it follows that

(7B-3) "if every set is in L,

then $\{(x, y) : x \leq_L y\}$ wellorders the universe of all sets".

This is a very strong, "global" and definable form of the Axiom of Choice for L, and we proved it in ZF_g^- (in fact in ZF^-)—but it does not quite mean the same thing as "$L \models \mathbf{AC}$"!

To see the subtle difference in meaning between the two claims in quotes, let us express (7B-3) in the language $\mathbb{FOL}(\in)$. Choose first a formula $\varphi_L(\mathbf{x}, \boldsymbol{\xi})$ of $\mathbb{FOL}(\in)$ by 7A.3 so that

$$x \in L_\xi \iff V \models \varphi_L[x, \xi] \tag{7B-4}$$

and let

$$V = L :\equiv (\forall \mathbf{x})(\exists \boldsymbol{\xi})\varphi_L(\mathbf{x}, \boldsymbol{\xi}). \tag{7B-5}$$

The formal sentence "$V = L$" expresses in $\mathbb{FOL}(\in)$ the proposition that *every (grounded) set is constructible.* Choose then another formula $\psi_L(\mathbf{x}, \mathbf{y})$ of $\mathbb{FOL}(\in)$ by 7A.8 such that

$$x \leq_L y \iff V \models \psi_L[x, y]$$

and set

$$\psi^* \iff \text{"}\{(\mathbf{x}, \mathbf{y}) : \psi_L(\mathbf{x}, \mathbf{y})\} \text{ is a wellordering of the universe"},$$

where it is easy to turn the symbolized English in quotes into a formal sentence of $\mathbb{FOL}(\in)$. Now (7B-3) is expressed by the formal sentence of $\mathbb{FOL}(\in)$

$$(V = L) \rightarrow \psi^*,$$

and what we would like to prove is that

$$L \models \psi^*. \tag{7B-6}$$

It is important here that Theorem 7A.8 was proved in ZF without appealing to **AC**. Since L is a model of ZF_g by 7A.7, it must also satisfy all the consequences of ZF_g and certainly

$$L \models (V = L) \rightarrow \psi^*. \tag{7B-7}$$

Now the hitch is that in order to infer (7B-6) from (7B-7), we must prove

$$L \models V = L \qquad \text{(Caution! Not proved yet).} \tag{7B-8}$$

This is what we are tempted to take as "obvious" in a sloppy reading of (7B-3). But is (7B-8) obvious?

By the definition of satisfaction and the construction of the sentence $V = L$ above, (7B-8) is equivalent to

(7B-9) $\quad$ for each $x \in L$, there exists $\xi \in L$ such that $L \models \varphi_L[x, \xi]$,

while what we know is

(7B-10) $\quad$ for each $x \in L$, there exists $\xi \in L$ such that $V \models \varphi_L[x, \xi]$.

Thus, to complete the proof of (7B-8) and verify that L satisfies the Axiom of Choice, we must prove that we can choose the formula $\varphi_L(\mathbf{x}, \boldsymbol{\xi})$ so that in addition to (7B-4), it also satisfies

(7B-11) $$V \models \varphi_L[x, \xi] \iff L \models \varphi_L[x, \xi],$$

when $x \in L$. In other words, we must show that *the basic condition of constructibility can be defined in* $\mathbb{FOL}(\in)$ *so that the model* L *recognizes that each of its members is constructible*.

The theory of *absoluteness* (for grounded classes) which we will develop to do this is the key to many other results, including the fact that $V = L$ implies the Generalized Continuum Hypothesis. We will study here the basic facts about absoluteness and then we will derive the consequences about L in the next section.

Definition 7B.1 (Absoluteness)**.** Let R be an n-ary condition on V, let $\phi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be a full extended $\mathbb{FOL}(\in)$-formula, and let $\mathcal{D}$ be a collection of transitive subclasses of V. We say that $\phi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ *defines* R *absolutely for* $M \in \mathcal{D}$ if

$$R(x_1, \ldots, x_n) \iff M \models \varphi[x_1, \ldots, x_n] \quad (M \in \mathcal{D}, x_1, \ldots, x_n \in M).$$

A condition R is *absolute for* $\mathcal{D}$ if it is defined by some formula absolutely for $M \in \mathcal{D}$. It is also common to call *absolute for* $\mathcal{D}$ the relevant formula $\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ of $\mathbb{FOL}(\in)$ which defines a condition absolutely for $\mathcal{D}$.

Notice that if $\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ defines R absolutely for $\mathcal{D}$, then in particular, for M, N in $\mathcal{D}$, if $M \subseteq N$ and $x_1, \ldots, x_n \in M$, then

$$M \models \varphi[x_1, \ldots, x_n] \iff N \models \varphi[x_1, \ldots, x_n].$$

In all the cases we will consider, the universe V of grounded sets will be in $\mathcal{D}$; then for each M in $\mathcal{D}$ and $x_1, \ldots, x_n \in M$, we have

$$\begin{aligned} R(x_1, \ldots, x_n) &\iff V \models \varphi[x_1, \ldots, x_n] \\ &\iff M \models \varphi[x_1, \ldots, x_n]. \end{aligned}$$

We express this by saying that R *is absolute for* M.

Following the same idea, an operation $F : C_1 \times \cdots \times C_n \to V$ (where $C_1, \dots, C_n$ are given classes) is *definable absolutely for* $\mathcal{D}$ or just *absolute for* $\mathcal{D}$, if three things hold.

(1) The classes $C_1, C_2, \dots, C_n$ are absolute for $\mathcal{D}$—i.e., each membership condition $x \in C_i$ is absolute for $\mathcal{D}$.

(2) If $M \in \mathcal{D}$ and $x_1 \in C_1 \cap M, \dots, x_n \in C_n \cap M$, then

$$F(x_1, \dots, x_n) \in M.$$

(3) There is a formula $\varphi(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y})$ of $\mathbb{FOL}(\in)$ such that for each $M \in \mathcal{D}$ and $x_1 \in C_1 \cap M, \dots, x_n \in C_n \cap M$,

$$F(x_1, \dots, x_n) = y \iff M \models \varphi[x_1, \dots, x_n, y].$$

A set c is *absolute for* $\mathcal{D}$ if for each $M \in \mathcal{D}$, $c \in M$ and the condition

$$R_c(x) \iff x = c$$

is absolute for $\mathcal{D}$.

We now come to the central metamathematical concept of T-absoluteness, where T is any *set theory*, e.g., ZF^-, ZF^-_g, ZF, ZFC, etc. We simplify the discussion a bit by collectively calling *notions* the relations and operations on V as well as the members of V (following Gödel).

Definition 7B.2 (T-absoluteness)**.** Let T be a set of $\mathbb{FOL}(\in)$-sentences—a *set theory*.

A *standard model* of T is any transitive, grounded class M (perhaps a set) such that $M \models T$; if in addition M contains all the ordinals, then M is an *inner model* of T. (By Theorem 7A.7, *L and each $L(A)$ are inner models of* ZF^-_g, and in ZF we proved that they are both inner models of ZF_g.)

A notion N is T*-absolute* if there exists a finite set $T^0 \subseteq T$ of axioms of T such that N is absolute for the collection $\mathcal{D}^0$ of standard models of T^0,

$$M \in \mathcal{D}^0 \iff M \text{ is transitive and } M \models T^0.$$

Notice that *if N is T-absolute and $T \subseteq T'$, then N is T'-absolute.* We are especially interested in ZF^-_g-absolute notions, which are then T-absolute for every axiomatic set theory stronger than ZF^-_g. Intuitively, a notion N is T-absolute if there is a formula of $\mathbb{FOL}(\in)$ which defines N in all standard models of some sufficiently large, finite part of T.

We will need to know that a good many notions are ZF^-_g-absolute, including all those defined in Theorems 6C.2 and 7A.1, and we start with the closure properties of the collection of T-absolute notions.

All but the last two parts of the next theorem have nothing to do with any particular set-theoretic principles—they are simple facts of logic.

Theorem 7B.3. *Let T be any set theory such that $V \models T$.*

(i) *The collection of T-absolute conditions contains $\in$ and $=$ and is closed under the propositional operations $\neg$, $\&$, $\vee$, $\Longrightarrow$, $\Longleftrightarrow$.*

(ii) *The collection of T-absolute operations is closed under addition and permutation of variables and under composition; each n-ary projection operation*

$$F(x_1, \dots, x_n) = x_i$$

is T-absolute.

(iii) *An object $c \in V$ is T-absolute if and only if each n-ary constant operation*

$$F(x_1, \dots, x_n) = c$$

is T-absolute.

(iv) *If $R \subseteq V^m$ and $F_1 : C_1 \times \cdots \times C_n \to V, \dots, F_m : C_1 \times \cdots \times C_n \to V$ are all T-absolute and*

$$P(x_1, \dots, x_n) \iff x_1 \in C_1 \,\&\, \cdots \,\&\, x_n \in C_n \\ \&\, R\big(F_1(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_n)\big),$$

then P is also T-absolute.

(v) *If $R \subseteq V^{n+1}$ is T-absolute and*

$$P(x_1, \dots, x_n, z) \iff (\exists y \in z) R(x_1, \dots, x_n, y),$$
$$Q(x_1, \dots, x_n, z) \iff (\forall y \in z) R(x_1, \dots, x_n, y),$$

then P and Q are also T-absolute.

(vi) *Suppose $P \subseteq V^{n+1}$ and $Q \subseteq V^{n+1}$ are both T-absolute, and there exists a finite $T^0 \subseteq T$ such that for each standard M, if $M \models T^0$ and $x_1, \dots, x_n \in M$, then*

$$(\exists y \in M) P(x_1, \dots, x_n, y) \iff (\forall y \in M) Q(x_1, \dots, x_n, y);$$

then the condition $R \subseteq V^n$ defined by

$$R(x_1, \dots, x_n) \iff (\exists y) P(x_1, \dots, x_n, y)$$

is T-absolute.

(vii) *Suppose $T \supseteq \mathsf{ZF}_g^-$. If $G : V^{n+1} \to V$ is T-absolute, then so is the operation $F : V^{n+1} \to V$ defined by*

$$F(x_1, \dots, x_n, w) = \{G(x_1, \dots, x_n, t) : t \in w\}.$$

Similarly with parameters, if $G : V^{n+m} \to V$ is T-absolute, so is

$$F(x_1, \dots, x_n, w_1, \dots, w_m) \\ = \{G(x_1, \dots, x_n, t_1, \dots, t_m) : t_1 \in w_1 \,\&\, \cdots \,\&\, t_n \in w_n\}.$$

(viii) *If* $T \supseteq \mathsf{ZF}_g^-$ *and* $R \subseteq V^{n+1}$ *is* T*-absolute, then so is the operation*

$$F(x_1, \dots, x_n, w) = \{t \in w : R(x_1, \dots, x_n, t)\} \quad (x_1, \dots , x_n, w \in V).$$

Proof. Parts (i) – (iv) are very easy, using the basic properties of the language $\mathbb{FOL}(\in)$.

For example if

$$R(x_1, \dots, x_n) \iff P(x_1, \dots, x_n) \,\&\, Q(x_1, \dots, x_n)$$

with P and Q given T-absolute conditions, choose finite $T^0 \subseteq \mathbf{ZF}$, $T^1 \subseteq T$ and formulas $\varphi(\mathbf{x}_1, \dots, \mathbf{x}_n)$, $\psi(\mathbf{x}_1, \dots, \mathbf{x}_n)$ of $\mathbb{FOL}(\in)$ such that

$$P(x_1, \dots, x_n) \iff M \models \varphi[x_1, \dots, x_n] \quad (M \models T^0,\ x_1, \dots, x_n \in M)$$

and for $M \models T^1$, $x_1, \dots, x_n \in M$,

$$Q(x_1, \dots, x_n) \iff M \models \psi[x_1, \dots, x_n].$$

It is clear that if $M \models T^0 \cup T^1$ and $x_1, \dots, x_n \in M$, then

$$R(x_1, \dots, x_n) \iff M \models \varphi[x_1, \dots, x_n] \,\&\, \psi[x_1, \dots, x_n],$$

so the formula $\varphi(\mathbf{x}_1, \dots, \mathbf{x}_n) \,\&\, \psi(\mathbf{x}_1, \dots, \mathbf{x}_n)$ defines R absolutely on all standard models of $T^0 \cup T^1$.

Suppose again that

$$F(x) = G\big(H_1(x), H_2(x)\big)$$

where G, H_1, H_2 are T-absolute and we have chosen one binary and two unary operations to simplify notation. Choose finite subsets T^G, T^1, T^2 of T and formulas $\psi(\mathbf{u}, \mathbf{v}, \mathbf{z})$, $\varphi_1(\mathbf{x}, \mathbf{u})$, $\varphi_2(\mathbf{x}, \mathbf{v})$ of $\mathbb{FOL}(\in)$ such that for $M \models T^G$ and $u, v, z \in M$ we have $G(u, v) \in M$ and

$$G(u, v) = z \iff M \models \psi[u, v, z]$$

and similarly with H_1, T^1 and $\varphi_1(\mathbf{x}, \mathbf{u})$, H_2, T^2 and $\varphi_2(\mathbf{x}, \mathbf{v})$. (It is easy to arrange that the free variables in these formulas are as indicated.) Now it is clear that if

$$M \models T^G \cup T^1 \cup T^2,$$

then

$$x \in M \Longrightarrow F(x) \in M$$

and for $x, z \in M$,

$$F(x) = z \iff M \models \chi[x, z]$$

where

$$\chi(\mathbf{x}, \mathbf{z}) \iff (\exists \mathbf{u})(\exists \mathbf{v})[\varphi_1(\mathbf{x}, \mathbf{u}) \,\&\, \varphi_2(\mathbf{x}, \mathbf{v}) \,\&\, \psi(\mathbf{u}, \mathbf{v}, \mathbf{z})].$$

Proof of (iv) is very similar to this.

(v) The argument is very similar to the proof of (i) in Lemma 7A.6 and we will omit it—the transitivity of M is essential here.

(vi) Choose a formula $\varphi(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y})$ and a finite $T^P \subseteq T$ such that for all standard $M \models T^P$ and $x_1, \dots, x_n \in M$,

$$P(x_1, \dots, x_n, y) \iff M \models \varphi[x_1, \dots, x_n, y]$$

and take

$$\chi(\mathbf{x}_1, \dots, \mathbf{x}_n) \iff (\exists \mathbf{y})\varphi(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y}).$$

If $M \models T^P \cup T^0$ and $x_1, \dots, x_n \in M$, then

$$\begin{aligned}
R(x_1, \dots, x_n) &\Longrightarrow (\exists y)P(x_1, \dots, x_n, y) \\
&\Longrightarrow (\forall y)Q(x_1, \dots, x_n, y) && \text{(since } V \models T^0\text{)} \\
&\Longrightarrow (\forall y \in M)Q(x_1, \dots, x_n, y) && \text{(obviously)} \\
&\Longrightarrow (\exists y \in M)P(x_1, \dots, x_n, y) && \text{(since } M \models T^0\text{)} \\
&\Longrightarrow \text{for some } y \in M,\ M \models \varphi[x_1, \dots, x_n, y] && \text{(since } M \models T^P\text{)} \\
&\Longrightarrow M \models (\exists \mathbf{y})\varphi[x_1, \dots, x_n, \mathbf{y}];
\end{aligned}$$

Conversely,

$$\begin{aligned}
M \models (\exists \mathbf{y})\varphi[x_1, \dots, x_n, \mathbf{y}] &\Longrightarrow (\exists y \in M)P(x_1, \dots, x_n, y) \\
&\Longrightarrow (\exists y)P(x_1, \dots, x_n, y) \\
&\Longrightarrow R(x_1, \dots, x_n),
\end{aligned}$$

so $\chi(\mathbf{x}_1, \dots, \mathbf{x}_n)$ defines R on all models of $T^P \cup T^0$ and hence R is T-absolute.

(vii) Suppose that if $M \models T^0$, then

$$x_1, \dots, x_n, t \in M \Longrightarrow G(x_1, \dots, x_n, t) \in M$$

and

$$G(x_1, \dots, x_n, t) = s \iff M \models \varphi[x_1, \dots, x_n, t, s].$$

Let ψ be the instance of the Replacement Axiom Scheme which concerns $\varphi(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{t}, \mathbf{s})$,

$$\begin{aligned}
\psi \iff (\forall \mathbf{x}_1) \cdots (\forall \mathbf{x}_n)(\forall \mathbf{w})\big\{&(\forall \mathbf{t})(\exists ! \mathbf{s})\varphi(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{t}, \mathbf{s}) \\
&\to (\exists \mathbf{z})(\forall \mathbf{s})\big[\mathbf{s} \in \mathbf{z} \leftrightarrow (\exists \mathbf{t})[\mathbf{t} \in \mathbf{w} \,\&\, \varphi(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{t}, \mathbf{s})]\big]\big\}
\end{aligned}$$

and take

$$T^1 = T^0 \cup \{\psi\}.$$

If $M \models T^1$ and $x_1, \dots, x_n, w \in M$, this means easily that there is some $z \in M$ so that for all $a \in M$,

$$\begin{aligned}
s \in z &\iff \text{for some } t \in w,\ M \models \varphi[x_1, \dots, x_n, t, s] \\
&\iff (\exists t \in w)[G(x_1, \dots, x_n, t) = s].
\end{aligned}$$

Since $M \models T^0$ and hence M is closed under G, this implies that in fact

$$\begin{aligned} z &= \{G(x_1, \dots, x_n, t) : t \in w\} \\ &= F(x_1, \dots, x_n, w), \end{aligned}$$

hence M is closed under F. Moreover, taking

$$\chi(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{w}, \mathbf{z}) \iff (\forall \mathbf{s})\big[\mathbf{s} \in \mathbf{z} \leftrightarrow (\exists \mathbf{t})[\mathbf{t} \in \mathbf{w} \,\&\, \varphi(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{t}, \mathbf{s})]\big],$$

is is clear that

$$F(x_1, \dots, x_n, w) = z \iff M \models \chi[x_1, \dots, x_n, w, z],$$

so F is T absolute.

The argument with $m > 1$ is similar.

(viii) Let

$$G(x_1, \dots, x_n, w, t) = \begin{cases} t & \text{if } R(x_1, \dots, x_n, t), \\ w & \text{if } \neg R(x_1, \dots, x_n, t). \end{cases}$$

This is T-absolute by (ii) and then by the hypothesis that $T \supseteq \mathsf{ZF}^-_g$, and (vii) (and (ii) again), the operation

$$F(x_1, \dots, x_n, w) = \{G(x_1, \dots, x_n, w, t) : t \in w\} \cap w$$

is also T-absolute. Clearly

$$\begin{aligned} s \in F(x_1, \dots, x_n, w) \iff\ & s \in w \,\&\, R(x_1, \dots, x_n, s) \\ & \vee [s = w \,\&\, w \in w \,\&\, (\exists t)\neg R(x_1, \dots, x_n, t)]; \end{aligned}$$

and since $w \in V$ so $w \notin w$,

$$s \in F(x_1, \dots, x_n, w) \iff s \in w \,\&\, R(x_1, \dots, x_n, s)$$

as required. ⊣

Corollary 7B.4. *The notions #1 – #21 of Theorem 6C.2 are all* ZF^-_g-*absolute.*

Proof is routine using the theorem and we will skip it. ⊣

Before proceeding to show the ZF^-_g-absoluteness of several other notions, it will be instructive to notice that *many natural and useful notions are not even* ZFC-*absolute*, cf. Problem x7.1*. Roughly speaking, no notion related to *cardinality* is ZFC-absolute.

The next result is fundamental.

Theorem 7B.5 (Mostowski's Theorem)**.** *The condition*

$$\mathrm{WF}(r) \iff r \text{ is a wellfounded relation}$$

is ZF^-_g-*absolute.*

Proof. Put

$$P(r,x) \iff \text{Relation}(r) \ \& \ [x=\emptyset \vee (\exists t\in x)(\forall s \in x)\langle s,t\rangle \notin r].$$

Clearly P is ZF^-_g-absolute and

$$\text{WF}(r) \iff (\forall x)P(r,x).$$

Similarly, let

$$\begin{aligned} Q(r,f) &\iff \text{Relation}(r)\,\&\,[f \text{ is a rank function for } r]\\ &\iff \text{Relation}(r) \ \& \ f:\text{Field}(r)\to \text{ON}\\ &\qquad \& \ (\forall x,y\in \text{Field}(r))[x<_r y \Longrightarrow f(x)\in f(y)]. \end{aligned}$$

Again Q is ZF^-_g-absolute (using the fact that ON is definable by a Σ_0 formula) and

$$\text{WF}(r) \iff (\exists f)Q(r,f).$$

Hence

$$(*) \qquad (\forall r)\Big[(\forall x)P(x,r) \iff (\exists f)Q(r,f)\Big].$$

This equivalence is Problem x6.26, and it can be proved in ZF^-_g.

Let θ be the formal sentence which expresses $(*)$, so that $\mathsf{ZF}^-_g \vdash \theta$. Let $T^* \subseteq \mathsf{ZF}^-_g$ be the finite set of ZF^-_g axioms used in the proof of θ, so that θ is true in all models of T^*, including all the standard models. Let T^0, T^1 be finite subsets of ZF^-_g such that P and Q are absolute for standard models of T^0 and T^1 respectively. It follows that if M is a standard model of $T^0\cup T^1\cup T^*$, then for $r\in M$

$$(**) \qquad (\forall x\in M)P(x,r) \iff (\exists f\in M)Q(r,f).$$

Now part (vi) of 7B.3 implies that $\text{WF}(r)$ is ZF^-_g-absolute. ⊣

Mostowski's proof is simple but typically metamathematical and generally causes uneasiness to people who encounter it for the first time. The subtle part of it is that *we do not need to identify the specific instances of replacement needed to prove θ*—we only need to notice that there are only finitely many of them, and then put them in T^*. In this instance, we could probably pinpoint these instances, but that would be the wrong way to go about understanding the proof: because this sort of argument is used repeatedly, in ever more complex situations where chasing the specific instances of replacement used would be practically impossible. The argument rests on the fact that *proofs are finite*, so that for any formal τ-theory T and any τ-sentence ϕ,

$$T\vdash\phi \Longrightarrow (\exists \text{ finite } T_0\subseteq T)[T_0\vdash \phi].$$

The same kind of metamathematical argument is needed in the proof of the next result.

Theorem 7B.6 (Absoluteness of ordinal recursion)**.** *Suppose the operation $G : V^{n+1} \to V$ is ZF^-_g-absolute, and let*

$$F : \mathrm{ON} \times V^n \to V$$

be the unique operation satisfying

$$F(\xi, x_1, \dots, x_n) = G\big(\{\langle \eta, F(\eta, x_1, \dots, x_n)\rangle : \eta < \xi\}, x_1, \dots, x_n\big);$$

then F is also ZF^-_g-absolute.

Proof. Assume G is absolute for all standard models of $T^0 \subseteq \mathsf{ZF}^-_g$. Go back to the proof of Theorem 6C.16 to recall that F is defined by an expression of the form

$$\begin{aligned} F(\xi, x_1, \dots, x_n) = w \iff (\exists h)\{&P(\xi, x_1, \dots, x_n, h) \,\&\, \mathrm{Function}(h) \\ &\& \, \xi \in \mathrm{Domain}(h) \,\&\, h(\xi) = w\}, \end{aligned}$$

where P is easily absolute for all models of T^0. Moreover, we can prove

$$(\forall \xi, x_1, \dots, x_n)(\exists h) P(\xi, x_1, \dots, x_n, h)$$

using only finitely many additional instances of the Axiom Scheme of Replacement, say those in $T^1 \subseteq \mathsf{ZF}^-_g$. Thus for every standard model M of $T^0 \cup T^1$ and $\xi, x_1, \dots, x_n$ in M we have $(\exists h \in M) P(\xi, x_1, \dots, x_n, h)$, which implies immediately that M is closed under F.

We can also prove easily in ZF^-_g (using only some finite $T^2 \subseteq \mathsf{ZF}^-_g$) that

$$\begin{aligned} &(\forall \xi, x_1, \dots, x_n, w)\big\{(\exists h)[P(\xi, x_1, \dots, x_n, h) \,\&\, \mathrm{Function}(h) \\ &\quad \& \, \xi \in \mathrm{Domain}(h) \,\&\, h(\xi) = w] \iff (\forall h)\big[[P(\xi, x_1, \dots, x_n, h) \\ &\qquad \& \, \mathrm{Function}(h) \,\&\, \xi \in \mathrm{Domain}(h)] \Longrightarrow h(\xi = w]\big\}; \end{aligned}$$

thus by part (vi) of 7B.3, the condition

$$R(\xi, x_1, \dots, x_n, w) \iff F(\xi, x_1, \dots, x_n) = w$$

is ZF^-_g-absolute and so F is ZF^-_g-absolute. ⊣

A special case of definition by recursion on ON is simple recursion on ω.

Corollary 7B.7. *Suppose $F(k, x_1, \dots, x_n)$ satisfies the recursion*

$$\begin{aligned} F(0, x_1, \dots, x_n) &= G(x_1, \dots, x_n) \\ F(k+1, x_1, \dots, x_n) &= G\big(F(k, x_1, \dots, x_n), k, x_1, \dots, x_n\big) \end{aligned}$$

where G_1 and G_2 are ZF^-_g-absolute. Then F is also ZF^-_g-absolute.

Proof. Define

$$G(f,k,x_1,\dots,x_n) = \begin{cases} G_1(x_1,\dots,x_n) & \text{if } m=0,\\ G_2\big(f(k-1,x_1,\dots,x_n),k-1,x_1,\dots,x_n\big) & \\ & \text{if } k\in\omega,\ k\neq 0,\\ 0 & \text{otherwise}\end{cases}$$

and verify easily that G is ZF^-_g-absolute and F is definable from G as in the theorem. ⊣

Corollary 7B.8. *All the conditions and operations #1 – #40 in Theorems 6C.2 and 7A.1 are* ZF^-_g*-absolute.*

Proof. Go back and reread the proofs of these theorems keeping in mind the results of this section. ⊣

7C. The basic facts about L

Let us start by collecting in one theorem the basic absoluteness facts about the constructible hierarchy that follow from the results of the preceding section.

Theorem 7C.1. (i) *The operation $\xi \mapsto L_\xi$ and the binary condition $x \in L_\xi$ are both* ZF^-_g*-absolute.*

(ii) *There is a canonical wellordering of L, $x \le_L y$ which is* ZF^-_g*-absolute and such that*

$$y \in L_\xi \,\&\, x \le_L y \Longrightarrow x \in L_\xi.$$

(iii) *The operation $(\xi, A) \mapsto L_\xi(A)$ and the ternary condition $x \in L_\xi(A)$ are both* ZF^-_g*-absolute.*

(iv) *The conditions $x \in L$ and $x \in L(A)$ are both absolute for inner models of some finite subset $T^0 \subseteq \mathsf{ZF}^-_g$.*

Proof. (i) and (ii) follow immediately from the definitions, 7B.8, 7B.6 and of course, the basic closure properties of ZF^-_g-absoluteness listed in 7B.3. Part (ii) also follows easily by examining the proof of 7A.8.

To prove (iv), let $\varphi_L(\mathbf{x},\boldsymbol{\xi})$ be a formula of $\mathbb{FOL}(\in)$ by (i) such that for some finite $T^0 \subseteq \mathsf{ZF}^-_g$ and any standard M

$$\text{(7C-12)} \qquad x \in L_\xi \iff M \models \varphi_L[x,\xi] \quad (M \text{ standard}, M \models T^0),$$

and set $\psi_L(\mathbf{x}) :\equiv (\exists\boldsymbol{\xi})\varphi_L(\mathbf{x},\boldsymbol{\xi})$. If M is an inner model of T^0 so that $M \models T^0$ and M contains all the ordinals, then for $x \in M$,

$$\begin{aligned} x \in L &\iff \text{for some } \xi,\ x \in L_\xi \\ &\iff \text{for some } \xi \in M,\ M \models \varphi_L[x,\xi] \\ &\iff M \models \psi_L[x]. \end{aligned}$$

The argument for $x \in L(A)$ is similar. ⊣

We are now in a position to prove (7B-8), that L "believes" that every set is constructible.

Fix once and for all a formula $\varphi_L(\mathbf{x}, \boldsymbol{\xi})$ and a finite $T^0 \subset \mathsf{ZF}^-_g$ so that (7C-12) holds and let "$V = L$" abbreviate the formal sentence of $\mathbb{FOL}(\in)$ which says that every set is constructible using this formula,

$$V = L :\equiv (\forall \mathbf{x})(\exists \boldsymbol{\xi})\varphi_L(\mathbf{x}, \boldsymbol{\xi}). \tag{7C-13}$$

We also construct a similar formula $V = L(\mathbf{a})$ with a free variable $\mathbf{a}$ which says that "every set is constructible from $\mathbf{a}$".

Theorem 7C.2. (i) $L \models V = L$.

(ii) *For each grounded set A, $L(A), \mathbf{a} := A \models V = L(\mathbf{a})$.*

Proof. Compute:

$$\begin{aligned} L \models V = L &\iff L \models (\forall \mathbf{x})(\exists \boldsymbol{\xi})\varphi_L(\mathbf{x}, \boldsymbol{\xi}) \\ &\iff \text{for each } x \in L, \text{ there exists } \xi \in L,\ L \models \varphi_L(x, \xi) \\ &\iff \text{for each } x \in L, \text{ there exists } \xi \in L,\ x \in L_\xi, \end{aligned}$$

and the last assertion is true by the definition of L and the fact that it contains all the ordinals. ⊣

This is a very basic result about L. One of its applications is that it allows us to prove theorems about L without constant appeal to metamathematical results and methods: we simply assume $V = L$ in addition to the axioms of ZF^-_g and any consequence of these assumptions must hold in L.

We also put down for the record the result about the Axiom of Choice in L which we discussed in the beginning of Section 7B.

Theorem 7C.3. *There is a formula $\psi_L(\mathbf{x}, \mathbf{y})$ of $\mathbb{FOL}(\in)$ such that*

$$L \models \text{“}\{(x, y) : \psi_L(x, y)\} \text{ is a wellordering of } V\text{”}.$$

*In particular, $L \models$ **AC**.*

Proof. If ψ^* is the formal sentence of $\mathbb{FOL}(\in)$ expressing the symbolized English in quotes, then by 7A.8 and the fact that $L \models \mathsf{ZF}^-_g$,

$$L \models V = L \rightarrow \psi^*$$

while by 7C.2 we have $L \models V = L$. ⊣

For the Generalized Continuum Hypothesis we need another basic fact about L which is also proved by absoluteness arguments. Its proof requires two general facts, not particularly related to L, which could have been included in Chapter 6.

The first of these is the natural generalization of Theorem 2B.1 to uncountable structures.

Lemma 7C.4 (The Downward Skolem-Löwenheim Theorem)**.** *If the universe B of a structure $\mathbf{B}$ of countable signature τ is wellorderable and $X \subseteq B$, then there exists an elementary substructure $\mathbf{A} \preceq \mathbf{B}$ such that $X \subseteq A$ and $|A| = \max(\aleph_0, |X|)$.*

PROOF. The assumption that B is wellorderable is needed to avoid appealing to the Axiom of Choice in the proof of Lemma 2B.4. Except for that, the required argument is a very minor modification of the proof of Theorem 2B.1. We enter it here in full, to avoid the need for extensive page-flipping.

Given $\mathbf{B}$ and $X \subseteq B$, fix some $y_0 \in B$, let

$$Y = X \cup \{y_0\} \cup \{c^{\mathbf{B}} \mid c \text{ a constant symbol}\},$$

so that Y is not empty (even if $X = \emptyset$ and there are no constants). Let $\mathcal{S}_\phi$ be a finite Skolem set for each formula ϕ, by Lemma 2B.4, and set

$$\mathcal{F} = \{f^{\mathbf{B}} \mid f \text{ is a function symbol in } \tau\} \cup \bigcup_\phi \mathcal{S}_\phi.$$

The set $\mathcal{F}$ of Skolem functions is countable, since there are countably many formulas. We define the sequence $n \mapsto A_n$ by the recursion

$$A_0 = Y, \quad A_{n+1} = A_n \cup \bigcup \{f(y_1, \dots, y_k) : f \in \mathcal{F}, y_1, \dots, y_k \in A_n\}$$

and set $A = \bigcup_{n\in\omega} A_n$. This is the universe of some substructure $\mathbf{A} \subseteq \mathbf{B}$ by Lemma 2B.2. Moreover, for each ϕ, A is closed under a Skolem set for ϕ, and so (2B-1) holds, which means that $\mathbf{A} \preceq \mathbf{B}$. Finally, to show that $|A| \leq \max(\aleph_0, |X|)$, we check by induction on n that

$$|A_n| \leq \max(\aleph_0, |X|) = \kappa, \tag{7C-14}$$

which in the end gives $|A| \leq \aleph_0 \cdot \kappa = \kappa$. The inequality (7C-14) is trivial at the base,

$$|A_0| = |Y| \leq |X| + 1 + \aleph_0 = \kappa,$$

and also in the inductive step: if k_f is the arity of each $f \in \mathcal{F}$, then

$$|A_{n+1}| \leq |A_n| + |\bigcup_{f\in\mathcal{F}} f[A_n^{k_f}]| \leq \kappa + \sum_{f\in\mathcal{F}} \kappa^{k_f} \leq \kappa + \aleph_0 \cdot \kappa = \kappa. \qquad \dashv$$

The second lemma we need is a version of the *Mostowski collapsing* construction, which we have covered in three, different forms in Theorem 6C.14 and Problems x6.17*, x6.18*.

Lemma 7C.5 (Mostowski Isomorphism Theorem)**.** *Suppose M is a (grounded) set which* (as a structure with $\in$) *satisfies the Axiom of Extensionality, i.e.,*

$$u = v \iff (\forall t \in M)[t \in u \iff t \in v] \qquad (u, v \in M).$$

Let $d_M : M \to d_M[M]$ be the Mostowski surjection of $\in\restriction M$, so that

$$d_M(u) = \{d_M(v) : v \in M \cap u\} \qquad (u \in M). \tag{7C-15}$$

Then $\overline{M} = d_M[M]$ is a transitive set, $d_M : M \rightarrowtail\!\!\!\!\twoheadrightarrow \overline{M}$ is an $\in$-isomorphism of $(M, \in)$ with $(\overline{M}, \in)$, and if $y \subseteq M$ is transitive, then $d_M(t) = t$ for every $t \in y$.

Proof. The unique function $d_M : M \to V$ satisfying (7C-15) is defined by wellfounded recursion, and its image is a transitive set, directly from (7C-15): because if $s \in d_M(u)$ for some $u \in M$, then

$$s = d_M(v) = \{d_M(t) : t \in M \cap v\}$$

for some $v \in M \cap u$ and so $s \subseteq d_M[M]$.

To prove that d_M is an injection, assume not and let u be an $\in$-minimal counterexample, so that for some $v \in M$, $v \neq u$,

$$d_M(u) = \{d_M(s) : s \in M \cap u\} = \{d_M(t) : t \in M \cap v\} = d_M(v).$$

It follows that if $s \in M \cap u$, then $d_M(s) = d_M(t)$ for some $t \in M \cap v$, so that by the choice of u, $s = t \in M \cap v$. Similarly, if $t \in M \cap v$, then $t \in M \cap u$. So $M \cap u = M \cap v$, and since M satisfies extensionality, $u = v$, which contradicts our assumption.

Finally, if d_M is not the identity on some transitive $y \subseteq M$, choose an $\in$-minimal $t \in y$ such that $d_M(t) \neq t$ and compute:

$$\begin{aligned} d_M(t) &= \{d_M(s) : s \in t\} && \text{(because } t \subseteq y \subseteq M\text{)} \\ &= \{s : s \in t\} && \text{(by the choice of } t\text{)} \\ &= t && \text{(because } t \subseteq y \subseteq M\text{),} \end{aligned}$$

which again contradicts our assumption. ⊣

In the context of the metamathematics of set theory (especially the study of L and other inner models), "the Mostowski Collapsing Lemma" most likely refers to this theorem. We used a different (standard but less common) name for it here, to avoid confusion. In any case, these two results (and Problems x6.17*, x6.18*) have different applications, but they are proved by the same method and they are all significant.

Theorem 7C.6 (The Condensation Lemma)**.** *There is a finite set of sentences $T^0 \subset \mathsf{ZF}_g^-$ such that with*

$$T^L = T^0 \cup \{V = L\}$$

the following hold.

(i) $L \models T^L$.

(ii) *If A is a transitive set and $A \models T^L$, then $A = L_\lambda$ for some limit ordinal λ.*

(iii) *For every infinite ordinal ξ and every set $x \in L$ such that $x \subseteq L_\xi$, there is some ordinal λ such that*

$$\xi \leq \lambda < \xi^+, \quad L_\lambda \models T^L, \text{ and } x \in L_\lambda.$$

Proof. Choose T^0 so that the operations $\xi \mapsto \xi + 1$, $\xi \mapsto L_\xi$, are absolute for the standard models of T^0 and the condition $x \in L_\xi$ is defined on all standard models of T^0 by the specific formula $\varphi_L(\mathbf{x}, \boldsymbol{\xi})$ which we used to construct the sentence $V = L$.

Clearly $L \models T^L$.

If A is a transitive set and $A \models T^L$, let

$$\lambda = \text{least ordinal not in } A$$

and notice that λ is a limit ordinal, since A is closed under the successor operation. Now

$$\xi < \lambda \Longrightarrow L_\xi \in A,$$

by the absoluteness of $\xi \mapsto L_\xi$, so

$$L_\lambda = \bigcup_{\xi<\lambda} L_\xi \subseteq A.$$

On the other hand, $A \models V = L$, so that

$$\text{for each } x \in A, \text{ there exists } \xi \in A,\ A \models \varphi_L[x, \xi]$$

i.e., (by the absoluteness of $\varphi_L(x, \xi)$), $A \subseteq L_\lambda$.

To prove (iii) suppose $x \subseteq L_\xi$ and $x \in L_\zeta$—where ζ may be a much larger ordinal than ξ. Using the Reflection Theorem 6D.7 on the hierarchy $\{L_\eta : \eta \in \text{ON}\}$ and the fact that $L \models T^L$, choose $\mu > \max(\zeta, \xi)$ such that $L_\mu \models T^L$. Now $x \in L_\mu$ and $L_\mu \models T^L$.

By the Downward Skolem-Löwenheim Theorem 7C.4 applied to the (well-orderable) structure $(L_\mu, \in)$, we can find an elementary substructure

$$(M, \in) \preceq (L_\mu, \in)$$

such that $L_\xi \subseteq M$, $x \in M$ and $|M| = |L_\xi| = |\xi|$ by x7.6. Since $(M, \in)$ is elementarily equivalent with $(L_\mu, \in)$, it satisfies in particular the Extensionality Axiom, so by the Mostowski Isomorphism Theorem 7C.5, there is a transitive set $\overline{M}$ and an $\in$-isomorphism

$$d : M \rightarrowtail\!\!\!\!\!\twoheadrightarrow \overline{M}.$$

Moreover, since the transitive set $y = L_\xi \cup \{x\} \subseteq M$, d is the identity on y and hence $x = d(x) \in \overline{M}$. Now $(L_\mu, \in) \models T^L$ and therefore the elementarily equivalent structure $(M, \in) \models T^L$, so that the isomorphic structure $(\overline{M}, \in) \models T^L$; by (ii) then,

$$\overline{M} = L_\lambda$$

for some λ and of course, $\lambda < \xi^+$, since $|\overline{M}| = |\xi|$. $\dashv$

From this key theorem we get immediately the Generalized Continuum Hypothesis for L.

Corollary 7C.7 (ZF)**.** *If $V = L$, then for each cardinal λ, $2^\lambda = \lambda^+$.*

PROOF. By the theorem, if $V = L$, then $\mathcal{P}(\lambda) \subseteq L_{\lambda^+}$, and hence

$$|\mathcal{P}(\lambda)| \leq |L_{\lambda^+}| = \lambda^+. \qquad \dashv$$

We should point out that the models $L(A)$ need not satisfy either the Axiom of Choice or the Continuum Hypothesis. For example, if in V truly $2^{\aleph_0} > \aleph_1$, then there is some surjection

$$\pi : \mathcal{N} \twoheadrightarrow \aleph_2$$

and obviously

$$L\big(\{\langle \alpha, \pi(\alpha)\rangle : \alpha \in \mathcal{N}\}\big) \models 2^{\aleph_0} \geq \aleph_2.$$

As another application of the basic Theorem 7C.1, we obtain intrinsic characterizations of the models L, $L(A)$.

Theorem 7C.8. *L is the smallest inner model of ZF^-_g and for each (grounded) set A, $L(A)$ is the smallest inner model of ZF^-_g which contains A.*

PROOF. Suppose M is an inner model of ZF^-_g and $A_0 \in M$. Since the operation

$$(\xi, A) \mapsto L_\xi(A)$$

is ZF^-_g-absolute, M is closed under this operation; since $A_0 \in M$ and every ordinal $\xi \in M$, we have $(\forall\xi)[L_\xi(A_0) \in M]$ so that $L(A_0) \subseteq M$. $\dashv$

We also put down for the record the relative consistency consequences of of the theory of constructible sets:

Theorem 7C.9. *If ZF is consistent, then so is the theory $\mathsf{ZF}_g + V = L$, and a fortiori the weaker theories ZFC, $\mathsf{ZFC} + \mathbf{GCH}$.*

PROOF. It is useful here to revert to the relativization notation of Definition 6D.6. The key observation is that for any $\mathbb{FOL}(\in)$ formula ϕ,

$$\text{if } \mathsf{ZF}_g + V = L \vdash \phi, \text{ then } \mathsf{ZF} \vdash (\phi)^L. \qquad \text{(7C-16)}$$

This is because $\mathsf{ZF} \vdash (\psi)^L$ for every axiom ψ of ZF_g by Theorem 7A.7; $\mathsf{ZF} \vdash (V = L)^L$ by Theorem 7C.2; and, pretty trivially,

$$\text{if } \psi_1, \dots, \psi_n \vdash \psi, \text{ then } (\psi_1)^M, \dots, (\psi_n)^M \vdash (\psi)^M,$$

for any definable class M, not just L. If $\mathsf{ZF} + V = L$ were inconsistent, then $\mathsf{ZF} + V = L \vdash \chi \mathbin{\&} \neg\chi$ for some χ for some χ, and then $\mathsf{ZF} \vdash (\chi)^L \mathbin{\&} \neg(\chi)^L$, so that ZF would also be inconsistent. $\dashv$

This is an example of a *finitistic relative consistency proof*: it can be formalized in a (very small) fragment of Peano arithmetic, but, more than that, it is generally recognized as a valid, constructive, combinatorial argument which assumes nothing about infinite objects beyond the usual properties of finite strings of symbols.

7D. $\Diamond$

Our (very limited) aim in this section is to introduce a basic principle of infinite combinatorics and prove that it holds in L. It was first formulated by Jensen to prove that L satisfies several propositions which are independent of ZFC, but we will not go into this here beyond a brief comment at the end: our main interest in $\Diamond$ is that its proof in L illustrates in a novel way many of the methods we have developed.

A *guessing sequence* (for ω_1) is any ω_1-sequence of functions on countable ordinals

$$s = \{s_\xi\}_{\xi\in\omega_1} \quad (s_\xi : \xi \to \xi). \tag{7D-17}$$

Definition 7D.1. $\Diamond$: *There exists a guessing sequence $s = \{s_\xi\}_{\xi\in\omega_1}$ such that for every $f : \omega_1 \to \omega_1$, there is at least one $\xi > 0$ such that $f \restriction \xi = s_\xi$.*

The *diamond principle* seems weak, but the next Proposition shows that it has considerable strength. For the proof, we will need to appeal to some simple properties of *pairing functions* on ordinals which we will leave for Problem x7.12.

Proposition 7D.2 (ZFC)**.** *If $\Diamond$ holds, then there is a guessing sequence $\{t_\xi\}_{\xi\in\omega_1}$ which guesses correctly $\aleph_1$-many restrictions of every $f : \omega_1 \to \omega_1$, i.e.,*

$$|\{\xi : f \restriction \xi = t_\xi\}| = \aleph_1 \quad (f : \omega_1 \to \omega_1).$$

Proof. Let $\{s_\xi\}_{\xi\in\omega_1}$ be a guessing sequence guaranteed by $\Diamond$, suppose $f : \omega_1 \to \omega_1$ is given, fix $\zeta < \omega_1$, and set

$$h_\zeta(\eta) = \langle f(\eta), \zeta\rangle \text{ so that } f(\eta) = (h_\zeta(\eta))_0.$$

Let $\xi(\zeta) > 0$ be such that

$$h_\zeta \restriction \xi(\zeta) = s_{\xi(\zeta)}.$$

This means that for every $\eta < \xi(\zeta)$, $f(\eta) = (s_{\xi(\zeta)})_0$; and it implies immediately that the sequence

$$t_\xi(\eta) = (s_\xi(\eta))_0 \le s_\xi(\eta) < \xi \quad (\xi < \omega_1, \eta < \xi)$$

guesses $f \restriction \xi(\zeta)$ correctly for every ζ. Moreover, these ordinals are all distinct, since

$$(s_{\xi(\zeta)}(0))_1 = \zeta,$$

so that we cannot have $\xi(\zeta_1) = \xi(\zeta_2) > 0$ when $\zeta_1 \neq \zeta_2$. ⊣

It is important in this proof, of course, to notice that the new guessing sequence $\{t_\xi\}_{\xi \in \omega_1}$ is defined directly from the one guaranteed by $\Diamond$, without reference to any specific f or ordinal ζ.

Corollary 7D.3 (ZFC). $\Diamond \Longrightarrow$ **CH**.

Proof. Fix a guessing sequence $\{t_\xi\}_{\xi \in \omega_1}$ which guesses correctly $\aleph_1$-many restrictions of every $f : \omega_1 \to \omega_1$, and for each $f : \omega \to \omega$ apply its characteristic property to the extension $\tilde{f} : \omega_1 \to \omega$ of f which is set $= 0$ for $\xi \geq \omega$. Let $\xi(f)$ be the least infinite ordinal such that $\tilde{f} \restriction \xi(f) = t_{\xi(f)}$; now $\xi(f)$ determines f uniquely, so that the map $f \mapsto \xi(f)$ is an injection of $(\omega \to \omega)$ into ω_1 and establishes the Continuum Hypothesis. ⊣

Theorem 7D.4 (ZFC). *If $V = L$, then $\Diamond$.*

Proof. We assume $V = L$ and define s_ξ by recursion on $\xi < \omega_1$, starting with (the irrelevant) $s_0 = \emptyset$. For $\xi > 0$, let

(7D-18) $s_\xi =$ the $\leq_L$-least function $h : \xi \to \xi$ such that
for every $\zeta < \xi, \zeta \neq 0$, $h \restriction \zeta \neq s_\zeta$,

with the understanding that if no h with the required property exists, then s_ξ is the constant 0 on ξ. Recall that by our general convention about "indexed sequences",

$$\{s_\xi\}_{\xi \in \omega_1} = s : \omega_1 \to (\omega_1 \to \omega_1),$$

i.e., s is a function, and $s_\xi = s(\xi)$ for every $\xi \in \omega_1$.

To prove that for every $f : \omega_1 \to \omega_1$, this sequence s guesses correctly $f \restriction \xi$ for at least one $\xi > 0$, assume that it does not, and let

(7D-19) $f =$ the $\leq_L$-least function $h : \omega_1 \to \omega_1$ such that
for every $\zeta < \omega_1, \zeta > 0$, $h \restriction \zeta \neq s_\zeta$.

Notice that by the Condensation Lemma, $s, f \in L_{\omega_2}$, cf. Problem x7.11.

A set $a \in L_{\omega_2}$ is definable (in L_{ω_2}) if there is a formula $\phi(\mathbf{x})$ such that

$$L_{\omega_2} \models (\exists! \mathbf{x})\phi(\mathbf{x}) \text{ and } L_{\omega_2} \models \phi[a].$$

We let

$$M = \{a \in L_{\omega_2} : a \text{ is definable in } L_{\omega_2}\}.$$

Lemma 1. $M \prec L_{\omega_2} \models \mathsf{ZF}^-_g$ and $\omega, \omega_1, s, f \in M$.

Proof. By Problem x7.10, $L_{\omega_2} \models \mathsf{ZF}^-_g$, and so all ZF^-_g-absolute notions are absolute for L_{ω_2}. In particular, the usual definitions of ω, ω_1 define these sets in L_{ω_2} and the formula which defines the canonical wellordering $\leq_L$ is also absolute for L_{ω_2}, and so we can interpret the definitions of s and f in L_{ω_2}; which means that $s, f \in M$.

To prove that $M \prec L_{\omega_2}$ by the basic test for elementary substructures Lemma 2A.3, it is enough to check that for every full extended formula $\phi(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y})$ and all $\vec{x} = x_1, \ldots, x_n \in M$,

$$\text{if there exists some } y \in L_{\omega_2} \text{ such that } L_{\omega_2} \models \phi[\vec{x}, y],$$
$$\text{then there exists some } z \in M \text{ such that } L_{\omega_2} \models \phi[\vec{x}, z].$$

This is immediate setting

$$z = \text{the } \leq_L\text{-least } y \in L_{\omega_2} \text{ such that } L_{\omega_2} \models \phi[\vec{x}, y]. \quad \dashv \text{ (Lemma 1)}$$

Let $d : M \rightarrowtail\!\!\!\!\twoheadrightarrow L_\lambda$ be the Mostowski isomorphism for M, so $\lambda < \omega_1$ and

$$d : M \rightarrowtail\!\!\!\!\twoheadrightarrow L_\lambda \models \mathsf{ZF}^-_g, \ (\forall y)[\mathrm{TC}(y) \subset M \Longrightarrow d(y) = y].$$

Lemma 2. If $F : L^n \to L$ is a ZF^-_g-absolute operation, then

$$x_1, \ldots, x_n \in M$$
$$\Longrightarrow F(x_1, \ldots, x_n) \in M \ \& \ d(F(x_1, \ldots, x_n)) = F(d(x_1), \ldots, d(x_n)).$$

Proof. Suppose $\phi(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y})$ defines F on every transitive model of ZF^-_g. In particular, $L_{\omega_2} \models (\forall \vec{\mathbf{x}})(\exists ! \mathbf{y})\phi(\vec{\mathbf{x}}, \mathbf{y})$, and so $M \models (\forall \vec{\mathbf{x}})(\exists ! \mathbf{y})\phi(\vec{\mathbf{x}}, \mathbf{y})$ which means that M is closed under F. Moreover, for $x_1, \ldots, x_n, y \in M$,

$$M \models \phi[x_1, \ldots, x_n, y] \iff L_\lambda \models \phi[d(x_1), \ldots, d(x_n), d(y)]$$
$$\iff F(d(x_1), \ldots, d(x_n)) = d(y),$$

the last because $L_\lambda \models \mathsf{ZF}^-_g$ and so $\phi(\vec{\mathbf{x}}, \mathbf{y})$ also defines F on it.$\dashv$ (Lemma 2)

Lemma 2 implies in particular that if $g \in M$, then $\mathrm{Domain}(g) \in M$ and for every $a \in \mathrm{Domain}(g) \cap M$,

$$d(g(a)) \in M \text{ and } d(g(a)) = d(g)(d(a)), \tag{7D-20}$$

simply because the operations

$$g \mapsto \mathrm{Domain}(g), \quad (g, a) \mapsto g(a)$$

are ZF^-_g-absolute. In particular, if $\xi < \omega_1$, then

$$\xi \in M \Longrightarrow f(\xi), s_\xi \in M, \text{ and } [\eta, \xi \in M \ \& \ \eta < \xi] \Longrightarrow s_\xi(\eta) \in M.$$

Lemma 3. If ξ is countable and $\xi \in M$, then $d(\xi) = \xi$.

Proof. We can prove in ZF^-_g that *every countable ordinal* ξ *is the image of some* $g : \omega \twoheadrightarrow \xi$, and so if ξ is definable in L_{ω_2}, then so is

$$g = \text{the } \le_L\text{-least } g : \omega \twoheadrightarrow \xi.$$

It follows that every $\eta < \xi$ is $g(n)$ for some $n \in \omega$ and hence definable in L_{ω_2}; and then $\xi + 1 = \mathrm{TC}(\xi) \subset M$, and so the Mostowski isomorphism d is the identity on $\xi + 1$ and gives $d(\xi) = \xi$. ⊣ (Lemma 3)

Lemma 4. If $\mu = d(\omega_1)$, *then* $d(f) = f \restriction \mu$ *and for* $\xi < \mu$, $d(s_\xi) = s_\xi$.

Proof. The key observation is that

$$\xi < \mu \iff [\xi \in M \ \&\ \xi < \omega_1].$$

This is because using Lemma 3,

$$\xi \in M \ \&\ \xi < \omega_1 \Longrightarrow \xi = d(\xi) \ \&\ d(\xi) < d(\omega_1) = \mu,$$

and on the other hand,

$$\begin{aligned}\xi < \mu \Longrightarrow & (\exists \eta)[\eta \in M \ \&\ \eta < \omega_1 \ \&\ \xi = d(\eta)] \\ & \Longrightarrow (\exists \eta)[\eta \in M \ \&\ \eta < \omega_1 \ \&\ \xi = \eta] \Longrightarrow \xi \in M \ \&\ \xi < \omega_1.\end{aligned}$$

In particular, $\xi < \mu \Longrightarrow d(\xi) = \xi$, and since $f \in M$ and $f(\xi) < \omega_1$, by (7D-20),

$$\xi < \mu \Longrightarrow f(\xi) = d(f(\xi)) = d(f)(d(\xi)) = d(f)(\xi),$$

i.e., $d(f) = f \restriction \mu$. Similarly,

$$\eta < \xi < \mu \Longrightarrow s_\xi(\eta) = d(s_\xi(\eta)) = d(s_\xi)(\eta),$$

and so for $\xi < \mu$, $d(s_\xi) = s_\xi$. ⊣ (Lemma 4)

We now consider the definition (7D-18) of s_μ: it is the unique $g : \mu \to \mu$ which satisfies the condition

$$\begin{aligned}\phi[g, \mu] \equiv g \text{ is the } \le_L\text{-least } h : \mu \to \mu \text{ such that} \\ \text{for every } \zeta < \mu, \zeta > 0, h \restriction \mu \neq s_\zeta.\end{aligned}$$

Since the formula $\phi(\mathbf{x}, \mathbf{y})$ is ZF^-_g-absolute and $s_\mu \in L_{\omega_2}$, this implies that

$$s_\mu \text{ is the unique } g \text{ such that } L_{\omega_2} \models \phi[g, \mu].$$

By the definition (7D-19) of f and the same reasoning,

$$f \text{ is the unique } h \text{ such that } L_{\omega_2} \models \phi[h, \aleph_1];$$

and so $M \models \phi[f, \aleph_1]$, hence $L_\lambda \models \phi[d(f), \mu]$. We now appeal again to the fact that $\phi(\mathbf{x}, \mathbf{y})$ is ZF^-_g-absolute: since $L_\lambda \models \mathsf{ZF}^-_g$, $N \models \phi[d(f), \mu]$ for every transitive $N \models \mathsf{ZF}^-_g$ which contains $d(f)$ and μ, and in particular,

$$L_{\omega_2} \models \phi[d(f), \mu].$$

So $s_\mu = d(f)$ and $d(f) = f \restriction \mu$ by Lemma 4, which contradicts the choice of f. ⊣

How large can we make the set of correct guesses

$$\{\xi > 0 : f \restriction \xi = t_\xi\}$$

for every $f : \omega_1 \to \omega_1$ by choosing cleverly the guessing sequence $\{t_\xi\}_{\xi\in\omega_1}$? We cannot (rather trivially) insure that this set is always a closed unbounded subset of ω_1, cf. Problem x7.13, but we can insure the next, best possible result.

A set $C \subseteq \omega_1$ is *stationary* if it intersects every closed, unbounded subset of ω_1.

Theorem 7D.5 (ZFC). *If $\Diamond$ holds, then there is a guessing sequence $\{t_\xi\}_{\xi\in\omega_1}$ such that for every $f : \omega_1 \to \omega_1$ the set $\{\xi > 0 : f \restriction \xi = t_\xi\}$ is stationary.*

Proof is left for Problem x7.14*. ⊣

This is about the strongest version of $\Diamond$ which is close to the formulation we chose as "primary", but there are many other equivalent propositions, each with its own uses and applications.

The Suslin Hypothesis. The order $(\mathbb{R}, \leq)$ on the real numbers can be characterized up to similarity by the following two properties which do not refer to the field structure of $\mathbb{R}$:

(1) $(X, \leq)$ is a linear ordering with no least or greatest element; it is *dense in itself*, i.e., $a < b \Longrightarrow (\exists x)[a < x < b]$; and it is *order complete*, i.e., every set $X \subseteq (a, b)$ contained in an open interval has a least upper bound and a greatest lower bound.

(2) $(X, \leq)$ is *separable*, i.e., there is a countable set $\mathbb{Q} \subset X$ which intersects every open interval (a, b).

Suslin's question was whether (2) can be replaced by the weaker

(2′) There is no uncountable set of disjoint open intervals in X.

Call $(X, \leq)$ a *Suslin line* if it satisfies (1) and (2′) but not (2).

Suslin Hypothesis. There is no Suslin line.

The Suslin Hypothesis is neither provable nor disprovable in ZFC. Both of these results were established by forcing techniques soon after Cohen's introduction of the method in 1963, and they were among the most important early results in forcing—especially the consistency of Suslin's Hypothesis. Soon afterwards Jensen proved that *there is a Suslin line in L*. His proof is combinatorial complex (and uses the intermediate notion of a *Suslin tree*) but the main tool for it was the proof of $\Diamond$ in L. It is fair to say that Jensen's theorem was the first, substantial result which started the modern development of *combinatorial set theory*, in and outside L.

7E. L and Σ^1_2

We finish this Chapter with some basic results of Shoenfield which relate the constructible universe to the analytical hierarchy developed in Sections ?? and ??. We will assume for simplicity ZFC as the underlying theory, although most of what we will prove can be established without the full versions of either the powerset axiom or the axiom of choice.

Theorem 7E.1. (i) *The set $\mathcal{N} \cap L$ of constructible members of Baire space is Σ^1_2.*

(ii) *The restriction of $\leq_L$ to $\mathcal{N}$ is a Σ^1_2-good wellordering of $\mathcal{N} \cap L$; i.e., it is a Σ^1_2 relation on $\mathcal{N}$, and if $P \subseteq \omega^n \times \mathcal{N}^\nu$ is in Σ^1_2, then so are the relations*

$$Q(\alpha, \vec{x}, \vec{\beta}) \iff \alpha \in L \;\&\; (\exists \beta \leq_L \alpha) P(\beta, \vec{x}, \vec{\beta}),$$
$$R(\alpha, \vec{x}, \vec{\beta}) \iff \alpha \in L \,\&\, (\forall \beta \leq_L \alpha) P(\beta, \vec{x}, \vec{\beta}).$$

(iii) *If $\mathcal{N} \subseteq L$, then $\mathcal{N}$ admits a Σ^1_2-good wellordering of rank $\aleph_1$.*

Proof. (i) is an easy consequence of (ii), but it is instructive to show (i) first.

First of all, we claim that if T^L is the finite set of sentences in the Condensation Lemma 7C.6, then

(7E-21)
$$\alpha \in L \iff \text{there exists a countable, transitive set } A \text{ such that } (A, \in) \models T^L \text{ and } \alpha \in A.$$

The implication ($\Longleftarrow$) in (7E-21) is immediate, because by Theorem 7C.6, if $(A, \in) \models T^L$, then $A = L_\lambda$ for some ordinal λ. For the other direction, notice that (as a set of pairs of natural numbers), each α is a subset of L_ω so by (iii) of 7C.6

$$\alpha \in L \iff \text{for some countable } \lambda,\ \alpha \in L_\lambda \text{ and } L_\lambda \models T^L.$$

The key idea of the proof is that the structures of the form $(A, \in)$ with countable transitive A can be characterized *up to isomorphism* by the version for sets of the Mostowski Collapsing Lemma in Problem x6.17*. In fact, if (M, E) is any structure with countable M and $E \subseteq M \times M$, then by x6.17*, immediately

$$(M, E) \text{ is isomorphic with some } (A, \in) \text{ where } A \text{ is countable, transitive}$$
$$\iff E \text{ is wellfounded and } (M, E) \models \text{“axiom of extensionality”};$$

thus

(7E-22)

$$\begin{aligned}\alpha\in L \iff{}& \text{there exists a countable, wellfounded structure}\\ & (M,E) \text{ such that } (M,E)\models \text{“axiom of}\\ & \text{extensionality”}, (M,E)\models T^L \text{ and } \alpha\in\overline{M} = \text{the}\\ & \text{unique transitive set such that } (M,E) \text{ is isomorphic}\\ & \text{with } (\overline{M},\in).\end{aligned}$$

To see how to express the last condition in a model-theoretic way, recall that the condition "$\alpha\in\mathcal{N}$" is ZF^-_g-absolute and choose some $\varphi_0(\boldsymbol{\alpha})$ such that for all transitive models M of some finite $T_0\subseteq\mathsf{ZF}^-_g$,

$$\alpha\in\mathcal{N} \iff M\models\varphi_0[\alpha].$$

Next define for each integer n a formula $\psi_n(\mathbf{x})$ which asserts that $\mathbf{x}=n$, by the recursion

$$\begin{aligned}\psi_0(\mathbf{x}) &\iff \mathbf{x}=0,\\ \psi_{n+1}(\mathbf{x}) &\iff (\exists\mathbf{y})[\psi_n(\mathbf{y})\,\&\,\mathbf{x}=\mathbf{y}\cup\{\mathbf{y}\}]\end{aligned}$$

and for each n, m, let

$$\psi_{n,m}(\boldsymbol{\alpha}) :\equiv (\exists\mathbf{x})(\exists\mathbf{y})[\psi_n(\mathbf{x})\,\&\,\psi_m(\mathbf{y})\,\&\,\langle\mathbf{x},\mathbf{y}\rangle\in\boldsymbol{\alpha}].$$

It follows that

(7E-23)

$$\begin{aligned}\alpha\in L \iff{}& \text{there exists a countable, wellfounded structure}\\ & (M,E) \text{ such that } (M,E)\models \text{“axiom of}\\ & \text{extensionality”}, (M,E)\models T^L \text{ and for some } a\in M,\\ & (M,E)\models\varphi_0[a] \text{ and for all } n, m,\\ & \alpha(n)=m \iff (M,E)\models\psi_{n,m}[a].\end{aligned}$$

Let

$$f(m,n) = \text{the code of the formula } \psi_{m,n}(\boldsymbol{\alpha}),$$

so that f is obviously a recursive function. Let also k_0 be the code of the conjunction of the sentences in T^L and the Axiom of Extensionality and let k_1 be the code of the formula $\varphi_0(\boldsymbol{\alpha})$ which defines $\alpha\in\mathcal{N}$; we are assuming that both in $\psi_{m,n}(\boldsymbol{\alpha})$ and in $\varphi_0(\boldsymbol{\alpha})$, the free variable $\boldsymbol{\alpha}$ is actually the first variable $\mathbf{v}_0$. It is now clear that with $u=\langle 2\rangle$ the code of the vocabulary for structures with just one binary relation,

$$\begin{aligned}\alpha \in L \iff (\exists\beta)\Big\{&\mathrm{Sat}(u,\beta,k_0,1)\\ &\&\,\{(t,s) : (\beta)_0(t) = (\beta)_0(s) = 1\,\&\,(\beta)_1(\langle t,s\rangle) = 1\}\\ &\qquad\text{is wellfounded}\\ &\&\,(\exists a)\Big[\mathrm{Sat}(u,\beta,k_1,\langle a\rangle)\\ &\qquad\&\,(\forall n)(\forall m)\big[\alpha(n) = m \iff \mathrm{Sat}\big(u,\beta,f(n,m),\langle\alpha\rangle\big)\big]\Big]\Big\}\end{aligned}$$

which implies directly that $L \cap \mathcal{N}$ is Σ^1_2, using the fact that wellfoundedness is Π^1_1.

To prove (ii), let $\psi_L(\mathbf{v}_0,\mathbf{v}_1)$ be a formula which defines the canonical wellordering of L absolutely on all transitive models of some finite $T^L_1 \subseteq \mathsf{ZF}^-_g$ (by (ii) of 7C.1) and let $S^L \subseteq \mathsf{ZF}^-_g$ be finite and large enough to include T^L_1, T^L, the Axiom of Extensionality and the set T_0 of part (i), chosen so that $\varphi_0(\boldsymbol{\alpha})$ defines $\boldsymbol{\alpha} \in \mathcal{N}$ on all transitive models of T_0. Using the key fact

$$\alpha \in L_\xi\,\&\,\beta \leq_L \alpha \Longrightarrow \alpha \in L_\xi$$

and Mostowski collapsing as above, we can verify directly that for $\alpha \in L$ and arbitrary $P \subseteq \mathcal{N} \times \mathcal{Z}$ (with $\mathcal{Z} = \omega^n \times \mathcal{N}^\nu$),

$$\begin{aligned}&(\forall\beta \leq_L \alpha)P(\beta,z)\\ &\quad\iff \text{there exists a countable, wellfounded structure}\\ &\qquad\quad (M,E) \models S^L \text{ and some } a \in M \text{ such that } (M,E) \models \varphi_0[a]\\ &\qquad\quad \text{and } (\forall n)(\forall m)[\alpha(n) = m \iff (M,E) \models \psi_{n,m}[a]]\\ &\qquad\quad \text{and } (\forall b)\big\{(M,E) \models \varphi_0[b]\,\&\,\psi_L(b,a) \Longrightarrow\\ &\qquad\qquad (\exists\beta)\big[(\forall n)(\forall m)[\beta(n) = m\\ &\qquad\qquad\qquad \iff (M,E) \models \psi_{n,m}[b]]\,\&\,P(\beta,z)\big]\big\}.\end{aligned}$$

If P is Σ^1_2, then it is easy to see that this whole expression on the right leads to a Σ^1_2 condition by coding the structures (M,E) by irrationals as above—the key being that the universal quantifier $\forall\beta$ has been turned to the number quantifier $\forall b$. ⊣

We put down the argument for (i) in considerable detail, because it illustrates a very useful technique for making analytical computations of conditions defined by set-theoretic constructions. For the next result we will do the opposite, i.e., we will give a set-theoretic construction for Σ^1_2 subsets of $\omega^n \times \mathcal{N}^\nu$ which will establish that (as conditions) they are absolute for L.

We show first a basic result, which has many applications beyond our immediate concern:

Theorem 7E.2 (Shoenfield's Lemma). *If $A \subseteq \mathcal{N}$ is Σ^1_2, then there exists a ZF^-_g-absolute operation*

$$\xi \mapsto T^\xi$$

which assigns to each ordinal $\xi \geq \omega$ a tree T^ξ on $\omega \times \xi$ such that the following holds, when λ is any uncountable ordinal:

$$\begin{aligned} \alpha \in A &\iff (\exists \xi \geq \omega)[T^\xi(a) \text{ is not wellfounded}] \\ &\iff (\exists \xi \geq \omega)[\xi < \omega_1 \,\&\, T^\xi(\alpha) \text{ is not wellfounded}] \\ &\iff T^\lambda(\alpha) \text{ is not wellfounded.} \end{aligned}$$

Proof. Choose a recursive, monotone R so that

$$\alpha \in A \iff (\exists \beta)(\forall \gamma)(\exists t) R\big(\overline{\alpha}(t), \overline{\beta}(t), \overline{\gamma}(t)\big),$$

and for all $\overline{\alpha}(n), \overline{\beta}(n), \overline{\gamma}(n)$,

$$\Big(\neg R(\overline{\alpha}(t), \overline{\beta}(t), \overline{\gamma}(t)) \;\&\; s < t\Big) \Longrightarrow \neg R(\overline{\alpha}(s), \overline{\beta}(s), \overline{\gamma}(s)).$$

It follows that for each α, β, the set of sequences

$$S^{\alpha,\beta} = \big\{(c_0, \dots, c_{s-1}) : (\forall t < s) \neg R\big(\overline{\alpha}(t), \overline{\beta}(t), \langle c_0, \dots, c_{t-1}\rangle\big)\big\}$$

is a tree and easily

$$\begin{aligned} &\text{(7E-1)} \quad \alpha \in A \iff (\exists \beta)\{S^{\alpha,\beta} \text{ is wellfounded}\} \\ &\iff (\exists \beta)(\exists f : S^{\alpha,\beta} \to \omega_1)\{\text{if } (c_0, \dots, c_{s-1}) \in S^{\alpha,\beta} \text{ and } t < s, \\ &\qquad\qquad \text{then } f(c_0, \dots, c_{t-1}) > f(c_0, \dots, c_{s-1})\}. \end{aligned}$$

In the computation below we will represent $S^{\alpha,\beta}$ by the set of codes in ω $\langle c_0, \dots, c_{s-1}\rangle$ of sequences $(c_0, \dots, c_{s-1}) \in S^{\alpha,\beta}$.

For each $\xi \geq \omega$, define first a tree S^ξ on $\omega \times \omega \times \xi$ as follows:

$$\begin{aligned} &\big((a_0, b_0, \xi_0), \dots, (a_{n-1}, b_{n-1}, \xi_{n-1})\big) \in S^\xi \iff \xi_0, \dots, \xi_{n-1} < \xi \\ &\& \; (\forall c_0, \dots, c_t, s < t)\Big[\neg R(\langle a_0, \dots, a_{t-1}\rangle, \langle b_0, \dots, b_{t-1}\rangle, \langle c_0, \dots, c_{t-1}\rangle)\Big] \\ &\qquad\qquad \Longrightarrow \xi_{\langle c_0, \dots, c_{s-1}\rangle} > \xi_{\langle c_0, \dots, c_{t-1}\rangle}. \end{aligned}$$

Notice that the operation

$$\xi \mapsto S^\xi$$

is clearly ZF^-_g-absolute and

$$\xi \leq \eta \Longrightarrow S^\xi \subseteq S^\eta.$$

Now set for any ξ, α,

$$S^\xi(\alpha) = \Big\{ \big((b_0, \xi_0), \dots, (b_{n-1}, \xi_{n-1})\big) : \Big(\big(\alpha(0), b_0, \xi_0\big), \dots, \big(\alpha(n-1), b_{n-1}, \xi_{n-1}\big) \Big) \in S^\xi \Big\}.$$

This is a tree on $\omega \times \xi$, the *tree of all attempts to prove that for some β, $S^{\alpha,\beta}$ is wellfounded with rank* $\leq \xi$: any infinite branch in $S^\xi(\alpha)$ provides a β and a rank function $f : S^{\alpha,\beta} \to \xi$. More precisely, we have the following two, simple facts:

(7E-2) $\qquad \alpha \in A \Longrightarrow (\exists \xi \in \omega_1)[S^\xi(\alpha) \text{ is not wellfoounded}],$

(7E-3) $\qquad S^\xi(\alpha) \text{ is not wellfounded} \Longrightarrow \alpha \in A \quad (\xi \text{ infinite}).$

To prove (7E-2) choose $\beta = (b_0, b_1, \dots)$ such that $S^{\alpha,\beta}$ is wellfounded, choose $f : S^{\alpha,\beta} \to \omega_1$ as in (7E-1), set $\xi_i = f(c_0, \dots, c_{s-1})$ if $i = \langle c_0, \dots, c_{s-1} \rangle$ for some $c_0, \dots, c_{s-1}$ and $\xi_i = 0$ otherwise. To prove (7E-3), choose an infinite branch $(b_0, \xi_0), (b_1, \xi_1), \dots$ in $S^\xi(\alpha)$, take $\beta = (b_0, b_1, \dots)$ and define $f : S^{\alpha,\beta} \to \xi$ by

$$f(c_0, \dots, c_{s-1}) = \xi_i \iff i = \langle c_0, \dots, c_{s-1} \rangle$$

so that it satisfies the defining condition in (7E-1).

Now (7E-2) and (7E-3) imply directly the assertions in the theorem taking $T^\xi = S^\xi$, except that S^ξ is a tree on $\omega \times (\omega \times \xi)$ rather than a tree on $\omega \times \xi$. To complete the proof, put

$$\begin{aligned} T^\xi = {} & \text{all initial segments of sequences of the form} \\ & \big((a_0, b_0), (a_1, \xi_0), (a_2, b_1), (a_3, \xi_1), \dots, (a_{2n}, b_n), (a_{2n+1}, \xi_n)\big) \\ & \text{such that} \\ & \big((a_0, b_0, \xi_0), (a_1, b_1, \xi_1), \dots, (a_n, b_n, \xi_n)\big) \in S^\xi \end{aligned}$$

so that T^ξ is a tree on $\omega \times \xi$ (because $\omega \subseteq \xi$) and easily, for any α,

$$T^\xi(\alpha) \text{ is not wellfounded} \iff S^\xi(\alpha) \text{ is not wellfounded}. \qquad \dashv$$

Theorem 7E.3 (Shoenfield's Theorem (I)). *Each Σ^1_2 set $A \subseteq \mathcal{N}$ is absolute as a condition for all standard models M of some finite $T_* \subseteq \mathsf{ZF}^-_g$ such that $\omega_1 \subseteq M$.*

In particular, every Σ^1_2 subset $A \subseteq \omega^n$ is constructible.

Proof. Suppose $A \subseteq \mathcal{N}$ is Σ^1_2 and by Shoenfield's Lemma, let $\varphi(\boldsymbol{\xi}, \mathbf{T})$ be a formula of $\mathbb{FOL}(\in)$ such that for all standard models M of some finite $T_1 \subseteq \mathsf{ZF}^-_g$,

$$\xi \in M \Longrightarrow T^\xi \in M,$$

$$T = T^\xi \iff M \models \varphi[\xi, T].$$

Notice also that the operation

$$(\alpha, T) \mapsto T(\alpha)$$

is easily ZF^-_g-absolute, so choose $\psi(\boldsymbol{\alpha}, \mathbf{S}, \mathbf{T})$ so that for all standard models M of some finite $T_2 \subseteq \mathsf{ZF}^-_g$,

$$\alpha, T \in M \Longrightarrow T(\alpha) \in M,$$
$$S = T(\alpha) \iff M \models \psi[\alpha, S, T].$$

Finally use Mostowski's Theorem 7B.5 to construct a formula $\chi(\mathbf{S})$ of $\mathbb{FOL}(\in)$ such that for all standard models M of some finite $T_3 \subseteq \mathsf{ZF}^-_g$ and $S \in M$,

$$S \text{ is wellfounded} \iff M \models \chi[S].$$

Now if M is any standard model of

$$T_* = T_1 \cup T_2 \cup T_3$$

such that $\omega_1 \subseteq M$, then by the lemma, for $\alpha \in M$

$$\begin{aligned}\alpha \in A \iff & \text{ there exists some } \xi \in M \text{ such that } T^{\xi}(\alpha) \text{ is not wellfounded}\\ \iff & \text{ there exists some } \xi \in M \text{ such that}\\ & M \models (\exists \mathbf{S})(\exists \mathbf{T})[\varphi(\xi, \mathbf{T}) \,\&\, \psi(\alpha, \mathbf{S}, \mathbf{T}) \,\&\, \neg\chi(\mathbf{S})]\\ \iff & M \models (\exists \boldsymbol{\xi})(\exists \mathbf{S})(\exists \mathbf{T})[\varphi(\boldsymbol{\xi}, \mathbf{T}) \,\&\, \psi(\alpha, \mathbf{S}, \mathbf{T}) \,\&\, \neg\chi(\mathbf{S})].\end{aligned}$$

To prove the second assertion, take $A \subseteq \omega$ for simplicity of notation, suppose

$$n \in A \iff P(n)$$

where P is Σ^1_2, and let $\psi(\mathbf{n})$ define P absolutely as in the first part, so that in particular

$$P(n) \iff L \models \psi[n].$$

The sentence

$$(\exists \mathbf{x})\big[\mathbf{x} \subseteq \omega \,\&\, (\forall \mathbf{n})[\mathbf{n} \in \mathbf{x} \iff \psi(\mathbf{n})]\big]$$

is a theorem of ZF^-_g and hence it holds in L. This implies that there is some $x \in L$ such that $x \subseteq \omega$ and for all n,

$$\begin{aligned} n \in x &\iff L \models \psi[n]\\ &\iff P(n)\\ &\iff n \in A;\end{aligned}$$

thus $x = A$ and $A \in L$. ⊣

To appreciate the significance of Shoenfield's Theorem, recall from the exercises of **??** that a formula $\theta(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m)$ of the language of second order arithmetic A^2 is Σ^1_n if

$$\theta(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m) \iff (\exists \boldsymbol{\beta}_1)(\forall \boldsymbol{\beta}_2)(\exists \boldsymbol{\beta}_3) \cdots (-\boldsymbol{\beta}_n)\varphi(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m, \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_n),$$

where $\varphi(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n)$ has no quantifiers over $\mathcal{N}$. It is clear that we can interpret these formulas over standard models of ZF_g^- simply by putting (for $\alpha_1, \ldots, \alpha_m \in M$),

$$M \models \theta(\alpha_1, \ldots, \alpha_m) \iff (\omega, \mathcal{N} \cap M, +, \cdot, \mathrm{ap}, 0, 1) \models \theta(\alpha_1, \ldots, \alpha_m),$$

i.e., by interpreting the quantifiers $\exists\boldsymbol{\beta}_i$, $\forall\boldsymbol{\beta}_i$ as ranging over the irrationals in M and using the standard interpretations for the operations $+$, $\cdot$, ap (which are ZF_g^--absolute by **??**) and the quantifiers $\exists n$, $\forall n$ (since ω is also ZF_g^--absolute and hence a member of M).

Theorem 7E.4 (Shoenfield's Theorem (II)). *[**??**] (i) If $\theta(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m)$ is a Σ^1_2 or Π^1_2 formula of second order arithmetic, then for every standard model M of ZF_g^- such that $\omega_1 \subseteq M$ and $\alpha_1, \ldots, \alpha_m \in M$,*

$$V \models \theta(\alpha_1, \ldots, \alpha_m) \iff M \models \theta(\alpha_1, \ldots, \alpha_m);$$

in particular, if $\alpha_1, \ldots, \alpha_m \in L$, then

$$V \models \theta(\alpha_1, \ldots, \alpha_m) \iff L \models \theta(\alpha_1, \ldots, \alpha_m).$$

*(ii) If we can prove a Σ^1_2 or Π^1_2 sentence θ by assuming in addition to the axioms in ZF_g^- the hypothesis $V = L$ (and its consequences **AC** and **GCH**), then θ is in fact true (i.e., $V \models \theta$).*

Proof. Take a Σ^1_2 sentence for simplicity of notation

$$\theta \iff (\exists\boldsymbol{\alpha})(\forall\boldsymbol{\beta})\varphi(\boldsymbol{\alpha}, \boldsymbol{\beta}),$$

and let

$$P(\alpha, \beta) \iff \mathsf{A}^2 \models \varphi(\alpha, \beta)$$

be the arithmetical pointset defined by the matrix of θ so that

$$\begin{aligned} V \models \theta &\iff (\exists\alpha)(\forall\beta)P(\alpha, \beta) \\ M \models \theta &\iff (\exists\alpha \in M)(\forall\beta \in M)P(\alpha, \beta). \end{aligned}$$

Using the Basis Theorem for Σ^1_2, **??**,

$$\begin{aligned} V \models \theta &\Longrightarrow (\exists\alpha)(\forall\beta)P(\alpha, \beta) \\ &\Longrightarrow (\exists\alpha \in \Delta^1_2)(\forall\beta)P(\alpha, \beta) && \text{(by ??)} \\ &\Longrightarrow (\exists\alpha \in M)(\forall\beta)P(\alpha, \beta) && \text{(by 7E.3)} \\ &\Longrightarrow (\exists\alpha \in M)(\forall\beta \in M)P(\alpha, \beta) && \text{(obviously)} \\ &\Longrightarrow M \models \theta. \end{aligned}$$

Conversely, assuming that $M \models \theta$, choose some $\alpha_0 \in M$ such that

$$(\forall\beta \in M)P(\alpha_0, \beta)$$

and assume towards a contradiction that

$$(\exists\beta)\neg P(\alpha_0, \beta);$$

by the Basis Theorem ?? again, we then have

$$\big(\exists\beta \in \Delta^1_2(\alpha_0)\big)\neg P(\alpha_0, \beta)$$

so that by 7E.3,

$$(\exists\beta \in M)\neg P(\alpha_0, \beta)$$

contradicting our assumption end establishing $(\forall\beta)P(\alpha_0, \beta)$, i.e., $V \models \theta$.

The second assertion follows immediately because if we can prove θ using the additional hypothesis $V = L$, then we know that $L \models \theta$ by 7C.2 and hence $V \models \theta$ by the first assertion. ⊣

This theorem is quite startling because so many of the propositions that we consider in ordinary mathematics are expressible by Σ^1_2 sentences—including all propositions of elementary or analytic number theory and most of the propositions of "hard analysis". The techniques in the proof of 7C.1 allow us to prove that many set theoretic propositions are also equivalent to Σ^1_2 sentences. Theorem 7E.2 assures us then that the truth or falsity of these "basic" propositions does not depend on the answers to difficult and delicate questions about the nature of sets like the continuum hypothesis; we might as well assume that $V = L$ in attempting to prove or disprove them.

Of course, in descriptive set theory we worry about propositions much more complicated than Σ^1_2 which may well have different truth values in L and in V.

7F. Problems for Chapter 7

Problem x7.1 (ZFC, The Countable Reflection Theorem)**.** Prove that for any sentence θ,

$$\theta \Longrightarrow (\exists M)[M \text{ is countable, transitive and } M \models \theta].$$

Hint: Use the Downward Skolem-Löwenheim Theorem 2B.1

Problem x7.2* (ZF_g)**.** None of the following notions is ZFC-absolute: $\mathcal{P}(\omega)$, $\mathrm{Card}(\kappa)$, $\mathbb{R}$, $x \mapsto \mathrm{Power}(x)$, $x \mapsto |x|$.
Hint: This follows quite easily in ZFC from the preceding problem. It can also be proved in ZF^-_g, with just a little more work.

Let us take up first a few simple exercises which will help clarify the definability notions we have been using.

Problem x7.3. Show that if $R(x_1, \ldots, x_n)$ is definable by a Σ_0 formula, then the condition

$$R^*(k_1, \ldots, k_n) \iff k_1 \in \omega \,\&\, \cdots \,\&\, k_n \in \omega \,\&\, R(k_1, \ldots, k_n)$$

is recursive.

A little thinking is needed for the next one.

Problem x7.4. Prove that the condition of satisfaction in #38 of 7A.1 is not definable by a Σ_0 formula.

Problem x7.5 (ZF)**.** Suppose that M is a grounded class, i.e., (by our definition) $M \subseteq \mathcal{V}$. Prove that

$$(\forall x \subseteq M)(\exists s \in M)(\forall t \in s)[t \notin x].$$

Note. This is trivial if we assume the Axiom of Foundation by which $\mathcal{V} = V$, so what is needed is to prove it without assuming foundation.

Problem x7.6 (ZF^-_g)**.** Show that for each infinite ordinal ξ, $|L_\xi| = |\xi|$.

Problem x7.7. Prove that

$$\mathsf{ZFC} \nvdash \text{“there exists a weakly inaccessible cardinal”}.$$

Problem x7.8 (ZF_g)**.** Prove that the set $E = \{\xi \in \omega_1 : L_\xi \prec L_{\omega_1}\}$ is closed and unbounded in ω_1.
Hint: Check first that if $\eta < \xi$ and $\eta, \xi \in E$, then $L_\eta \prec L_\xi$.

Definition 7F.1. For each cardinal κ, we set

$$\mathrm{HC}(\kappa) = \{x : |\mathrm{TC}(x)| < \kappa\}. \tag{7F-1}$$

So the sets of hereditarily finite and hereditarily countable sets introduced in Definition 6C.8 are respectively $\mathrm{HC}(\aleph_0)$ and $\mathrm{HC}(\aleph_1)$ with this notation. The sets in $\mathrm{HC}(\kappa)$ are *hereditarily of cardinality* $< \kappa$.

Problem x7.9 (ZFC)**.** Prove that if κ regular, then $\mathrm{HC}(\kappa) \models \mathsf{ZF}^-_g$.

Problem x7.10 (ZF_g)**.** (a) Prove that for every cardinal κ,

$$L_\kappa = \mathrm{HC}(\kappa) \cap L.$$

Infer that if κ is regular, then $L_\kappa \models \mathsf{ZF}^-_g$.

(b) Prove that $\mathsf{ZF}_g \nvdash (L_{\aleph_\omega} \vdash \mathsf{ZF}^-_g)$.

Problem x7.11 (ZF)**.** Prove that for every infinite ordinal ξ,

$$(\xi \to L_{\xi^+}) \cap L \subset L_{\xi^+}.$$

Problem x7.12 (Ordinal pairing functions)**.** Define a binary operation

$$(\eta, \zeta) \mapsto \langle \eta, \zeta \rangle \in \mathrm{ON}$$

on pairs of ordinal to ordinals with the following properties:

(1) $\langle \eta, \zeta \rangle = \langle \eta', \zeta' \rangle \iff \eta = \eta' \ \&\ \zeta = \zeta'$, i.e., $\langle\ \rangle$ is injective.
(2) Every ordinal is $\langle \eta, \zeta \rangle$ for some η, ζ, i.e., $\langle\ \rangle$ is surjective.
(3) For every infinite cardinal κ, if $\eta, \zeta < \kappa$ then $\langle \eta, \zeta \rangle < \kappa$.
(4) For all η, ζ, $\eta \leq \langle \eta, \zeta \rangle$, $\zeta \leq \langle \eta, \zeta \rangle$.

We denote the inverse functions by $(\ )_0, (\ )_1$ so that for every ξ,

$$\xi = \langle (\xi)_0, (\xi)_1 \rangle.$$

Hint: For each cardinal κ, define on $\kappa \times \kappa$ the relation

$$\begin{aligned}(\eta, \zeta) \leq_\kappa (\eta', \zeta') \iff &\max(\eta, \zeta) < \max(\eta', \zeta') \\ &\vee \max(\eta, \zeta) = \max(\eta', \zeta') \ \&\ \eta < \eta' \\ &\vee \max(\eta, \zeta) = \max(\eta', \zeta') \ \&\ \eta = \eta' \ \&\ \zeta \leq \zeta',\end{aligned}$$

check that it is a wellordering with rank κ and let

$$\langle\ \rangle_\kappa : \kappa \times \kappa \rightarrowtail\!\!\!\!\twoheadrightarrow \kappa$$

be the (unique) similarity. Let $\langle\ \rangle = \bigcup_{\kappa \in \mathrm{Card}} \langle\ \rangle_\kappa$. Only (4) requires some thinking.

Problem x7.13. Prove that there is no guessing sequence $\{t_\xi\}_{\xi \in \omega_1}$ such that for every $f : \omega_1 \to \omega$ the set $\{\xi : f \restriction \xi = t_\xi\}$ is closed and unbounded in ω_1.

Problem x7.14 ($\mathsf{ZFC} + V = L$)**.** Suppose U is a non-principal ultrafilter on ω_1. Prove that there is no guessing sequence $\{t_\xi\}_{\xi \in \omega_1}$ such that for every $f : \omega_1 \to \omega_1$, $\{\xi : f \restriction \xi = t_\xi\} \in U$.

Problem x7.15* (ZFC)**.** Prove that if $\Diamond$ holds, then there is a guessing sequence $\{t_\xi\}_{\xi \in \omega_1}$ such that for every $f : \omega_1 \to \omega_1$, the set $\{\xi : f \restriction \xi = t_\xi\}$ is stationary (Theorem 7D.5).

Hint: Take $t_\xi(\eta) = (s_\xi(\eta))_0$, where $\{s_\xi\}_{\xi \in \omega_1}$ is supplied by $\Diamond$ and $(\zeta)_0$ is the first projection of a coding of triples below ω_1, i.e., some $\langle\ \rangle : \omega_1^3 \rightarrowtail\!\!\!\!\twoheadrightarrow \omega_1$ such that for all ξ, $\xi = \langle (\xi)_0, (\xi)_1), (\xi)_2 \rangle$ and $(\xi)_i \leq \xi$. (There are many other proofs.)

Definition 7F.2 (Σ_1)**.** A formula is Σ_1 if it is of the form

$$(\exists \mathbf{y})\phi \text{ where } \phi \text{ is } \Sigma_0,$$

and a condition $R(x_1, \dots, x_n)$ is Σ_1 in a theory T if it is defined by a full extended formula $\phi(\mathbf{x}_1, \dots, \mathbf{x}_n)$ such that for some Σ_1 full extended formula $\phi^*(\mathbf{x}_1, \dots, \mathbf{x}_n)$,

$$T \vdash \phi(\mathbf{x}_1, \dots, \mathbf{x}_n) \leftrightarrow \phi^*(\mathbf{x}_1, \dots, \mathbf{x}_n).$$

An operation $F : V^n \to V$ is Σ_1 in a theory T if

$$F(x_1, \dots, x_n) = w \iff V \models \phi[x_1, \dots, x_n, w]$$

with a formula $\phi(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{w})$ which is Σ_1 in T and such that

$$T \vdash (\forall \vec{\mathbf{x}})(\exists! \mathbf{w})\phi(\vec{\mathbf{x}}, \mathbf{w}).$$

A condition R is Δ_1 in a theory T if both R and $\neg R$ are Σ_1 in T.

Problem x7.16. Prove that the conditions $x \in L$ and $x \leq_L y$ are both Σ_1 in ZF_g^-.

Problem x7.17. Prove that if $F : V^n \to V$ is Σ_1 in a theory T, then the condition

$$R(\vec{x}, w) \iff F(x_1, \dots, x_n) = w$$

is Δ_1 in T.

Definition 7F.3 (Collection)**.** An instance of the *Collection Scheme* is any formula of the form

$$(\forall x \in z)(\exists y)\phi \Longrightarrow (\exists w)(\forall x \in z)(\exists y \in w)\phi$$

where w is chosen so that it does not occur free in ϕ. It is an instance of Σ_1-*Collection* if ϕ is a Σ_1 formula.

Problem x7.18. Prove the Collection Scheme in ZF_g.

Problem x7.19. Prove that the collection of conditions which are Σ_1 in ZF_g^- + Collection contains all Σ_0 conditions and is closed under the positive propositional operations $\&, \vee$, the restricted quantifiers $(\forall x \in y), (\exists x \in y)$, existential quantification $(\exists x)$, and the substitution of operations which are Σ_1 in ZF_g^- + Collection, i.e., the scheme

$$P(\vec{x}) \iff R(F_1(\vec{x}), \dots, F_m(\vec{x})).$$

Show also that the collection of operations which are Σ_1 in ZF_g^- + Collection is closed under composition,

$$F(\vec{x}) = G(F_1(\vec{x}), \dots, F_m(\vec{x})).$$

Infer the same closure properties for the collection of notions which are Σ_1 in ZF_g.

Problem x7.20. Prove that all the notions #1 – #40 defined in Theorems 6C.2, 7A.1 are Σ_1 in ZF_g^- + Collection, and so also Σ_1 in ZF_g.