Appendix 2: $V = L$, $\Diamond$, and Souslin's Hypothesis

The Axiom of Constructibility settles most interesting set-theoretic questions. A number of them can be answered using Jensen's combinatorial principle $\Diamond$. $\Diamond$ is the assertion that there is a sequence $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ (i.e., a function $\alpha \mapsto A_\alpha$ with domain ω_1) such that each $A_\alpha \subseteq \alpha$ and such that, for any $A \subseteq \omega_1$ and any closed, unbounded subset C of ω_1,

$$(\exists \alpha \in C)\, A \cap \alpha = A_\alpha\,.$$

This formulation of $\Diamond$ is different from, but equivalent to, that in the course notes. We will prove our version of $\Diamond$ from $V = L$, and we will deduce the negation of Souslin's Hypothesis from $\Diamond$.

Exercise. Assume $V = L$.

(a) Show that if $(y; \in) \preceq (L_{\omega_2}; \in)$ then $y \cap \omega_1$ is transitive.

(b) Show that

$$\{\alpha < \omega_1 \mid L_{\alpha+1} \cap \mathcal{P}(\omega) \subseteq L_\alpha\}$$

has a subset that is closed and unbounded in ω_1.

Hint. If $\alpha < \omega_1$, then α is countable. For (a), begin with the definition of "α is countable." For (b), observe that $L_{\omega_1+1} \cap \mathcal{P}(\omega) \subseteq L_{\omega_1}$. Build a chain of length ω_1 of elementary submodels of L_{ω_1+1} and then apply Mostowski collapse.

Theorem 1. $V = L \to \Diamond$.

Proof. Assume $V = L$. We define A_α by recursion. For α not a limit ordinal, set $A_\alpha = \emptyset$. Assume that α is limit ordinal and that A_β is defined for $\beta < \alpha$. Let ρ_α be the least ordinal ρ such that there are A and C belonging to L_ρ such that $A \subseteq \alpha$, C is a closed, unbounded subset of α, and

$$(\forall \beta \in C)\, A \cap \beta \neq A_\beta$$

if such a ρ exists. In this case let A_α and C_α be the lexicographically least A and C (using $<_L$). If ρ_α does not exist, let $A_\alpha = \emptyset$.

Suppose that $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ does not witness that $\Diamond$ holds. Let ρ be the least ordinal such that some counterexample sets A and C belong to L_ρ. Let A and C be the lexicographically least such pair (again using $<_L$). Note that $\rho < \omega_2$.

Let $(y;\in) \prec (L_{\omega_2};\in)$ with y countable and with

$$\{\omega_1, \rho, A, C, \langle A_\alpha \mid \alpha < \omega_1\rangle\} \subseteq y\,.$$

Let z and π be such that z is transitive and $\pi : (y;\in) \cong (z;\in)$. Let $\delta < \omega_1$ be such that $z = L_\delta$.

Let $\alpha = \pi(\omega_1)$. By part (a) of Exercise above, we have that $\alpha \subseteq y$. It follows that

(i) $A \cap \alpha = \pi(A)$;

(ii) $C \cap \alpha = \pi(C)$;

(iii) $\langle A_\beta \mid \beta < \alpha\rangle = \pi(\langle A_\beta \mid \beta < \omega_1\rangle)$.

Using (i)–(iii), the definitions of ρ, A, and C, and the fact that π^{-1} is an elementary embedding of $(L_\delta;\in)$ into $(L_{\omega_2};\in)$, we get that $\pi(\rho)$, $A\cap\alpha$, and $C \cap \alpha$ satisfy in L_δ the definitions of ρ_α, A_α, and C_α respectively. Thus

(a) $\pi(\rho) = \rho_\alpha$;

(b) $A \cap \alpha = A_\alpha$;

(c) $C \cap \alpha = C_\alpha$.

Since $C\cap\alpha = \pi(C)$, $C\cap\alpha$ is an unbounded subset of α. Since C is closed, it follows that $\alpha \in C$. This fact and (b) contradict the definitions of A and C. □

One of the earliest applications of $\Diamond$ was to show that *Souslin's Hypothesis* fails in L.

To state Souslin's Hypothesis, we need some definitions. Let R be a linear ordering of a set X. If every R-bounded subset of X has a least upper bound, then $(X;R)$ is said to be *complete*. If every set of disjoint open (in the obvious sense) R-intervals is countable, then $(X;R)$ is *ccc*: *satisfies the countable chain condition*. Give X the order topology: the basic open sets are the open intervals. If X has a countable dense subset then $(X;R)$ is *separable*.

The set $\mathbb{R}$ of reals, with its usual ordering, is—up to isomorphism—the unique separable, complete, dense linear ordering without endpoints. Souslin's hypothesis says this characterization continues to hold when "separable" is replaced by "ccc." Clearly the failure of Souslin's Hypothesis is equivalent to the existence of a *Souslin line*, a ccc, complete, dense linear ordering that is not separable.

The existence of a Souslin line is can be shown equivalent to the existence of a *Souslin tree*: a $(T;\lhd)$ such that

(1) $\lhd$ is a partial ordering of T;

(2) For all $x \in T$, $\{y \in T \mid x \lhd y\}$ is wellordered by $\lhd$;

(3) $|T| = \aleph_1$;

(4) $(T; \lhd)$ has no uncountable branches and no uncountable antichains.

Here a *branch* is a maximal subset of T linearly ordered by $\lhd$, and an *antichain* is a set of pairwise $\lhd$-incomparable elements of T.

Conditions (1) and (2) define the (set-theoretic) concept of a *tree*. Let us call a tree $(T; \lhd)$ *ultranormal* if

(i) $T \subseteq \omega_1$;

(ii) for β and $\gamma \in T$, $\beta \lhd \gamma \to \beta < \gamma$;

(iii) T has a $\lhd$-least element;

(iv) For each $\alpha < \omega_1$, the set of all $\beta \in T$ such that $\mathrm{level}(\beta) = \alpha$ is countable, where $\mathrm{level}(\beta)$ is the $\lhd$ order type of $\{\gamma \in T \mid \gamma \lhd \beta\}$;

(v) if $\beta \in T$ then β has infinitely many immediate successors with respect to $\lhd$;

(vi) for each $\beta \in T$ and each α such that $\mathrm{level}(\beta) < \alpha < \omega_1$, there is a $\gamma \in T$ such that $\mathrm{level}(\gamma) = \alpha$ and $\beta \lhd \gamma$;

(vii) if β and γ are elements of T with the same limit level and the same $\lhd$-predecessors, then $\beta = \gamma$.

Lemma 1. *If there is an ultranormal Souslin tree, then there is a Souslin line.*

Proof. We first observe that it is enough to construct a ccc, dense, linear ordering $(X; R)$ that is not separable. If we have such an $(X; R)$, then we can let X' be the set of all *Dedekind cuts* in $(X; R)$, i.e., the set of all bounded initial segments of $(X; R)$ without R-greatest elements, and we can let $x' \, R' \, y' \leftrightarrow x' \subseteq y'$. Clearly $(X'; R')$ a linear ordering. The function $x \mapsto \{y \in X \mid y \, R \, x\}$ embeds $(X; R)$ into $(X'; R')$ and has dense range. Therefore $(X'; R')$ is dense, ccc, and not separable. If A is an R'-bounded subset of X', then $\bigcup A$ is the R'-least upper bound of A; hence $(X'; R')$ is complete.

Let $(T; \lhd)$ be an ultranormal Souslin tree. Let

$$X = \{b \mid b \text{ is a branch of } T\}\,.$$

To define an ordering R on X, let us first fix, for each $\beta \in T$, an ordering $<_\beta$ of the the immediate successors of β with respect to $\lhd$. By (iv) and (v), we can—and do—make $<_\beta$ isomorphic to the standard ordering of the rationals. Let b and b' be distinct branches of $(T; \lhd)$. By (vii), there is a $\lhd$-greatest β that belongs to both b and b'. Let γ and γ' be the immediate $\lhd$-successors of β that belong to b and b' respectively. Define

$$b\,R\,b' \;\leftrightarrow\; \gamma <_\beta \gamma'.$$

It is easy to see that R is a linear ordering of X. Suppose that I is an open interval of $(X; R)$. let $I = (b, b')$. Define β, γ, and γ' as in the preceding paragraph. Let δ_I be such that $\gamma <_\beta \delta_I <_\beta \gamma'$. Observe that every branch containing δ_I belongs to the interval I. Observe also that if I_1 and I_2 are disjoint intervals, then δ_{I_1} and δ_{I_2} are $\lhd$-incomparable. The first fact implies that the $(X; R)$ is a dense ordering, and the second fact implies that $(X; R)$ has the ccc. For non-separability, let B be any countable subset of X. Since every member of B is countable, $\bigcup_{b\in B} b$ is countable. Let $\alpha \in T$ be $>$ every member of this countable set. Then the set of branches containing α is a neighborhood witnessing that B is not dense. □

Theorem 2. *If $\diamondsuit$ holds, then there is an ultranormal Souslin tree.*

Proof. Let $\langle A_\alpha \mid \alpha < \omega_1\rangle$ witness that $\diamondsuit$ holds.

We will define an ultranormal tree $(T; \lhd)$ by transfinite recursion. More precisely, we will define for each $\alpha < \omega_1$ a tree $(T_\alpha; \lhd_\alpha)$, and we will arrange that

(a) for $\alpha' < \alpha < \omega_1$, $T_{\alpha'}$ is the set of all elements of T_α of $\lhd_\alpha$-level $\leq \alpha'$, and $\lhd_{\alpha'}$ is the restriction of $\lhd_\alpha$ to T'_α;

(b) for $\alpha < \omega_1$, (i)-(vii) hold with $(T_\alpha; \lhd_\alpha)$ replacing $(T; \lhd)$ and with the $\alpha + 1$ replacing ω_1..

We will then let $T = \bigcup_{\alpha<\omega_1} T_\alpha$ and $\lhd = \bigcup_{\alpha<\omega_1} \lhd_\alpha$. The only task that will remain to us is the verification that $(T; \lhd)$ satisfies condition (4) in the definition of a Souslin tree.

Let $\alpha < \omega_1$ and assume that $(T_{\alpha'}; \lhd_{\alpha'})$ is defined for $\alpha' < \alpha$ in such a way that (a) and (b) are not violated.

If $\alpha = 0$ let $T_0 = \{0\}$ and stipulate that 0 does not bear $\lhd_0$ to itself.

If $\alpha = \alpha' + 1$ for some α', then assign to the ordinals $\beta \in T_{\alpha'}$ of level α' disjoint countable infinite sets $B_\beta \subseteq \omega_1$. Do this so that $\beta < \gamma \notin T_{\alpha'}$ for each $\gamma \in B_\beta$. Let

$$T_\alpha = T_{\alpha'} \cup \bigcup \{B_\beta \mid \beta \in T_{\alpha'} \wedge \text{level}(\beta) = \alpha'\}\,.$$

Let

$$\lhd_\alpha = \lhd'_\alpha \cup \{\langle\beta,\gamma\rangle \mid \beta \in T_{\alpha'} \,\wedge\, \text{level}(\beta) = \alpha' \wedge \gamma \in B_\beta\}\,.$$

Assume that α is a limit ordinal. The plan is to make sure that if A_α is a maximal antichain in $\bigcup_{\alpha'<\alpha} T_{\alpha'}$, then A_α is a maximal antichain in T. This is called "sealing off" A_α. As we will see later, the fact that $\langle A_\alpha \mid \alpha < \omega_1\rangle$ witnesses $\Diamond$ will guarantee that every maximal antichain in T is sealed off at some stage.

Let $\langle\alpha_i \mid i \in \omega\rangle$ be a strictly increasing sequence of ordinals with supremum α. Let

$$\begin{array}{rcll} T^*_\alpha &=& \bigcup_{\alpha'<\alpha} T_{\alpha'} & (= \;\bigcup_{i\in\omega} T_{\alpha_i});\\ \lhd^*_\alpha &=& \bigcup_{\alpha'<\alpha} \lhd_{\alpha'} & (= \;\bigcup_{i\in\omega} \lhd_{\alpha_i}).\end{array}$$

For $\beta \in T^*_\alpha$, define $\langle\beta_i \mid i \in \omega\rangle$ by recursion as follows. If A_α is not a maximal antichain in the tree $(T^*_\alpha; \lhd^*_\alpha)$ or if there is a $\xi \in A_\alpha$ such that $\xi \lhd^*_\alpha \beta$, then set $\beta_0 = \beta$. Otherwise there is a $\xi \in A_\alpha$ such that $\beta \lhd^*_\alpha \xi$. Let β_0 be some such ξ. If $\text{level}(\beta_i) \geq \alpha_i$, then let $\beta_{i+1} = \beta_i$. If $\text{level}(\beta_i) < \alpha_i$, let $\beta_{i+1} \in T_{\alpha_i}$ be such that $\beta_i \lhd_{\alpha_i} \beta_{i+1}$ and $\text{level}(\beta_{i+1}) = \alpha_i$. (Such a β_{i+1} exists by condition (vi) on $(T_{\alpha_i}; \lhd_{\alpha_i})$.) Let b_β be the unique branch containing all the β_i. Let $\mathcal{B}_\alpha$ be the set of all the b_β for $\beta \in T^*_\alpha$. For each $b \in \mathcal{B}_\alpha$, let γ_b be a countable ordinal γ such that $\gamma \notin T^*_\alpha$ and $\gamma >$ every member of b. Make sure that the function $b \mapsto \gamma_b$ is one-one. Let

$$T_\alpha = T^*_\alpha \cup \{\gamma_b \mid b \in \mathcal{B}_\alpha\}\,.$$

Let

$$\lhd_\alpha = \lhd^*_\alpha \cup \{\langle\delta,\gamma_b\rangle \mid (b \in \mathcal{B}_\alpha \,\wedge\, \delta \in b)\}\,.$$

To verify that $(T; \lhd)$ satisfies condition (4), we first show that if $(T; \lhd)$ has an uncountable branch then it has an uncountable antichain. Let b be an uncountable branch. By condition (v), each $\beta \in b$ has an immediate $\lhd$-successor that does not belong to b. Let

$$A = \{\gamma \mid \gamma \notin b \,\wedge\, (\exists\beta \in b)\,\gamma \text{ is an immediate } \lhd\text{-successor of } \beta\}\,.$$

The uncountable set A is clearly an antichain of $(T; \lhd)$.

Since every antichain can be extended to a maximal antichain, it suffices to prove that $(T; \lhd)$ has no uncountable maximal antichains.

Let A be a maximal antichain of $(T; \lhd)$. For limit $\alpha < \omega_1$, let $(T^*_\alpha; \lhd^*_\alpha)$ be defined as above. Note that T^*_α is the set of $\beta \in T$ such that, with respect to $\lhd$, $\text{level}(\beta) < \alpha$. Note also that $\lhd^*_\alpha$ is just the restriction of $\lhd$ to T^*_α.

Let C be the set of all limit $\alpha < \omega_1$ such that

(a) $T^*_\alpha = T \cap \alpha$;

(b) $A \cap \alpha$ is a maximal antichain of $(T^*_\alpha; \lhd^*_\alpha)$.

We will prove that C is closed and unbounded in ω_1.

By the definition of T^*_α, it is clear that $\{\alpha \mid T^*_\alpha = T \cap \alpha\}$ is closed in ω_1. To show that C is closed, it is therefore enough to show that the set of all α that satisfy (b) is closed in ω_1. Suppose that $\langle \alpha_i \mid i \in \omega \rangle$ is a strictly increasing sequence of countable ordinals such that for each i, $A \cap \alpha_i$ a maximal antichain of $(T^*_{\alpha_i}; \lhd^*_{\alpha_i})$. Let $\alpha = \bigcup_{i \in \omega} \alpha_i$. Let $\beta \in T^*_\alpha$. For any sufficiently large $i \in \omega$, $\beta \in T^*_{\alpha_i}$. Thus β is comparable with some $\gamma \in A \cap \alpha_i \subseteq A \cap \alpha$. This shows that $A \cap \alpha$ is a maximal antichain in $(T^*_\alpha; \lhd^*_\alpha)$.

For $\alpha < \omega_1$, let

$$\begin{aligned} f(\alpha) &= \mu\delta\,(\forall \beta \in T^*_\alpha)\,\beta < \delta; \\ g(\alpha) &= \mu\delta\,(\forall \beta \in T^*_\alpha)(\exists \gamma \in A \cap \delta)\,\gamma \text{ is } \lhd\text{-comparable with } \beta. \end{aligned}$$

That $f(\alpha)$ and $g(\alpha)$ are defined for every α follows from the fact that T^*_α is countable (by (iv)) and the fact that A is an maximal antichain of $(T; \lhd)$. By an argument like one in the proof of Reflection, the set C' of all countable ordinals closed under f and g is an unbounded subset of ω_1. By (ii), $T \cap \alpha \subseteq T^*_\alpha$ for every $\alpha < \omega_1$. Therefore every $\alpha \in C'$ satisfies (a) and (b).

Since $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ witnesses the truth of $\Diamond$, let $\alpha \in C$ be such that $A \cap \alpha = A_\alpha$. By (b), A_α is a maximal antichain of T^*_α. By the definition of $\mathcal{B}_\alpha$, every $b \in \mathcal{B}_\alpha$ contains a member of A_α. For $b \in \mathcal{B}_\alpha$, every member of b is $\lhd_\alpha \gamma_b$ and so is $\lhd \gamma_b$. Hence for each $b \in \mathcal{B}_\alpha$ there is a $\xi \in A_\alpha$ such that $\xi \lhd \gamma_b$. If $\beta \in T \setminus T_\alpha$, then there is a b such that $\gamma_b \lhd \beta$. Putting all these facts together, we get that every element of T is $\lhd$-comparable with some element of A_α. In other words, A_α—i.e., $A \cap \alpha$—is a maximal antichain of T. But this means that $A = A \cap \alpha$. Hence A is countable. □