Appendix 1

We correct the course text's definition of *ordinal number*, define a few additional notions, reorder the text's treatment of ω, and give alternative proofs of a few facts. Nothing except the definition of ordinal number is inconsistent with the course text.

A *strict wellordering* is a strict linear ordering r such that every nonempty subset x of Field(r) has an r-least element: a $y \in x$ such that $\forall z \in x \langle z, y\rangle \notin r$.

A *wellordering* is a (reflexive) linear ordering r such that the difference $r \setminus \{\langle y, y\rangle \mid \langle y, y\rangle \in r\}$ is a strict wellordering.

If r is a (set or class) relation and A is a class, the $r \restriction A = \{\langle x, y\rangle \in r \mid x \in A \wedge y \in A\}$.

An *ordinal number* is a transitive set α such that $\in \restriction \alpha$ is a strict wellordering. Here $\in$ is the class $\{\langle x, y\rangle \mid x \in y\}$.

ON is the class of all ordinal numbers.

A set z is *inductive* if $0 \in z$ and $\forall y \in z\ y' \in z$. Here $0 = \emptyset$ and $y' = y \cup \{y\}$. Let Induct be the class of all inductive sets.

Define $\omega = \bigcap \text{Induct} = \{x \mid \forall z(z \text{ inductive} \rightarrow x \in z)\}$.

The Axiom of Infinity says that Induct $\neq \emptyset$. Let $z^* \in$ Induct. Then

$$\omega = \bigcap \text{Induct} = z^* \cap \bigcap \text{Induct}.$$

By Comprehension, ω is a set.

Proposition 6C.3 (Induction Principle). *If z is an inductive subset of ω, then $z = \omega$.*

Proposition 6C.4. (1) *If $x \in \omega$ then $x = 0$ or $x = k'$ for some $k \in \omega$.*
(2) *ω is transitive.*

Proof. For (1) Apply the Induction Principle with $\{x \in \omega \mid x = 0 \vee (\exists k \in \omega)\, x = k'\}$ as z.

For (2), we apply the Induction Principle with $\{n \in \omega \mid n \subseteq \omega\}$ as z. Since $0 = \emptyset$, $0 \in z$. Assume that $n \in \omega$ and $n \in z$. If $m \in n'$ then $m \in n$ or $m = n$. If $m \in n$, then $m \in \omega$ because $n \subseteq \omega$. If $m = n$ then $m \in \omega$ since $n \in \omega$. □

Proposition 6C.4$\frac{1}{2}$. *Every member of ω is an ordinal number.*

Proof. Let $z = \{k \in \omega \mid k \text{ is an ordinal number}\}$.

Clearly 0 is transitive and $\in\restriction 0$ is a strict wellordering. Hence $0 \in z$.

Assume that $n \in z$. Since $n' = n \cup \{n\}$, $\bigcup n' = \bigcup n \cup n = n \subseteq n'$. Thus n' is transitive. To see that $\in\restriction n'$ is a strict wellordering, note the following two facts.

(i) For all $m \in n$, $\langle m, n\rangle$ belongs to $\in\restriction n'$

(ii) For all $m \in n'$ $\langle n, m\rangle$ does not belong to $\in\restriction n'$.

(ii) cannot fail for $m = n$ because $n \in n$ contradicts the irreflexiveness of $\in\restriction n$. If (ii) fails for $m \in n$, then $n \in m \in n$ and the transitivity of n implies that $n \in n$. Together (i) and (ii) show that $\in\restriction n'$ is just the wellordering $\in\restriction n$ with one element added at the end. One can easily check that this implies that $\in\restriction n'$ is a wellordering. □

Proposition 6C.4$\frac{3}{4}$. *Let α and β be ordinal numbers. Then*

$$\alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha.$$

Proof. We will show the following:

(a) If $\alpha \not\subseteq \beta$ then $\alpha \cap \beta$ is the $\in\restriction\alpha$-least member of $\alpha \setminus \beta$.

(b) If $\beta \not\subseteq \alpha$ then $\alpha \cap \beta$ is the $\in\restriction\beta$-least member of $\beta \setminus \alpha$.

Assume (a) and (b). Then $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$, and

$$\begin{aligned} \alpha \subsetneq \beta \quad &\Rightarrow \quad \alpha = \alpha \cap \beta \in \beta; \\ \beta \subsetneq \alpha \quad &\Rightarrow \quad \beta = \alpha \cap \beta \in \alpha; \\ \alpha \subseteq \beta \wedge \beta \subseteq \alpha \quad &\Rightarrow \quad \alpha = \beta. \end{aligned}$$

We prove (a). The proof of (b) is similar. Let u be the $\in\restriction\alpha$-least member of $\alpha \setminus \beta$. We show that $u \subseteq \alpha \cap \beta$ and $\alpha \cap \beta \subseteq u$.

Assume $v \in u$. By the transitivity of α, $v \in \alpha$. If $v \notin \beta$, then v contradicts the $\in\restriction\alpha$-minimality of u, so $v \in \alpha \cap \beta$.

Assume $v \in \alpha \cap \beta$. Since $\in\restriction\alpha$ is a linear ordering, $u \in v$ or $u = v$ or $v \in u$. Both $u \in v$ and $u = v$ imply the contradiction that $u \in \beta$. Hence $v \in u$. □

Theorem 6C.5 *ω is an ordinal number.*

Proof. By part (2) of Proposition 6.4, we need only show that $\in\restriction\omega$ is a strict wellordering.

If $n \in n \in \omega$, then $\in \restriction n$ is not irreflexive. This shows that $\in\restriction\omega$ is irreflexive.

If $l \in m \in n \in \omega$, then the transitivity of n implies that $l \in n$. Hence $\in\restriction\omega$ is transitive.

By Propositions 6C.4$\frac{1}{2}$ and 6C.4$\frac{3}{4}$, $\in\restriction\omega$ is connected.

Let x be a non-empty subset of ω. Let $k \in x$. If $k \cap x = \emptyset$, then k is the $\in\restriction\omega$-least member of x. If $k \cap x \neq \emptyset$, then the $\in\restriction k$-least member of x is the $\in\restriction\omega$-least member of x, for if $i \in j \cap x$ then $i \in k$ by the transitivity of k. $\Box$