Mathematics 220A Fall 2009

FINAL EXAMINATION

Due 5:00 PM, Friday, December 11

1. Give a deduction witnessing the following:

$$\{\neg\exists v_1 \neg F(v_1)\} \vdash \exists v_1\, F(v_1).$$

You may treat $(\phi \to \psi) \to (\neg\psi \to \neg\phi)$ and $(\neg\psi \to \neg\phi) \to (\phi \to \psi)$ as axiom schemas, but otherwise you must give an actual deduction: no steps omitted; no derived rules or meta-rules; etc.

2. Suppose we replaced our Exists Elimination rule with the following additional axiom schema:

$$((\phi \to \psi) \to ((\exists v)\phi \to \psi))$$
for v not occurring free in ψ.

Would Soundness still hold? Would Completeness still hold? Prove your answers.

3. Assume that τ is a countable signature containing a binary relation symbol P. Let $\mathbf{A}$ be an countably infinite τ-structure such that $P^{\mathbf{A}}$ is an equivalence relation. Prove that at least one of the following holds:

(i) There is some positive integer n such that there are infinitely many n-element equivalence classes of $P^{\mathbf{A}}$.

(ii) There is an elementary extension $\mathbf{B}$ of $\mathbf{A}$ such that $P^{\mathbf{B}}$ is an equivalence relation with an infinite equivalence class.

4. For each of the following classes or linear orderings, determine whether it is basic elementary, elementary or neither and prove your answer:

(1) The class $\mathcal{W}$ of wellorderings.

(2) The class $\mathcal{W}^c$ of linear orderings which are not wellorderings.

You may use the following characterization of the linear orderings that are wellorderings: A linear ordering $(A, \leq)$ is a wellordering if and only if there is no infinite, descending chain $x_0 > x_1 > \cdots$.

5. Let τ be a countable signature and let T be a τ-theory. Assume that all countable models of T are $\aleph_0$-saturated. Prove that all models of T are $\aleph_0$-saturated.

6. Let $T = \mathrm{Th}(\mathbb{N}, <)$. For models $\mathbf{A}$ of T, let $\sim$ be the equivalence relation defined by setting $a_1 \sim a_2$ just in case there are only finitely many elements of A between—in the obvious sense—a_1 and a_2. For $a \in A$, let $\bar{a}$ be the equivalence class of a. Let $<^*_{\mathbf{A}}$ be the ordering of equivalence classes induced by $<^{\mathbf{A}}$.

(a) Sketch a proof that every countable model $\mathbf{A}$ of T has an elementary extension $\mathbf{B}$ such that, for all elements a_1 and a_2 of A with $\bar{a}_1 <^*_{\mathbf{A}} \bar{a}_2$, there is a $b \in B$ with $\bar{a} <^*_{\mathbf{B}} \bar{b} <^*_{\mathbf{B}} \bar{a}_2$.

(b) Use part (a) and Problem 2A.4 to show that every countable model of A has an elementary extension $\mathbf{B}$ such that the ordering $<^*_{\mathbf{B}}$ is dense.

7. Consider the following set of formulas with just v free in the language of orderings:

$$\Phi(v) = \{\exists u_1(u_1 < v),\ \exists u_1 \exists u_2(u_1 < u_2 < v), \ldots\}.$$

Let T be the theory of dense linear orderings with a least element and no greatest element. Prove that $\Phi(v)$ is a partial type of T, and determine (with proofs) whether each of the following claims is true or false:

(a) There is a unique complete type of T extending $\Phi(v)$

(b) $\Phi(v)$ is principal.

(c) $\Phi(v)$ is realized in some model of T.

(d) $\Phi(v)$ is realized in every model of T.

8. Let τ be a countable signature. T be a complete, consistent τ-theory that is not $\aleph_0$-categorical.

(a) Let $\mathbf{A}$ be a countable model of T and let $Y \subseteq A$ be finite. Assume that $\mathrm{Th}(\mathbf{A}_Y)$ has an atomic model $\mathbf{B}^*$. (A structure $\mathbf{B}$ is *atomic* if $\mathbf{B}$ realizes no non-principal types of $\mathrm{Th}(\mathbf{B})$.) Prove that $\mathbf{B}^*$ is not $\aleph_0$-saturated, and use this fact to show that $\mathbf{B}^*$'s τ-reduct is not $\aleph_0$-saturated. You may use the fact that a structure $\mathbf{B}$ is $\aleph_0$-saturated if and only if, for every n and every finite $X \subseteq B$, $\mathbf{B}$ realizes every n-type of $\mathrm{Th}(\mathbf{B}_X)$.

(b) Prove, without making the assumption of (a), that there are at least three isomorphism types of countable models of T.

Hint. For (a), first use Theorem 2C.10 and Compactness to show that $\mathrm{Th}(\mathbf{A}_Y)$ has infinitely many n-types for some n. Then apply Theorem 2C.9.

For (b), assume that there are no more than two isomorphism types of countable models of T. Show that T has a countable, $\aleph_0$-saturated model $\mathbf{A}$. Prove that the assumption of (a) holds for all finite subsets of A. Apply (a) to a set $Y = \{a_1, \ldots, a_n\}$, where $\mathrm{type}_{\mathbf{A}}(a_1, \ldots, a_n)$ is a non-principal n-type of T.