5 Deduction in First-Order Logic

The system $\mathbf{FOL}_{\mathrm{C}}$.

Let C be a set of constant symbols. $\mathbf{FOL}_{\mathrm{C}}$ is a system of deduction for the language $\mathcal{L}^{\#}_{\mathrm{C}}$.

Axioms: The following are axioms of $\mathbf{FOL}_{\mathrm{C}}$.

(1) All tautologies.

(2) Identity Axioms:

   (a) $t = t$
   for all terms t;

   (b) $t_1 = t_2 \rightarrow (A(x; t_1) \rightarrow A(x; t_2))$
   for all terms t_1 and t_2, all variables x, and all formulas A such that there is no variable y occurring in t_1 or t_2 such that there is free occurrence of x in A in a subformula of A of the form $\forall yB$.

(3) Quantifier Axioms:

$$\forall xA \rightarrow A(x; t)$$

for all formulas A, variables x, and terms t such that there is no variable y occurring in t such that there is a free occurrence of x in A in a subformula of A of the form $\forall yB$.

Rules of Inference:

Modus Ponens (MP) $$\frac{A\,,\ (A \rightarrow B)}{B}$$

Quantifier Rule (QR) $$\frac{(A \rightarrow B)}{(A \rightarrow \forall xB)}$$

provided the variable x does not occur free in A.

Discussion of the axioms and rules.

(1) We would have gotten an equivalent system of deduction if instead of taking all tautologies as axioms we had taken as axioms all instances (in $\mathcal{L}^{\#}_{\mathrm{C}}$) of the three schemas on page 16. All instances of these schemas are tautologies, so the change would have not have increased what we could

deduce. In the other direction, we can apply the proof of the Completeness Theorem for **SL** by thinking of all sententially atomic formulas as sentence letters. The proof so construed shows that every tautology in $\mathcal{L}^{\#}_{\mathrm{C}}$ is deducible using MP and schemas (1)–(3). Thus the change would not have decreased what we could deduce.

(2) Identity Axiom Schema (a) is self-explanatory. Schema (b) is a formal version of the *Indiscernibility of Identicals*, also called *Leibniz's Law*.

(3) The Quantifer Axiom Schema is often called the schema of *Universal Instantiation*. Its idea is that whatever is true of a all objects in the domain is true of whatever object t might denote. The reason for the odd-looking restriction is that instances where the restriction fails do not conform to the idea. Here is an example. Let A be $\exists v_2\, v_1 \neq v_2$, let x be v_1 and let t be v_2. The instance of the schema would be

$$\forall v_1 \exists v_2\, v_1 \neq v_2 \rightarrow \exists v_2\, v_2 \neq v_2.$$

The antecedent is true in all models whose domains have more than one element, but the consequent is not satisfiable.

(MP) Modus ponens is the rule we are familiar with from the system **SL**.

(QR) As we shall explain later, the Quantifier Rule is not a valid rule. The reason it will be legitimate for us to use it as a rule is that we shall allow only sentences as premises of our deductions. How this works will be explained in the proof of the Soundness Theorem.

Deductions: A *deduction* in $\mathbf{FOL}_{\mathrm{C}}$ from a set Γ of sentences is a finite sequence $\mathbf{D}$ of formulas such that whenever a formula A occurs in the sequence $\mathbf{D}$ then at least one of the following holds.

(1) $A \in \Gamma$.

(2) A is an axiom.

(3) A follows by modus ponens from two formulas occurring earlier in the sequence $\mathbf{D}$ or follows by the Quantifier Rule from a formula occurring earlier in $\mathbf{D}$.

A *deduction in* $\mathbf{FOL}_{\mathrm{C}}$ *of a formula* A *from a set* Γ of sentences is a deduction $\mathbf{D}$ in $\mathbf{FOL}_{\mathrm{C}}$ from Γ with A on the last line of $\mathbf{D}$. We write $\Gamma \vdash_{\mathbf{FOL}_{\mathrm{C}}} A$ and say A is *deducible* in $\mathbf{FOL}_{\mathrm{C}}$ from Γ to mean that there is a deduction in $\mathbf{FOL}_{\mathrm{C}}$ of A from Γ. We write $\vdash_{\mathbf{FOL}_{\mathrm{C}}} A$ for $\emptyset \vdash_{\mathbf{FOL}_{\mathrm{C}}} A$.

Announcement. For the rest of this section, we shall omit subscripts "$\mathbf{FOL}_\mathrm{C}$." and phrases "in $\mathbf{FOL}_\mathrm{C}$" except in contexts where we are considering more than one set C.

In order to avoid dealing directly with long formulas and long deductions, it will be useful to begin by justifying some derived rules.

Lemma 5.1. *Assume that $\Gamma \vdash A_i$ for $1 \leq i \leq n$ and $\{A_1, \ldots, A_n\} \models_{\mathrm{sl}} B$. Then $\Gamma \vdash B$. (See page 36 for the definition of $\models_{\mathrm{sl}}$.)*

Proof. If we string together deductions witnessing that $\Gamma \vdash A_i$ for each i, then we get a deduction from Γ in which each A_i is a line. The fact that $\{A_1, \ldots, A_n\} \models_{\mathrm{sl}} B$ gives us that the formula

$$(A_1 \to A_2 \to \cdots A_n \to B)$$

is a tautology. Appending this formula to our deduction and applying MP n times, we get B. □

Lemma 5.1 justifies a derived rule, which we call SL. A formula B follows from formulas $A_1, \ldots, A_n$ by SL iff

$$\{A_1, \ldots, A_n\} \models_{\mathrm{sl}} B.$$

Lemma 5.2. *If $\Gamma \vdash A$ then $\Gamma \vdash \forall x A$.*

Proof. Assume that $\Gamma \vdash A$. Begin with a deduction from Γ with last line A. Use SL to get the line $(p_0 \vee \neg p_o) \to A$. Now apply QR to get $(p_0 \vee \neg p_o) \to \forall x A$. Finally use SL to get $\forall x A$. □

Lemma 5.2 justifies a derived rule, which we call Gen:

$$\text{Gen} \qquad \frac{A}{\forall x A}$$

Lemma 5.3. *For all formulas A and B,*

$$\vdash \forall x(A \to B) \to (\forall x A \to \forall x B).$$

Proof. Here is an abbreviated deduction.

1.	$\forall x(A \to B) \to (A \to B)$	QAx
2.	$\forall x A \to A$	QAx
3.	$(\forall x(A \to B) \wedge \forall x A) \to B$	1,2; SL
4.	$(\forall x(A \to B) \wedge \forall x A) \to \forall x B$	3; QR
5.	$\forall x(A \to B) \to (\forall x A \to \forall x B)$	4; SL

□

Lemma 5.4. *For all formulas A,*

$$\vdash \exists x \forall y A \to \forall y \exists x A.$$

Proof. Here is an abbreviated deduction.

1.	$\forall y A \to A$	QAx
2.	$\neg A \to \neg \forall y A$	1; SL
3.	$\forall x(\neg A \to \neg \forall y A)$	2; Gen
4.	$\forall x(\neg A \to \neg \forall y A) \to (\forall x \neg A \to \forall x \neg \forall y A)$	Lemma 5.3
5.	$\forall x \neg A \to \forall x \neg \forall y A$	3,4; MP
6.	$\neg \forall x \neg \forall y A \to \neg \forall x \neg A$	5; SL
	$[\exists x \forall y A \to \exists x A]$	
7.	$\exists x \forall y A \to \forall y \exists x A$	6; QR

□

Exercise 5.1. Show that $\vdash (\exists v_1\, P^1 v_1 \to \exists v_2\, P^1 v_2)$.

Exercise 5.2. Show that $\{\forall v_1\, P^1 v_1\} \vdash \exists v_1\, P^1 v_1$.

Lemma 5.5. *If* $\Gamma \vdash (A \to B)$ *then* $\Gamma \vdash (\forall\, x A \to \forall x\, B)$.

Proof. Start with a deduction from Γ with last line $(A \to B)$. Use Gen to get the line $\forall x(A \to B)$. Then apply Lemma 5.3 and MP. □

Theorem 5.6 (Deduction Theorem). *Let Γ be a set of sentences, let A be a sentence, and let B be a formula. If $\Gamma \cup \{A\} \vdash B$ then $\Gamma \vdash (A \to B)$.*

Proof. The proof is similar to the proof of the Deduction Theorem for **SL**. Assume that $\Gamma \cup \{A\} \vdash B$. Let **D** be a deduction of B from $\Gamma \cup \{A\}$. We prove that

$$\Gamma \vdash (A \to C)$$

for every line C of **D**. Assume that this is false. Consider the first line C of **D** such that $\Gamma \nvdash (A \to C)$.

Assume that C either belongs to Γ or is an axiom. Then $\Gamma \vdash C$ and $(A \to C)$ follows from C by SL. Hence $\Gamma \vdash (A \to C)$.

Assume next that C is A. Since $A \to A$ is a tautology, $\Gamma \vdash (A \to A)$.

Assume next that C follows from formulas E and $(E \to C)$ by MP. These formulas are on earlier lines of **D** than C. Since C is the first "bad" line of **D**, $\Gamma \vdash A \to E$ and $\Gamma \vdash A \to (E \to C)$. Since

$$\{(A \to E), (A \to (E \to C))\} \models_{\text{sl}} (A \to C),$$

Lemma 5.1 gives us that $\Gamma \vdash (A \to C)$.

Finally assume that C is $(E \to \forall x F)$ and that C follows by QR from an earlier line $(E \to F)$ of **D**. Since C is the first "bad" line of **D**, $\Gamma \vdash A \to (E \to F)$. Starting with a deduction from Γ of $A \to (E \to F)$, we can get a deduction from Γ of $A \to (E \to \forall x F)$ as follows.

...	...	...
...	...	...
...	...	...
n	$A \to (E \to F)$	...
$n+1.$	$(A \wedge E) \to F$	n; SL
$n+2.$	$(A \wedge E) \to \forall x F$	$n+1$; QR
$n+3.$	$A \to (E \to \forall x F)$	$n+2$; SL

Note that the variable x has no free occurrences in A because A is a sentence, and we know that it has no free occurrences in E because we know that QR was used in **D** to get $E \to \forall x F$ from $E \to F$.

This contradiction completes the proof that the "bad" line C cannot exist. Applying this fact to the last line of **D**, we get that $\Gamma \vdash (A \to B)$. $\Box$

A set Γ of sentences of $\mathcal{L}_{\mathrm{C}}^{\#}$ is *inconsistent* in $\mathbf{FOL}_{\mathrm{C}}$ if there is a formula B such that $\Gamma \vdash_{\mathbf{FOL}_{\mathrm{C}}} B$ and $\Gamma \vdash_{\mathbf{FOL}_{\mathrm{C}}} \neg B$. Otherwise Γ is *consistent*.

Theorem 5.7. *Let Γ and Δ be sets of sentences, let A and $A_1, \ldots, A_n$ be sentences, and let B be a formula.*

(1) $\Gamma \cup \{A\} \vdash B$ if and only if $\Gamma \vdash (A \to B)$.

(2) $\Gamma \cup \{A_1, \ldots, A_n\} \vdash B$ if and only if $\Gamma \vdash (A_1 \to \ldots \to A_n \to B)$.

(3) Γ is consistent if and only if there is some formula C such that $\Gamma \nvdash C$.

(4) If $\Gamma \vdash C$ for all $C \in \Delta$ and if $\Delta \vdash B$, then $\Gamma \vdash B$.

Proof. The proof is like the proof of Theorem 2.2, except that we may now use the derived rule SL instead of the particular axioms and rules of the system **SL**. $\Box$

A system **S** of deduction for $\mathcal{L}_{\mathrm{C}}^{\#}$ is *sound* if, for all sets Γ of sentences and all formulas A, if $\Gamma \vdash_{\mathbf{S}} A$ then $\Gamma \models A$. A system **S** of deduction for $\mathcal{L}_{\mathrm{C}}^{\#}$ is *complete* if, for all sets Γ of sentences and all formulas A, if $\Gamma \models A$ then $\Gamma \vdash_{\mathbf{S}} A$.

Remark. These definitions are like the definitions of soundness and completeness of systems for $\mathcal{L}$, except that the new definitions require Γ to consist of sentences, not just formulas. We hereby make the analoguous definitions for our other languages.

Exercise 5.3. Prove that all instances of Identity Axiom Schema (b) are valid.

Exercise 5.4. Prove that all instances of the Quantifier Axiom Schema are valid.

Hint for Exercises 5.3 and 5.4: For terms t^* and t and variables x, let $t^*(x;t)$ be the result of replacing the occurrences of x in t^* by occurrences of t.

Let $\mathfrak{M}$ be a model and and let s be variable assignment. Let x be variable and let t be a term. Assume that $s(x) = \mathrm{den}^s_{\mathfrak{M}}(t)$. Prove by induction on length that, for all terms t^*,

$$\mathrm{den}^s_{\mathfrak{M}}(t^*) = \mathrm{den}^s_{\mathfrak{M}}(t^*(x;t)).$$

Next prove by induction on length that, for all formulas A, if A, x, and t satisfy the restriction in the statement of the Quantifier Axiom Schema, then

$$v^s_{\mathfrak{M}}(A) = v^s_{\mathfrak{M}}(A;t).$$

Theorem 5.8 (Soundness). *The systems* $\mathbf{FOL}_{\mathrm{C}}$ *are sound.*

Proof. The proof is similar to the proof of soundness for **SL** (Theorem 2.4). Let $\mathbf{D}$ be a deduction in $\mathbf{FOL}_{\mathrm{C}}$ of a formula A from a set Γ of sentences. We shall show that, for every line C of $\mathbf{D}$, $\Gamma \models C$. Applying this to the last line of $\mathbf{D}$, this will give us that $\Gamma \models A$.

Assume that what we wish to show is false. Let C be the first line of $\mathbf{D}$ such that $\Gamma \not\models C$.

Using Exercises 5.3 and 5.4, it is easy to see that all the axioms are valid. It follows that the cases that $C \in \Gamma$, that C is an axiom, and that C follows by MP from earlier lines of $\mathbf{D}$, are just like the corresponding cases in the proof of Theorem 2.4.

The only remaining case is that C is $B \to \forall x E$ and C follows by QR from an earlier line $B \to E$ of $\mathbf{D}$. Since C is the first "bad" line of $\mathbf{D}$, $\Gamma \models B \to E$. Let $\mathfrak{M} = (\boldsymbol{D}, v, \chi)$ be any model and let s be any variable assignment. We assume that $v^s_{\mathfrak{M}}(\Gamma) = \mathbf{T}$ (i.e., that $v^s_{\mathfrak{M}}(H) = \mathbf{T}$ for each $H \in \Gamma$), and we show that $v^s_{\mathfrak{M}}(B \to \forall x E) = \mathbf{T}$. For this, we assume that

$v^s_{\mathfrak{M}}(B) = \mathbf{T}$ and we show that $v^s_{\mathfrak{M}}(\forall x E) = \mathbf{T}$. Let d be any element of $\boldsymbol{D}$ and let s' be any variable assignment that agrees with s except that $s'(x) = d$. We must show that $v^{s'}_{\mathfrak{M}}(E) = \mathbf{T}$. Since Γ is a set of sentences, $v^{s'}_{\mathfrak{M}}(\Gamma) = \mathbf{T}$. Since the variable x does not occur free in B, $v^{s'}_{\mathfrak{M}}(B) = \mathbf{T}$. Since $\Gamma \models B \to E$, it follows that $v^{s'}_{\mathfrak{M}}(E) = \mathbf{T}$ □

Lemma 5.9. *Let Γ be a set of sentences of $\mathcal{L}^{\#}_{\mathrm{C}}$ consistent in* $\mathbf{FOL}_{\mathrm{C}}$ *and let A be a sentence of $\mathcal{L}^{\#}_{\mathrm{C}}$. Then either $\Gamma \cup \{A\}$ is consistent in* $\mathbf{FOL}_{\mathrm{C}}$ *or $\Gamma \cup \{\neg A\}$ is consistent in* $\mathbf{FOL}_{\mathrm{C}}$.

Proof. The proof is like that of Lemma 2.5. □

Lemma 5.10. *Let Γ be set of sentences of $\mathcal{L}^{\#}_{\mathrm{C}}$ consistent in* $\mathbf{FOL}_{\mathrm{C}}$. *Let C^* be a set gotten from C by adding infinitely many new constants. There is a set Γ^* of sentences of $\mathcal{L}^{\#}_{\mathrm{C}^*}$ such that*

(1) $\Gamma \subseteq \Gamma^*$;

(2) Γ^* *is consistent in* $\mathbf{FOL}_{\mathrm{C}^*}$;

(3) for every sentence A of $\mathcal{L}^{\#}_{\mathrm{C}^*}$, either A belongs to Γ^* or $\neg A$ belongs to Γ^*;

(4) Γ^* *is Henkin.*

Proof. Let $c_0, c_1, c_2, \ldots$ be all the constants of $\mathcal{L}^{\#}_{\mathrm{C}^*}$. Let

$$A_0,\ A_1,\ A_2,\ A_3,\ \ldots$$

be the list (defined in the proof of Lemma 4.8) of all the sentences of $\mathcal{L}^{\#}_{\mathrm{C}^*}$. As in that proof we define, by recursion on natural numbers, a function that associates with each natural number n a set Γ_n of formulas.

Let $\Gamma_0 = \Gamma$.

As in the proofs of Lemmas 3.5, 4.2, and 4.8, we shall make sure that, for each n, at most two sentences belong to Γ_{n+1} but not to Γ_n. As in the earlier proofs, it follows that for each n only finitely many of the new constants occur in sentences in Γ_n.

We define Γ_{n+1} from Γ_n in two steps. For the first step, let

$$\Gamma'_n = \begin{cases} \Gamma_n \cup \{A_n\} & \text{if } \Gamma_n \cup \{A_n\} \text{ is consistent in } \mathbf{FOL}_{\mathrm{C}^*}; \\ \Gamma_n \cup \{\neg A_n\} & \text{otherwise.} \end{cases}$$

Let $\Gamma_{n+1} = \Gamma'_n$ unless both of the following hold.

(a) $\neg A_n \in \Gamma'_n$.

(b) A_n is $\forall x_n B_n$ for some variable x_n and formula B_n.

Suppose that both (a) and (b) hold. Let i_n be the least i such that the constant c_i does not occur in any formula belonging to Γ'_n. Such an i must exist, since only finitely many of the infinitely many new constants occur in sentences in Γ'_n. Let

$$\Gamma_{n+1} = \Gamma'_n \cup \{\neg B_n(x_n; c_{i_n})\}.$$

Let $\Gamma^* = \bigcup_n \Gamma_n$.

Because $\Gamma = \Gamma_0 \subseteq \Gamma^*$, Γ^* has property (1).

We prove by mathematical induction that Γ_n is consistent for each n.

Γ_0 (i.e., Γ) is consistent in $\mathbf{FOL}_{\mathrm{C}}$ by hypothesis, but we must prove that it is consistent in $\mathbf{FOL}_{\mathrm{C}^*}$. Observe that any deduction $\mathbf{D}$ from Γ in $\mathbf{FOL}_{\mathrm{C}^*}$ of a formula of $\mathcal{L}^{\#}_{\mathrm{C}}$ can be turned into a deduction from Γ in $\mathbf{FOL}_{\mathrm{C}}$ of the same formula: just replace the new constants occurring in $\mathbf{D}$ by distinct variables that do not occur in $\mathbf{D}$. It follows easily that Γ is inconsistent in $\mathbf{FOL}_{\mathrm{C}}$ if it is inconsistent in $\mathbf{FOL}_{\mathrm{C}^*}$.

Assume that Γ_n is consistent in $\mathbf{FOL}_{\mathrm{C}^*}$. Lemma 5.9 implies that Γ'_n is consistent. If $\Gamma_{n+1} = \Gamma'_n$, then Γ_{n+1} is consistent. Assume then that $\Gamma_{n+1} = \Gamma'_n \cup \{\neg B_n(x_n; c_{i_n})\}$ and, in order to derive a contradiction, assume that Γ_{n+1} is not consistent. By Theorem 5.7, every formula of $\mathcal{L}^{\#}_{\mathrm{C}^*}$ is deducible from Γ_{n+1} in $\mathbf{FOL}_{\mathrm{C}^*}$. Hence $\Gamma_{n+1} \vdash_{\mathbf{FOL}_{\mathrm{C}^*}} (p_0 \wedge \neg p_0)$. In other words,

$$\Gamma'_n \cup \{\neg B_n(x_n; c_{i_n})\} \ \vdash_{\mathbf{FOL}_{\mathrm{C}^*}} \ (p_0 \wedge \neg p_0).$$

By the Deduction Theorem,

$$\Gamma'_n \vdash_{\mathbf{FOL}_{\mathrm{C}^*}} \neg B_n(x_n; c_{i_n}) \rightarrow (p_0 \wedge \neg p_0).$$

Let $\mathbf{D}$ be a deduction from Γ'_n in $\mathbf{FOL}_{\mathrm{C}^*}$ with last line $\neg B_n(x_n; c_{i_n}) \rightarrow (p_0 \wedge \neg p_0)$. Let y be a variable not occurring in $\mathbf{D}$. Let $\mathbf{D}'$ come from d by replacing every occurrence of c_{i_n} by an occurrence of y. Since c_{i_n} does not occur Γ'_n or in $\neg B_n$, $\mathbf{D}'$ is a deduction from Γ'_n in $\mathbf{FOL}_{\mathrm{C}^*}$ with last line $\neg B_n(x_n; y) \rightarrow (p_0 \wedge \neg p_0)$. We can turn $\mathbf{D}'$ into a deduction from Γ'_n in

$\mathbf{FOL}_{\mathrm{C}^*}$ with last line $\neg\forall x_n B_n \to (p_0 \wedge \neg p_0)$ as follows.

...	...	...
...	...	...
...	...	...
$n.$	$\neg B_n(x_n; y) \to (p_0 \wedge \neg p_0)$	...
$n+1.$	$\neg(p_0 \wedge \neg p_0) \to B_n(x_n; y)$	n; SL
$n+2.$	$\neg(p_0 \wedge \neg p_0) \to \forall y B_n(x_n; y)$	$n+1$; QR
$n+3.$	$\forall y B_n(x_n; y) \to B_n$	QAx
$n+4.$	$\neg(p_0 \wedge \neg p_0) \to B_n$	$n+2$,$n+3$; SL
$n+5.$	$\neg(p_0 \wedge \neg p_0) \to \forall x_n B_n$	$n+4$; QR
$n+6.$	$\neg\forall x_n B_n \to (p_0 \wedge \neg p_0)$	$n+5$; SL

This shows that $\Gamma'_n \vdash_{\mathbf{FOL}_{\mathrm{C}^*}} \neg\forall x_n B_n \to (p_0 \wedge \neg p_0)$. But $\Gamma'_n = \Gamma \cup \{\neg\forall x_n B_n\}$, so it follows that $\Gamma'_n \vdash_{\mathbf{FOL}_{\mathrm{C}^*}} (p_0 \wedge \neg p_0)$. By SL, we get the contradiction that Γ'_n is inconsistent in $\mathbf{FOL}_{\mathrm{C}^*}$.

As in the proof of Lemma 2.6, the consistency of all the Γ_n implies that consistency of Γ^*. Hence Γ^* has property (2).

Because either A_n or $\neg A_n$ belongs to Γ_{n+1} for each n and because each $\Gamma_{n+1} \subseteq \Gamma^*$, Γ^* has property (3).

If $A_n \notin \Gamma^*$, then $A_n \notin \Gamma_{n+1}$ and so $\neg A_n \in \Gamma_{n+1}$. But this implies that $\neg B_n(x_n; c_{i_n}) \in \Gamma_{n+1} \subseteq \Gamma^*$ if $A_n = \forall x_n B_n$. Hence Γ^* has property (4). □

Exercise 5.5. Show that

$$\{\forall v_1 \forall v_2 (P^2 v_1 v_2 \vee P^2 v_2 v_1)\} \vdash \forall v_1 P^2 v_1 v_1.$$

Exercise 5.6. Show that

$$\vdash \forall v_1 \exists v_2 F^1 v_1 = v_2.$$

Exercise 5.7. Let c_1 and c_2 be constants. Show that

$$\{c_1 = c_2\} \vdash c_2 = c_1.$$

Lemma 5.11. *Let Γ^* be a set of sentences of a language $\mathcal{L}^{\#}_{\mathrm{C}^*}$ having properties (2), (3), and (4) described in the statement of Lemma 5.10. Then Γ^* is satisfiable.*

Proof. As in the proof of Lemma 2.7, it follows from (2) and (3) that Γ^* is deductively closed: for all sentences A, if $\Gamma^* \vdash A$ then $A \in \Gamma^*$.

As in the proofs of Lemmas 4.4 and 4.9, we shall define a model whose domain is a set of equivalence classes of constants. As in the proof of Lemma 4.4, let R be the relation on C* defined by

$$Rc_1c_2 \text{ holds iff } c_1 = c_2 \in \Gamma^*.$$

We shall prove that R is an equivalence relation on C*.

For reflexivity, we must show that $c = c$ belongs to Γ^* for all members c of C*. Since $c = c$ is an instance of Identity Axiom Schema (a), $\vdash c = c$ and so $\Gamma^* \vdash c = c$. By deductive closure, $c = c \in \Gamma^*$.

For symmetry, we must show that, for all members c_1 and c_2 of Γ^*, if $c_1 = c_2 \in \Gamma^*$, then $c_2 = c_1 \in \Gamma^*$. Assume that $c_1 = c_2 \in \Gamma^*$. By Exercise 5.7, $\Gamma^* \vdash c_2 = c_1$. By deductive closure, $c_2 = c_1 \in \Gamma^*$.

Before proving transitivity, we show that

$$\{c_1 = c_2,\, c_2 = c_3\} \vdash c_1 = c_3$$

for any constants c_1, c_2, and c_3.

1.	$c_1 = c_2$	Premise
2.	$c_2 = c_3$	Premise
3.	$c_2 = c_1$	1; Exercise 5.7
4.	$c_2 = c_1 \to (c_2 = c_3 \to c_1 = c_3)$	IdAx(b)
5.	$c_1 = c_3$	2,3,4; SL

For transitivity, we must show that, for all members c_1, c_2, and c_3 of Γ^*, if $c_1 = c_2 \in \Gamma^*$ and $c_2 = c_3 \in \Gamma^*$, then $c_1 = c_3 \in \Gamma^*$. Assume that $c_1 = c_2 \in \Gamma^*$ and $c_2 = c_3 \in \Gamma^*$. By what we have just proved, $\Gamma^* \vdash c_1 = c_3$. By deductive closure, $c_1 = c_3 \in \Gamma^*$.

We define a model $\mathfrak{M} = (\boldsymbol{D}, v, \chi)$ exactly as in the proof of Lemma 4.9, that is:

(i) $\boldsymbol{D} = \{[c]_R \mid c \in \mathrm{C}^*\}$.

(ii) (a) $v(p_i) = \mathbf{T}$ if and only if $p_i \in \Gamma^*$.

(b) $v((P_i^n, [c_1]_R, \ldots, [c_n]_R)) = \mathbf{T}$ if and only if $P_i^n c_1 \ldots c_n \in \Gamma^*$.

(iii) (a) $\chi(c) = [c]_R$ for each $c \in \mathrm{C}^*$.

(b) $\chi((F_i^n, [c_1]_R, \ldots, [c_n]_R)) = [c]_R$ if and only if $F_i^n c_1 \ldots c_n = c \in \Gamma^*$.

We must show that the definitions given in clauses (ii)(b) and (iii)(b) do not depend on the choice of elements of equivalence classes. In the case of clause (iii)(b), we need to show something additional. (See below.)

A special case of the proof that clause (iii)(b) is independent of the choice of elements of equivalence classes is Exercise 5.8, and the proof for the general case is merely an elaboration of the proof for the special case. The case of (ii)(b) is a bit simpler.

The additional fact we to show concerning clause (iii)(b) is that, for all F_i^n and all $c_1, \ldots c_n$, that there is a c such that

$$F_i^n c_1 \ldots c_n = c \in \Gamma^*.$$

Suppose there is no such c. By property (3) of Γ^*,

$$F_i^n c_1 \ldots c_n \neq c \in \Gamma^*$$

for all $c \in C^*$. By property (4) of Γ^*,

$$\forall v_1 F_i^n c_1 \ldots c_n \neq v_1 \in \Gamma^*.$$

Since

$$\forall v_1 F_i^n c_1 \ldots c_n \neq v_1 \rightarrow F_i^n c_1 \ldots c_n \neq F_i^n c_1 \ldots c_n$$

is an instance of the Quantifier Axiom Schema,

$$\Gamma^* \vdash F_i^n c_1 \ldots c_n \neq F_i^n c_1 \ldots c_n.$$

But $F_i^n c_1 \ldots c_n \neq F_i^n c_1 \ldots c_n$ is an instance of Identity Axiom Schema (a), and so Γ^* is inconsistent, contradicting property (2) of Γ^*.

Let P be the property of being a sentence A such that

$$v_{\mathfrak{M}}(A) = \mathbf{T} \text{ if and only if } A \in \Gamma^*.$$

We prove by induction on length that every sentence has property P.

The case of atomic sentences is like that case in the proof of Lemma 4.9, except for one change. Recall that in proving atomic cases (i)(b) and (i)(c), we first used induction on length to demonstrate that all terms without variables have property Q, where t has property Q if and only if, for every $c \in \mathbf{C}^*$,

$$\text{if } \mathrm{den}_{\mathfrak{M}}(t) = [c]_R \text{ then } c = t \in \Gamma^*.$$

In the course of this demonstration, we got a contradiction from the assumption that $\Delta \subseteq \Gamma^*$, where

$$\Delta = \{c_1 = t_1, \ldots, c_n = t_n,\ F_i^n c_1 \ldots c_n = c,\ c \neq F_i^n t_1 \ldots t_n\}.$$

This assumption contradicted the hypothesis that Γ^* was finitely satisfiable. What we need to show in our new context is that it contradicts the hypothesis that Γ^* is consistent. Obviously $\Delta \vdash c \neq F_i^n t_1 \dots t_n$, so $\Delta \vdash F_i^n t_1 \dots t_n \neq c$. Thus it is enough to show that $\Delta \vdash F_i^n t_1 \dots t_n = c$.

1.	$c_1 = t_1$	Premise
..	$\dots$	$\dots$
..	$\dots$	$\dots$
..	$\dots$	$\dots$
n.	$c_n = t_n$	Premise
$n+1$.	$c_1 = t_1 \rightarrow$ $(F_i^n c_1 c_2 \dots c_n = c \rightarrow F^n t_1 c_2 \dots c_n = c)$	IdAx(b)
..	$\dots$	$\dots$
..	$\dots$	$\dots$
..	$\dots$	$\dots$
$2n$.	$t_n = c_n \rightarrow$ $(F_i^n t_1 t_2 \dots t_{n-1} c_n = c \rightarrow F^n t_1 t_2 \dots t_{n-1} t_n = c)$	IdAx(b)
$2n+1$.	$F_i^n t_1 \dots t_n = c$	1,...,2n; SL

Cases cases (ii) and (iii) of the proof that all formulas have property P are like the corresponding cases in the proof of Lemma 2.7.

Case (iv) is like the corresponding case in the proof of Lemma 4.9, except for one change. The last step in case (iv) proof was to show that

$$\text{for all } c \in \mathrm{C}^*,\ B(x;c) \in \Gamma^* \quad \text{iff} \quad \forall x B \in \Gamma^*.$$

The "if" part of this "iff" was proved using the fact that Γ^* was finitely satisfiable. In the new context, we must prove it using the fact that Γ^* is consistent. To do this, assume that $\forall x B \in \Gamma^*$ and let $c \in \mathrm{C}^*$. Notice that the sentence

$$\forall x B \rightarrow B(x;c)$$

is an instance of the Quantifier Axiom Schema. Thus $\Gamma^* \vdash B(x;c)$. By deductive closure, $B(x;c) \in \Gamma^*$.

As in our earlier proofs, we have in particular that $v_{\mathfrak{M}}(A) = \mathbf{T}$ for every member of A of Γ^*, and this means we have shown that Γ^* is satisfiable. □

Theorem 5.12. *Let Γ be a consistent set of sentences of $\mathcal{L}^{\#}_{\mathrm{C}}$. Then Γ is satisfiable, i.e., true in a model for $\mathcal{L}^{\#}_{(C)}$.*

Proof. By Lemma 5.10, let Γ^* have properties (1)–(3) of that lemma. By Lemma 5.11, Γ^* is satisfiable (true in a model $\mathcal{L}^{\#}_{(C)}$). As in the proof of Theorem 3.7 Γ is true in a model for $\mathcal{L}^{\#}_{(C)}$. □

Theorem 5.13 (Completeness). *Let* Γ *be a set of sentences of* $\mathcal{L}_{\mathrm{C}}^{\#}$ *and let* A *be a formula of* $\mathcal{L}_{\mathrm{C}}^{\#}$ *such that* $\Gamma \models A$. *Then* $\Gamma \vdash_{\mathbf{FOL}_{\mathrm{C}}} A$. *In other words,* $\mathbf{FOL}_{\mathrm{C}}$ *is complete.*

Proof. Since $\Gamma \models A$, for every model $\mathfrak{M}$ and every variable assignment s, if Γ is true in $\mathfrak{M}$, then $v_{\mathfrak{M}}^{s}(A) = \mathbf{T}$. Let $x_1, \ldots, x_n$ be all the variables occurring free in A. Let $\mathfrak{M}$ be any model in which Γ is true. For every variable assignment s, $v_{\mathfrak{M}}^{s}(A) = \mathbf{T}$. This means that $\forall x_1 \ldots \forall x_n A$ is true in $\mathfrak{M}$. Thus

$$\Gamma \models \forall x_1 \ldots \forall x_n A.$$

Since $\Gamma \models \forall x_1 \ldots \forall x_n A$, $\Gamma \cup \{\neg \forall x_1 \ldots \forall x_n A\}$ is not satisfiable. By Theorem 5.12, $\Gamma \cup \{\neg \forall x_1 \ldots \forall x_n A\}$ is inconsistent. Let B be a formula such that $\Gamma \cup \{\neg \forall x_1 \ldots \forall x_n A\} \vdash B$ and $\Gamma \cup \{\neg \forall x_1 \ldots \forall x_n A\} \vdash \neg B$. By the Deduction Theorem, $\Gamma \vdash (\neg \forall x_1 \ldots \forall x_n A \to B)$ and $\Gamma \vdash \neg \forall x_1 \ldots \forall x_n A \to \neg B)$. By SL, $\Gamma \vdash \forall x_1 \ldots \forall x_n A$. Using the Quantifier Axiom Schema and MP n times, we get that $\Gamma \vdash A$. □

Exercise 5.8. In the proof of Lemma 5.11, clause (iii)(b) of the definition of the model $\mathfrak{M}$ says that

$$\chi((F_i^n, [c_1]_R, \ldots, [c_n]_R)) = [c]_R \quad \text{iff} \quad F_i^n c_1 \ldots c_n = c \in \Gamma^*.$$

Show, in the special case $n = 2$ and $i = 0$, that this definition does not depend on the choice of elements of equivalence classes. In other words, assume that

(1) $[c_1]_R = [c_1']_R$ and $[c_2]_R = [c_2']_R$;

(2) $F^2 c_1 c_2 = c \in \Gamma^*$ and $F^2 c_1' c_2' = c' \in \Gamma^*$,

and prove that

$$[c]_R = [c']_R.$$