Solutions to Exercises 5.3, 5.4, 5.5, and 5.6

Exercise 5.3. Assume that t is $u_1 \cdot u_2$. Let $j_1 = (u_1)_{\mathfrak{N}}$ and let $j_2 = (u_2)_{\mathfrak{N}}$. By the fact that u_1 and u_2 are shorter than t,
(a) $\mathrm{Q} \models u_1 = \mathbf{S}^{j_1}\mathbf{0}$
(b) $\mathrm{Q} \models u_2 = \mathbf{S}^{j_2}\mathbf{0}$.

By mathematical induction, we show that $\mathrm{Q} \models \mathbf{S}^{j_1}\mathbf{0} \cdot \mathbf{S}^{k}\mathbf{0} = \mathbf{S}^{j_1 \cdot k}\mathbf{0}$ for every $k \geq 0$. For $k = 0$, this is given by Axiom (7). Assume that it is true for k. By Axiom (8),

$$\mathrm{Q} \models \mathbf{S}^{j_1}\mathbf{0} \cdot \mathbf{S}^{k+1}\mathbf{0} = \mathbf{S}^{j_1}\mathbf{0} \cdot \mathbf{S}^{k}\mathbf{0} + \mathbf{S}^{j_1}\mathbf{0}.$$

Hence our induction hypothesis gives

$$\mathrm{Q} \models \mathbf{S}^{j_1}\mathbf{0} \cdot \mathbf{S}^{k+1}\mathbf{0} = \mathbf{S}^{j_1 \cdot k}\mathbf{0} + \mathbf{S}^{j_1}\mathbf{0}.$$

Applying Axiom (5) and applying Axiom (6) j_1 times, we get that

$$\mathrm{Q} \models \mathbf{S}^{j_1}\mathbf{0} \cdot \mathbf{S}^{k+1}\mathbf{0} = \mathbf{S}^{j_1 \cdot k + j_1}\mathbf{0}.$$

Since $j_1 \cdot k + j_1 = j_1 \cdot (k+1)$, this means that

$$\mathrm{Q} \models \mathbf{S}^{j_1}\mathbf{0} \cdot \mathbf{S}^{k+1}\mathbf{0} = \mathbf{S}^{j_1 \cdot (k+1)}\mathbf{0}.$$

Applying what we have proved by induction in the case $k = j_2$, we get that

$$\mathrm{Q} \models \mathbf{S}^{j_1}\mathbf{0} \cdot \mathbf{S}^{j_2}\mathbf{0} = \mathbf{S}^{j_1 \cdot j_2}\mathbf{0}.$$

This, together with (a) and (b), implies that $Q \models u_1 \cdot u_2 = \mathbf{S}^{j_1 \cdot j_2}\mathbf{0}$.

Exercise 5.4. For addition, we use primitive recursion, with I_1^1 as f and with $g(a_1, a_2, a_3) = S(I_3^3(a_1, a_2, a_3))$. ($g$ comes by composition, with S as f and I_3^3 as g_1.) We show by induction that the resulting h is addition:

$$h(a, 0) = f(a) = I_1^1(a) = a = a + 0.$$

If $h(a, b) = a + b$, then

$$h(a, S(b)) = g(a, b, h(a, b)) = S(I_3^3(a, b, h(a, b)) = S(h(a, b)) = S(a+b) = a+S(b).$$

For multiplication, we use primitive recursion, with the constant 1-argument function with value 0 as f and with $g(a_1, a_2, a_3) = I_3^3(a_1, a_2, a_3) + I_1^3(a_1, a_2, a_3)$. ($g$ comes by composition with $m = 2$ and $n = 3$.) We show by induction that the resulting h is multiplication:

$$h(a, 0) = f(a) = 0 = a \cdot 0;$$

If $h(a, b) = a \cdot b$,

$$\begin{aligned} h(a, S(b)) &= g(a, b, h(a, b)) = I_3^3(a, b, h(a, b)) + I_1^3(a, b, h(a, b)) \\ &= h(a, b) + a = a \cdot b + a = a \cdot S(b). \end{aligned}$$

For the factorial function, we use primitive recursion with the constant 0-argument function with value 1 as f and with $g(a_1, a_2) = I_2^2(a_1, a_2) \cdot S(I_1^2(a_1, a_2))$. ($g$ comes by composition with $m = n = 2$.) We show by induction the resulting h is the factorial function:

$$h(0) = 1 = 0!;$$

If $h(a) = a!$, then

$$\begin{aligned} h(S(a)) &= g(a, h(a)) = I_2^2(a, h(a)) \cdot S(I_1^2(a, h(a))) \\ &= h(a) \cdot S(a) = n! \cdot S(n) = (S(n))!. \end{aligned}$$

Exercise 5.5. First assume that $\varphi(v_, \ldots, v_n)$ represents the relation R in Q. Let $\psi(v_1, \ldots, v_{n+1})$ be

$$(\varphi(v_1, \ldots, v_n) \wedge v_{n+1} = \mathbf{S0}) \vee (\neg\varphi(v_1, \ldots, v_n) \wedge v_{n+1} = \mathbf{0}).$$

For any $a_1, \ldots, a_n$,

$$\begin{array}{rcl} R(a_1, \ldots, a_n) & \Rightarrow & \mathrm{Q} \models \varphi(\mathbf{S}^{a_1}\mathbf{0}, \ldots, \mathbf{S}^{a_n}\mathbf{0}) \\ & \Rightarrow & \mathrm{Q} \models (\varphi(\mathbf{S}^{a_1}\mathbf{0}), \ldots, \mathbf{S}^{a_n}\mathbf{0}) \wedge \mathbf{S0} = \mathbf{S0} \wedge \mathbf{S0} \neq \mathbf{0}) \\ & \Rightarrow & \mathrm{Q} \models (\psi(\mathbf{S}^{a_1}\mathbf{0}, \ldots, \mathbf{S}^{a_n}\mathbf{0}, \mathbf{S0}) \wedge \neg\psi(\mathbf{S}^{a_1}\mathbf{0}, \ldots, \mathbf{S}^{a_n}\mathbf{0}, \mathbf{0})); \\ \neg R(a_1, \ldots, a_n) & \Rightarrow & \mathrm{Q} \models \neg\varphi(\mathbf{S}^{a_1}\mathbf{0}, \ldots, \mathbf{S}^{a_n}\mathbf{0}) \\ & \Rightarrow & \mathrm{Q} \models (\neg\varphi(\mathbf{S}^{a_1}\mathbf{0}, \ldots, \mathbf{S}^{a_n}\mathbf{0}) \wedge \mathbf{0} = \mathbf{0} \wedge \mathbf{0} \neq \mathbf{S0}) \\ & \Rightarrow & \mathrm{Q} \models (\psi(\mathbf{S}^{a_1}\mathbf{0}, \ldots, \mathbf{S}^{a_n}\mathbf{0}, \mathbf{0}) \wedge \neg\psi(\mathbf{S}^{a_1}\mathbf{0}, \ldots, \mathbf{S}^{a_n}\mathbf{0}, \mathbf{S0})) \end{array}$$

Since $\models \forall v_{n+1}(\psi(\mathbf{S}^{a_1}\mathbf{0}, \ldots, \mathbf{S}^{a_n}\mathbf{0}, v_{n+1}) \rightarrow (v_{n+1} = \mathbf{S0} \vee v_{n+1} = \mathbf{0}))$, we get that $\mathrm{Q} \models \forall v_{n+1}(\psi(\mathbf{S}^{a_1}\mathbf{0}, \ldots, \mathbf{S}^{a_n}\mathbf{0}) \leftrightarrow v_{n+1} = \mathbf{S}^{K_R(a, \ldots, a_n)}\mathbf{0})$.

Now assume that $\varphi(v_1, \ldots, v_{n+1})$ represents K_R in Q. Let $\psi(v_1, \ldots, v_n)$ be $\varphi(v_1, \ldots, v_n, \mathbf{S0})$. For any $a_1, \ldots, a_n$,

$$\begin{array}{rcl} R(a_1, \ldots, a_n) & \Rightarrow & K_R(a_1, \ldots, a_n) = 1 \\ & \Rightarrow & \mathrm{Q} \models \varphi(\mathbf{S}^{a_1}\mathbf{0}, \ldots, \mathbf{S}^{a_n}\mathbf{0}, \mathbf{S0}) \\ & \Rightarrow & \mathrm{Q} \models \psi(\mathbf{S}^{a_1}\mathbf{0}, \ldots, \mathbf{S}^{a_n}\mathbf{0}); \\ \neg R(a_1, \ldots, a_n) & \Rightarrow & K_R(a_1, \ldots, a_n) \neq 1 \\ & \Rightarrow & \mathrm{Q} \models \neg\varphi(\mathbf{S}^{a_1}\mathbf{0}, \ldots, \mathbf{S}^{a_n}\mathbf{0}, \mathbf{S0}) \\ & \Rightarrow & \mathrm{Q} \models \neg\psi(\mathbf{S}^{a_1}\mathbf{0}, \ldots, \mathbf{S}^{a_n}\mathbf{0}). \end{array}$$

Exercise 5.6. We can get g from f using composition:

$$g(a_1, a_2) = f(I_2^2(a_1, a_2), I_1^2(a_1, a_2)).$$