Solutions for 2nd Midterm

1. Assume (a). Let Σ be a set of sentences and let φ be a formula. Assume that $\Sigma \models \varphi$. Since φ is true under every variable assignment in every model in which Σ is true, the universal closure τ of φ is true in every model in which Σ is true. Thus $\Sigma \cup \{\neg\tau\}$ is a set of sentences that is not satisfiable. By (a), let Δ' be a finite subset of $\Sigma \cup \{\neg\tau\}$ that is not satisfiable. Let $\Delta = \Delta' \cap \Sigma$. In every model in which Δ is true, τ is true. Hence $\Delta \models \tau$. Since $\tau \models \varphi$, $\Delta \models \varphi$.

Remark. Exam problem 1 is just the "if" part of homework problem Exercise 4.3, with consistency replaced by the property of having every finite subset satisfiable and with "$\Sigma \vdash \tau$" replaced by "There is a finite subset of Σ that $\models \tau$." The solution for the one is like the solution for the other.

2. The sentence $\exists v_1 \exists v_2\, v_1 \neq v_2$ is false in all models whose universe has only one element. Hence $\not\models \exists v_1 \exists v_2\, v_1 \neq v_2$. By Soundness, $\nvdash \exists v_1 \exists v_2\, v_1 \neq v_2$.

3. To prove that pred is primitive recursive, let be f be 0, i.e., the zero-argument function with value 0, and let $g = I_1^2$. Both are primitive recursive. Note that

$$\begin{aligned} &\mathrm{pred}(0) = 0 = f; \\ &\mathrm{pred}(S(b)) = b = g(b, \mathrm{pred}(b)). \end{aligned}$$

To prove that $\dot{-}$ is primitive recursive, let $f = I_1^1$ and let $g(a,b,c) = \mathrm{pred}(I_1^3(a,b,c))$. Both f and g are primitive recursive, the latter by closure under Composition. Furthermore,

$$\begin{aligned} &a \dot{-} 0 = a = f(0); \\ &a \dot{-} S(b) = \mathrm{pred}(a \dot{-} b) = g(a, b, a \dot{-} b). \end{aligned}$$

4. Let $\varphi(v_1, v_2)$ represent f (or just its graph) in Q. We may assume that neither v_1 nor v_2 occurs bound in $\varphi(v_1, v_2)$. Let $\psi(v_1, v_2) = \varphi(v_2, v_1)$ (i.e, = the result of replacing the free occurrences of v_1 in φ by occurrences of v_2 and vice-versa. To see that ψ represents the graph of f^{-1} in Q, note that

$$\psi(\mathbf{S}^{a_1}\mathbf{0}, \mathbf{S}^{a_2}\mathbf{0}) \text{ is the same sentence as } \varphi(\mathbf{S}^{a_2}\mathbf{0}, \mathbf{S}^{a_1}\mathbf{0}).$$

Remark. The answer, "If $\varphi(v_1, v_2)$ represents the graph of f in Q, then $\varphi(v_2, v_1)$ represents the graph of f^{-1} in Q," would have gotten full credit for the problem.

5. Define a relation div by $\mathrm{div}(a, b) \Leftrightarrow a$ divides b. Since

$$\mathrm{div}(a, b) \Leftrightarrow \exists c(c \leq b \wedge a \cdot c = b),$$

closure under bounded quantification implies that div is representable in Q. (Instead of introducing div, one could uses its bounded quantifier definition.) For any (a, b, c), $\mathrm{LCM}(a, b, c) \Leftrightarrow$ (i) or (ii) below.

(i) $(a = 0 \vee b = 0) \wedge c = 0)$;
(ii) $(c > 0 \wedge \mathrm{div}(a, c) \wedge \mathrm{div}(b, c) \wedge \forall d(0 < d < c \Rightarrow (\neg\mathrm{div}(a, d) \vee \neg\mathrm{div}(b, d)))$.

This, together with closure complement, union, intersection, and bounded quantification, implies that LCM is representable in Q.

For a full proof—which was not required—one would have to show that the set of (a,b,c) that satisfy (i) and the set of (a,b,c) that satisfy (ii) are both representable in Q. In the case of (i), this follows by closure under intersection and union from the fact that $\{(a, b, c) \mid a = 0\}$, $\{(a, b, c) \mid b = 0\}$, and $\{(a, b, c) \mid c = 0\}$ are all representable in Q, and this follows from the fact that $\{d \mid d = 0\}$ is representable in Q. The case of (ii) is similar, except that to show that

$$\{(a, b, c) \mid \forall d(0 < d < c \Rightarrow (\neg\mathrm{div}(a, d) \vee \neg\mathrm{div}(b, d)))\}$$

is representable in Q, one has to show the representability of the set of (a, b, c, d) such that $0 < d < c \Rightarrow (\neg\mathrm{div}(a, d) \vee \neg\mathrm{div}(b, d)))$ and then use closure under bounded quantification.