Solutions to Exercises 6.6, 6.7, and 6.8

Exercise 6.6. The new deductive system would be complete but not sound. It is complete because the old system is complete and every deduction in the old system is a deduction in the new system. To see that the new system is not sound, note that $(v_1 = v_2 \rightarrow v_1 = v_2)$ is a tautology, and so it is deducible from the empty set in both old and new systems. The new Quantifier Rule allows us to deduce $(v_1 = v_2 \rightarrow \forall v_2\, v_1 = v_2)$ from the empty set. This is clearly not a valid formula.

Exercise 6.7. Let $\mathcal{L}^*$ come from $\mathcal{L}$ by adding infinitely many new constants. Let C be the set of all the new constants. Let Σ be

$$\mathrm{Th}(\mathfrak{A}) \cup \{(c \neq c' \wedge c \sim c') \mid c \in C \wedge c' \in C \wedge c \neq c'\}.$$

Let Δ be any finite subset of Σ. Extend $\mathfrak{A}$ to a model for $\mathcal{L}^*$. There is an equivalence class of $\mathfrak{A}$ that has at least as many members as there are members of C that occur in Δ. Extend $\mathfrak{A}$ to a model for $\mathcal{L}^*$ by assigning the members of C that occur in Δ to distinct members of this equivalence class and assigning the other members of C arbitrarily. Clearly Δ is true in this model. By Compactness, there is a model $\mathfrak{B}*$ in which Σ is true. Let $\mathfrak{B}$ be the reduct of $\mathfrak{B}^*$ to $\mathfrak{L}$.

Exercise 6.8. Let $\mathcal{L}'$ be gotten from $\mathcal{L}$ by removing the symbol P. By completeness for our deductive system for $\mathcal{L}'$, it is enough to show that $\Sigma \models \tau$ in $\mathcal{L}'$. To show this, let $\mathfrak{A}'$ be a model for $\mathcal{L}'$ in which Σ is true. Extend $\mathfrak{A}'$ to a model $\mathfrak{A}$ for $\mathcal{L}$ by assigning some relation as $P_{\mathfrak{A}}$. Since P does not occur in Σ, Σ is true in $\mathfrak{A}$. Since By soundness for our deductive system for $\mathcal{L}$, $\Sigma \models \tau$ in $\mathcal{L}$. Hence τ is true in $\mathfrak{A}$. Since P does not occur in τ, τ is true in $\mathfrak{A}'$.