Solutions to Exercises 5.1 and 5.2

Exercise 5.1. Since the axioms of Q are true in \( \mathfrak{N} \),

\[ Q \models \sigma \Rightarrow \sigma \text{ is true in } \mathfrak{N}, \]

for every sentence of \( \mathcal{L}^A \). Thus in both this exercise and Exercise 5.2 we only have to prove the \( \Leftarrow \) part of the biconditionals.

Every atomic sentence is, for some variable-free terms \( t_1 \) and \( t_2 \), either \( t_1 = t_2 \) or \( t_1 < t_2 \). For such terms, let \( j_1 = (t_1)\mathfrak{N} \) and \( j_2 = (t_2)\mathfrak{N} \). By Lemma 5.3,

\[ Q \models t_1 = S^{j_1}0 \text{ and } Q \models t_2 = S^{j_2}0. \]

Thus

\[ Q \models (t_1 = t_2 \Leftrightarrow S^{j_1}0 = S^{j_2}0) \text{ and } Q \models (t_1 < t_2 \Leftrightarrow S^{j_1}0 < S^{j_2}0). \]

If \( t_1 = t_2 \) is true in \( \mathfrak{N} \), then \( S^{j_1}0 = S^{j_2}0 \). If \( t_1 < t_2 \) is true in \( \mathfrak{N} \), then

Lemma 5.2 implies that \( Q \models S^{j_1}0 < S^{j_2}0 \). Thus we get that if \( \sigma \) is an atomic sentence true in \( \mathfrak{N} \) then \( Q \models \sigma \).

To prove that this is also true of negations of atomic sentences, we have to prove

(a) If \( j_1 \neq j_2 \) then \( Q \models S^{j_1}0 \neq S^{j_2}0 \);
(b) If \( j_1 \nless j_2 \) then \( Q \models S^{j_1}0 \nless S^{j_2}0 \).

We first prove (a). Assume \( j_1 \neq j_2 \). Either \( j_1 < j_2 \) or \( j_2 < j_1 \). We’ll assume \( j_1 < j_2 \). The other case is similar. By \( j_1 \) applications of Axiom (2), we get that

\[ Q \models (S^{j_1}0 = S^{j_2}0 \rightarrow 0 = S^{j_2-j_1}0). \]

But Axiom (1) implies that \( Q \models 0 \neq S^{j_2-j_1}0 \).

To prove (b), assume that \( j_1 \nless j_2 \). By (a),

\[ Q \models S^{j_1}0 \neq S^k0 \]

for every \( k < j_2 \). By Axiom (3) and Lemma 5.2, \( Q \models S^{j_1}0 \nless S^{j_2}0 \).

Exercise 5.2. The \textit{length*} of a formula is like the length of a formula except that the length of atomic formulas is counted as 1 (as if atomic formulas were just single symbols). The \textit{complexity} of a formula is defined as follows. The
complexity of an atomic formula is 0. The complexities of \( \neg \varphi \) and \( \forall x \varphi \) are one more than the complexity of \( \varphi \). The complexity of \( (\varphi \rightarrow \psi) \) is one more than the maximum of the complexity of \( \varphi \) and the complexity of \( \psi \).

(a) Let \( P \) be the property of being a \( \Delta_0 \) sentence \( \sigma \) such that

\[
\begin{align*}
\sigma & \text{ is true in } \mathfrak{N} \Rightarrow Q \models \sigma; \\
\sigma & \text{ is false in } \mathfrak{N} \Rightarrow Q \models \neg \sigma.
\end{align*}
\]

We show by induction on complexity \([\text{induction on length}^*]\) that all \( \Delta_0 \) sentences have property \( P \).

Let \( \sigma \) be a \( \Delta_0 \) sentence, and assume that all \( \Delta_0 \) sentences of smaller complexity \([\text{smaller length}^*]\) have property \( P \).

The case that \( \sigma \) is an atomic sentence is Exercise 5.1. The cases that \( \sigma \) is \( \neg \tau \) and that \( \sigma \) is \( (\rho \rightarrow \tau) \) are easy. Assume that \( \sigma \) is

\[
\forall x(x < t \rightarrow \psi).
\]

(The other bounded quantification case is similar.) Let \( k = t_{\mathfrak{N}} \).

Assume first that \( \sigma \) is true in \( \mathfrak{N} \). For each \( j < k \), \( \psi(x; S^j \mathbf{0}) \) is true in \( \mathfrak{N} \). These sentences have smaller complexity \([\text{smaller length}^*]\) than \( \sigma \), and so \( Q \models \) each of them. By Lemma 5.2, \( Q \models \sigma \).

Now assume that \( \sigma \) is false in \( \mathfrak{N} \). For some \( j < k \), \( \psi(x; S^j \mathbf{0}) \) is false in \( \mathfrak{N} \), and so \( Q \models \neg \psi(x; S^j \mathbf{0}) \). By Lemma 5.2, \( Q \models \neg \sigma \).

(b) Let \( \sigma \) be \( \exists x_1 \cdots \exists x_n \psi \), with \( \psi \) \( \Delta_0 \). Assume that \( \sigma \) is true in \( \mathfrak{N} \). For some numbers \( k_1, \ldots, k_n \), the \( \Delta_0 \) sentence

\[
\psi(x_1; S^{k_1} \mathbf{0}) \cdots (x_n; S^{k_n} \mathbf{0})
\]

is true in \( \mathfrak{N} \). By part (a) of this exercise,

\[
Q \models \psi(x_1; S^{k_1} \mathbf{0}) \cdots (x_n; S^{k_n} \mathbf{0}).
\]