Solutions to Exercises 5.1 and 5.2

Exercise 5.1. Since the axioms of Q are true in $\mathfrak{N}$,

$$\mathrm{Q} \models \sigma \Rightarrow \sigma \text{ is true in } \mathfrak{N},$$

for every sentence of $\mathcal{L}^A$. Thus in both this exercise and Exercise 5.2 we only have to prove the $\Leftarrow$ part of the biconditionals.

Every atomic sentence is, for some variable-free terms t_1 and t_2, either $t_1 = t_2$ or $t_1 < t_2$. For such terms, let $j_1 = (t_1)_{\mathfrak{N}}$ and $j_2 = (t_2)_{\mathfrak{N}}$. By Lemma 5.3,

$$\mathrm{Q} \models t_1 = \mathbf{S}^{j_1}\mathbf{0} \;\text{ and }\; \mathrm{Q} \models t_2 = \mathbf{S}^{j_2}\mathbf{0}.$$

Thus

$$\mathrm{Q} \models (t_1 = t_2 \leftrightarrow \mathbf{S}^{j_1}\mathbf{0} = \mathbf{S}^{j_2}\mathbf{0}) \;\text{ and }\; \mathrm{Q} \models (t_1 < t_2 \leftrightarrow \mathbf{S}^{j_1}\mathbf{0} < \mathbf{S}^{j_2}\mathbf{0}).$$

If $t_1 = t_2$ is true in $\mathfrak{N}$, then $\models \mathbf{S}^{j_1}\mathbf{0} = \mathbf{S}^{j_2}\mathbf{0}$. If $t_1 < t_2$ is true in $\mathfrak{N}$, then Lemma 5.2 implies that $\mathrm{Q} \models \mathbf{S}^{j_1}\mathbf{0} < \mathbf{S}^{j_2}\mathbf{0}$. Thus we get that if σ is an atomic sentence true in $\mathfrak{N}$ then $\mathrm{Q} \models \sigma$.

To prove that this as also true of negations of atomic sentences, we have to prove

(a) If $j_1 \neq j_2$ then $\mathrm{Q} \models \mathbf{S}^{j_1}\mathbf{0} \neq \mathbf{S}^{j_2}\mathbf{0}$;

(b) If $j_1 \not< j_2$ then $\mathrm{Q} \models \mathbf{S}^{j_1}\mathbf{0} \not< \mathbf{S}^{j_2}\mathbf{0}$.

We first prove (a). Assume $j_1 \neq j_2$. Either $j_1 < j_2$ or $j_2 < j_1$. We'll assume $j_1 < j_2$. The other case is similar. By j_1 applications of Axiom (2), we get that

$$\mathrm{Q} \models (\mathbf{S}^{j_1}\mathbf{0} = \mathbf{S}^{j_2}\mathbf{0} \rightarrow \mathbf{0} = \mathbf{S}^{j_2 - j_1}\mathbf{0}).$$

But Axiom (1) implies that $\mathrm{Q} \models \mathbf{0} \neq \mathbf{S}^{j_2 - j_1}\mathbf{0}$.

To prove (b), assume that $j_1 \not< j_2$. By (a),

$$\mathrm{Q} \models \mathbf{S}^{j_1}\mathbf{0} \neq \mathbf{S}^{k}\mathbf{0}$$

for every $k < j_2$. By Axiom (3) and Lemma 5.2, $\mathrm{Q} \models \mathbf{S}^{j_1}\mathbf{0} \not< \mathbf{S}^{j_2}\mathbf{0}$.

Exercise 5.2. The *length** of a formula is like the length of a formula except that the length of atomic formulas is counted as 1 (as if atomic formulas were just single symbols). The *complexity* of a formula is defined as follows. The complexity of an atomic formula is 0. The complexities of $\neg\varphi$ and $\forall x\varphi$ are one more than the complexity of φ. The complexity of $(\varphi \to \psi)$ is one more than the maximum of the complexity of φ and the complexity of ψ.

(a) Let P be the property of being a Δ_0 sentence σ such that

$$\begin{aligned}&\sigma \text{ is true in } \mathfrak{N} \Rightarrow \mathrm{Q} \models \sigma;\\ &\sigma \text{ is false in } \mathfrak{N} \Rightarrow \mathrm{Q} \models \neg\sigma.\end{aligned}$$

We show by induction on complexity [induction on length*] that all Δ_0 sentences have property P.

Let σ be a Δ_0 sentence, and assume that all Δ_0 sentences of smaller complexity [smaller length*] have property P.

The case that σ is an atomic sentence is Exercise 5.1. The cases that σ is $\neg\tau$ and that σ is $(\rho \to \tau)$ are easy. Assume that σ is

$$\forall x(x < t \to \psi).$$

(The other bounded quantification case is similar.) Let $k = t_{\mathfrak{N}}$.

Assume first that σ is true in $\mathfrak{N}$. For each $j < k$, $\psi(x; \mathbf{S}^j\mathbf{0})$ is true in $\mathfrak{N}$. These sentences have smaller complexity [smaller length*] than σ, and so $\mathrm{Q} \models$ each of them. By Lemma 5.2, $\mathrm{Q} \models \sigma$.

Now assume that σ is false in $\mathfrak{N}$. For some $j < k$, $\psi(x; \mathbf{S}^j\mathbf{0})$ is false in $\mathfrak{N}$, and so $\mathrm{Q} \models \neg\psi(x; \mathbf{S}^j\mathbf{0})$. By Lemma 5.2, $\mathrm{Q} \models \neg\sigma$.

(b) Let σ be $\exists x_1 \cdots \exists x_n \psi$, with ψ Δ_0. Assume that σ is true in $\mathfrak{N}$. For some numbers $k_1, \ldots, k_n$, the Δ_0 sentence

$$\psi(x_1; \mathbf{S}^{k_1}\mathbf{0}) \cdots (x_n; \mathbf{S}^{k_n}\mathbf{0})$$

is true in $\mathfrak{N}$. By part (a) of this exercise,

$$\mathrm{Q} \models \psi(x_1; \mathbf{S}^{k_1}\mathbf{0}) \cdots (x_n; \mathbf{S}^{k_n}\mathbf{0}).$$