Solutions for 1st Midterm

1. Note that

$$\begin{array}{ll} \forall x\varphi \text{ is valid} & \\ \Leftrightarrow & \text{for all } \mathfrak{A},\ \text{for all } s,\ \mathrm{tv}^s_{\mathfrak{A}}(\forall x\varphi) = \mathbf{T} \\ \Leftrightarrow & \text{for all } \mathfrak{A},\ \text{for all } a \in A, \text{ for all } s \text{ with } s(x) = a,\ \mathrm{tv}^s_{\mathfrak{A}}(\varphi) = \mathbf{T} \\ \Leftrightarrow & \text{for all } \mathfrak{A},\ \text{for all } a \in A, \text{ for all } s, \text{ if } c_{\mathfrak{A}} = a \text{ then } \mathrm{tv}^s_{\mathfrak{A}}(\varphi(x;c)) = \mathbf{T} \\ \Leftrightarrow & \varphi(x;c) \text{ is valid.} \end{array}$$

The third $\Leftrightarrow$ is justified as follows.

$\Rightarrow$: Given $\mathfrak{A}$, a, and s, let s' agree with s except that $s'(x) = c_{\mathfrak{A}}$. Since $\mathrm{tv}^{s'}_{\mathfrak{A}}(\varphi) = \mathbf{T}$ and $\mathrm{den}^{s'}_{\mathfrak{A}}(x) = \mathrm{den}^{s}_{\mathfrak{A}}(c)$, $\mathrm{tv}^s_{\mathfrak{A}}(\varphi(x;c)) = \mathbf{T}$.

$\Leftarrow$: Given $\mathfrak{A}$, a, and s, let $\mathfrak{A}'$ be like $\mathfrak{A}$ except that $c_{\mathfrak{A}'} = a$. Since $\mathrm{tv}^s_{\mathfrak{A}'}(\varphi(x:c)) = \mathbf{T}$ and $\mathrm{den}^s_{\mathfrak{A}'}(c) = \mathrm{den}^s_{\mathfrak{A}}(x)$, $\mathrm{tv}^s_{\mathfrak{A}}(\varphi) = \mathbf{T}$.

2. (a) $\Gamma \not\models \varphi$. Let $A = \{a_1, a_2\}$ with a_1 and a_2 distinct. Set $f_{\mathfrak{A}}(a_1) = f_{\mathfrak{A}}(a_2) = a_1$ and let $Q_{\mathfrak{A}} = \{(a_1, a_1), (a_2, a_1)\}$.

(b) $\Gamma \models \varphi$. Let $\mathfrak{A}$ be any model in which Γ is true. There must be elements a_1 and a_2 of A such that $a_1 \in P_{\mathfrak{A}}$ if and only if $a_2 \notin P_{\mathfrak{A}}$. Thus, for any b_1 assigned to v_1, one or the other of a_1 and a_2 can be assigned to v_2 so as to make $(Pv_1 \leftrightarrow \neg Pv_2)$ true.

3. (a) $(\forall v_1 \exists v_2 Q v_1 v_2 \to \exists v_2 Q v_2 v_2)$ is not valid. It is false in a model $\mathfrak{A}$ such that $A = \{d_1, d_2\}$ with d_1 and d_2 distinct objects and $Q_{\mathfrak{A}} = \{(d_1, d_2), (d_2, d_1)\}$. $(\forall v_1 \exists v_2 Q v_1 v_2 \to \exists v_2 Q v_2 v_2)$ is also not an instance of the Quantifier Axiom Schema. This is because the occurrence of v_1 for which v_2 is substituted is in the subformula $\forall v_2 Q v_1 v_2$.

(b) $(\forall v_1 \forall v_2 Q v_1 v_2 \to \forall v_2 Q f v_3 v_2)$ is valid and is an instance of the Quantifier Axiom Schema.

4. (a)

1.	$\forall v_2 P v_2 \to P v_1$	QAx
2.	$\forall v_2 P v_2 \to \forall v_1 P v_1$	1; QR
3.	$\neg \forall v_1 P v_1 \to \neg \forall v_2 P v_2$	2; SL

(b) By the Deduction Theorem, it is enough to show that

$$\{(\forall v_1 (P v_1 \to p)\} \vdash (\exists v_1 P v_1 \to p).$$

1.	$\forall v_1(Pv_1 \rightarrow p)$	Premise
2.	$\forall v_1(Pv_1 \rightarrow p) \rightarrow (Pv_1 \rightarrow p)$	QAx
3.	$Pv_1 \rightarrow p$	1,2; MP
4.	$\neg p \rightarrow \neg Pv_1$	3; SL
5.	$\neg p \rightarrow \forall v_1 \neg Pv_1$	4; QR
6	$\neg \forall v_1 \neg Pv_1 \rightarrow p$	5; SL
	$[\exists v_1 Pv_1 \rightarrow p]$	