Research Statement

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My research interests are as follows:

1. **Nonlinear partial differential equations (PDEs) and applications.** I focused on:
   (1) The deterministic and stochastic Zakharov-Kuznetsov (ZK) equations. Basic mathematical theory was established for the initial-boundary value problem.
   (2) Numerical analysis for stochastic geophysical fluid models. A time discrete approximation was constructed for the stochastic primitive equations of the atmosphere and oceans.

2. **Mathematical modeling, computation and applied analysis.** I am currently working on:
   Stochastic-statistical models of criminal behavior. Statistical agent-based models for residential burglary were improved and analyzed with the application of probability, PDE, and stochastic analysis.

## 1 Deterministic and Stochastic ZK Equations

My PhD thesis mainly focused on the deterministic and stochastic ZK equation with multiplicative noise in a bounded domain in space dimensions 2 and 3 ([39, 89, 104, 105]).

The ZK equation has caught considerable attention in the recent decades ([4, 24, 31, 32, 60, 66, 84, 88]). It is the long-wave small-amplitude limit of the Euler-Poisson system of cold plasmas uniformly magnetized in the x direction ([59, 65, 110]). It also is a multi-dimensional extension of the Korteweg-de Vries (KdV) equation with mixed features, e.g., partial hyperbolicity and anisotropy. PDEs with mixed features play an important role in many areas, such as physics, chemistry, and mechanics ([13, 17, 18, 51, 52, 73, 80, 87, 92, 112]). The addition of noise into the ZK equation is also meaningful as stochastic random waves generally capture a closer picture of the physics of fluid dynamics.

The study of the deterministic ZK equation started with the Cauchy problem in the unbounded domain. It was established that the solutions are locally well-posed in the Sobolev space $H^s(\mathbb{R}^2)$, for $s > 1/2$ ([43]), and globally well-posed in $H^3(\mathbb{R}^2)$, for $s > 1$ ([178]). Later, for computation and controllability purposes, the initial-boundary value problems were considered, mainly in a half space [31] or a strip ([4, 32, 60, 88]). There are much fewer results for the equation posed in a bounded domain, and as far as we know, except for our works, the strong solutions in a bounded domain in 3D have not been addressed, locally or globally in time. There have been many works on the stochastic KdV equation [22, 23, 35, 48, 67, 111], mainly with additive noise. The stochastic ZK equation in 2D or 3D driven by multiplicative noise however has not been studied.

In the deterministic setting, we established the existence (3D) and uniqueness (2D) of the weak solution, and local and global existence of strong solutions (3D), while in the stochastic setting we proved the existence of martingale solutions (3D) and existence and uniqueness of the pathwise solution (2D).

The major challenge here is how to handle the mixed features of the equation, including partial hyperbolicity, nonlinearity, anisotropy, and stochasticity. This is different from the classical models in fluid dynamics, such as the KdV equation, the Navier-Stokes equation, and the Euler equation. Thus a different treatment is required. We introduced an idea of splitting up the dissipative and dispersion effect and use them for different purposes. Also throughout our works we rely on energy estimates, while harmonic analysis is the typical method used for the hyperbolic-type equations. Only recently have researchers started to explore what energy estimates can do for this type of equations [81].

### 1.1 Deterministic ZK equation

We consider the ZK equation posed in a bounded domain $\mathcal{M}$:

$$\frac{\partial u}{\partial t} + \Delta \frac{\partial u}{\partial x} + c \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} = f. \quad (1)$$

Here, $u$ represents the ion density, $c > 0$ is the sound velocity, $f$ is a given deterministic forcing term, $u = u(x, x^+, t)$, $x^+ = y$ or $x^+ = (y, z)$, $\Delta u = u_{xx} + u_{yy}$, and $\Delta^2 u = u_{yy} + u_{yy} + u_{zz}$. We assume the domain $\mathcal{M} = (0, 1)^d \times (-L, L)^d$, $d = 1$ or $2$, is a rectangle or parallelepiped in $\mathbb{R}^n$, $n = 2$ or $3$. For the $y$ and $z$ boundaries, we assume either the Dirichlet or the periodic boundary conditions. For $x = 0, 1$, two types of boundary conditions will be assumed for different purposes:

$$u(0, x^+, t) = u(1, x^+, t) = u_x(1, x^+, t) = 0, \quad (2)$$


or

\[
\begin{aligned}
(\Pi) \quad \frac{\partial^\ell u}{\partial x^\ell}(0, x^+, t) = \frac{\partial^\ell u}{\partial x^\ell}(1, x^-, t), \quad \ell = 0, 1, 2.
\end{aligned}
\]

Roughly speaking we assume Type (I) boundary condition when using the dissipative effect and assume Type (II) boundary condition when using the dispersion effect.

### 1.1.1 Existence (3D) and uniqueness (2D) of weak solutions

Assuming (2) we first established the existence and uniqueness of weak solutions:

**Theorem 1.** Assume that \( u_0 \in L^2(M) \) and \( f \in L^\infty(0, T; L^2(M)) \). Then the initial-boundary value problem possesses a solution \( u \) in 3D such that

\[
u \in L^\infty(0, T; L^2(M)) \cap L^2(0, T; H^1_\perp(M)).
\]

In 2D, the solution \( u \) is unique.

The proof of Theorem 1 relies on the following parabolic regularization:

\[
\frac{\partial u^\epsilon}{\partial t} + \Delta u^\epsilon + \epsilon \frac{\partial u^\epsilon}{\partial x} + u^\epsilon \frac{\partial u^\epsilon}{\partial x} + \epsilon \left( \frac{\partial^4 u^\epsilon}{\partial x^4} + \frac{\partial^4 u^\epsilon}{\partial y^4} + \frac{\partial^4 u^\epsilon}{\partial z^4} \right) = f.
\]

We use energy estimates to derive compactness of the sequence \( u^\epsilon \) and pass to the limit on (5) as \( \epsilon \to 0 \).

There are mainly three difficulties in the proof. The first one is how to derive a bound on \( u^\epsilon \) in \( L^2(0, T; H^1(M)) \) independent of \( \epsilon \). The issue is that the linear operator \( \Delta u_x \) in (1) is not coercive because of the extra derivative in \( x \). To solve this problem, we multiply (5) by \( xu^\epsilon \), integrate over \( M \) and integrate by parts. The idea is that intuitively the multiplication of \( x \) reduces the extra derivative in \( x \). The boundary condition plays a key role here, as it contributes the necessary dissipative effect. The second issue is to pass to the limit on \( u^\epsilon \) at \( x = 1 \). This requires a preliminary result showing that the trace \( u_x \big|_{x=1} \) is well defined for a weak solution \( u \) and the trace depends continuously on \( u \). We established this trace result by extending the singular perturbation argument in [88]. The third difficulty comes up in the proof of the uniqueness of the weak solution. The difference of two weak solutions can not be estimated directly due to the insufficient regularity. We solve this problem by observing that the linearized ZK equation possesses a unique weak solution satisfying an energy estimate. In space dimension 2 this well-posedness result applies to the difference of weak solutions and thus the energy estimate follows as desired.

### 1.1.2 Local existence of strong solutions in 3D

Still assuming boundary condition (2), we also established the existence of a strong solution locally in time:

**Theorem 2.** In space dimensions two and three, there exists a local strong solution to the ZK equation on some time interval \([0, T_\ast)\), where \( T_\ast \) is positive and depends only on the data, such that

\[
\nabla u, u_{yy}, u_{zz}, u_t \in L^\infty(0, T_\ast; L^2(M)),
\]

and

\[
u_t \in L^2(0, T_\ast; H^1(M)).
\]

Also all the spatial derivatives of the third order of \( u \) belong to \( L^2(0, T_\ast; L^2(M)) \), except for \( u_{xxy} \) and \( u_{zzz} \).

The proof of Theorem 2 also depends on the stochastic parabolic regularization (5).

The main issue here is the anisotropy of the nonlinear term \( uu_x \) in the \( x \) direction, which makes the inequalities for energy estimates difficult to handle. To this end, we observe that \( |u_x|_{L^2(M)} \) can be dominated by \( |u_\|_{L^\infty(M)} \) (Lemma 3.2, [104]). Motivated by this observation, we differentiate (5) in time, and multiply (5) with \( u_x^\epsilon + xu^\epsilon \), integrate over \( M \) and integrate by parts, and the inequality that follows gives \( u_t \in L^\infty([0, T_\ast]; L^2(M)) \) and \( \nabla u_t \in L^2(0, T_\ast; L^2(M)) \). Thus we have overcome the difficulty brought about by the anisotropicity.

### 1.1.3 Global existence of strong solutions in 3D

We change the boundary conditions and assume (3) instead of (2). In this case we established the existence of solutions continuously evolving in \( H^1(M) \) in space dimensions 2 and 3 ([105]):
Theorem 3. In space dimensions two and three, the initial and boundary value problem for the ZK equation possesses a solution \( u \) such that

\[
    u \in C([0,T]; H^1(M)) \cap W^{3,3/2}(0,1)_x; H^{-1}(0,T; H^{-4}((-L,L)^d)), \quad d = 1, 2.
\]  

To prove Theorem 3, we first observe that (1) possesses a hamiltonian structure and two invariants:

\[
    M(t) = \int_M u^2(t) \, dM = M(0),
\]

\[
    H(t) = \frac{1}{2} \int_M |\nabla u(t)|^2 - \frac{u^3(t)}{3} \, dM = H(0).
\]

The Sobolev imbedding theorem in 3D is sufficient to handle (10). Then we derived an analogue of (9) and (10) by multiplying (5) with \( u^\epsilon \) and then with \(-2\Delta u^\epsilon - (u^\epsilon)^2\), integrating over \( M \) and integrating by parts.

We do not know if Theorem 3 can be established if we assume (2) instead of (3). In that setting, there are several obstacles. Firstly, as in the case of the 3D Navier-Stokes equation, the nonlinear term cannot be controlled by the Sobolev imbedding. Then, due to the partial hyperbolicity, the \( L^6(M) \) estimation used in [14] does not work here either. Additionally (2) will bring high order boundary terms into (9) and (10).

It is difficult to make both the dissipative and dispersion effects useful at the same time due to a certain competition between them. For instance, we cannot assume both Type (I) and (II) boundary conditions simultaneously. As a result, the uniqueness of the strong solution in space dimension 3 in Theorem 3 is still open.

1.2 Stochastic ZK equation

In order to model the randomness of the waves, we consider the stochastic ZK equation with a multiplicative white noise ([39])

\[
    du + (\Delta u_x + cu_x + uu_x) \, dt = f \, dt + \sigma(u) \, dW(t).
\]

Here \( \sigma \) is a uniformly bounded and Lipschitz operator between suitable spaces, \( W(t) \) is a cylindrical Wiener process defined on an auxiliary Hilbert space, and \( f \) is a deterministic external forcing term.

Notions of solutions are different in the PDE and the probabilistic sense. Here ‘weak’ solutions are martingale solutions and ‘strong’ solutions are pathwise solutions. For the martingale solutions, the stochastic basis (the probability space, the filtration and the Wiener process) is not specified in the beginning and is viewed as part of the unknown. In contrast for the pathwise solutions, the stochastic basis is fixed in advance as part of the assumptions.

1.2.1 Existence (3D) and uniqueness (2D) of martingale solutions

We assume (2) and derived the existence of martingale solutions which are also weak in the PDE sense:

Theorem 4. In space dimensions 2 and 3 there exists a martingale solution with the same regularity in time and space as that of the weak (in the PDE sense) solutions in Theorem 1. This martingale solution is unique in space dimension 2.

Theorem 4 is an extension of Theorem 1 from the deterministic to the stochastic setting. We use a stochastic parabolic approximation analogue to (5) and derive the bounds of \( u^\epsilon \) independent of \( \epsilon \), which implies the tightness and relativeness compactness of the sequence.

As in Theorem 1, there are two major challenges in the proof. The first one is how to handle the boundary condition in the stochastic setting. The trace results derived previously in the deterministic case ([89]) do not apply to \( u^\epsilon \) in the stochastic parabolic approximation, since \( u^\epsilon \) is rough in time. Therefore we focus instead on \( U^\epsilon := \int_0^t u^\epsilon(s) \, ds \), which has enough regularity in time. In this way we derived a stochastic trace result analogous to the deterministic one.

The second challenge is how to derive an energy estimate for the difference of two martingale solutions. The idea in the deterministic case can not be used directly since the deterministic Gronwall lemma does not apply in the stochastic setting. The stochastic Gronwall lemma derived in [38] does not apply either, since the resulting stopping times are difficult to handle. Therefore we established a stochastic Gronwall adapted to our problem where the stopping times are largely avoided.
1.2.2 Existence and uniqueness of pathwise solutions in $2D$

With the same assumptions as in Theorem 4, in space dimension 2 we passed from the martingale solutions in Theorem 4 to pathwise solutions:

**Theorem 5.** There exists a unique pathwise solution in $2D$ with the same regularity in time and space as that of the weak (in the PDE sense) solution in Theorem 1.

We apply the extension of the Yamada-Watanabe theorem ([109]) in the infinite dimensional space (see [45]), which basically implies that the pathwise solution exists whenever there exists a unique martingale solution. Thus Theorem 5 follows from Theorem 4.

2 Numerical analysis for stochastic geophysical fluid models

We have established the implicit Euler schemes to approximate the stochastic equations of geophysical fluid dynamics ([40]), based on the explicit schemes established for the deterministic equations (see e.g. [71, 96]). This is a first step towards numerical analysis in the stochastic setting for general fluid equations. Besides the numerical analysis of the Euler scheme, we also obtained a new theoretical result, namely, the existence of solutions which are weak in both PDE and probabilistic sense ([40]).

2.1 Implicit Euler scheme

The limit system we would like to approximate is the following abstract stochastic evolution equation

$$dU + (A U + B(U) + E U) \, dt = (\ell + \xi(U)) \, dt + \sigma(U) \, dW,$$

where $E U$ accounts for the Coriolis forces, $\xi$ accounts for the difference between a Stratonovich and an Itô type noise, and $W$ is a cylindrical Wiener process. This abstract equation covers a wide class of stochastic geophysical evolution equations driven by a broad class of nonlinear, state dependent white noise, e.g., the stochastic primitive equations for the oceans, the atmosphere, and the coupled oceanic-atmospheric system. We constructed an implicit Euler time discretization scheme to approximate (12) as follows:

$$\frac{U^n_N - U^{n-1}_N}{\Delta t} + A U^n_N + B(U^n_N) + E U^n_N = \ell^n_N + \xi(t^n_N, U^n_N) + \sigma_N(t^{n-1}_N, U^{n-1}_N, \eta^n_N) / \Delta t,$$  \hspace{1cm} (13)

where

$$\Delta t = T/N, \quad t^n_N = n \Delta t, \quad \text{for } n = 0, 1, \ldots, N,$$

$$\eta^n_N = \eta^n_N = W(t_n) - W(t_{n-1}), \quad \text{for } n = 1, \ldots, N.$$

2.2 Solving the adaptiveness issue of the scheme

The major challenge in our construction is how to ensure the adaptiveness of the scheme, which does not occur in the deterministic setting. While classical arguments involving the Brouwer fixed point theorem can be used to establish the existence of sequences satisfying the implicit scheme, it is crucial that these sequences are adapted to the noise. To address this issue we rely on a specifically chosen filtration derived with the canonical Wiener space ([50]), and the application of a measurable selection theorem in [8] (see also [15, 58]) and universal Radon measurability ([25, 90]). The adaptiveness issue also occurs when we construct the continuous time stochastic processes based on the scheme (13). To deal with this issue we set a time lag in the stochastic processes. This is different from the classical methods in the deterministic setting ([71, 96]).

3 Stochastic-statistical models of criminal behavior

My postdoctoral research focuses on the stochastic-statistical agent-based models of criminal behavior for residential burglaries.

Residential crime is one of the toughest issues in modern society. A quantitative, informative, and applicable model of criminal behavior is needed to assist law enforcement.

In the past ten years applied mathematicians have been working in the burgeoning area of crime modeling and prediction (see e.g. [1, 5, 6, 7, 9, 11, 34, 42, 55, 56, 61, 68, 69, 70, 72, 75, 76, 77, 83, 85, 86, 93, 94, 98, 99, 102, 103, 113]), since the seminal work [94] on the mathematics of agent-based models
for residential burglary. Roughly speaking, there are two classes of crime models. Class I is statistical in nature aiming to predict the patterns of observed events. Class II is agent-based and describes the actions of individuals that lead to aggregate pattern formation. Class II also plays a more significant role in application for the law-enforcement to understand the feedback between treatment and hotspots of crimes localized in time and space.

Our works have made progress to the agent-based models in two ways. (1) In one space dimension, we assumed that the movement patterns of the criminals involve truncated Lévy distributions for the jump length ([82]), other than classical random walks ([94]), or Lévy flights without truncation ([16]). This is the first time that the truncated Lévy flights have been applied in crime modeling. Furthermore (2), in two space dimensions, we used the Poisson clocks to govern the time steps of the evolution of the model [106], rather than a discrete time Markov chain with deterministic time increments in [94]. Poisson clocks are particularly suitable to model the times at which arrivals enter a system. Moreover, this modification brings the model into the mathematical framework of Markov pure jump processes, interacting particle systems, and stochastic differential equations with Lévy noise (see e.g. [2, 26, 57, 62, 63, 64]).

Now we are working on the incorporation of both modifications ([108]) and the application of independent Poisson clocks for each agent ([107]).

### 3.1 Crime modeling with truncated Lévy flights

Agent-based models could be used for prediction if all model parameters were known. However, movement patterns of individual criminals are difficult to track. Therefore it is imperative to identify the simplest class of universal models for criminal movement. The article [94] used a biased random walk, that is, short hops. Since it is well-known that people foraging in an environment are more likely to move according to a Lévy flight than a random walk, later in one space dimension the Lévy flight is adopted in this model [16]. In reality criminals move in confined areas with a max jump length. Hence we diverted the analysis to truncated Lévy flights with the truncation size representing the maximum mobility of an agent. An analogue mean-field continuum model is derived which yields local Laplace diffusion. This suggests that local diffusion models are universal for continuum limits of this problem. There is excellent agreement between the continuum model and the agent-based simulations under a wide range of parameters. We also considered incorporating another group of agents with different group maximal mobility, e.g. the law-enforcement agents, into the model.

#### 3.1.1 Discrete model

The system is defined on a one-dimensional grid with constant lattice spacing ℓ and periodic boundary conditions. Attracted to each grid k is a vector (nk(t), Ak(t)), representing the number of criminals and the “attractiveness” at site k at time t. The attractiveness refers to the burglar’s beliefs about the vulnerability and value of the target site. We assume that Ak(t) = A0k + Bk(t), where A0k is a positive static background term, and Bk(t) is the dynamic term associated with repeat and near-repeat victimization, which will be elaborated shortly.

The system evolves only at discrete time steps t = nδt, n ∈ N, δt > 0, and unfolds starting with some initial distribution of the criminals and attractiveness field, (n0, A0). At each time step, every criminal either commits a burglary with probability pk(t) = 1 − e−Ak(t)δt, and is removed immediately, or he moves to another site according to a truncated Lévy distribution biased towards areas with a high attractiveness. The relative weight of him moving from site i to k, i ≠ k, is set as

\[
\begin{align*}
   w_{i\rightarrow k}(t) = \begin{cases} 
   \frac{A_k(t)}{\mu|\bar{i} - \bar{k}|^{\mu}}, & 1 \leq |i - k| \leq L, \\
   0, & \text{otherwise,}
   \end{cases}
\end{align*}
\]

(14)

Here μ ∈ (1,3) is the exponent of the underlying power law of the Lévy distribution, and L ∈ N is the truncation size. The probability qi→k(t) of an agent jumping from site i to k is obtained by the renormalization of wi→k(t). Next at each site a new agent is replaced with rate γ. Finally the attractiveness field gets updated according to the repeat victimization and the broken windows effect ([12, 36, 46]) as follows:

\[
B_k(t + \delta t) = \left[1 - \eta \right]B_k(t) + \frac{\eta}{2} \left(B_{k-1}(t) + B_{k+1}(t) \right) \left(1 - \omega \delta t \right) + \theta \beta t A_k(t) n_k(t).
\]

(15)

Here η ∈ (0,1) is a constant measuring the significance of the near-repeat victimization effect, θ is the increase in Bk for each burglary event, ω is the decay rate, and Akn_k δt is the average number of burglary events occurred during the time interval \((t, t + \delta t)\).
3.1.2 Continuum model

An analogue mean-field continuum model based on the discrete model was derived, where the local Laplace diffusion appears rather than fractional diffusion. Fractional diffusion appears in the continuum Lévy-flight model [16]. This replacement happens essentially because the infinitesimal generator of the truncated Lévy flight is a local operator. Specifically we have

\[
\begin{align*}
\frac{\partial A}{\partial t} &= \frac{l^2}{2\delta t} A_{xx} - \omega (A - A^0) + \theta n A, \\
\frac{\partial n}{\partial t} &= D(\mu, L) \nabla \cdot \left( \nabla n - \frac{2n}{A} \nabla A \right) - An + \gamma,
\end{align*}
\]

(16)

where

\[
D(\mu, L) = \frac{l^2}{\delta t} \sum_{k=1}^{L} k^{2-\mu}. \tag{17}
\]

When \( L = 1 \), (17) implies that \( D(\mu, L) = \frac{l^2}{2\delta t} \) and thus the continuum truncated Lévy-flight model coincides with the continuum random-walk model (see (3.2) and (3.5), [94]).

3.1.3 Incorporation of law enforcement agents

In the field there is another essential component that affects the criminal behavior, namely, the presence of law enforcement agents. We incorporate their effects into the model and assume that they also take the truncated Lévy flights. Their maximum mobility can be different than that of the criminal agents. In the previous works, the law-enforcement and criminal agents all follow the random walks ([49, 113]).

Discrete model

Let \( \psi_k(t) \) be the number of the police agents at site \( k \) at time \( t \), and \( \tilde{A}_k(t) \) be the attractiveness perceived by the criminals in the presence of the police agents. As in [49], letting \( \chi \) be a given constant measuring the effectiveness of police patrol, we assume

\[
\tilde{A}_k(t) := e^{-\chi \psi_k(t)} A_k(t). \tag{18}
\]

The probability of a criminal agent choosing to burglarize at site \( k \) and jumping from site \( i \) to site \( k \) are defined respectively in the same way as \( p_k(t) \) and \( q_{i\rightarrow k}(t) \) (see Section 3.1.1), except for \( A_k \) substituted with \( \tilde{A}_k \). The evolution equation of \( B_k(t) \) is the same as (15) but with \( \tilde{A}_k \) replacing \( A_k \). The law enforcement agents follow a truncate Lévy flight biased according to \( \tilde{A}_k \), with mobility parameters \( \tilde{\mu} \) and \( \tilde{L} \). The probability of a law enforcement agent moving from site \( i \) to site \( k \) is defined in the same way as \( q_{i\rightarrow k}(t) \) (Section 3.1.1), but with \( \tilde{\mu} \) and \( \tilde{L} \) replacing \( \mu \) and \( L \) respectively.

Continuum model

The continuum equations are derived in a similar way as (16), and also have a Brownian dynamics with modified diffusion coefficients:

\[
\begin{align*}
\frac{\partial A}{\partial t} &= \frac{l^2}{2\delta t} A_{xx} - \omega (A - A^0) + \theta n \tilde{A}, \\
\frac{\partial n}{\partial t} &= D(\mu, L) \nabla \cdot \left( \nabla n - \frac{2n}{A} \nabla \tilde{A} \right) - \tilde{A} n + \gamma, \\
\frac{\partial \psi}{\partial t} &= D(\tilde{\mu}, \tilde{L}) \nabla \cdot \left( \nabla \psi - \frac{2\psi}{A} \nabla \tilde{A} \right).
\end{align*}
\]

(19)

The continuum model is validated through simulations as excellent agreement with the agent-based simulations is demonstrated.

3.2 Crime modeling with a Poisson clock

We extended the agent-based model in [94] by applying a Poisson clock. It comes form the Poisson processes which are intensively studied and used in areas concerning random phenomena such as biology,
ecology, economics, and physics (see e.g. [3, 10, 21, 27, 41, 79]). The time increments are independent exponentially distributed random variables, rather than a fixed number as in [94]. In this setting, the martingale approach to the theory of Markov processes is applicable. As a result we expressed the model as a semimartingale ([47, 74]) which is a summation of the predictable or deterministic component and the stochastic or unpredictable component. This martingale formulation provides us with the basic tool to systematically analyze the model. A hydrodynamic-limit-type continuum model and a scaling property concerning the criminal population are derived based on the martingale formulation. The scaling property provides an explanation to the “finite size effects” observed in the simulations, where transient hotspots generally occur with low total criminal population. In real life the crime population is of finite size and thus the theory is quite relevant to real crime statistics.

3.2.1 Discrete model

The general assumptions are analogous to those in Section 3.1.1. The domain is a 2D lattice with lattice spacing \( \ell \) and the periodic boundary conditions. The grid points are denoted as \( s = (s_1, s_2) \) and the whole collection of the grid points is denoted as \( S \). The vector \( (n_s(t), A_s(t)) \) represents the number of criminals and the “attractiveness” at site \( s \) at time \( t \), where \( A_s(t) = A^0_s + B_s(t) \). \( A^0_s \) is a positive static background term, and \( B_s(t) \) is a dynamic term.

The system starts with some given initial condition \( (n_s(0), A_s(0)) \) and evolves according to a Poisson clock with rate \( D\ell^{-2} \), where \( D \) is an absolute constant independent of \( \ell \). Whenever the Poisson clock rings at some time \( t^- \), then at time \( t \), the following things happen one by one. Firstly, each agent chooses to burglarize with the probability \( p_s(t) = 1 - e^{-A_s(t^-)\ell^2/D} \), and will be immediately removed from the system. If not, he moves following a biased random walk, jumping from site \( s \) to one of the neighboring sites, say \( k \), with the probability \( q_{s \rightarrow k}(t) = A_k(t^-)/\sum_{s'} q_{s' \rightarrow s} A_{s'}(t^-) \), where \( s' \sim s \) indicates all of the neighboring sites of \( s \). Then at each site with probability \( e^{\ell^2/D} \), a new agent will be replaced. Finally, the attractiveness field gets updated in a way similar to (15), but with the diffusion taking place in space dimension two.

The simulation exhibits the same hotspot dynamic regimes with that of the deterministic-time-step model ([94]) under equivalent parameters, namely, spatial homogeneity, dynamic hotspots and stationary hotspots.

3.2.2 Martingale formulation

Let \( B(t) := \{B_s(t) : s \in S\} \) and \( n(t) := \{n_s(t) : s \in S\} \). For an arbitrary stationary scalar field \( \phi = \{\phi_s : s \in S\} \), we define

\[
\langle (B(t), n(t)), \phi \rangle := \left( \ell^2 \sum_s B_s \phi_s, \ell^2 \sum_s n_s \phi_s \right).
\]

Then \( \langle (B(t), n(t)), \phi \rangle \) is a Markov pure jump process. Its martingale formulation can be derived based on its infinitesimal means and variances:

**Theorem 6.** The Markov process \( \langle (B(t), n(t)), \phi \rangle \) can be expressed as a semimartingale:

\[
\begin{align*}
\langle B(t), \phi \rangle &= \langle A^0 - A^0, \phi \rangle + \int_0^t \mathcal{G}_1 \left( \langle (B(s), n(s)), \phi \rangle \right) ds + \mathcal{M}_1 \left( \langle (B(t), n(t)), \phi \rangle \right), \\
\langle n(t), \phi \rangle &= \langle n_0, \phi \rangle + \int_0^t \mathcal{G}_2 \left( \langle (B(s), n^f(s)), \phi \rangle \right) ds + \mathcal{M}_2 \left( \langle (B(t), n(t)), \phi \rangle \right).
\end{align*}
\]

where \( \mathcal{G}_i \left( \langle (B(t), n(t)), \phi \rangle \right) \), \( i = 1, 2 \), are the infinitesimal means at time \( t \), and \( \mathcal{M}_i \left( \langle (B(t), n(t)), \phi \rangle \right) \) are martingales, whose variances can be derived from the infinitesimal variances \( \mathcal{V}_i \left( \langle (B(t), n(t)), \phi \rangle \right) \), \( i = 1, 2 \), as follows:

\[
\text{Var} \left( \mathcal{M}_i \left( \langle (B(t), n(t)), \phi \rangle \right) \right) = \int_0^t \mathbb{E} \left[ \mathcal{V}_i \left( \langle (B(r), n(r)), \phi \rangle \right) \right] dr, \quad i = 1, 2.
\]

The infinitesimal means and variances were computed explicitly using classical methods.

3.2.3 Continuum model

Formal estimations demonstrate that the stochastic component of the martingale formulation has a lower order of magnitude than the grid lattice. Therefore it is reasonable to set the continuum version of
the deterministic component as the continuum analogue of the discrete model, which yields:

\[
\begin{aligned}
\frac{\partial B}{\partial t} &= \frac{\eta D}{4} \Delta B - \omega B + \theta n A, \\
\frac{\partial n}{\partial t} &= \frac{D}{4} \nabla \cdot \left( \nabla n - \frac{2n}{A} \nabla A \right) - n A + \gamma.
\end{aligned}
\]  

This is the same as the continuum deterministic-time-step model (see (3.2), (3.5) in [94]). The idea here is used also in the derivation of the hydrodynamic limit of interacting particle systems, which has been extensively studied in e.g. [30, 44, 53, 54, 91, 95, 97, 101, 100].

### 3.2.4 Scaling property of the martingale formulation

With the martingale formulation we also derived a scaling property concerning the total criminal population.

**Theorem 7.** Fixing \( \ell, \bar{\theta}, \bar{\Gamma} > 0, \phi \in \mathbb{R}^S, \) and \((B, n) \in (\mathbb{R}_+)^S \times \mathbb{N}^S, \) for every \( q \in (0, D/\ell^2 \bar{\Gamma}) \), and for \( i = 1, 2 \) we have

\[
\begin{aligned}
\mathcal{G}_i ((B(t), n(t)), \phi)) &\bigg|_{B(t) = B, n(t) = qn, \theta = \frac{\ell}{q}, \gamma = q \bar{\Gamma}} = \mathcal{G}_i ((B(t), n(t)), \phi)) \bigg|_{B(t) = B, n(t) = n, \theta = \bar{\theta}, \gamma = \bar{\Gamma}}, \\
\sqrt{V_1 ((B(t), n(t)), \phi))} &\bigg|_{B(t) = B, n(t) = qn, \theta = \frac{\ell}{q}, \gamma = q \bar{\Gamma}} = \sqrt{X_1 + \frac{1}{q} Y_1}, \\
\frac{1}{q} \sqrt{V_2 ((B(t), n(t)), \phi))} &\bigg|_{B(t) = B, n(t) = qn, \theta = \frac{\ell}{q}, \gamma = q \bar{\Gamma}} = \sqrt{X_2 + \frac{1}{q} \left( 1 - \frac{\ell^2 \bar{\Gamma}}{D} \right) Y_2 + 1 q Z_2},
\end{aligned}
\]

where \( X_1, X_2, Y_1, Y_2 \) and \( Z_2 \) are positive numbers whose values only depend on \( \ell, B, n, \phi, \bar{\theta} \) and \( \bar{\Gamma} \), and are independent of \( q \).

This scaling theorem roughly says that firstly the deterministic component of the martingale formulation does not change when \( q \) changes, and secondly the rescaled stochastic component increases as \( q \) decreases.

### 3.2.5 Finite size effects and the scaling property

In the simulations, the same finite size effects appear which were first observed in [94]. That is, while the behavioral regimes of the discrete model exhibit both the transient and stationary hotspots, the continuum model shows only the stationary ones. Also the smaller the initial criminal population is, the more transient the hotspots are in the discrete model. We quantitatively analyzed the finite size effects based on the scaling property and the martingale formulation.

We first built a mathematical framework for the finite size effects by quantifying them. We use statistics like the discrete Fisher information (Section 3.2, [20]; see also [19, 28, 29, 33]) to measure the degree of hotspot transience. Simulations show that our choice of the statistics is proper and they coincide with our previous qualitative observations.

Equipped with the quantification, we provide a theory to the finite size effects. From Theorem 6 and 7, we infer that starting with the equilibrium, at the initial time, the components of the martingale formulation behave differently towards the scaling of the initial criminal population. The deterministic component does not scale with the initial criminal population, while the stochastic component increases as the initial criminal population decreases. We conjecture that these properties will prevail in time and our argument can be bootstrapped.

We simulated the rescaled standard deviations as in (25) and (26) directly, and the output supports our conjecture.

### 4 Current and future projects

**Global existence of smooth solutions for the ZK equation**

We hope to extend Theorem 3 and demonstrate that this strong solution is unique in 3D.

**Stability and the CFL condition of the stochastic Navier-Stokes equation**

We study the stability and consistency of a class of numerical schemes (both explicit and semi-implicit) for the 2D and 3D stochastic Navier-Stokes equations driven by a state dependent noise ([37]). Here the space and time are both discretized.
Crime modeling with poison clocks and truncated Lévy flights

We are now working on further progress with the agent-based models, such as applying independent Poisson clocks for each of the agents ([107]), and incorporating both the truncated Lévy distribution and Poisson clocks ([108]).

Stochastic-statistical multi-layer models of interacting particles

We plan to work on the multi-layer modeling of the interacting particles which would be applicable to data science, biology, chemistry, physics, social sciences, etc.

References


