Sample Midterm 2 (Solutions)

1) Let $a_n = (-1)^n, \forall n \in \mathbb{N}$

$b_n = (-1)^{n+1}, \forall n \in \mathbb{N}$

Then $a_{k+1} = (-1)^{k+1} = b_k$ $\quad a_n \neq b_n, \forall n \in \mathbb{N}$

$b_{k+1} = (-1)^{k+1+1} = (-1)^k = a_k$

So $(b_k)_k$ is the subsequence $(a_{k+1})_k$ of $(a_n)_n$

$(a_k)_k$ is the subsequence $(b_{k+1})_k$ of $(b_n)_n$

2) a) $f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ $\quad f(0) = 0$.

Let $a_n = \frac{1}{2n\pi + \frac{\pi}{2}} \longrightarrow 0$ as $n \to +\infty$, but $f(a_n) = 1 \to 1$

so $f(a_n)$ does not conv. to $f(0)$

Therefore f is not conti in 0

b) i) If $0 < a < b$ or $a < b < 0$ then f is continuous on $[a, b]$, and therefore has the intermediate value property.

ii) If $a < 0 < b$, and y is so that

$-1 \le f(a) < y < f(b) \le 1$ or $-1 \le f(b) < y < f(a) \le 1$

then ~~$y = \sin \frac{1}{\omega}$~~

let ~~$y = \sin \frac{1}{x_n}$~~ $x_n = \frac{1}{2n\pi + \arcsin y}, \forall n \in \mathbb{N}$

For n large enough we have that $x_n \in (a, b)$

So $f(x_n) = \sin \frac{1}{x_n} = \sin(2n\pi + \arcsin y) = \sin(\arcsin y) =$

$= y$

3) (i) See the lecture. (More precisely Thm 19.4)

(ii) $f : (0,1) \longrightarrow \mathbb{R}$, $f(x) = \frac{1}{x^{10}}$

let $x_n = \frac{1}{n} \longrightarrow 0 \Rightarrow (x_n)$ Cauchy - Sequence

$f(x_n) = n^{10} \longrightarrow +\infty \Rightarrow (f(x_n))_{n=1}^{\infty}$ is not a Cauchy-Sequence.

Therefore f cannot be uniformly continuous

4) Assume that there is a function $f : \mathbb{R} \to \mathbb{R}$ continuous and with the property that $(f \circ f)(x) = -x, \forall x \in \mathbb{R}$

Then such a function has to be one-to-one

$$\left(f(x) = f(y) \Rightarrow f(f(x)) = f(f(y)) \Leftrightarrow -x = -y \Rightarrow \right.$$
$$\left. \Rightarrow x = y \right).$$

But if a function is one to one and continuous is strictly monotonic. So $f \circ f$ is strictly increasing. Since $-x$ is strictly decreasing, it cannot equal $f \circ f$.