1. Assume you draw 10 cards from a standard 52 card deck without replacement.

(a) What’s the probability the 10th card is black?
(b) If you know the first nine drawn are not black, what is the probability the 10th card is black?
(c) Same question only now you know there is at most one black chosen in the previous nine.
(d) What’s the probability that half are red and half are black?
(e) What’s the probability you have at least one face card?
(f) What’s the probability you have drawn all four kings and all four aces?

Solution:

(a) 1/2. Remember the probability that any card in our deck is black is 1/2.
(b) After we have chosen 9 non-black cards, there are 43 cards left over and 26 of these 43 are black since we haven’t gotten any so far. Hence, our solution is: 26/43
(c) This is hard. Using the definition of conditional probability, one can find:

\[
\frac{\binom{26}{9} \cdot 26 + \binom{26}{8} \cdot \binom{26}{1} \cdot 25}{\binom{52}{10}}
\]

Here \( A = \) “get 10th black” and \( B = \) “at most one card black in previous 9”. So \( P(A|B) = P(A \cap B)/P(B) \). Let’s focus on \( P(A \cap B) \), i.e. the numerator:

\[
\frac{\binom{26}{9} \cdot 26 + \binom{26}{8} \cdot \binom{26}{1} \cdot 25}{\binom{52}{10}}
\]

We can either get no blacks or exactly one black on the first nine draws. Let’s define the following:

\( C = \) choosing the first 9 cards so they are not black and the 10th card is black
\( D = \) choosing exactly one black in first nine and 10th card black

Observe that:

\[ A \cap B = C \cup D \]
and moreover, $C \cap D = \emptyset$. Hence,

$$P(A \cap B) = P(C \cup D) = P(C) + P(D) - P(C \cap D) = P(C) + P(D)$$

Let’s examine $|C|$ (that is, the size of $C$ or the number of ways it can occur). The number of ways to get no blacks on the first nine cards is $\binom{26}{9}$. Then we pick our 10th card, and there are 26 black cards left. Hence, $|C| = \binom{26}{9} \cdot 26$. Similarly, $|D| = \binom{26}{1} \cdot \binom{26}{8} \cdot 25$. We conclude:

$$P(C) = \frac{\binom{26}{9} \cdot 26}{\binom{52}{10}}$$

$$P(D) = \frac{\binom{26}{1} \cdot \binom{26}{8} \cdot 25}{\binom{52}{10}}$$

$$\implies P(A \cap B) = P(C) + P(D) = \frac{\binom{26}{9} \cdot 26 + \binom{26}{8} \cdot \binom{26}{1} \cdot 25}{\binom{52}{10}}$$

An identical argument is used to calculate $P(B)$.

(d) $\left(\frac{26}{5}\right)^2 \cdot \frac{26}{52} \cdot \frac{10}{10}$

(e) There are 16 face cards. Face cards include A, K, Q, J. We will compute the complement:

$$P(\text{get at least one face card}) = 1 - P(\text{no face cards}) = 1 - \frac{\binom{36}{10}}{\binom{52}{10}}$$

(f) $\frac{44}{52} \cdot \frac{2}{52} \cdot \frac{10}{10}$

2. (Weisbart) I have three colored cards. Two of the cards are painted green on each side, and one is painted orange on one side and green on the other. Suppose you are shown a single face of a card and that face is green. Find the probability that the other side is orange?
Solution:

P(one side orange | the side you see is black). How many black sides are there? 5. How many of those will reveal an opposite side orange? 1. Hence, the answer is 1/5.

If we want to use the definition of conditional probability, observe that:

\[
P(\text{the side you don't see is orange} | \text{the side you see is green}) = P(A|B) = \frac{P(A \cap B)}{P(B)}
\]

\[
= \frac{5 \cdot \frac{1}{5}}{5/6}
\]

\[
= \frac{1}{5}
\]

Remark. Most people mistakenly think that the answer is 1/3.

3. (Goldbring) You roll two die. Determine if the following events are independent: A = “the sum of two die is odd” and B = “the second die is a 3.” Prove your answer.

Solution:

First, note that if the sum of the two die is odd, then one die must be even and one die must be odd (convince yourself of this!). Hence we can rewrite the probability of A:

\[
P(A) = P([ (\text{the first dies is odd}) \text{ and } (\text{the second die is even})] \text{ OR } [ (\text{the first die is even}) \text{ and } (\text{the second dies is odd})])
\]

\[
= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}
\]

\[
= \frac{1}{4} + \frac{1}{4}
\]

\[
= \frac{1}{2}
\]

Clearly, \(P(B) = \frac{1}{6}\).

Lastly, we compute: \(P(A \cap B)\). Observe the only way we can get an odd sum and roll a 3 on the second die is if we roll (2, 3) or (4, 3) or (6, 3). Hence, \(P(A \cap B) = \frac{3}{36} = \frac{1}{12}\).

We see that \(P(A \cap B) = P(A)P(B)\).

So, A and B independent.

4. Suppose you are sleeping over at your friends house and you both have to wake up early in the morning for the Black Friday sale at Best Buy. It is still dark outside. You reach into a sock drawer with 10 socks (half are polka-dotted and half striped). You reach in and pull out 4 socks. Find the probability you obtained two pairs of matching socks. To clarify, you
would be glad if you picked two striped socks and two polka-dotted socks or all four of the same color. However, you would be mildly disappointed if you pulled out 3 striped socks and a single polka-dotted sock.

Solution:

Let’s define our sample space $\Omega$. We set $\Omega = \{ \text{ways to choose 4 socks from 10} \}$. Here, we assume that order we choose socks doesn’t matter. Observe that:

$$|\Omega| = \binom{10}{4}$$

In order to get two matching pairs:

- 4 striped: the number ways to get 4 striped is $\binom{5}{4}$
- 4 polka dotted: the number of ways to get 4 striped is $\binom{5}{4}$
- 2 polka dotted and 2 striped: the number of ways to get 2 striped and 2 Polka-dotted is $\left[\binom{5}{2}\right]^2$

$$P(\text{two matching pairs}) = P(4 \text{ striped}) + P(4 \text{ Polka dotted}) + P(2 \text{ striped and 2 polka dotted})$$

$$= \frac{\binom{5}{4}}{\binom{10}{4}} + \frac{\binom{5}{4}}{\binom{10}{4}} + \frac{\left[\binom{5}{2}\right]^2}{\binom{10}{4}}$$

5. (Hoel) A factory owner knows that 5% of all the bolts he produces are defective. However, he has the following guarantee on his product: “If you buy a box of 10,000 bolts with at least $n$ defective bolts, then we will replace your box at no extra coooooost!!!”

(a) Using Chebychev’s inequality, determine $n$ so that the factory owner need only refund at most 1% of the boxes he sells.

(b) Using the Central Limit theorem, approximate the probability that the probability there are more than $n$ defective bolts. You may leave your answer in terms of the CDF of the standard Normal Distribution. Also, clearly state where you use of the Central Limit Theorem.

Solution:

(a) Let $X = \text{the number of bolts defective in a box}$. Let $\mu = E(X) = 500$. Define $\varepsilon$ so that $n = \mu + \varepsilon$ where $\varepsilon > 0$ is a whole number:

$$P(X \geq n) = P(X \geq \mu + \varepsilon) < P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2} \leq .01$$
\[ \sigma^2 = \text{Var}(X) = (10,000)(.05)(.95). \] Hence, \( \varepsilon^2 = \frac{(10,000)(.05)(.95)}{(0.1)} \). Plugging into a calculator gives us that \( \varepsilon = 217.9 \). To make \( \varepsilon \) a whole number we choose it to be \( \varepsilon = 218 \). Hence, \( n = \mu + \varepsilon = 500 + 218 = 718 \).

(b) The Central Limit Theorem States that:

**Theorem.** Suppose \( X_1, X_2, \ldots \) are i.i.d. \((\text{independent, identically distributed})\) with mean \( E(X_i) = \mu \) and \( \text{Var}(X_i) = \sigma^2 < \infty \). Define \( S_k = \sum_{i=1}^{k} X_i \), then:

\[
P \left( \frac{S_k - k\mu}{\sqrt{k\sigma^2}} \leq x \right) \rightarrow P(X_{\text{normal}} < x) = F_{\text{normal}}(x)
\]

where \( X_{\text{normal}} \) is THE standard normally distributed random variable.

For our scenario, we label each screw, \( i = 1, 2, \ldots \) and set:

\[
X_i = \begin{cases} 
1 & \text{ith screw is defective} \\
0 & \text{otherwise}
\end{cases}
\]

Then, if we set \( X = \# \) of screws that are defective, then \( X = S_{10,000} = \sum_{i=1}^{10,000} X_i \). Note that:

\[
\sigma^2 = .05 \cdot .95 \\
E(X_i) = .05
\]

We conclude from the CLT:

\[
P(X \leq 718) = P \left( \frac{X - (.05)(10,000)}{\sqrt{10,000 \cdot .05 \cdot .95}} \leq \frac{718 - (.05)(10,000)}{\sqrt{10,000 \cdot .05 \cdot .95}} \right) \approx F_{\text{normal}} \left( \frac{718 - (.05)(10,000)}{\sqrt{10,000 \cdot .05 \cdot .95}} \right).
\]

So, \( P(X > 718) = 1 - F_{\text{normal}} \left( \frac{718 - (.05)(10,000)}{\sqrt{10,000 \cdot .05 \cdot .95}} \right) \).

Remark (sometimes, we may inspect \( P(X > 717.5) \) instead since \( X \) is discrete, this may give us a marginally better estimate. This is known as the Histogram method).

6. (a) Let \( T \) be a non-negative random variable with hazard rate function \( \lambda(t) \). Assume \( S(t) = P(T > t) = e^{-\int_0^t \lambda(x)dx} \) and \( \lim_{t \to \infty} S(t) = 0 \). Show that \( E(T) = \int_0^\infty S(t)dt \). [Hint 1: You do not need the fact that \( P(T > t) = e^{-\int_0^t \lambda(x)dx} \)][Hint 2: This was on your homework]

(b) Suppose that \( \lambda(t) = a^2t \) is the survival rate function of a non-negative random variable \( T \), where \( a \) is a fixed constant. If \( E(T) = 1 \), find \( a \). You may use that \( \int_0^\infty e^{xz}dx = \frac{\sqrt{\pi}}{2} \).

**Solution:**

(a) This was a homework Problem from 12.5. Use integration by parts.
(b) 
\[ E(T) = \int_0^\infty S(t) dt = \int_0^\infty e^{a^2t^2} dt = 1 \]
\[ \Rightarrow \frac{1}{a} \int_0^\infty e^{a^2u^2} du = 1 \]
\[ \Rightarrow \frac{1}{a} \cdot \frac{\sqrt{\pi}}{2} = 1 \] Since \( \int_0^\infty e^{u^2} du = 1 \)

Hence, \( a = \sqrt{\pi}/2 \).

7. Let \( X \) be a continuous random variable with CDF:
\[
F(x) = \begin{cases} 
0 & x \leq 0 \\
\frac{x}{4} & 0 \leq x \leq C \\
1 & x \geq C 
\end{cases}
\]

(a) Find \( a \) so that \( \text{Var}(aX) = 1 \).
(b) Find all \( b \) so that \( P(X = b) = 0 \).

Solution:
(a) Since \( F \) is a CDF, \( F \) must be continuous. Hence,
\[
\frac{C}{4} = \lim_{x \to C^-} F(x) = \lim_{x \to C^+} F(x) = 1
\]
Hence, \( C = 4 \).

Now, observe that \( X \) is uniformly distributed on \([0, 4]!\) You can see this by noting the distribution function is that of a uniform random variable OR you could find the PDF and see that it must be:
\[
f(x) = \begin{cases} 
\frac{1}{4} & 0 \leq x \leq 4 \\
0 & \text{otherwise}
\end{cases}
\]

Hence, \( \text{Var}(X) = \frac{(4-0)^2}{12} = \frac{4}{3} \). Thus, from the problem we have \( 1 = \text{Var}(aX) \). So, \( 1 = \text{Var}(aX) = a^2\text{Var}(X) = a^2 \cdot \frac{4}{3} \). Solving for \( a \), we have \( a = \sqrt{3}/2 \).

(b) Any \( b \) works! This follows as \( X \) is continuous.

8. (Towsner/Goldbring) Let \( X \) be a uniformly distributed Random Variable on \([1, 2]\). Find \( E\left(\frac{1}{X}\right) \).

Solution:
Since \( X \) is uniform, it’s PDF is \( f(x) = 1 \) on \([1, 2]\) and 0 elsewhere.
\[
E\left(\frac{1}{X}\right) = \int_{-\infty}^\infty \frac{1}{x} \cdot f(x) dx = \int_1^2 \frac{1}{x} \cdot 1 dx = \ln 2
\]
9. (Pitman) Suppose that you buy two lightbulbs, one fluorescent and the other incandescent. The fluorescent bulb lasts on average 3 years, while the incandescent one lasts on average for 2 years. Assume the lifetime of each bulb is exponentially distributed.

(a) What is the median lifetime of each of lightbulbs (in other words, find $t$ so that $P(T > t) = .5$, where $T =$ the lifetime of some bulb)?

(b) Suppose the fluorescent has not burned out before it’s median lifetime. What’s the probability it lasts another median lifetime after that?

(c) If you connect the lightbulbs in a series (this means the circuit fails if either one lightbulb fails), how long do you expect the circuit to last?

(d) Say you choose one of the lightbulbs and stick the bulb into a lamp, and see the bulb hasn’t burned out after 1 year of operation. What is the probability it was the fluorescent bulb?

**Solution:**

Let $T_1 =$ lifetime of fluorescent with parameter $\frac{1}{3}$ and $T_2 =$ lifetime of incandescent with parameter $\frac{1}{2}$.

(a) We have that $P(T_2 > t) = e^{-\frac{1}{2}t} = \frac{1}{2}$. Solving for $t$ gives us $t = 2 \ln 2$, and similarly for $T_1$, we find $t = 3 \ln 2$.

(b) By the memoryless property, this is $\frac{1}{2}$.

(c)

$$P(\min(T_1, T_2) > t) = P(T_1 > t)P(T_2 > t) = e^{-t/3} \cdot e^{-t/2} = e^{-\frac{5}{6}t}$$

So the series circuit is exponential! This is because it has a survival function that is of the form $e^{-\lambda_{\text{series}}t}$ and $\lambda_{\text{series}} = \frac{5}{6}$, so the expected lifetime is $\frac{6}{5}$.

(d) Draw out a chart (or use Bayes formula):

$$\frac{\frac{1}{2}e^{-\frac{1}{3}}}{\frac{1}{2}e^{-\frac{1}{3}} + \frac{1}{2}e^{-\frac{1}{2}}}$$

10. Suppose that the number of calls into Pleasantville Police Station is Poisson random variable and on average they receive 2 phone calls per hour. However, since Pleasantville has not renovated their police station for a long time, if they receive over 3 calls in an hour, there phone system shuts down and must be rebooted. However, on every hour (12, 1, 2, etc), their system resets permitting 3 new phone calls in the following hour. For instance, if we get 2 phone calls between 12:30 and 1; two phone calls between 1 and 1:30; and furthermore these are the only phone calls between 12-2, the system does NOT shut down. On the other hand, if we get 4 calls between 1 and 2, the system collapses. On average, how many hours does their phone system last before requiring a reboot?

**Solution:**

$X =$ number of calls in an hour. From the info given in the problem, this is Poisson with $\lambda = 2$. 
\( Y = \) number of hours until the system needs rebooting. Clearly, this random variable is geometric with \( p = P(X \geq 4) \).

We compute: \( P(X \geq 4) = 1 - P(X < 4) = 1 - e^{-2}\left[1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right] \). Set this number to be \( p \).

We seek \( E(Y) = 1/p \), since \( Y \) is geometric!

11. Suppose that a community of 10 couples each wants a son. Each couple will not stop bearing children until they get one. Let \( X \) be the random variable that denotes the number of children in this community. Assume boys and girls are equally likely to be born by any couple.

(a) Find \( P(X = 11) \) [If you are not sure how to proceed, then you may want to see 12.4.76]
(b) Find the expected number of children.
(c) Find the expected number of girls.

Solution:

(a) We need 9 out of the 10 couples to have only 1 child (i.e. 1 boy) and exactly one couple to have two children (1 boy and one girl). How many ways can this be done? \( \binom{10}{9} \).

Hence,

\[
P(X = 11) = \binom{10}{9} \cdot (0.5)^{11}
\]

(b) Let \( X_i \) = the number of children the \( i \)th couple has, then \( X = \sum_{i=1}^{10} X_i \). Hence, \( E(X) = E(\sum_{i=1}^{10} X_i) = \sum_{i=1}^{10} E(X_i) = \sum_{i=1}^{10} 2 \), since each \( X_i \) is geometric with \( p = 0.5 \) and \( E(X_i) = 1/0.5 = 2 \).

We conclude that \( E(X) = 2 + \ldots + 2 = 20 \)

(c) Number of girls = number of children – number of boys = 20 – 10 = 10.

12. (Louider) Suppose a student has precisely 20 minutes to get to UCLA to take the final exam for 3c. He lives in Santa Monica and doesn’t have a car, so he must either take the Big Blue Bus or the Metro to get to campus. The Metro bus arrival is an exponentially distributed random variable with average arrival time 10 minutes. The ride takes exactly 10 minutes. The Big Blue Bus will arrives 5 minutes from when this student gets to the bus stop, but it’s travel time is a uniformly distributed random variable on \([5, 25]\). Both buses drop the Student directly in front of his classroom.

(a) What is the probability the student will make it on time to his exam if he chooses the Metro?
(b) The Big Blue Bus?
(c) Which bus has a greater probability of getting him to class on time? (You may use \( 1/e \approx 0.36 \).)
(d) Find the expected arrival time of each bus.
(e) Find the Variance of the arrival time of each bus.

Solution:

(a) Let \( X \) = the amount of time the student must wait for the Metro. If we are to arrive at UCLA on time then, we must have \( X \leq 10 \) since the travel time takes 10 minutes regardless of when the bus comes to the student’s apartment. Note that \( \lambda = \frac{1}{10} \).

Hence, \( P(X < 10) = 1 - e^{-1} \) using relevant formulae for exponential distributions.

(b) Let \( Y \) = the travel time of the Big Blue Bus. Since the Big Blue Bus comes 5 minutes after the student comes to the bus stop, we need \( Y < 15 \). Hence, \( P(Y < 15) = \frac{10}{20} = \frac{1}{2} \) using relevant formulae for uniform distributions.

(c) Metro

(d) The average arrival time for the Metro = waiting time for Metro + travel time for Metro = \( E(X) + 10 = 10 + 10 = 20 \). Similarly, we have the average arrival time for the big blue bus is \( 5 + E(Y) = 5 + \frac{25+5}{2} = 20 \).

(e) Metro has variance is \( \text{Var}(X + 10) = \text{Var}(X) = \frac{1}{\lambda^2} = 100 \), while Big Blue bus has variance of the arrival time is: \( \text{Var}(5 + Y) = \text{Var}(Y) = (25 - 5)^2/12 < 100 \).

Note even though the expected arrival time is 20 for both buses, the Metro has a greater variance and the probability that you arrive on time turns out to be greater!