

GRID DIAGRAMS AND HEEGAARD FLOER INVARIANTS

CIPRIAN MANOLESCU, PETER S. OZSVÁTH, AND DYLAN P. THURSTON

ABSTRACT. We give combinatorial descriptions of the Heegaard Floer homology groups for arbitrary three-manifolds (with coefficients in $\mathbb{Z}/2\mathbb{Z}$). The descriptions are based on presenting the three-manifold as an integer surgery on a link in the three-sphere, and then using a grid diagram for the link. We also give combinatorial descriptions of the mod 2 Ozsváth-Szabó mixed invariants of closed four-manifolds, in terms of grid diagrams.

1. INTRODUCTION

Starting with the seminal work of Donaldson [3], gauge theory has found numerous applications in low-dimensional topology. Its role is most important in dimension four, where the Donaldson invariants [4], and later the Seiberg-Witten invariants [26, 27, 28], were used to distinguish between homeomorphic four-manifolds that are not diffeomorphic. More recently, Ozsváth and Szabó introduced Heegaard Floer theory [19, 18], an invariant for low-dimensional manifolds inspired by gauge theory, but defined using methods from symplectic geometry. Heegaard Floer invariants in dimension three are known to detect Thurston norm [16] and fiberedness [12]. In dimension four one has the mixed invariants constructed in [21]. The mixed invariants are conjecturally the same as the Seiberg-Witten invariants and, further, they are known to share many of their properties: in particular, they are able to distinguish homeomorphic four-manifolds with different smooth structures. In a different direction, there also exist Heegaard Floer invariants for null-homologous knots and links in three-manifolds (see [17, 24, 22]), which have applications to knot theory.

One feature shared by the Donaldson, Seiberg-Witten, and Heegaard Floer invariants is that their original definitions are based on counting solutions to some nonlinear partial differential equations. This makes it difficult to exhibit algorithms which, given a combinatorial presentation of the topological object (for example, a triangulation of the manifold, or a diagram of the knot), calculate the respective invariant. The first such algorithms started to appear in 2006, in the setting of Heegaard Floer theory. Sarkar and Wang [25] gave an algorithm for calculating the hat version of Heegaard Floer homology of three-manifolds. The corresponding maps induced by simply connected cobordisms were calculated in [8]. In a different direction, all versions of the Heegaard Floer homology for links in S^3 were found to be algorithmically computable using *grid diagrams*, see [10]. See also [14, 13] for other developments. This progress notwithstanding, a combinatorial description of smooth, closed four-manifold invariants remained elusive.

Our aim in this paper is to present combinatorial descriptions of all (completed) versions of Heegaard Floer homology for three-manifolds, as well as of the mixed invariants of closed four-manifolds. It should be noted that we only work with invariants defined over the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. However, we expect a sign refinement of our descriptions to be possible.

Our strategy is to use (toroidal) grid diagrams for links, and to represent three- and four-manifolds in terms of surgeries on those links. Grid diagrams were previously used in [10] to give a combinatorial description of link Floer homology. From here one automatically obtains a

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combinatorial description of the Heegaard Floer homology of Dehn surgeries on knots in S^3 , since these are known to be determined by the knot Floer complex, cf. [23, 15]. Since every three-manifold can be expressed as surgery on a link in S^3 , it is natural to pursue a similar approach for links.

In a forthcoming paper [9], the Heegaard Floer homology of an integral surgery on a link is described in terms of some data associated to Heegaard diagrams for the link and its sublinks. In particular, we can consider this description in the case where the Heegaard diagrams come from a grid diagram for the link. In the setting of grids, the problem of computing the Heegaard Floer homology of surgeries boils down to counting isolated pseudo-holomorphic polygons in the symmetric product of the grid torus. Pseudo-holomorphic bigons of index one are easy to count, as they correspond bijectively to empty rectangles on the grid, cf. [10]. In general, one needs to count $(k + 2)$ -gons of index $1 - k$ that relate the Floer complex of the grid to those of its destabilizations at k points, where $k \geq 0$.

Just as in the case of bigons, one can associate to each polygon a certain object on the grid, which we call an *enhanced domain*. Roughly, an enhanced domain consists of an ordinary domain on the grid plus some numerical data at each destabilization point. (See Definition 3.1 below.) Further, when the enhanced domain is associated to a pseudo-holomorphic polygon, it satisfies certain positivity conditions. (See Definition 3.3 below.) Thus, the problem of counting pseudo-holomorphic polygons reduces to finding the positive enhanced domains of the respective index, and counting the number of their pseudo-holomorphic representatives.

When $k = 1$, this problem is almost as simple as in the case $k = 0$. Indeed, the only positive enhanced domains for $k = 1$ are the “snail-like” ones used to construct the destabilization map in [11, Section 3.2], see Figure 4 below. It is not hard to check that each such domain has exactly one pseudo-holomorphic representative, modulo 2.

The key fact that underlies the $k = 1$ calculation is that in that case there exist no positive enhanced domains of index $-k$ corresponding to destabilization at k points. Hence, the positive domains of index $1 - k$ are what is called indecomposable: in particular, the respective counts of pseudo-holomorphic representatives depend only on the topology of the enhanced domain (i.e. they are independent of its conformal structure). Unfortunately, this fails to be true for $k \geq 2$: there exist positive enhanced domains of index $-k$, so the counts in index $1 - k$ depend on the almost complex structure on the symmetric product of the grid.

Nevertheless, we know that different almost complex structures, even though they may produce different counts for the particular domains, do give rise to chain homotopy equivalent Floer complexes for the surgeries on the link. Let us think of an almost complex structure as a way of assigning a count (zero or one, modulo 2) to each positive enhanced domain on the grid, in a compatible way. Of course, there may exist other such assignments: we refer to an assignment that satisfies certain compatibility conditions as a *formal complex structure*. (See Definition 4.7 below.) Each formal complex structure produces a model for the Floer complex of the link surgery. If two formal complex structures are homotopic (in a suitable sense, see Definition 4.9), the resulting Floer complexes are chain homotopy equivalent. Suppose now we could show that all formal complex structures on a grid are homotopic. Then, we would easily arrive at a combinatorial formulation for the Floer homology of surgeries. Indeed, one could pick an arbitrary formal complex structure on the grid, which is a combinatorial object; and the homology of the resulting complex would be the right answer, whether or not the formal structure came from an actual almost complex structure on the symmetric product.

The question of whether or not any two formal complex structures on a grid G are homotopic basically reduces to whether or not a certain cohomology group $HE^d(G)$ vanishes in degrees $d < \min\{0, 2 - k\}$. Here, $HE^*(G)$ is the cohomology of a complex generated by positive enhanced domains, modulo periodic domains. (See Definition 4.7.) Computer experimentation suggests the following:

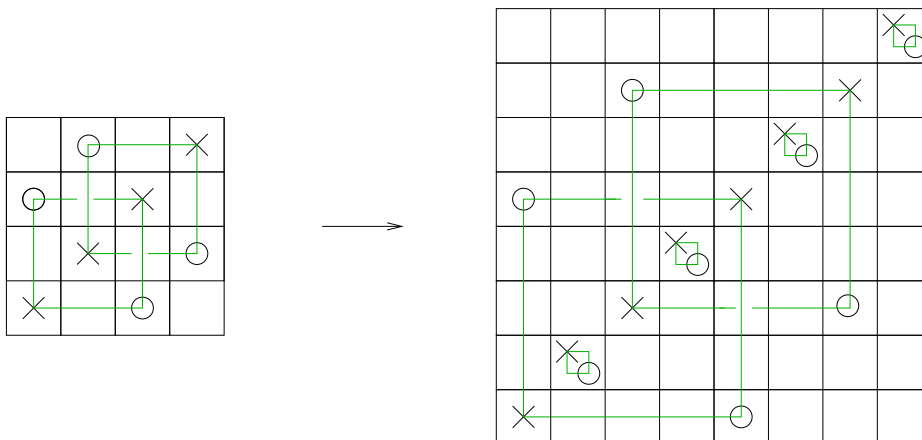


FIGURE 1. **The sparse double.** On the left we show a grid diagram G for the Hopf link. On the right is the corresponding sparse double $G_{\#}$.

Conjecture 1.1. *For any toroidal grid diagram G representing a link L , we have $HE^d(G) = 0$ for any $d < 0$.*

We can prove only a weaker version of this conjecture, but one that is sufficient for our purposes. We replace the grid G by a larger one, called its *sparse double*: see Figure 1. If G represents the link L , then $G_{\#}$ represents the union of L with some unknots. As such, one can still compute the Floer homologies of surgeries on L using $G_{\#}$ rather than G . We can show that $HE_{\#}^d(G_{\#}) = 0$ for $d < \min\{0, 2 - k\}$, where $HE_{\#}^*$ is a suitable modification of HE^* . As a consequence, we obtain:

Theorem 1.2. *Let $\vec{L} \subset S^3$ be an oriented link with a framing Λ . Let $S_{\Lambda}^3(L)$ be the manifold obtained by surgery on S^3 along (L, Λ) , and let \mathfrak{u} be a Spin^c structure on $S_{\Lambda}^3(L)$. Then, given as input a grid diagram for L , the Heegaard Floer homology groups $\mathbf{HF}^-(S_{\Lambda}^3(L), \mathfrak{u})$, $\mathbf{HF}^{\infty}(S_{\Lambda}^3(L), \mathfrak{u})$, $HF^+(S_{\Lambda}^3(L), \mathfrak{u})$ with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ are algorithmically computable.*

The algorithm alluded to in the statement of Theorem 1.2 is given in its proof. In the above statement, \mathbf{HF}^- and \mathbf{HF}^{∞} are the completed versions of the usual Heegaard Floer homology groups HF^- and HF^{∞} , respectively. The groups \mathbf{HF}^- and \mathbf{HF}^{∞} are constructed from complexes defined over the power series rings $\mathbb{F}[[U]]$ and $\mathbb{F}[[U, U^{-1}]]$, respectively, cf. [9, Section 2], compare also [6]. More precisely, the respective chain complexes are

$$\mathbf{CF}^- = CF^- \otimes_{\mathbb{F}[[U]]} \mathbb{F}[[U]], \quad \mathbf{CF}^{\infty} = CF^{\infty} \otimes_{\mathbb{F}[[U, U^{-1}]]} \mathbb{F}[[U, U^{-1}]].$$

One could also define a completed version of HF^+ in a similar way; but in that case, since multiplication by U is nilpotent on CF^+ , the completed version is the same as the original one.

Moving to four dimensions, in [9] it is shown that every closed four-manifold with $b_2^+ > 1$ admits a description as a four-colored link with a framing. This description is called a link presentation, see Definition 2.7 below.

Theorem 1.3. *Let X be a closed four-manifold with $b_2^+(X) > 1$, and \mathfrak{s} a Spin^c structure on X . Given as input a link presentation $(\vec{L} = L_1 \cup L_2 \cup L_3 \cup L_4, \Lambda)$ for X and a grid diagram for L , one can algorithmically compute the mixed invariant $\Phi_{X, \mathfrak{s}}$ with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.*

Although Theorems 1.2 and 1.3 give combinatorial procedures for calculating the respective invariants, in practice our algorithms are very far from being effective. This is especially true since we doubled the size of the grid in order to arrive at $G_{\#}$. However, one could just assume the truth of Conjecture 1.1, and try to do calculations with G . This is still not effective, and it remains an

interesting challenge to modify the algorithm in such a way as to make it more suitable for actual calculations. We remark that, in the case of the hat version of knot Floer homology, the algorithm from [10] has been implemented on computers, see [1, 2, 5]; at present, one can calculate these groups for links with grid number up to 13.

This paper is organized as follows. In Section 2 we state the results from [9] about how \mathbf{HF}^- of an integral surgery on a link L can be expressed in terms of counts of holomorphic polygon counts on the symmetric product of a grid. We also give the variant of this description using the sparse double. In Section 3 we define enhanced domains, and associate one to each homotopy class of polygons on the symmetric product. We also introduce the positivity condition on enhanced domains, which is necessary in order for the domain to have holomorphic representatives. In Section 4 we associate to a grid G the complex $CE^*(G)$ whose generators are positive enhanced domains modulo periodic domains. We show that, if Conjecture 1.1 is true, then all possible ways of assigning holomorphic polygon counts to enhanced domains are homotopic. We explain how this would lead to a combinatorial description of the Heegaard Floer invariants. In Section 5 we prove the weaker (but sufficient for our purposes) version of the conjecture, in which we use the sparse double $G_\#$ instead of G .

Throughout the paper we work over the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

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2. THE SURGERY THEOREM APPLIED TO GRID DIAGRAMS

Our main goal here is to present the statement of Theorem 2.4 below, which expresses the Heegaard Floer homology \mathbf{HF}^- of an integral surgery on a link in terms of a grid diagram for a link (or, more precisely, in terms of holomorphic polygon counts on a symmetric product of the grid). We also state similar results for the cobordism maps on \mathbf{HF}^- induced by two-handle additions (Theorem 2.5), for the other completed versions of Heegaard Floer homology (Theorem 2.6), and for the mixed invariants of four-manifolds (Proposition 2.9). Moreover, we state an analogue of Theorem 2.4 that uses the sparse double of a grid (Theorem 2.10); this plays an essential role in the combinatorial descriptions that will be presented later in this paper.

The proofs of all the results from this section are given in [9].

2.1. Hyperboxes of chain complexes. We start by summarizing some homological algebra from [9, Section 3].

When f is a function, we denote its n^{th} iterate by $f^{\circ n}$, i.e. $f^{\circ 0} = id$, $f^{\circ 1} = f$, $f^{\circ(n+1)} = f^{\circ n} \circ f$. For $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$ a collection of nonnegative integers, we set

$$\mathbb{E}(\mathbf{d}) = \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \mid \varepsilon_i \in \{0, 1, \dots, d_i\}, i = 1, \dots, n\}.$$

In particular, $\mathbb{E}_n = \mathbb{E}(1, \dots, 1) = \{0, 1\}^n$ is the set of vertices of the n -dimensional unit hypercube.

For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{E}(\mathbf{d})$, set

$$\|\varepsilon\| = \varepsilon_1 + \dots + \varepsilon_n.$$

We can view the elements of $\mathbb{E}(\mathbf{d})$ as vectors in \mathbb{R}^n . There is a partial ordering on $\mathbb{E}(\mathbf{d})$, given by $\varepsilon' \leq \varepsilon \iff \forall i, \varepsilon'_i \leq \varepsilon_i$. We write $\varepsilon' < \varepsilon$ if $\varepsilon' \leq \varepsilon$ and $\varepsilon' \neq \varepsilon$. We say that two multi-indices $\varepsilon, \varepsilon'$ with $\varepsilon \leq \varepsilon'$ are *neighbors* if $\varepsilon' - \varepsilon \in \mathbb{E}_n$, i.e. none of their coordinates differ by more than one.

We define an n -dimensional hyperbox of chain complexes $\mathcal{H} = ((C^\varepsilon)_{\varepsilon \in \mathbb{E}(\mathbf{d})}, (D^\varepsilon)_{\varepsilon \in \mathbb{E}_n})$ of size $\mathbf{d} \in \mathbb{N}^n$ to consist of a collection of \mathbb{Z} -graded vector spaces

$$(C^\varepsilon)_{\varepsilon \in \mathbb{E}(\mathbf{d})}, \quad C^\varepsilon = \bigoplus_{* \in \mathbb{Z}} C^{\varepsilon, *},$$

together with a collection of linear maps

$$D^\varepsilon : C_*^{\varepsilon^0} \rightarrow C_{*+\|\varepsilon\|-1}^{\varepsilon^0+\varepsilon},$$

defined for all $\varepsilon^0 \in \mathbb{E}(\mathbf{d})$ and $\varepsilon \in \mathbb{E}_n$ such that $\varepsilon^0 + \varepsilon \in \mathbb{E}(\mathbf{d})$ (i.e. the multi-indices of the domain and the target are neighbors). The maps D^ε are required to satisfy the relations

$$(1) \quad \sum_{\varepsilon' \leq \varepsilon} D^{\varepsilon-\varepsilon'} \circ D^{\varepsilon'} = 0,$$

for all $\varepsilon \in \mathbb{E}_n$. If $\mathbf{d} = (1, \dots, 1)$, we say that \mathcal{H} is a *hypercube of chain complexes*.

Note that D^ε in principle also depends on ε_0 , but we omit that from notation for simplicity. Further, if we consider the total complex

$$C_* = \bigoplus_{\varepsilon \in \mathbb{E}(\mathbf{d})} C_{*+\|\varepsilon\|}^\varepsilon,$$

we can think of D^ε as a map from C_* to itself, by extending it to be zero when is not defined. Observe that $D = \sum D^\varepsilon : C_* \rightarrow C_{*-1}$ is a chain map.

Let $\mathcal{H} = ((C^\varepsilon)_{\varepsilon \in \mathbb{E}(\mathbf{d})}, (D^\varepsilon)_{\varepsilon \in \mathbb{E}_n})$ be an n -dimensional hyperbox of chain complexes. Let us imagine the hyperbox $[0, d_1] \times \dots \times [0, d_n]$ to be split into $d_1 d_2 \dots d_n$ unit hypercubes. At each vertex ε we see a chain complex $(C^\varepsilon, D^{(0, \dots, 0)})$. Associated to each edge of one of the unit hypercubes is a chain map. Along the two-dimensional faces we have chain homotopies between the two possible ways of composing the edge maps, and along higher-dimensional faces we have higher homotopies.

There is a natural way of turning the hyperbox \mathcal{H} into an n -dimensional hypercube $\hat{\mathcal{H}} = (\hat{C}^\varepsilon, \hat{D}^\varepsilon)_{\varepsilon \in \mathbb{E}_n}$, which we called the *compressed hypercube* of \mathcal{H} . The compressed hypercube has the property that along its i^{th} edge we see the composition of all the d_i edge maps on the i^{th} axis of the hyperbox.

In particular, if $n = 1$, then \mathcal{H} is a string of chain complexes and chain maps:

$$C^{(0)} \xrightarrow{D^{(1)}} C^{(1)} \xrightarrow{D^{(1)}} \dots \xrightarrow{D^{(1)}} C^{(d)},$$

and the compressed hypercube $\hat{\mathcal{H}}$ is

$$C^{(0)} \xrightarrow{(D^{(1)})^{od}} C^{(d)}.$$

For general n and $\mathbf{d} = (d_1, \dots, d_n)$, the compressed hypercube $\hat{\mathcal{H}}$ has at its vertices the same complexes as those at the vertices of the original hyperbox \mathcal{H} :

$$\hat{C}^{(\varepsilon_1, \dots, \varepsilon_n)} = C^{(\varepsilon_1 d_1, \dots, \varepsilon_n d_n)}, \quad \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{E}_n.$$

If along the i^{th} coordinate axis in \mathcal{H} we have the edge maps $f_i = D^{(0, \dots, 0, 1, 0, \dots, 0)}$, then along the respective axis in $\hat{\mathcal{H}}$ we see $f_i^{od_i}$. Given a two-dimensional face of \mathcal{H} with edge maps f_i and f_j and chain homotopies $f_{\{i,j\}}$ between $f_i \circ f_j$ and $f_j \circ f_i$, to the respective compressed face in $\hat{\mathcal{H}}$ we assign the map

$$\sum_{k_i=1}^{d_i} \sum_{k_j=1}^{d_j} f_i^{o(k_i-1)} \circ f_j^{o(k_j-1)} \circ f_{\{i,j\}} \circ f_j^{o(d_j-k_j)} \circ f_i^{o(d_i-k_i)},$$

which is a chain homotopy between $f_i^{od_i} \circ f_j^{od_j}$ and $f_j^{od_j} \circ f_i^{od_i}$. The formulas for what we assign to the higher-dimensional faces in $\hat{\mathcal{H}}$ are more complicated, but they always involve sums of compositions of maps in \mathcal{H} .

Let ${}^0\mathcal{H}$ and ${}^1\mathcal{H}$ be two hyperboxes of chain complexes, of the same size $\mathbf{d} \in \mathbb{N}^n$. A *chain map* $F : {}^0\mathcal{H} \rightarrow {}^1\mathcal{H}$ is defined to be a collection of linear maps

$$F_{\varepsilon^0}^\varepsilon : {}^0C_*^{\varepsilon^0} \rightarrow {}^1C_{*+\|\varepsilon\|}^{\varepsilon^0+\varepsilon},$$

satisfying

$$\sum_{\varepsilon' \leq \varepsilon} (D_{\varepsilon^0 + \varepsilon'}^{\varepsilon - \varepsilon'} \circ F_{\varepsilon^0}^{\varepsilon'} + F_{\varepsilon^0 + \varepsilon'}^{\varepsilon - \varepsilon'} \circ D_{\varepsilon^0}^{\varepsilon'}) = 0,$$

for all $\varepsilon^0 \in \mathbb{E}(\mathbf{d})$, $\varepsilon \in \mathbb{E}_n$ such that $\varepsilon^0 + \varepsilon \in \mathbb{E}(\mathbf{d})$. In particular, F gives an ordinary chain map between the total complexes 0C and 1C .

Starting with from here, we can define chain homotopies and chain homotopy equivalences between hyperboxes by analogy with the usual notions between chain complexes.

The construction of $\hat{\mathcal{H}}$ from \mathcal{H} is natural in the following sense. Given a chain map $F : {}^0\mathcal{H} \rightarrow {}^1\mathcal{H}$, there is an induced chain map $\hat{F} : {}^0\hat{\mathcal{H}} \rightarrow {}^1\hat{\mathcal{H}}$ between the respective compressed hypercubes. Moreover, if F is a chain homotopy equivalence, then so is \hat{F} .

2.2. Chain complexes from grid diagrams. Consider an oriented, ℓ -component link $\vec{L} \subset S^3$. We denote the components of \vec{L} by $\{L_i\}_{i=1}^\ell$. Let

$$\mathbb{H}(L)_i = \frac{\text{lk}(L_i, L - L_i)}{2} + \mathbb{Z} \subset \mathbb{Q}, \quad \mathbb{H}(L) = \bigoplus_{i=1}^\ell \mathbb{H}(L)_i,$$

where lk denotes linking number. Further, let

$$\bar{\mathbb{H}}(L)_i = \mathbb{H}(L)_i \cup \{-\infty, +\infty\}, \quad \bar{\mathbb{H}}(L) = \bigoplus_{i=1}^\ell \bar{\mathbb{H}}(L)_i.$$

Let G be a toroidal grid diagram representing the link \vec{L} , as in [10, 11]. We follow the notation from [11]: G consists of a torus \mathcal{T} , viewed as a square in the plane with the opposite sides identified, and split into n annuli (called rows) by n horizontal circles $\alpha_1, \dots, \alpha_n$, and into n other annuli (called columns) by n vertical circles β_1, \dots, β_n . Further, we are given $2n$ markings on the torus, of two types: X and O , such that each row and each column contains exactly one O marking and one X marking. We then draw horizontal arcs between the markings in the same row (oriented to go from the O to the X), and vertical arcs between the markings in the same column (oriented to go from the X to the O). Letting the vertical arcs be overpasses whenever they intersect the horizontal arcs, we obtain a planar diagram for the link \vec{L} . The number n is called the grid number of G .

We let $\mathbf{S} = \mathbf{S}(G)$ be the set of matchings between the horizontal and vertical circles in G . Any $\mathbf{x} \in \mathbf{S}$ admits a Maslov grading $M(\mathbf{x}) \in \mathbb{Z}$ and an Alexander multi-grading given by

$$A_i(\mathbf{x}) \in \mathbb{H}(L)_i, \quad i \in \{1, \dots, \ell\}.$$

For $\mathbf{x}, \mathbf{y} \in \mathbf{S}$, we let $\text{Rect}^\circ(\mathbf{x}, \mathbf{y})$ be the space of empty rectangles between \mathbf{x} and \mathbf{y} , as in [11]. Specifically, a *rectangle* from \mathbf{x} to \mathbf{y} is an embedded rectangle r in the torus, whose lower left and upper right corners are coordinates of \mathbf{x} , and whose lower right and upper left corners are coordinates of \mathbf{y} ; and moreover, if all other coordinates of \mathbf{x} coincide with all other coordinates of \mathbf{y} . We say that the rectangle is *empty* if its interior contains none of the coordinates of \mathbf{x} (or \mathbf{y}). For $r \in \text{Rect}^\circ(\mathbf{x}, \mathbf{y})$, we denote by $O_j(r)$ and $X_j(r) \in \{0, 1\}$ the number of times O_j (resp. X_j) appears in the interior of r . Let \mathbb{O}_i and \mathbb{X}_i be the set of O 's (resp. X 's) belonging to the i^{th} component of the link. We fix orderings of the elements of \mathbb{O}_i and \mathbb{X}_i , for all i . The i^{th} coordinate of the Alexander multi-grading is characterized uniquely up to an overall additive constant by the property that

$$A_i(\mathbf{x}) - A_i(\mathbf{y}) = \sum_{j \in \mathbb{X}_i} X_j(r) - \sum_{j \in \mathbb{O}_i} O_j(r),$$

where here r is any rectangle from \mathbf{x} to \mathbf{y} .

For $i \in \{1, \dots, \ell\}$ and $s \in \overline{\mathbb{H}}(L)_i$, we set

$$E_s^i(r) = \begin{cases} \sum_{j \in \mathbb{O}_i} O_j(r) & \text{if } A_i(\mathbf{x}) \leq s, A_i(\mathbf{y}) \leq s \\ (s - A_i(\mathbf{x})) + \sum_{j \in \mathbb{X}_i} X_j(r) & \text{if } A_i(\mathbf{x}) \leq s, A_i(\mathbf{y}) \geq s \\ \sum_{j \in \mathbb{X}_i} X_j(r) & \text{if } A_i(\mathbf{x}) \geq s, A_i(\mathbf{y}) \geq s \\ (A_i(\mathbf{x}) - s) + \sum_{j \in \mathbb{O}_i} O_j(r) & \text{if } A_i(\mathbf{x}) \geq s, A_i(\mathbf{y}) \leq s. \end{cases}$$

Alternatively, we can write

$$(2) \quad E_s^i(r) = \max(s - A_i(\mathbf{x}), 0) - \max(s - A_i(\mathbf{y}), 0) + \sum_{j \in \mathbb{X}_i} X_j(r)$$

$$(3) \quad = \max(A_i(\mathbf{x}) - s, 0) - \max(A_i(\mathbf{y}) - s, 0) + \sum_{j \in \mathbb{O}_i} O_j(r).$$

In particular, $E_{-\infty}^i(r) = \sum_{j \in \mathbb{X}_i} X_j(r)$ and $E_{+\infty}^i(r) = \sum_{j \in \mathbb{O}_i} O_j(r)$.

Given $\mathbf{s} = (s_1, \dots, s_\ell) \in \overline{\mathbb{H}}(L)$, we define an associated chain complex $\mathfrak{A}^-(G, \mathbf{s}) = \mathfrak{A}^-(G, s_1, \dots, s_\ell)$ as follows. $\mathfrak{A}^-(G, \mathbf{s})$ is the free module over $\mathcal{R} = \mathbb{F}[[U_1, \dots, U_\ell]]$ generated by \mathbf{S} , endowed with the differential:

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbf{S}} \sum_{r \in \text{Rect}^\circ(\mathbf{x}, \mathbf{y})} U_1^{E_{s_1}^1(r)} \cdots U_\ell^{E_{s_\ell}^\ell(r)} \mathbf{y}.$$

Note that we can view $\mathfrak{A}^-(G, \mathbf{s})$ as a suitably modified Floer chain complex in $\text{Sym}^n(\mathcal{T})$, equipped with Lagrangian tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_n$, the product of horizontal circles, and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_n$, the product of vertical circles. The notation \mathbf{s} indicates the way we count powers of U 's. Empty rectangles are the same as holomorphic strips of index one in $\text{Sym}^n(\mathcal{T})$, compare [10].

Note that $\mathfrak{A}^-(G, \mathbf{s})$ can be equipped with a \mathbb{Z} -grading $\mu_{\mathbf{s}}$ such that the differential ∂ decreases $M_{\mathbf{s}}$ by one. Indeed, when none of the values s_i is $-\infty$, we can set the grading on generators to be

$$(4) \quad \mu_{\mathbf{s}}(\mathbf{x}) = M(\mathbf{x}) - 2 \sum_{i=1}^{\ell} \max(A_i(\mathbf{x}) - s_i, 0),$$

and let each U_i decrease grading by 2. When some of the values s_i are $-\infty$, we replace the corresponding expressions $\max(A_i(\mathbf{x}) - s_i, 0)$ by $A_i(\mathbf{x})$ in the Equation (4).

2.3. Destabilization at a set of markings. Let Z be one of the markings (an X or an O) on the grid diagram G . We define a subset $J(Z) \subset \overline{\mathbb{H}}(L)$ as follows. If $Z \in \mathbb{O}_i$ for some component L_i , set

$$J(Z) = \{(s_1, \dots, s_\ell) \in \overline{\mathbb{H}}(L) \mid s_i = +\infty\}.$$

If $Z \in \mathbb{X}_i$, set

$$J(Z) = \{(s_1, \dots, s_\ell) \in \overline{\mathbb{H}}(L) \mid s_i = -\infty\}.$$

For $\mathbf{s} \in J(Z)$, note that the other markings in the same column or row as Z are irrelevant for the computation of the chain complex $\mathfrak{A}^-(G, \mathbf{s})$.

Next, consider a set of markings $\mathcal{Z} = \{Z_1, \dots, Z_k\}$. We say that \mathcal{Z} is *consistent* if, for any i , at most one of the sets $\mathcal{Z} \cap \mathbb{O}_i$ and $\mathcal{Z} \cap \mathbb{X}_i$ is nonempty. If \mathcal{Z} is consistent, we set

$$J(\mathcal{Z}) = \bigcap_{i=1}^k J(Z_i).$$

Let us define a new set of curves $\beta^{\mathcal{Z}} = \{\beta_j^{\mathcal{Z}} \mid j = 1, \dots, n\}$ on the torus \mathcal{T} . First, assume that \mathcal{Z} has strictly fewer than n markings. Let j_i be the index corresponding to the vertical circle β_{j_i} just to the left of a marking $Z_i \in \mathcal{Z}$. We let $\beta_{j_i}^{\mathcal{Z}}$ be a circle encircling Z_i and intersecting β_{j_i} , as well as the α curve just below Z_i , in two points each; in other words, $\beta_{j_i}^{\mathcal{Z}}$ is obtained from β_{j_i} by handlesliding it over the vertical curve just to the right of Z_i . For those j that are not j_i for

any $Z_i \in \mathcal{Z}$, we let $\beta_j^{\mathcal{Z}}$ be a curve that is isotopic to β_j , intersects β_j in two points, and does not intersect any of the other beta curves.

In the special case when \mathcal{Z} has cardinality n , so that it contains one marking in each column, the definition above does not make sense, because we cannot do handleslides over all the markings. To fix this, we pick some $Z_0 \in \mathcal{Z}$, and let $\beta^{\mathcal{Z}}$ consist of curves isotopic to those in $\beta^{\mathcal{Z} \setminus \{Z_0\}}$.

For any consistent collection \mathcal{Z} , we denote

$$\mathbb{T}_{\beta}^{\mathcal{Z}} = \beta_1^{\mathcal{Z}} \times \cdots \times \beta_n^{\mathcal{Z}} \subset \text{Sym}^n(\mathcal{T}).$$

The fact that $\mathbf{s} \in J(\mathcal{Z})$ implies that there is a well-defined Floer chain complex $\mathfrak{A}^{-}(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta}^{\mathcal{Z}}, \mathbf{s})$, where the differentials take powers of the U_i 's according to \mathbf{s} , generalizing the constructions $\mathfrak{A}^{-}(G, \mathbf{s})$ in a natural manner. More precisely, $\mathfrak{A}^{-}(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta}^{\mathcal{Z}}, \mathbf{s})$ is generated over \mathcal{R} by $\mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}^{\mathcal{Z}}$, with differential given by

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}^{\mathcal{Z}}} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi) = 1\}} U_1^{E_{s_1}^1(\phi)} \cdots U_{\ell}^{E_{s_{\ell}}^{\ell}(\phi)} \mathbf{y},$$

where $\pi_2(\mathbf{x}, \mathbf{y})$ is the set of homology classes of Whitney disks from \mathbf{x} to \mathbf{y} , compare [19], and the functions E_s^i are as defined in Equation (2) or (3).

When we have two collections of markings $\mathcal{Z}, \mathcal{Z}'$ such that $\mathcal{Z} \cup \mathcal{Z}'$ is consistent, we require that $\beta_i^{\mathcal{Z}}$ and $\beta_i^{\mathcal{Z}'}$ intersect in exactly two points. As such, there is always a maximal degree intersection point $\Theta^{\mathcal{Z}, \mathcal{Z}'} \in \mathbb{T}_{\beta}^{\mathcal{Z}} \cap \mathbb{T}_{\beta}^{\mathcal{Z}'}$.

For each consistent collection of markings $\mathcal{Z} = \{Z_1, \dots, Z_m\}$, and each $\mathbf{s} \in J(\mathcal{Z})$, we define an m -dimensional hypercube of chain complexes

$$\mathcal{H}_{\mathbf{s}}^{\mathcal{Z}} = (C_{\mathbf{s}}^{\mathcal{Z}, \varepsilon}, D_{\mathbf{s}}^{\mathcal{Z}, \varepsilon})_{\varepsilon \in \mathbb{E}_m}$$

as follows. For $\varepsilon \in \mathbb{E}_m = \{0, 1\}^m$, we let

$$\mathcal{Z}^{\varepsilon} = \{Z_i \mid \varepsilon_i = 1\}$$

and set

$$C_{\mathbf{s}, * }^{\mathcal{Z}, \varepsilon} = \mathfrak{A}^{-}(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta}^{\mathcal{Z}^{\varepsilon}}, \mathbf{s}).$$

For simplicity, we denote $\Theta^{\mathcal{Z}^{\varepsilon}, \mathcal{Z}^{\varepsilon'}}$ by $\Theta_{\varepsilon, \varepsilon'}^{\mathcal{Z}}$. We write $\varepsilon \prec \varepsilon'$ whenever $\varepsilon, \varepsilon' \in \mathbb{E}_m$ are immediate successors, i.e. $\varepsilon < \varepsilon'$ and $\|\varepsilon' - \varepsilon\| = 1$. For a string of immediate successors $\varepsilon = \varepsilon^0 \prec \varepsilon^1 \prec \dots \prec \varepsilon^k = \varepsilon'$, we let

$$(5) \quad D_{\mathbf{s}}^{\mathcal{Z}, \varepsilon^0 \prec \varepsilon^1 \prec \dots \prec \varepsilon^k} : C_{*}^{\mathcal{Z}, \varepsilon} \rightarrow C_{*+k-1}^{\mathcal{Z}, \varepsilon'}$$

$$D_{\mathbf{s}}^{\mathcal{Z}, \varepsilon^0 \prec \varepsilon^1 \prec \dots \prec \varepsilon^k}(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}^{\mathcal{Z}^{\varepsilon^k}}} \sum_{\{\phi \in \pi_2(\mathbf{x}, \Theta_{\varepsilon^0, \varepsilon^1}^{\mathcal{Z}}, \dots, \Theta_{\varepsilon^{k-1}, \varepsilon^k}^{\mathcal{Z}}, \mathbf{y}) \mid \mu(\phi) = 1 - k\}} (\#\mathcal{M}(\phi)) \cdot U_1^{E_{s_1}^1(\phi)} \cdots U_{\ell}^{E_{s_{\ell}}^{\ell}(\phi)} \mathbf{y}$$

be the map defined by counting isolated pseudo-holomorphic polygons in the symmetric product $\text{Sym}^n(\mathcal{T})$. Here,

$$\pi_2(\mathbf{x}, \Theta_{\varepsilon^0, \varepsilon^1}^{\mathcal{Z}}, \dots, \Theta_{\varepsilon^{k-1}, \varepsilon^k}^{\mathcal{Z}}, \mathbf{y})$$

denotes the set of homotopy classes of polygons with edges on $\mathbb{T}_{\alpha}, \mathbb{T}_{\beta}^{\mathcal{Z}^{\varepsilon^0}}, \dots, \mathbb{T}_{\beta}^{\mathcal{Z}^{\varepsilon^k}}$, in this cyclic order, and with the specified vertices. The number $\mu(\phi) \in \mathbb{Z}$ is the Maslov index, and $\mathcal{M}(\phi)$ is the moduli space of holomorphic polygons in the class ϕ . The Maslov index has to be $1 - k$ for the expected dimension of the moduli space of polygons to be zero. This is because the conformal automorphism group of a disk with $k + 2$ marked points has dimension $(k + 2) - 3 = k - 1$. Note that this definition of μ is different from the one in [20, Section 4.2], where it denoted expected dimension.

In the special case $k = 0$, we need to divide $\mathcal{M}(\phi)$ by the action of \mathbb{R} by translations; the resulting $D_{\mathbf{s}}^{\mathcal{Z}, \varepsilon^0}$ is the usual differential ∂ .

Define

$$(6) \quad D_{\mathbf{s}}^{\mathcal{Z}, \varepsilon} : C_{*}^{\mathcal{Z}, \varepsilon^0} \rightarrow C_{*+k-1}^{\mathcal{Z}, \varepsilon^0 + \varepsilon}, \quad D_{\mathbf{s}}^{\mathcal{Z}, \varepsilon} = \sum_{\varepsilon^0 \prec \varepsilon^1 \prec \dots \prec \varepsilon^k = \varepsilon^0 + \varepsilon} D_{\mathbf{s}}^{\mathcal{Z}, \varepsilon^0 \prec \varepsilon^1 \prec \dots \prec \varepsilon^k}.$$

For simplicity, we write $D_{\mathbf{s}}^{\mathcal{Z}}$ for the longest map $D_{\mathbf{s}}^{\mathcal{Z}, (1, \dots, 1)}$ in the hypercube $\mathcal{H}_{\mathbf{s}}^{\mathcal{Z}}$. Note that $D_{\mathbf{s}}^{\mathcal{Z}, \varepsilon}$ is the same as the longest map $D_{\mathbf{s}}^{\mathcal{Z}(\varepsilon)}$ in a sub-hypercube of $\mathcal{H}_{\mathbf{s}}^{\mathcal{Z}}$.

Lemma 2.1. *For any consistent collection $\mathcal{Z} = \{Z_1, \dots, Z_m\}$ and $\mathbf{s} \in J(\mathcal{Z})$, the resulting $\mathcal{H}_{\mathbf{s}}^{\mathcal{Z}} = (C_{\mathbf{s}}^{\mathcal{Z}, \varepsilon}, D_{\mathbf{s}}^{\mathcal{Z}, \varepsilon})_{\mathbf{s} \in \mathbb{E}_m}$ is a hypercube of chain complexes.*

Compare [19, Lemma 9.7] and [20, Lemma 4.3].

Remark 2.2. For future reference, it is helpful to introduce a different notation for the maps (5) and (6). First, let us look at (5) in the case $k = m$, $\varepsilon^0 = (0, \dots, 0)$ and $\varepsilon^m = (1, \dots, 1)$. A string of immediate successors $\varepsilon^0 \prec \dots \prec \varepsilon^m$ is the same as a re-ordering $(Z_{\sigma(1)}, \dots, Z_{\sigma(m)})$ of $\mathcal{Z} = \{Z_1, \dots, Z_m\}$, according to the permutation σ in the symmetric group S_m such that

$$\mathcal{Z}^{\varepsilon^i} = \mathcal{Z}^{\varepsilon^{i-1}} \cup \{Z_{\sigma(i)}\}.$$

We then write:

$$D_{\mathbf{s}}^{(Z_{\sigma(1)}, \dots, Z_{\sigma(m)})} = D_{\mathbf{s}}^{\mathcal{Z}, \varepsilon^0 \prec \varepsilon^1 \prec \dots \prec \varepsilon^k}.$$

In particular, we let $D^{(\mathcal{Z})} = D^{(Z_1, \dots, Z_m)}$ be the map corresponding to the identity permutation. Further, we write $D_{\mathbf{s}}^{\mathcal{Z}}$ for the longest map $D_{\mathbf{s}}^{\mathcal{Z}, (1, \dots, 1)}$ in the hypercube $\mathcal{H}_{\mathbf{s}}^{\mathcal{Z}}$, that is,

$$(7) \quad D_{\mathbf{s}}^{\mathcal{Z}} = \sum_{\sigma \in S_m} D_{\mathbf{s}}^{(Z_{\sigma(1)}, \dots, Z_{\sigma(m)})}.$$

Observe that $D_{\mathbf{s}}^{\mathcal{Z}}$, unlike $D_{\mathbf{s}}^{(\mathcal{Z})}$, is independent of the ordering of \mathcal{Z} . Observe also that an arbitrary map $D_{\mathbf{s}}^{\mathcal{Z}, \varepsilon}$ from $\mathcal{H}_{\mathbf{s}}^{\mathcal{Z}}$ is the same as the longest map $D_{\mathbf{s}}^{\mathcal{Z}(\varepsilon)}$ in a sub-hypercube of $\mathcal{H}_{\mathbf{s}}^{\mathcal{Z}}$. In this notation, the result of Lemma 2.1 can be written as:

$$\sum_{\mathcal{Z}' \subseteq \mathcal{Z}} D_{\mathbf{s}}^{\mathcal{Z} \setminus \mathcal{Z}'} \circ D_{\mathbf{s}}^{\mathcal{Z}} = 0.$$

2.4. Sublinks and projection maps. Suppose that $M \subseteq L$ is a sublink. We choose an orientation on M (possibly different from the one induced from \vec{L}), and denote the corresponding oriented link by \vec{M} . We let $I_+(\vec{L}, \vec{M})$ (resp. $I_-(\vec{L}, \vec{M})$) to be the set of indices i such that the component L_i is in M and its orientation induced from \vec{L} is the same as (resp. opposite to) the one induced from \vec{M} .

For $i \in \{1, \dots, \ell\}$, we define a map $p_i^{\vec{M}} : \overline{\mathbb{H}}(L)_i \rightarrow \overline{\mathbb{H}}(L)_i$ by

$$p_i^{\vec{M}}(s) = \begin{cases} +\infty & \text{if } i \in I_+(\vec{L}, \vec{M}), \\ -\infty & \text{if } i \in I_-(\vec{L}, \vec{M}), \\ s & \text{otherwise.} \end{cases}$$

Then, for $\mathbf{s} = (s_1, \dots, s_{\ell}) \in \overline{\mathbb{H}}(L)$, we set

$$p^{\vec{M}}(\mathbf{s}) = (p_1^{\vec{M}}(s_1), \dots, p_{\ell}^{\vec{M}}(s_{\ell})).$$

and define a projection map

$$\mathcal{P}_{\mathbf{s}}^{\vec{M}} : \mathfrak{A}^-(G, \mathbf{s}) \rightarrow \mathfrak{A}^-(G, p^{\vec{M}}(\mathbf{s}))$$

by

$$(8) \quad \mathcal{P}_{\mathbf{s}}^{\vec{M}} \mathbf{x} = \prod_{i \in I_+(\vec{L}, \vec{M})} U_i^{\max(A_i(\mathbf{x}) - s_i, 0)} \cdot \prod_{i \in I_-(\vec{L}, \vec{M})} U_i^{\max(s_i - A_i(\mathbf{x}), 0)} \cdot \mathbf{x}.$$

provided the exponents are finite, that is, $s_i \neq -\infty$ for all $i \in I_+(\vec{L}, \vec{M})$, and $s_i \neq +\infty$ for all $i \in I_-(\vec{L}, \vec{M})$. These conditions will always be satisfied when we consider projection maps in this paper.

Equations (2) and (3) imply that $\mathcal{P}_{\mathbf{s}}^{\vec{M}}$ is a chain map.

Let N be the complement of the sublink M in L . We define a reduction map

$$(9) \quad \psi^{\vec{M}} : \overline{\mathbb{H}}(L) \longrightarrow \overline{\mathbb{H}}(N)$$

as follows. The map $\psi^{\vec{M}}$ depends only on the summands $\overline{\mathbb{H}}(L)_i$ of $\overline{\mathbb{H}}(L)$ corresponding to $L_i \subseteq N$. Each of these L_i 's appears in N with a (possibly different) index j_i , so there is a corresponding summand $\overline{\mathbb{H}}(N)_{j_i}$ of $\overline{\mathbb{H}}(N)$. We then set

$$\psi_i^{\vec{M}} : \overline{\mathbb{H}}(L)_i \rightarrow \overline{\mathbb{H}}(N)_{j_i}, \quad s_i \rightarrow s_i - \frac{\text{lk}(L_i, \vec{M})}{2},$$

where L_i is considered with the orientation induced from L , while \vec{M} is with its own orientation. We then define $\psi^{\vec{M}}$ to be the direct sum of the maps $\psi_i^{\vec{M}}$, pre-composed with projection to the relevant factors. Note that $\psi^{\vec{M}} = \psi^{\vec{M}} \circ p^{\vec{M}}$.

2.5. Destabilized complexes. Let $\mathcal{Z} = \{Z_1, \dots, Z_m\}$ be a consistent set of markings, and pick $\mathbf{s} \in J(\mathcal{Z})$. In Section 2.3 we introduced a Floer complex $\mathfrak{A}^-(\mathbb{T}_\alpha, \mathbb{T}_\beta^{\mathcal{Z}}, \mathbf{s})$. Our next goal is to describe this complex combinatorially.

Let $L(\mathcal{Z}) \subseteq L$ be the sublink consisting of those components L_i such that at least one of the markings on L_i is in \mathcal{Z} . We orient $L(\mathcal{Z})$ as $\vec{L}(\mathcal{Z})$, such that a component L_i is given the orientation coming from \vec{L} when $\mathcal{Z} \cap \mathbb{O}_i \neq \emptyset$, and is given the opposite orientation when $\mathcal{Z} \cap \mathbb{X}_i \neq \emptyset$. Moreover, we let $L((\mathcal{Z})) \subseteq L(\mathcal{Z})$ be the sublink consisting of the components L_i such that either all X or all O markings on L_i are in \mathcal{Z} .

Consider the grid diagram $G^{\mathcal{Z}}$ obtained from G by deleting the rows and columns containing the markings in \mathcal{Z} , and further deleting the markings in \mathbb{X}_i when $\mathcal{Z} \cap \mathbb{O}_i \neq \emptyset$, and deleting the markings in \mathbb{O}_i when $\mathcal{Z} \cap \mathbb{X}_i \neq \emptyset$. Note that $G^{\mathcal{Z}}$ is not quite an ordinary grid diagram in the sense of [10, 11], because it may have rows and columns with only one marking. Nevertheless, it is a multi-pointed Heegaard diagram for the link $L - L(\mathcal{Z})$ (with the orientation induced from \vec{L}), in the sense of [9, Section 4.1]. We can still define complexes $\mathfrak{A}^-(G^{\mathcal{Z}}, \mathbf{s})$ for $\mathbf{s} \in J(\mathcal{Z})$, and holomorphic disks of index one in the symmetric product are still in one-to-one correspondence with rectangles on $G^{\mathcal{Z}}$.

For $Z \in \mathcal{Z}$, let $j(Z) \in \{1, \dots, \ell\}$ be such that $L_{j(Z)}$ is the component of L containing Z , and let $j(Z)'$ correspond to the component $L_{j(Z)'}$ containing the markings in the row exactly under the row through Z . If Z has less than n markings, we define a complex

$$\mathcal{K}(Z) = \bigotimes_{Z \in \mathcal{Z}} \left(\mathcal{R} \xrightarrow{U_{j(Z)} - U_{j(Z)'}} \mathcal{R} \right).$$

If \mathcal{Z} has one marking in each column of G , we pick $Z_0 \in \mathcal{Z}$ as in Section 2.3 and set

$$\mathcal{K}(\mathcal{Z}) = \bigotimes_{Z \in \mathcal{Z}, Z \neq Z_0} \left(\mathcal{R} \xrightarrow{U_{j(Z)} - U_{j(Z)'}} \mathcal{R} \right).$$

Then, for a suitable choice of almost complex structure J on the symmetric product $\text{Sym}^n(T)$, we have an isomorphism

$$(10) \quad \Psi_{\mathbf{s}}^{\mathcal{Z}} : \mathfrak{A}^-(\mathbb{T}_\alpha, \mathbb{T}_\beta^{\mathcal{Z}}, \mathbf{s}) \rightarrow \mathfrak{A}^-(G^{\mathcal{Z}}, \psi^{\vec{L}(\mathcal{Z})}(\mathbf{s}))[[\{U_i\}_{L_i \subseteq L((\mathcal{Z}))}]] \otimes_{\mathcal{R}} \mathcal{K}(\mathcal{Z}).$$

Here, the square brackets mean adjoining the variables U_i , in order to make the respective group into a module over $\mathcal{R} = \mathbb{F}[[U_1, \dots, U_\ell]]$. The map $\psi^{\vec{L}(\mathcal{Z})} : \overline{\mathbb{H}}(L) \longrightarrow \overline{\mathbb{H}}(L - L(\mathcal{Z}))$ is as in Equation (9).

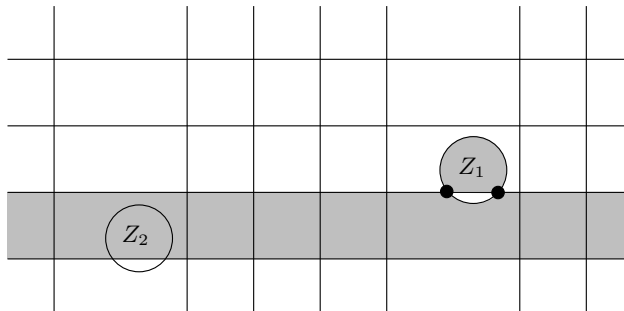


FIGURE 2. **The complex $\mathcal{K}(\mathcal{Z})$.** The figure shows part of a grid diagram with some arcs on the α and $\beta^{\mathcal{Z}}$ curves drawn. There are two intersection points (marked as bullets) between the alpha curve below the marking $Z_1 \in \mathcal{Z}$, and the corresponding beta curve. There are two differentials going from the left to the right generator: a bigon containing Z_1 and an annulus containing Z_2 , both drawn shaded in the diagram. This produces one of the factors in the definition of the complex $\mathcal{K}(\mathcal{Z})$.

To get an understanding of Equation (10), note that there is a clear one-to-one correspondence between the generators of each side. The differentials correspond likewise: on the left hand side, apart from rectangles, we also have for example annuli as the one shown in Figure 2, which give rise to the factors in the complex $\mathcal{K}(\mathcal{Z})$.

For a generic almost complex structure J on the symmetric product, we still have a map $\Psi_{\mathbf{s}}^{\mathcal{Z}}$, but this is a chain homotopy equivalence rather than an isomorphism.

A particular instance of the discussion above appears when we have a sublink $M \subseteq L$, with some orientation \vec{M} . We then set

$$\mathcal{Z}(\vec{M}) = \bigcup_{i \in I_+(\vec{L}, \vec{M})} \mathbb{O}_i \cup \bigcup_{i \in I_-(\vec{L}, \vec{M})} \mathbb{X}_i.$$

Note that $\vec{L}(\mathcal{Z}(\vec{M})) = \vec{M}$. In this setting, the destabilized grid diagram $G^{L-M} = G^{\mathcal{Z}(\vec{M})}$ is obtained from G by eliminating all rows and columns on which M is supported. It is an ordinary toroidal grid diagram as in [11], representing the link $L - M$.

For simplicity, we denote

$$\mathcal{K}(\vec{M}) = \mathcal{K}(\mathcal{Z}(\vec{M})), \quad J(\vec{M}) = J(\mathcal{Z}(\vec{M})),$$

and, for $\mathbf{s} \in J(\vec{M})$,

$$(11) \quad \Psi_{\mathbf{s}}^{\vec{M}} = \Psi_{\mathbf{s}}^{\mathcal{Z}(\vec{M})} : \mathfrak{A}^-(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta}^{\mathcal{Z}(\vec{M})}, \mathbf{s}) \rightarrow \mathfrak{A}^-(G^{L-M}, \psi^{\vec{M}}(\mathbf{s}))[[\{U_i\}_{L_i \subseteq M}]] \otimes_{\mathcal{R}} \mathcal{K}(\vec{M}).$$

2.6. Destabilization of a sublink. Let $M \subseteq L$ be a sublink, endowed with an arbitrary orientation \vec{M} . For any $\mathbf{s} \in J(\vec{M}) = J(\mathcal{Z}(\vec{M}))$, we construct a hyperbox of chain complexes $\mathcal{H}_{\mathbf{s}}^{\vec{L}, \vec{M}}$, as follows. Order the components of M according to their ordering as components of L :

$$M = L_{i_1} \cup \cdots \cup L_{i_m}, \quad i_1 < \cdots < i_m.$$

For $j = 1, \dots, m$, let us denote $M_j = L_{i_j}$ for simplicity, and equip M_j with the orientation \vec{M}_j induced from \vec{M} . Then $\mathcal{Z}(\vec{M}_j)$ is either \mathbb{O}_{i_j} or \mathbb{X}_{i_j} . In either case, we have an ordering of its elements, so we can write

$$\mathcal{Z}(\vec{M}_j) = \{Z_1^{\vec{M}_j}, \dots, Z_{d_j}^{\vec{M}_j}\},$$

where d_j is the cardinality of $\mathcal{Z}(\vec{M}_j)$.

The hyperbox $\mathcal{H}_s^{\vec{L}, \vec{M}}$ is m -dimensional, of size $\mathbf{d}^M = (d_1, \dots, d_m)$. For each multi-index $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \mathbb{E}(\mathbf{d}^M)$, we let $\mathcal{Z}(\vec{M})^\varepsilon \subseteq \mathcal{Z}(\vec{M})$ be the collection of markings

$$\mathcal{Z}(\vec{M})^\varepsilon = \bigcup_{j=1}^m \{Z_1^{\vec{M}_j}, \dots, Z_{\varepsilon_j}^{\vec{M}_j}\}.$$

We then let

$$\beta^\varepsilon = \beta^{\mathcal{Z}(\vec{M})^\varepsilon}$$

be the collection of beta curves destabilized at the points of $\mathcal{Z}(\vec{M})^\varepsilon$. For each ε , consider the Heegaard diagram $\mathcal{H}_\varepsilon^{\vec{L}, \vec{M}} = (\mathcal{T}, \alpha, \beta^\varepsilon)$, with the z basepoints being the markings in \mathbb{X}_i for $L_i \not\subseteq M$, and the w basepoints being the markings in $\mathcal{Z}(\vec{M})$, together with those in \mathbb{O}_i for $L_i \not\subseteq M$. This diagram represents the link $\vec{L} - M$.

At each vertex $\varepsilon \in \mathbb{E}(\mathbf{d}^M)$ we place the Floer complex

$$C_s^{\vec{L}, \vec{M}, \varepsilon} = \mathfrak{A}^-(\mathbb{T}_\alpha, \mathbb{T}_\beta^{\mathcal{Z}(\vec{M})^\varepsilon}, \mathbf{s}),$$

and along the faces we have linear maps

$$D_s^{\vec{L}, \vec{M}, \varepsilon} = D_s^{\mathcal{Z}(\vec{M})^\varepsilon, \varepsilon}, \quad \varepsilon \in \mathbb{E}_m \subseteq \mathbb{E}(\mathbf{d}^M),$$

where $D_s^{\mathcal{Z}(\vec{M})^\varepsilon, \varepsilon}$ are as in (6) above.

We compress the hyperbox of Floer complexes associated to $\mathcal{H}_s^{\vec{L}, \vec{M}}$, cf. Section 2.1, and define

$$(12) \quad \hat{D}_s^{\vec{M}} : \mathfrak{A}^-(G, \mathbf{s}) \rightarrow \mathfrak{A}^-(\mathbb{T}_\alpha, \mathbb{T}_\beta^{\mathcal{Z}(\vec{M})}, \mathbf{s}).$$

to be the longest diagonal map in the compressed hypercube $\hat{\mathcal{H}}_s^{\vec{L}, \vec{M}}$.

For example, when $M = L_i$ is a single component, the map $\hat{D}_s^{\vec{M}}$ is a composition of the triangle maps corresponding to handleslides over the basepoints in $\mathcal{Z}(\vec{M})$, in the given order. When M has several components, it is a composition of more complicated polygon maps, corresponding to chain homotopies (of higher order) between compositions of the handleslide maps.

Note that for each $\mathbf{s} \in \mathbb{H}(L)$, we have $p^{\vec{M}}(\mathbf{s}) \in J(\mathcal{Z}(\vec{M}))$ by definition. Therefore, by composing the maps (8), (12) and (11) (the latter two taken with respect to $p^{\vec{M}}(\mathbf{s})$ rather than \mathbf{s}), we obtain a map

$$\begin{aligned} \Phi_s^{\vec{M}} : \mathfrak{A}^-(G, \mathbf{s}) &\longrightarrow \mathfrak{A}^-(G^{L-M}, \psi^{\vec{M}}(\mathbf{s}))[[\{U_i\}_{L_i \subseteq M}]] \otimes_{\mathcal{R}} \mathcal{K}(\vec{M}), \\ \Phi_s^{\vec{M}} &= \Psi_{p^{\vec{M}}(\mathbf{s})}^{\vec{M}} \circ \hat{D}_{p^{\vec{M}}(\mathbf{s})}^{\vec{M}} \circ \mathcal{P}_s^{\vec{M}}, \end{aligned}$$

defined for any $\mathbf{s} \in \overline{\mathbb{H}}(L)$.

For simplicity, let us denote

$$(13) \quad D_s^{\vec{M}} = \Psi_s^{\vec{M}} \circ \hat{D}_s^{\vec{M}}.$$

One can easily check the following:

Lemma 2.3. *Let $M_1, M_2 \subseteq L$ be two disjoint sublinks, with orientations \vec{M}_1 and \vec{M}_2 . For any $\mathbf{s} \in J(\vec{M}_1)$, we have:*

$$(14) \quad \mathcal{P}_{\psi^{\vec{M}_1}(\mathbf{s})}^{\vec{M}_2} \circ D_s^{\vec{M}_1} = D_{p^{\vec{M}_2}(\mathbf{s})}^{\vec{M}_1} \circ \mathcal{P}_s^{\vec{M}_2}.$$

For any \vec{M} and $\mathbf{s} \in \overline{\mathbb{H}}(L)$, by applying Lemma 2.3 and the properties of compression from Section 2.1, we get:

$$(15) \quad \sum_{\vec{M}_1 \amalg \vec{M}_2 = \vec{M}} \Phi_{\psi^{\vec{M}_1}(\mathbf{s})}^{\vec{M}_2} \circ \Phi_{\mathbf{s}}^{\vec{M}_1} = 0,$$

where \vec{M}_1 and \vec{M}_2 are considered with the orientations induced from \vec{M} .

2.7. The surgery theorem. Let us fix a framing Λ for the link \vec{L} . For a component L_i of L , we let Λ_i be its induced framing, thought of as an element in $H_1(Y - L)$. This last group can be identified with \mathbb{Z}^ℓ using the basis of oriented meridians for the components. Under this identification, for $i \neq j$, the j^{th} component of the vector Λ_i is the linking number between L_i and L_j . The i^{th} component of Λ_i is its framing coefficient.

Given a sublink $N \subseteq L$, we let $\Omega(N)$ be the set of all possible orientations on N . For $\vec{N} \in \Omega(N)$, we let

$$\Lambda_{\vec{L}, \vec{N}} = \sum_{i \in I_-(\vec{L}, \vec{N})} \Lambda_i \in \mathbb{Z}^\ell.$$

We consider the \mathcal{R} -module

$$\mathcal{C}^-(G, \Lambda) = \bigoplus_{M \subseteq L} \prod_{\mathbf{s} \in \mathbb{H}(L)} \left(\mathfrak{A}^-(G^{L-M}, \psi^M(\mathbf{s}))[\{\{U_i\}_{L_i \subseteq M}\}] \right) \otimes_{\mathcal{R}} \mathcal{K}(M),$$

where ψ^M simply means $\psi^{\vec{M}}$ with \vec{M} being the orientation induced from the one on \vec{L} .

We equip $\mathcal{C}^-(G, \Lambda)$ with a boundary operator \mathcal{D}^- as follows.

For $\mathbf{s} \in \mathbb{H}(L)$ and $\mathbf{x} \in \left(\mathfrak{A}^-(G^{L-M}, \psi^M(\mathbf{s}))[\{\{U_i\}_{L_i \subseteq M}\}] \right) \otimes_{\mathcal{R}} \mathcal{K}(M)$, we set

$$\begin{aligned} \mathcal{D}^-(\mathbf{s}, \mathbf{x}) &= \sum_{N \subseteq L-M} \sum_{\vec{N} \in \Omega(N)} (\mathbf{s} + \Lambda_{\vec{L}, \vec{N}}, \Phi_{\mathbf{s}}^{\vec{N}}(\mathbf{x})) \\ &\in \bigoplus_{N \subseteq L-M} \bigoplus_{\vec{N} \in \Omega(N)} \left(\mathfrak{A}^-(G^{L-M-N}, \psi^{M \cup \vec{N}}(\mathbf{s}))[\{\{U_i\}_{L_i \subseteq M \cup N}\}] \right) \otimes_{\mathcal{R}} \mathcal{K}(M \cup N) \subset \mathcal{C}^-(G, \Lambda). \end{aligned}$$

According to Equation (15), $\mathcal{C}^-(G, \Lambda)$ is a chain complex.

Let $H(L, \Lambda) \subseteq \mathbb{Z}^\ell$ be the lattice generated by $\Lambda_i, i = 1, \dots, \ell$. The complex $\mathcal{C}^-(G, \Lambda)$ splits into a direct product of complexes $\mathcal{C}^-(G, \Lambda, \mathbf{u})$, according to equivalence classes $\mathbf{u} \in \mathbb{H}(L) : H(L, \Lambda)$. Further, the space of equivalence classes $\mathbb{H}(L) : H(L, \Lambda)$ can be canonically identified with the space of Spin^c structures on the three-manifold $S_\Lambda^3(L)$ obtained from S^3 by surgery along the framed link (L, Λ) .

Given a Spin^c structure \mathbf{u} on $Y_\Lambda(L)$, we set

$$\mathfrak{d}(\mathbf{u}) = \gcd_{\xi \in H_2(Y_\Lambda(L); \mathbb{Z})} \langle c_1(\mathbf{u}), \xi \rangle,$$

where $c_1(\mathbf{u})$ is the first Chern class of the Spin^c structure. Thinking of \mathbf{u} as an equivalence class in $\mathbb{H}(L)$, we can find a function $\nu : \mathbf{u} \rightarrow \mathbb{Z}/\mathfrak{d}(\mathbf{u})\mathbb{Z}$ with the property that

$$\nu(\mathbf{s} + \Lambda_i) \equiv \nu(\mathbf{s}) + 2s_i,$$

for any $i = 1, \dots, \ell$ and $\mathbf{s} = (s_1, \dots, s_\ell) \in \mathbf{u}$. The function ν is unique up to the addition of a constant. For $\mathbf{s} \in \mathbb{H}(L)$ and $\mathbf{x} \in \left(\mathfrak{A}^-(G^{L-M}, \psi^M(\mathbf{s}))[\{\{U_i\}_{L_i \subseteq M}\}] \right) \otimes_{\mathcal{R}} \mathcal{K}(M)$, let

$$\mu(\mathbf{s}, \mathbf{x}) = \mu_{\mathbf{s}}^M(\mathbf{x}) + \nu(\mathbf{s}) - |M|,$$

where $\mu_{\mathbf{s}}^M = \mu_{\psi^M(\mathbf{s})}$ is as in Equation (4), and $|M|$ denotes the number of components of M . Then μ gives a relative $\mathbb{Z}/\mathfrak{d}(\mathbf{u})\mathbb{Z}$ -grading on the complex $\mathcal{C}^-(G, \Lambda, \mathbf{u})$. The differential \mathcal{D}^- decreases μ by one modulo $\mathfrak{d}(\mathbf{u})$.

Theorem 2.4. *Fix a grid diagram G for an oriented, ℓ -component link \vec{L} in S^3 , and fix a framing Λ of L . Then, for every $\mathbf{u} \in \text{Spin}^c(S_\Lambda^3(L))$, we have an isomorphism of relatively graded $\mathbb{F}[[U]]$ -modules:*

$$(16) \quad H_*(\mathcal{C}^-(G, \Lambda, \mathbf{u}), \mathcal{D}^-) \cong \mathbf{HF}_*^-(S_\Lambda^3(L), \mathbf{u}) \otimes_{\mathbb{F}} H_*(T^{n-\ell}),$$

where n is the grid number of G .

Observe that the left hand side of Equation (16) is a priori a module over $\mathcal{R} = \mathbb{F}[[U_1, \dots, U_\ell]]$. Part of the claim of Theorem 2.4 is that all the U_i 's act the same way, so that we have a $\mathbb{F}[[U]]$ -module.

2.8. Maps induced by surgery. Let $L' \subseteq L$ be a sublink, with the orientation \vec{L}' induced from \vec{L} .

Denote by $W = W_\Lambda(L', L)$ the cobordism from $S_{\Lambda|_{L'}}^3(L')$ to $S_\Lambda^3(L)$ given by surgery on $L - L'$, framed with the restriction of Λ . Let $H(L, \Lambda|_{L'}) \subseteq \mathbb{Z}^\ell$ be the sublattice generated by the framings Λ_i , for $L_i \subseteq L'$. There is an identification:

$$\text{Spin}^c(W_\Lambda(L', L)) \cong \mathbb{H}(L) : H(L, \Lambda|_{L'})$$

under which the natural projection

$$\pi^{L, L'} : (\mathbb{H}(L) : H(L, \Lambda|_{L'})) \longrightarrow (\mathbb{H}(L) : H(L, \Lambda))$$

corresponds to restricting the Spin^c structures to $S_\Lambda^3(L)$, while the map

$$\psi^{L-L'} : (\mathbb{H}(L) : H(L, \Lambda|_{L'})) \rightarrow (\mathbb{H}(L') : H(L', \Lambda|_{L'}))$$

corresponds to restricting them to $S_{\Lambda|_{L'}}^3(L')$.

Observe that, for every equivalence class $\mathbf{t} \in \mathbb{H}(L) : H(L, \Lambda|_{L'})$,

$$\mathcal{C}^-(G, \Lambda)^{L', \mathbf{t}} = \bigoplus_{L-L' \subseteq M \subseteq L} \prod_{\{\mathbf{s} \in \mathbb{H}(L) \mid [\mathbf{s}] = \mathbf{t}\}} \left(\mathfrak{A}^-(G^{L-M}, \psi^M(\mathbf{s}))[\{\{U_i\}_{L_i \subseteq M}\}] \right) \otimes_{\mathcal{R}} \mathcal{K}(M),$$

is a subcomplex of $\mathcal{C}^-(G, \Lambda, \pi^{L, L'}(\mathbf{t})) \subseteq \mathcal{C}^-(G, \Lambda)$. This subcomplex is chain homotopy equivalent to

$$\mathcal{C}^-(G^{L'}, \Lambda^{L'}, \psi^{L-L'}(\mathbf{t})) \otimes H^*(T^{(n-n')-(\ell-\ell')}),$$

where

$$\mathcal{C}^-(G^{L'}, \Lambda^{L'}, \psi^{L-L'}(\mathbf{t})) = \bigoplus_{M' \subseteq L'} \prod_{\{\mathbf{s}' \in \mathbb{H}(L') \mid [\mathbf{s}'] = \psi^{L-L'}(\mathbf{t})\}} \left(\mathfrak{A}^-(G^{L'-M'}, \psi^{M'}(\mathbf{s}'))[\{\{U_i\}_{L_i \subseteq M'}\}] \right) \otimes_{\mathcal{R}'} \mathcal{K}(M')$$

and \mathcal{R}' is the power series ring in the U_i variables from L' . The chain homotopy equivalence is induced by taking M to $M' = M - (L - L')$, \mathbf{s} to $\mathbf{s}' = \psi^{\vec{L}-\vec{L}'}(\mathbf{s})$, and getting rid of the U_i variables from $L - L'$ via relations coming from $\mathcal{K}(L - L')$.

Theorem 2.4 implies that the homology of $\mathcal{C}^-(G, \Lambda)^{L', \mathbf{t}}$ is isomorphic to

$$\mathbf{HF}_*^-(S_{\Lambda'}^3(L'), \mathbf{t}|_{S_{\Lambda'}^3(L')}) \otimes H_*(T^{n-\ell}).$$

In [21], the authors associated a map $F_{W, \mathbf{s}}^-$ to any cobordism W between connected three-manifolds, and Spin^c structure \mathbf{t} on that cobordism. In the case when the cobordism W consists only of two-handles (i.e. is given by integral surgery on a link), the following theorem gives a way of understanding the map $F_{W, \mathbf{t}}^-$ in terms of grid diagrams:

Theorem 2.5. *Let $\vec{L} \subset S^3$ be an ℓ -component link, $L' \subseteq L$ a sublink, G a grid diagram for \vec{L} of grid number n , and Λ a framing of L . Set $W = W_\Lambda(L', L)$. Then, for any $\mathfrak{t} \in \text{Spin}^c(W) \cong \mathbb{H}(L) : H(L, \Lambda|_{L'})$, the following diagram commutes:*

$$\begin{array}{ccc} H_*(\mathcal{C}^-(G, \Lambda)^{L', \mathfrak{t}}) & \longrightarrow & H_*(\mathcal{C}^-(G, \Lambda, \pi^{L, L'}(\mathfrak{t}))) \\ \cong \downarrow & & \downarrow \cong \\ \mathbf{HF}_*^-(S_\Lambda^3(L'), \mathfrak{t}|_{S_\Lambda^3(L')}) \otimes H_*(T^{n-\ell}) & \xrightarrow{F_{W, \mathfrak{t}}^- \otimes \text{Id}} & \mathbf{HF}_*^-(S_\Lambda^3(L), \mathfrak{t}|_{S_\Lambda^3(L)}) \otimes H_*(T^{n-\ell}). \end{array}$$

Here, the top horizontal map is induced from the inclusion of chain complexes, while the two vertical isomorphisms are the ones from Theorem 2.4.

2.9. Other versions. The chain complex $\mathcal{C}^-(G, \Lambda, \mathfrak{u})$ was constructed so that the version of Heegaard Floer homology appearing in Theorem 2.4 is \mathbf{HF}^- . We now explain how one can construct similar chain complexes $\hat{\mathcal{C}}(G, \Lambda, \mathfrak{u})$, $\mathcal{C}^+(G, \Lambda, \mathfrak{u})$ and $\mathcal{C}^\infty(G, \Lambda, \mathfrak{u})$, corresponding to the theories \widehat{HF} , HF^+ and \mathbf{HF}^∞ .

The chain complex $\hat{\mathcal{C}}(G, \Lambda, \mathfrak{u})$ is simply obtained from $\mathcal{C}^-(G, \Lambda)$ by setting one of the variables U_i equal to zero. Its homology computes $\widehat{HF}(S_\Lambda^3(L), \mathfrak{u}) \otimes H^*(T^{n-\ell})$.

The chain complex $\mathcal{C}^\infty(G, \Lambda, \mathfrak{u})$ is obtained from $\mathcal{C}^-(G, \Lambda, \mathfrak{u})$ by inverting all the U_i variables. It is a vector space over the field of semi-infinite Laurent polynomials

$$\mathcal{R}^\infty = \mathbb{F}[[U_1, \dots, U_\ell; U_1^{-1}, \dots, U_\ell^{-1}]] = (U_1, \dots, U_\ell)^{-1} \mathcal{R}.$$

In other words, \mathcal{R}^∞ consists of those power series in U_i 's that are sums of monomials with degrees bounded from below.

Note that $\mathcal{C}^-(G, \Lambda, \mathfrak{u})$ is a subcomplex of $\mathcal{C}^\infty(G, \Lambda, \mathfrak{u})$. We denote the respective quotient complex by $\mathcal{C}^+(G, \Lambda, \mathfrak{u})$. Theorems 2.4 and 2.5 admit the following extension:

Theorem 2.6. *Fix a grid diagram G (of grid number n) for an oriented, ℓ -component link \vec{L} in S^3 , and fix a framing Λ of L . Choose also an ordering of the components of \vec{L} , as well as of the O and X markings on the grid G . Set $V = H_*(T^{n-\ell})$. Then, for every $\mathfrak{u} \in \text{Spin}^c(S_\Lambda^3(L)) \cong \mathbb{H}(L) : H(L, \Lambda)$, there are vertical isomorphisms and horizontal long exact sequences making the following diagram commute:*

$$\begin{array}{ccccccc} \dots \rightarrow H_*(\mathcal{C}^-(G, \Lambda, \mathfrak{u})) & \longrightarrow & H_*(\mathcal{C}^\infty(G, \Lambda, \mathfrak{u})) & \longrightarrow & H_*(\mathcal{C}^+(G, \Lambda, \mathfrak{u})) & \rightarrow \dots \\ & & \downarrow \cong & & \downarrow \cong & \\ \dots \rightarrow \mathbf{HF}_*^-(S_\Lambda^3(L), \mathfrak{u}) \otimes V & \longrightarrow & \mathbf{HF}_*^\infty(S_\Lambda^3(L), \mathfrak{u}) \otimes V & \longrightarrow & HF_*^+(S_\Lambda^3(L), \mathfrak{u}) \otimes V & \rightarrow \dots \end{array}$$

Furthermore, the maps in these diagrams behave naturally with respect to cobordisms, in the sense that there are commutative diagrams analogous to those in Theorem 2.5, involving the cobordism maps $F_{W, \mathfrak{t}}^-, F_{W, \mathfrak{t}}^\infty, F_{W, \mathfrak{t}}^+$.

2.10. Mixed invariants of closed four-manifolds. We recall the definition of the closed four-manifold invariant from [21]. Let X be a closed, oriented four-manifold with $b_2^+(X) \geq 2$. By puncturing X in two points we obtain a cobordism W from S^3 to S^3 . We can cut W along a three-manifold N so as to obtain two cobordisms W_1, W_2 with $b_2^+(W_i) > 0$; further, the manifold N can be chosen such that $\delta H^1(N; \mathbb{Z}) \subset H^2(W; \mathbb{Z})$ is trivial. (If this is the case, N is called an *admissible cut*.) Let \mathfrak{t} be a Spin^c structure on X and $\mathfrak{t}_1, \mathfrak{t}_2$ its restrictions to W_1, W_2 . In this situation, the cobordism maps

$$F_{W_1, \mathfrak{t}_1}^- : \mathbf{HF}^-(S^3) \rightarrow \mathbf{HF}^-(N, \mathfrak{t}|_N)$$

and

$$F_{W_2, \mathfrak{t}_2}^+ : HF^+(N, \mathfrak{t}|_N) \rightarrow HF^+(S^3)$$

factor through $HF_{\text{red}}(N, \mathfrak{t}|_N)$, where

$$HF_{\text{red}} = \text{Coker}(\mathbf{HF}^\infty \rightarrow HF^+) \cong \text{Ker}(\mathbf{HF}^- \rightarrow \mathbf{HF}^\infty).$$

By composing them we obtain the mixed map

$$F_{W, \mathfrak{t}}^{\text{mix}} : \mathbf{HF}^-(S^3) \rightarrow HF^+(S^3),$$

which changes degree by the quantity

$$d(\mathfrak{t}) = \frac{c_1(\mathfrak{t})^2 - 2\chi(X) - 3\sigma(X)}{4}.$$

Let Θ_- be the maximal degree generator in $\mathbf{HF}^-(S^3)$. Clearly the map $F_{W, \mathfrak{t}}^{\text{mix}}$ can be nonzero only when $d(\mathfrak{t})$ is even and nonnegative. If this is the case, the value

$$(17) \quad \Phi_{X, \mathfrak{t}} = U^{d(\mathfrak{t})/2} \cdot F_{W, \mathfrak{t}}^{\text{mix}}(\Theta_-) \in HF_0^+(S^3) \cong \mathbb{F}$$

is an invariant of the four-manifold X and the Spin^c structure \mathfrak{t} . It is conjecturally the same as the Seiberg-Witten invariant.

Definition 2.7. *Let X be a closed, oriented four-manifold with $b_2^+(X) \geq 2$. A link presentation for X consists of a link $L \subset S^3$, a decomposition of L as a disjoint union*

$$L = L_1 \amalg L_2 \amalg L_3 \amalg L_4,$$

and a framing Λ for L (with restrictions Λ_i to $L_i, i = 1, \dots, 4$) with the following properties:

- $S_{\Lambda_1}^3(L_1)$ is a connected sum of m copies of $S^1 \times S^2$, for some $m \geq 0$. We denote by W_1 the cobordism from S^3 to $\#^m(S^1 \times S^2)$ given by m one-handle attachments;
- $S_{\Lambda_1 \cup \Lambda_2 \cup \Lambda_3}^3(L_1 \cup L_2 \cup L_3)$ is a connected sum of m' copies of $S^1 \times S^2$, for some $m' \geq 0$. We denote by W_4 the cobordism from $\#^{m'}(S^1 \times S^2)$ to S^3 given by m' three-handle attachments;
- If we denote by W_2 resp. W_3 the cobordisms from $S_{\Lambda_1}^3(L_1)$ to $S_{\Lambda_1 \cup \Lambda_2}^3(L_1 \cup L_2)$, resp. from $S_{\Lambda_1 \cup \Lambda_2}^3(L_1 \cup L_2)$ to $S_{\Lambda_1 \cup \Lambda_2 \cup \Lambda_3}^3(L_1 \cup L_2 \cup L_3)$, given by surgery on L_2 resp. L_3 (i.e. consisting of two-handle additions), then

$$W = W_1 \cup W_2 \cup W_3 \cup W_4$$

is the cobordism from S^3 to S^3 obtained from X by deleting two copies of B^4 ;

- The manifold $N = S_{\Lambda_1 \cup \Lambda_2}^3(L_1 \cup L_2)$ is an admissible cut for W , i.e. $b_2^+(W_1 \cup W_2) > 0, b_2^+(W_3 \cup W_4) > 0$, and $\delta H^1(N) = 0$ in $H^2(W)$.

It is proved in [9, Lemma 8.7] that any closed, oriented four-manifold X with $b_2^+(X) \geq 2$ admits a link presentation.

Definition 2.8. *Let X be a closed, oriented four-manifold with $b_2^+(X) \geq 2$. A grid presentation Γ for X consists of a link presentation $(L = L_1 \cup L_2 \cup L_3 \cup L_4, \Lambda)$ for X , together with a grid presentation for L .*

The four-manifold invariant $\Phi_{X, \mathfrak{t}}$ can be expressed in terms of a grid presentation Γ for X as follows. Using Theorem 2.6, we can express the maps $F_{W_2, \mathfrak{t}_{W_2}}^-$ and $F_{W_3, \mathfrak{t}_{W_3}}^+$ (or, more precisely, their tensor product with the identity on $V = H_*(T^{n-\ell})$) in terms of counts of holomorphic polygons on a symmetric product of the grid. We can combine these maps using their factorization through HF_{red} , and obtain a mixed map

$$F_{W_2 \cup W_3, \mathfrak{t}_{W_2 \cup W_3}}^{\text{mix}} : \mathbf{HF}^-(\#^m(S^1 \times S^2)) \rightarrow HF^+(\#^{m'}(S^1 \times S^2)).$$

By counting holomorphic polygons on the symmetric product of the grid, we can construct a similar map

$$F_{\Gamma, \mathfrak{t}}^{\text{mix}} : H_*(\mathcal{C}^-(G, \Lambda)^{L_1, \mathfrak{t}_{W_2 \cup W_3 \cup W_4}}) \rightarrow H_*(\mathcal{C}^+(G, \Lambda)^{L_1 \cup L_2 \cup L_3, \mathfrak{t}_{W_4}}).$$

Theorem 2.6 implies that $F_{\Gamma, \mathfrak{t}}^{\text{mix}}$ is the same as $F_{W_2 \cup W_3, \mathfrak{t}|_{W_2 \cup W_3}}^{\text{mix}} \otimes \text{Id}_V$, up to compositions with isomorphisms on both the domain and the target. Note, however, that at this point we do not know how to identify elements in the domains (or targets) of the two maps in a canonical way. For example, we know that there is an isomorphism

$$(18) \quad H_*(\mathcal{C}^-(G, \Lambda)^{L_1, \mathfrak{t}|_{W_2 \cup W_3 \cup W_4}}) \cong \mathbf{HF}^-(\#^m(S^1 \times S^2)) \otimes V,$$

but it is difficult to say exactly what the isomorphism is. Nevertheless, both $\mathbf{HF}^-(\#^m(S^1 \times S^2))$ and V have unique maximal degree elements Θ_{max}^m and Θ_V , respectively. We can identify what $\Theta_{\text{max}}^m \otimes \Theta_V$ corresponds to on the left hand side of (18) by simply computing degrees. Let us denote the respective element by

$$\Theta_{\text{max}}^\Gamma \in H_*(\mathcal{C}^-(G, \Lambda)^{L_1, \mathfrak{t}|_{W_2 \cup W_3 \cup W_4}}).$$

The following proposition says that one can decide whether $\Phi_{X, \mathfrak{t}} \in \mathbb{F}$ is zero or one from information coming from a grid presentation Γ :

Proposition 2.9. *Let X be a closed, oriented four-manifold X with $b_2^+(X) \geq 2$, with a Spin^c structure \mathfrak{t} with $d(\mathfrak{t}) \geq 0$ even. Let Γ be a grid presentation for X . Then $\Phi_{X, \mathfrak{t}} = 1$ if and only if $U^{d(\mathfrak{t})/2} \cdot F_{\Gamma, \mathfrak{t}}^{\text{mix}}(\Theta_{\text{max}}^\Gamma)$ is nonzero.*

2.11. Sparse doubles. We state here a variant of the surgery theorem that will be useful to us later.

Given a grid diagram G of grid number n , we define its *sparse double* $G_\#$ to be the grid diagram obtained from G by adding n additional rows R_2, R_4, \dots, R_{2n} , interspersed between all adjacent rows of G (now denoted R_1, \dots, R_{2n-1}), and also adding n additional columns C_2, C_4, \dots, C_{2n} similarly; then introducing an X and O marking in each square of the form $R_{2i} \cap C_{2i}$. See Figure 1.

If G represents the ℓ -component link \vec{L} , then its sparse double $G_\#$ represents the disjoint union $\vec{L}_\#$ of \vec{L} and n unknots, all unlinked. Thus,

$$L_\# = L \amalg \mathcal{U}_n,$$

where \mathcal{U}_n is the n -component unlink. Let $\mathcal{Z}(\mathcal{U}_n)$ be the set of newly introduced O markings, corresponding to \mathcal{U}_n . Thus $J(\mathcal{Z}(\mathcal{U}_n))$ is the set of values $\mathbf{s} \in \overline{\mathbb{H}}(L_\#)$ such that the values corresponding to the unlink are all $+\infty$.

Note that $\mathbb{H}(L_\#)_i = \mathbb{H}(L)_i$ whenever i corresponds to any of the original ℓ components of L . Therefore, there is an isomorphism:

$$\overline{\mathbb{H}}(L) \cong J(\mathcal{Z}(\mathcal{U}_n)) \subset \overline{\mathbb{H}}(L_\#),$$

given by adjoining values $+\infty$ for the n new components. We view $\overline{\mathbb{H}}(L)$ as a subset of $\overline{\mathbb{H}}(L_\#)$ via this isomorphism.

Let $\mathcal{R}_\#$ be the free polynomial ring in $n + \ell$ variables, associated to $G_\#$. We view \mathcal{R} as a module over $\mathcal{R}_\#$, on which the U variables corresponding to \mathcal{U}_n act trivially. Consider the \mathcal{R} -module

$$\mathcal{C}_\#^-(G, \Lambda) = \bigoplus_{M \subseteq L} \prod_{\mathbf{s} \in \mathbb{H}(L)} \left((\mathfrak{A}^-(G_\#^{L_\# - M}, \psi^M(\mathbf{s}))[\{\{U_i\}_{L_i \subseteq M}\}]) \otimes_{\mathcal{R}_\#} \mathcal{R} \right) \otimes_{\mathcal{R}} \mathcal{K}(M),$$

equipped with a boundary operator $D_\#$ as follows. Given $\mathbf{s} \in \mathbb{H}(L)$ and

$$\mathbf{x} \in \left((\mathfrak{A}^-(G_\#^{L_\# - M}, \psi^M(\mathbf{s}))[\{\{U_i\}_{L_i \subseteq M}\}]) \otimes_{\mathcal{R}_\#} \mathcal{R} \right) \otimes_{\mathcal{R}} \mathcal{K}(M),$$

the corresponding differential is given by the formula

$$\begin{aligned} \mathcal{D}_{\#}^{-}(\mathbf{s}, \mathbf{x}) &= \sum_{N \subseteq L-M} \sum_{\vec{N} \in \Omega(N)} (\mathbf{s} + \Lambda_{\vec{L}, \vec{N}}, \Phi_{\mathbf{s}}^{\vec{N}}(\mathbf{x})) \\ &\in \bigoplus_{N \subseteq L-M} \bigoplus_{\vec{N} \in \Omega(N)} \left((\mathfrak{A}^{-}(G_{\#}^{L-M-N}, \psi^{M \cup \vec{N}}(\mathbf{s}))[\{U_i\}_{L_i \subseteq M \cup N}]) \otimes_{\mathcal{R}_{\#}} \mathcal{R} \right) \otimes_{\mathcal{R}} \mathcal{K}(M \cup N). \end{aligned}$$

In other words, the complex $(\mathcal{C}_{\#}^{-}(G, \Lambda), \mathcal{D}_{\#}^{-})$ only consists of destabilization maps at the components of $L_{\#}$ which are in L . Just like the other complexes of this form, it splits into a direct product of $\mathbb{Z}/\mathfrak{d}(\mathbf{u})\mathbb{Z}$ -graded factors $\mathcal{C}_{\#}^{-}(G, \Lambda, \mathbf{u})$, over $\mathbf{u} \in \text{Spin}^c(S_{\Lambda}^3(L))$.

Theorem 2.10. *Fix a grid diagram G for an oriented, ℓ -component link \vec{L} in S^3 , and fix a framing Λ of L . Let $G_{\#}$ be the sparse double of G . Then, for every $\mathbf{u} \in \text{Spin}^c(S_{\Lambda}^3(L))$, we have an isomorphism of relatively graded $\mathbb{F}[[U]]$ -modules:*

$$H_*(\mathcal{C}_{\#}^{-}(G, \Lambda, \mathbf{u}), \mathcal{D}_{\#}^{-}) \cong \mathbf{HF}_*^{-}(S_{\Lambda}^3(L), \mathbf{u}) \otimes_{\mathbb{F}} H_*(T^{2n-\ell}),$$

where n is the grid number of G .

Theorems 2.5, 2.6 and Proposition 2.9 all admit straightforward analogues in terms of the sparse double.

3. ENHANCED DOMAINS OF HOLOMORPHIC POLYGONS

3.1. Domains and shadows. In the construction of the complex $\mathcal{C}^{-}(G, \Lambda)$ in Section 2.7, the only non-combinatorial ingredients were the holomorphic polygon counts in the definition of the maps $D_{\mathbf{s}}^{\mathcal{Z}}$. (We use here the notation from Remark 2.2.) According to Equation (7), the maps $D_{\mathbf{s}}^{\mathcal{Z}}$ are in turn summations of maps of the form $D_{\mathbf{s}}^{(\mathcal{Z})}$, where (\mathcal{Z}) denotes an ordering of a consistent collection of markings \mathcal{Z} .

Let $(\mathcal{Z}) = (Z_1, \dots, Z_k)$. For simplicity, we assume that $k < n$. (The case $k = n$ can be treated similarly, with minor modifications due to the fact that we do not handleslide the last curve.)

The maps

$$D_{\mathbf{s}}^{(\mathcal{Z})} : \mathfrak{A}^{-}(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta}^{\mathcal{Z}^0}, \mathbf{s}) \rightarrow \mathfrak{A}^{-}(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta}^{\mathcal{Z}^0 \cup \mathcal{Z}}, \mathbf{s}),$$

correspond to destabilization (in the given order) at a set of markings \mathcal{Z} , starting with a diagram G already destabilized at a base set of markings \mathcal{Z}^0 . Our goal is to get as close as possible to a combinatorial description of these maps. In light of the isomorphisms (10), we can assume without loss of generality that $\mathcal{Z}_0 = \emptyset$. The difficulty lies in understanding the counts of pseudo-holomorphic polygons $\#\mathcal{M}(\phi)$, where

$$\phi \in \pi_2(\mathbf{x}, \Theta^{\emptyset, \{Z_1\}}, \Theta^{\{Z_1\}, \{Z_1, Z_2\}}, \dots, \Theta^{\{Z_1, \dots, Z_{k-1}\}, \{Z_1, \dots, Z_k\}}, \mathbf{y})$$

is a homotopy class of $(2+k)$ -gons with edges on

$$\mathbb{T}_{\alpha}, \mathbb{T}_{\beta}, \mathbb{T}_{\beta}^{\{Z_1\}}, \mathbb{T}_{\beta}^{\{Z_1, Z_2\}}, \dots, \mathbb{T}_{\beta}^{\{Z_1, \dots, Z_k\}},$$

in this cyclic order. The Maslov index $\mu(\phi)$ is required to be $1 - k$.

As in [19, Definition 2.13], every homotopy class ϕ has an associated *domain* $D(\phi)$ on the surface \mathcal{T} . The domain is a linear combination of regions, i.e. connected components of the complement in \mathcal{T} of all the curves $\alpha, \beta, \beta^{\{Z_1\}}, \beta^{\{Z_1, Z_2\}}, \dots, \beta^{\{Z_1, \dots, Z_k\}}$. If ϕ admits a holomorphic representative, then $D(\phi)$ is a linear combination of regions, all appearing with nonnegative coefficients.

Let us mark an asterisk in each square of the grid diagram G . When we construct the new beta curves $\beta^{\{Z_1\}}, \beta^{\{Z_1, Z_2\}}, \dots, \beta^{\{Z_1, \dots, Z_k\}}$ (all obtained from the original β curves by handleslides), we make sure that each beta curve encircling a marking does not include an asterisk, and also that whenever we isotope a beta curve to obtain a new beta curve intersecting it in two points, these

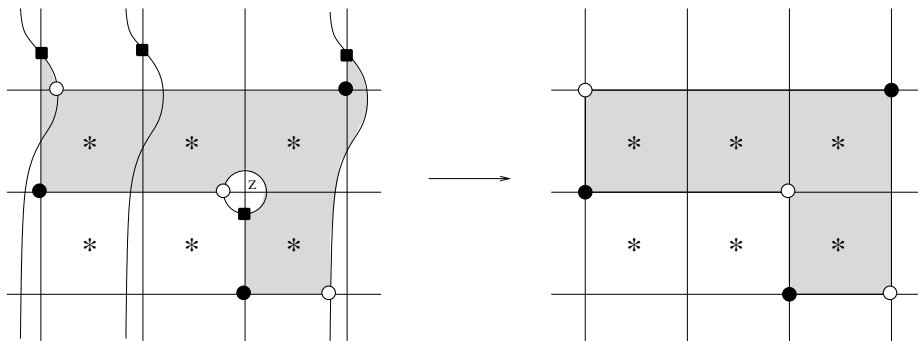


FIGURE 3. **The domain of a triangle and its shadow.** On the left, we show the domain $D(\phi)$ of a homotopy class ϕ of triangles in $\text{Sym}^n(\mathcal{T})$. The alpha and beta curves are the horizontal and vertical straight lines, respectively, while the curved arcs (including the one encircling Z) are part of the $\beta^{\{Z\}}$ curves. The black squares mark components of $\Theta^{\emptyset, \{Z\}} \in \mathbb{T}_\beta \cap \mathbb{T}_\beta^{\{Z\}}$. On the right, we show the shadow $Sh(\phi)$.

isotopies do not cross the asterisks. Then, the regions on \mathcal{T} fall naturally into two types: *small* regions, which do not contain asterisks, and *large* regions, which do. We define the *shadow* $Sh(R)$ of a large region R to be the square in G containing the same asterisk as R ; and the shadow of a small region to be the empty set.

If

$$D = \sum a_i R_i$$

is a linear combination of regions, with $a_i \in \mathbb{Z}$, we define its shadow to be

$$Sh(D) = \sum a_i Sh(R_i).$$

The *shadow* $Sh(\phi)$ of a homotopy class ϕ is defined as $Sh(D(\phi))$. See Figure 3.

We will study homotopy classes ϕ by looking at their shadows, together with two additional pieces of data as follows.

First, each homotopy class ϕ corresponds to $(2+k)$ -gons where one vertex is an intersection point $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta^Z$. In particular, \mathbf{y} contains exactly one of the two intersection points between the alpha curve just below the marking Z_i and the beta curve encircling Z_i , for $i = 1, \dots, k$. Set $\epsilon_i = 0$ if that point is the left one, and $\epsilon_i = 1$ if it is the right one.

Second, for $i = 1, \dots, k$, let $\rho_i \in \mathbb{Z}$ be the multiplicity of the domain $D(\phi)$ at the marking Z_i .

Then, we define the *enhanced domain* $E(\phi)$ associated to ϕ to be the triple $(Sh(\phi), \epsilon(\phi), \rho(\phi))$ consisting of the shadow $Sh(\phi)$, the collection $\epsilon(\phi) = (\epsilon_1, \dots, \epsilon_k)$ corresponding to \mathbf{y} , and the set of multiplicities $\rho(\phi) = (\rho_1, \dots, \rho_k)$.

3.2. Grid diagrams marked for destabilization. We now turn to studying enhanced domains on their own. For simplicity, we will assume that all the markings Z_1, \dots, Z_k relevant for our destabilization procedure are O 's.

Consider a toroidal grid diagram G of grid number n . We ignore the X 's, and consider the n O 's with associated variables U_1, \dots, U_n . Note that, unlike in Section 2.2, here we use one variable for each O , rather than one for each link component.

Some subset of the O 's, corresponding to indices $i_1, \dots, i_k \in \{1, \dots, n\}$, are marked for destabilization. (These are the markings Z_1, \dots, Z_k from the previous subsection.) The corresponding points on the lower left of each O_{i_j} are denoted p_j and called *destabilization points*.

Let $C^-(G) = \mathbf{CF}^-(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ be the chain complex freely generated over $\mathbb{F}[[U_1, \dots, U_n]]$ by the $n!$ elements of $\mathbf{S}(G)$, namely the n -tuples of intersection points on the grid. Its differential ∂ counts

rectangles, while keeping track of the U 's:

$$\partial^- \mathbf{x} = \sum_{\mathbf{y} \in \mathbf{S}(G)} \sum_{r \in \text{Rect}^o(\mathbf{x}, \mathbf{y})} U_1^{O_1(r)} \dots U_n^{O_n(r)} \mathbf{y}.$$

Each generator $\mathbf{x} \in \mathbf{S}(G)$ has a well-defined homological grading $M(\mathbf{x}) \in \mathbb{Z}$. Note that $M(U_i \cdot \mathbf{x}) = M(\mathbf{x}) - 2$.

Let G' be the destabilized diagram. There is a similar complex $C^-(G')$. We identify $\mathbf{S}(G')$ with a subset of $\mathbf{S}(G)$ by adjoining to them the destabilization points.

The set of *enhanced generators* $\mathbf{ES}(G)$ consists of pairs (\mathbf{y}, ϵ) , where $\mathbf{y} \in \mathbf{S}(G')$ and $\epsilon = (\epsilon_1, \dots, \epsilon_k)$ is a collection of markings (L or R) at each destabilization point. We set $\epsilon_j = 0$ for an L marking and $\epsilon_j = 1$ for an R marking. The homological grading of an enhanced generator is

$$M(\mathbf{y}, \epsilon) = M(\mathbf{y}) + \sum_{j=1}^k \epsilon_j.$$

We consider the enhanced destabilized complex $EC^-(G)$ whose generators are $\mathbf{ES}(G)$. It is formed as the tensor product of k mapping cones

$$(19) \quad C^-(G') \rightarrow C^-(G')$$

given by the map $U_{i_j} - U_{i'_j}$ from L to R , where $U_{i'_j}$ corresponds to the O in the row directly below the destabilization point p_j . For a suitable choice of almost-complex structure on the symmetric product, there is a natural isomorphism

$$EC^-(G) \cong \mathbf{CF}^-(\mathbb{T}_\alpha, \mathbb{T}_\beta^{\{O_{i_1}, \dots, O_{i_k}\}}),$$

with the caveat that on the right hand side we count different U variables than we did in Section 2.3.

It was shown in [11] that in the case $k = 1$ there is a quasi-isomorphism

$$(20) \quad F : C^-(G) \rightarrow EC^-(G)$$

given by counting snail-like domains as in Figure 4.

Definition 3.1. *Given a domain D on the α - β grid (that is, a linear combination of squares), we let $O_i(D)$ be the multiplicity of D at O_i . We define an enhanced domain (D, ϵ, ρ) to consist of:*

- *A domain D on the grid between points $\mathbf{x} \in \mathbf{S}(G)$ and $\mathbf{y} \in \mathbf{S}(G')$ (in particular, the final configuration contains all destabilization points);*
- *A set of markings $\epsilon = (\epsilon_1, \dots, \epsilon_k)$ at each destabilization point (so that (\mathbf{y}, ϵ) is an enhanced generator); and*
- *A set of integers $\rho = (\rho_1, \dots, \rho_k)$, one for each destabilization point.*

We call ρ_j the *real multiplicity* at O_{i_j} . The number $t_j = O_{i_j}(D)$ is called the *total multiplicity*, and the quantity $f_j = t_j - \rho_j$ is called the *fake multiplicity* at O_{i_j} . The reason for this terminology is that, if D is the shadow of a holomorphic $(k+2)$ -gon, then the real multiplicity ρ_j is the multiplicity of the original polygon at O_{i_j} . On the other hand, the fake multiplicity (if positive) is the multiplicity that appears in the shadow even though it does not come from the polygon. For example, in the domain pictured in Figure 3, the fake multiplicity at the Z marking is one.

Consider the full collection of real multiplicities (N_1, \dots, N_n) , where $N_{i_j} = \rho_j$ for $j \in \{1, \dots, k\}$, and $N_i = O_i(D)$ when O_i is not one of the O 's used for destabilization. We say that the enhanced domain (D, ϵ, ρ) goes from the generator \mathbf{x} to $U_1^{N_1} \dots U_n^{N_n} \cdot (\mathbf{y}, \epsilon)$. These are called the initial and final point of the enhanced domain, respectively.

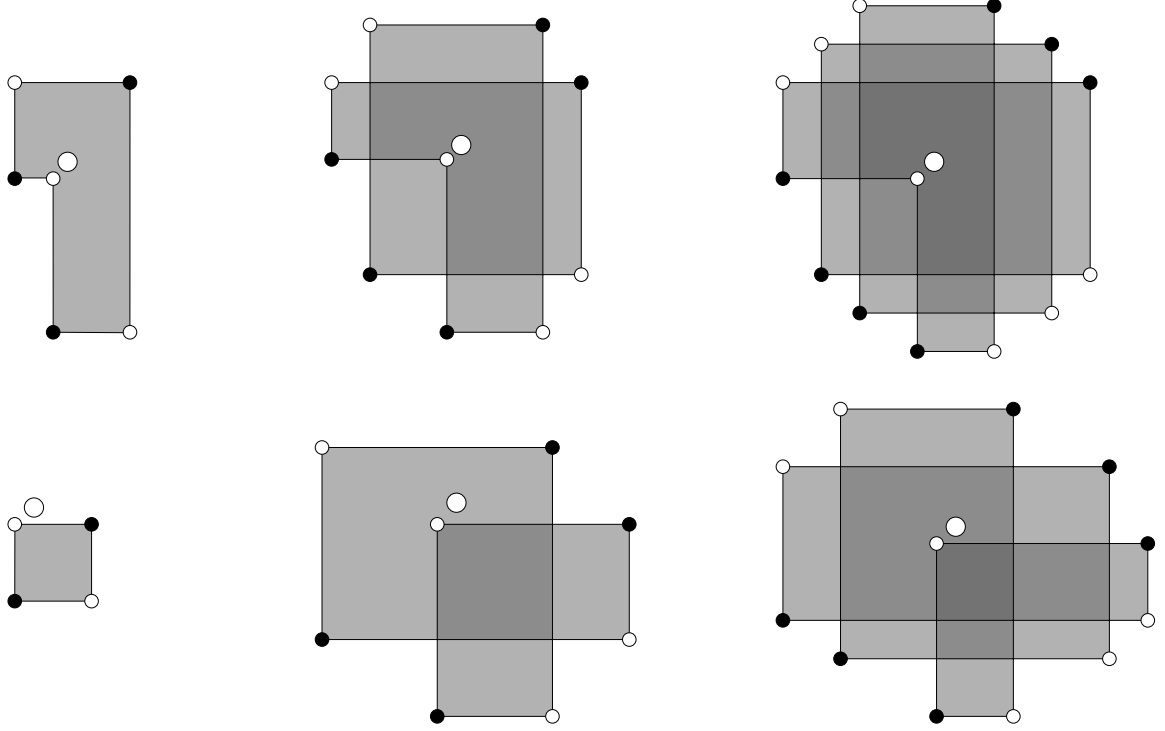


FIGURE 4. **Domains for destabilization at one point.** We label initial points by dark circles, and terminal points by empty circles. The top row lists domains of type L (i.e., ending in an enhanced generator with an L marking), while the second row lists some of type R . Darker shading corresponds to higher local multiplicities. The domains in each row (top or bottom) are part of an infinite series, corresponding to increasing complexities. The series in the first row also contains the trivial domain of type L , not shown here.

We define the index of an enhanced domain to be

$$\begin{aligned}
 I(D, \epsilon, \rho) &= M(\mathbf{x}) - M(U_1^{N_1} \dots U_n^{N_n} \cdot (\mathbf{y}, \epsilon)) \\
 &= M(\mathbf{x}) - M(\mathbf{y}) - \sum_{j=1}^k \epsilon_j + 2 \sum_{j=1}^k N_j \\
 (21) \quad &= M(\mathbf{x}) - M(U_1^{O_1(D)} \dots U_n^{O_n(D)} \cdot \mathbf{y}) - \sum_{j=1}^k (\epsilon_j + 2f_j) \\
 &= I(D) - \sum_{j=1}^k (\epsilon_j + 2f_j).
 \end{aligned}$$

Here $I(D)$ is the ordinary Maslov index of D , given by Lipshitz's formula [7, Corollary 4.3]:

$$(22) \quad I(D) = \sum_{x \in \mathbf{x}} n_x(D) + \sum_{y \in \mathbf{y}} n_y(D),$$

where $n_p(D)$ denotes the average multiplicity of D in the four quadrants around the point p . (Lipshitz's formula in the reference has an extra term, the Euler measure of D , but this is zero in our case because D can be decomposed into rectangles.)

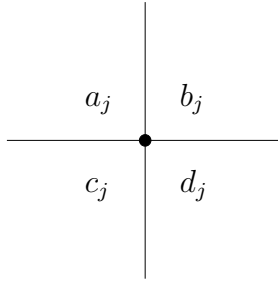


FIGURE 5. **Local multiplicities around a destabilization point.** The marked point in the middle is the destabilization point (p_j for left destabilization and q_j for right destabilization).

For example, consider the domains in Figure 4. We turn them into enhanced domains by adding the respective marking (L or R) as well as choosing the real multiplicity at the destabilization point to be zero. Then all those domains have index zero, regardless of how many (non-destabilized) O 's they contain and with what multiplicities.

Lemma 3.2. *Suppose that (D, ϵ, ρ) is the enhanced domain associated to a homotopy class ϕ of $(k+2)$ -gons as in Section 3.2. Then the Maslov indices match: $\mu(\phi) = I(D, \epsilon, \rho)$.*

Proof. Note that $I(D, \epsilon, \rho)$ is the difference in Maslov indices between \mathbf{x} (in the original grid diagram) and $\tilde{\mathbf{y}} = U_1^{N_1} \dots U_n^{N_n} \cdot (\mathbf{y}, \epsilon)$ (in the destabilized diagram), cf. Equation (21). Because $\mu(\phi)$ is additive with respect to pre- or post-composition with rectangles, we must have $\mu(\phi) = I(D, \epsilon, \rho) + C$, where C is a constant that only depends on the grid. To compute C , consider the trivial enhanced domain with $D = 0, \epsilon = (0, \dots, 0), \rho = (0, \dots, 0)$. This is associated to a class ϕ whose support is a disjoint union of n polygons, all of whom are $(k+2)$ -gons with three acute angles and $k-1$ obtuse angles. See Figure 6 for an example of a quadrilateral of this type. It is easy to check that $\mu(\phi) = 0$, which implies $C = 0$. \square

Given an enhanced domain (D, ϵ, ρ) , we denote by a_j, b_j, c_j, d_j the multiplicities of D in the four squares around p_j , as in Figure 5. In particular, $b_j = t_j$ is the total multiplicity there. Note that, if $p_j \notin \mathbf{x}$, then

$$(23) \quad a_j + d_j = b_j + c_j + 1,$$

while if $p_j \in \mathbf{x}$ then

$$(24) \quad a_j + d_j = b_j + c_j.$$

Definition 3.3. *We say that the enhanced domain (D, ϵ, ρ) is positive if D has nonnegative multiplicities everywhere in the grid, and, furthermore, for every $j \in \{1, \dots, k\}$, we have*

$$(25) \quad \begin{aligned} a_j &\geq f_j, & b_j &\geq f_j, \\ c_j &\geq f_j + \epsilon_j - 1, & d_j &\geq f_j + \epsilon_j. \end{aligned}$$

Observe that the second of the above inequalities reads $b_j - f_j = \rho_j \geq 0$. Thus, in a positive enhanced domain, all real multiplicities ρ_j are nonnegative.

Lemma 3.4. *Suppose that (D, ϵ, ρ) is the enhanced domain associated to a homotopy class ϕ of $(k+2)$ -gons as in Section 3.2, and that the homotopy class ϕ admits at least one holomorphic representative. Then (D, ϵ, ρ) is positive, in the sense of Definition 3.3.*

Proof. If a homotopy class ϕ has at least one holomorphic representative (for suitable choices of almost-complex structure on the symmetric product, see for example Lemma 3.2 of [19]). It follows

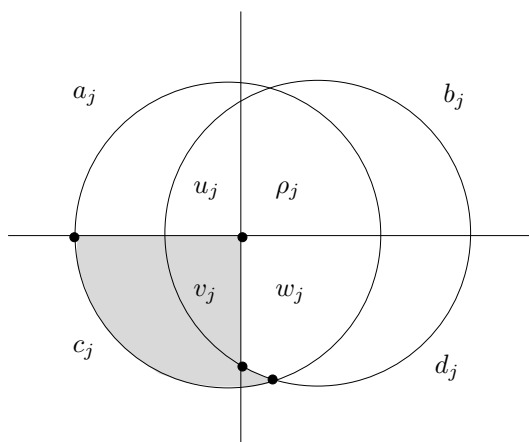


FIGURE 6. **Local multiplicities in detail.** This is a more detailed version of Figure 5, in which we show two other types of beta curves: $\beta^{\{Z_1\}}$ and $\beta^{\{Z_1, Z_2\}}$. We mark the local multiplicity in each region. The shaded region is the domain of a quadrilateral of index zero.

now that D has nonnegative multiplicities everywhere in the grid and that $\rho_j = b_j - f_j \geq 0$. It remains to check the three other relations.

For concreteness, let us consider the case $k = 2$, with $\epsilon_j = 0$, cf. Figure 6. The two circles encircling the destabilization point in the middle are of the type $\beta^{\{Z_1\}}$ (the right one) and $\beta^{\{Z_1, Z_2\}}$ (the left one).

If we are given the domain of a holomorphic quadrilateral (an example could be the shaded one in Figure 6), at most of the intersection points the alternating sum of nearby multiplicities is zero. The only exceptions are the four bulleted points in the figure, which correspond to the vertices of the quadrilateral, and where the alternating sum of nearby multiplicities is ± 1 . By adding up suitable such relations between local multiplicities, we obtain

$$\begin{aligned} a_j - f_j &= a_j + \rho_j - b_j &= u_j &\geq 0, \\ c_j - f_j + 1 &= c_j + \rho_j - b_j + 1 = v_j &\geq 0, \\ d_j - f_j &= d_j + \rho_j - b_j &= w_j &\geq 0. \end{aligned}$$

as desired. The proof in the $\epsilon_j = 1$ case, or for other values of k , is similar. \square

Note that an enhanced domain $E = (D, \epsilon, \rho)$ is determined by its initial point \mathbf{x} and final point $\tilde{\mathbf{y}} = U_1^{N_1} \dots U_n^{N_n} \cdot (\mathbf{y}, \epsilon)$ up to the addition of linear combinations of *periodic domains* of the following form:

- A column minus the row containing the same O_i ;
- A column containing one of the O_{i_j} 's used for destabilization.

When we add a column of the latter type, the total multiplicity t_j (and hence also the fake multiplicity f_j) increase by 1. The real multiplicity ρ_j does not change.

Definition 3.5. *We say that the pair $[\mathbf{x}, \tilde{\mathbf{y}}]$ is positive if there exists a positive enhanced domain with \mathbf{x} and $\tilde{\mathbf{y}}$ as its initial and starting points.*

3.3. Positive domains of negative index. Figure 7 shows some examples of positive enhanced domains of negative index, in a twice destabilized grid. We see that the index can get as negative as we want, even if we fix the number of destabilizations at two (but allow the size of the grid to change).

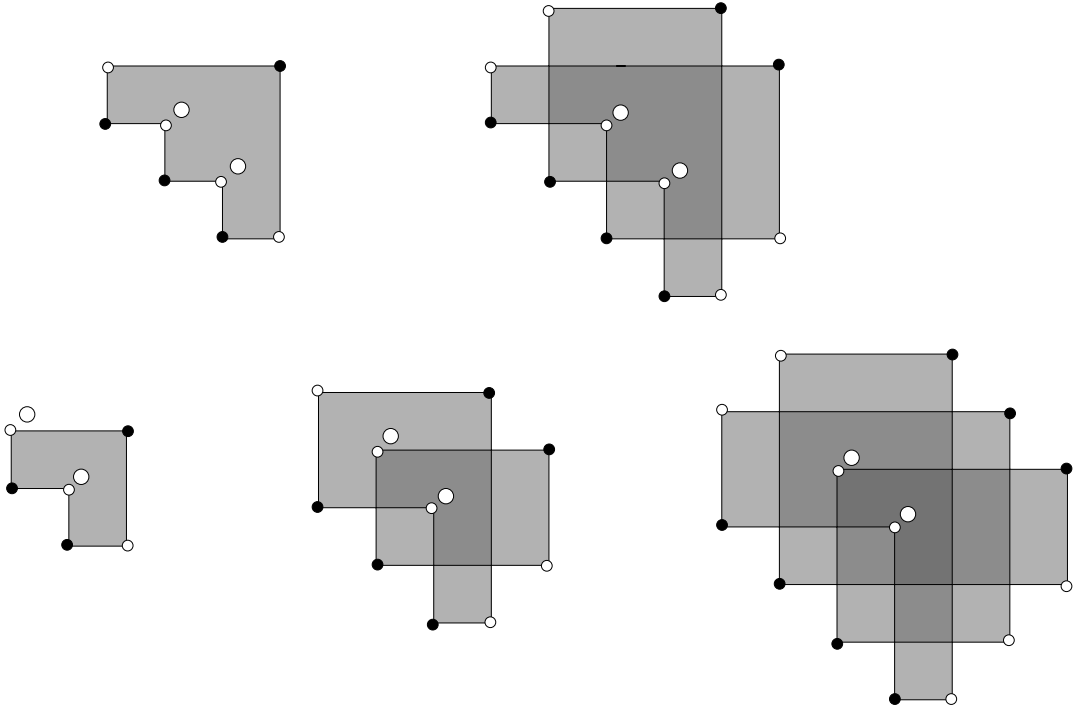


FIGURE 7. **Positive domains of negative index.** We destabilize at two O 's, marked by the two larger circles in the picture. The top row shows two positive domains of type LL , the first of index -1 , the second of index -2 . The second row shows three positive domains of type RL , of indices -1 , -2 and -3 , respectively. Darker shading corresponds to higher local multiplicities.

However, this phenomenon is impossible for one destabilization:

Proposition 3.6. *Let G be a toroidal grid diagram with only one O marked for destabilization. Then any positive enhanced domain has nonnegative index. Furthermore, if the enhanced domain is positive and has index zero, then it is of one of the types from the sequences in Figure 4.*

Proof. Let us first consider a positive enhanced domain (D, ϵ, ρ) going between the generators \mathbf{x} and $\tilde{\mathbf{y}} = U_1^{N_1} \dots U_n^{N_n} \cdot (\mathbf{y}, \epsilon)$. With the notations of Section 3.2, we have $k = 1$, $\epsilon_1 \in \{0, 1\}$ (corresponding to the type of the domain, L or R), $a_1, b_1 = t_1, c_1, d_1$ are the local multiplicities around the destabilization point, and ρ_1 and $f_1 = t_1 - \rho_1$ are the real and fake multiplicities there. Note that $t_1 \geq f_1$.

Without loss of generality we can assume that every row or column contains at least one square where the multiplicity of D is zero; otherwise we could subtract that row or column and obtain another positive domain, whose index is at least as large. Indeed, if the row or column does not contain the destabilization point p_1 on its boundary, then subtracting it decreases the index by 2. If we subtract the row just to the left of p_1 , or the column just below p_1 , by simultaneously increasing ρ_1 by one we can preserve the positivity conditions and leave the index the same. If we subtract the row or column through the O marked for destabilization, while leaving ρ unchanged, the positivity and the index are again unaffected.

We say that a row or a column is *special* if \mathbf{x} and \mathbf{y} intersect it in different points. Let m be the number of special rows; it is the same as the number of special columns. Note that, as we move across a row (or column), the multiplicity of the domain D can only change by ± 1 , and it can do

that only if the row (or column) is special. This is because of our assumption that we have zeros in every row and column.

Let us look at the column containing the destabilized O . The domain has multiplicity d_1 in the spot right below O , and it has multiplicity zero on some other spot on that column. We can move from O to the multiplicity zero spot either by going up or down. As we go in either direction, we must encounter at least d_1 special rows. This means that the total number of special rows is at least $2d_1$. Using the second inequality in (25), we get

$$(26) \quad m \geq 2d_1 \geq 2(f_1 + \epsilon_1).$$

The ordinary index of the domain D is given by Equation (22), involving the sums of the average local multiplicities of D at the points of \mathbf{x} and \mathbf{y} . One such point is the destabilization point p_1 which is part of \mathbf{y} . Using the relations in Equation (25), we find that the average vertex multiplicity there is

$$(27) \quad \frac{a_1 + b_1 + c_1 + d_1}{4} \geq f_1 + \frac{2\epsilon_1 - 1}{4}.$$

On the other hand, apart from the destabilization point, \mathbf{x} and \mathbf{y} together have either $2m - 1$ (if $p_1 \notin \mathbf{x}$) or $2m$ (if $p_1 \in \mathbf{x}$) corner vertices, where the average multiplicity has to be at least $1/4$. Together with Equations (26) and (27), this implies that

$$(28) \quad I(D) \geq \frac{2m - 1}{4} + f_1 + \frac{2\epsilon_1 - 1}{4}.$$

Using the formula for the index of an enhanced domain, together with Equations (26) and (28), we obtain

$$(29) \quad I(D, \epsilon, \rho) = I(D) - \epsilon_1 - 2f_1 \geq \frac{m}{2} - f_1 - \frac{\epsilon_1}{2} - \frac{1}{2} \geq \frac{\epsilon_1 - 1}{2}.$$

Since the index is an integer, we must have $I(D, \epsilon, \rho) \geq 0$. Equality happens only when D has average vertex multiplicity $1/4$ at all its corners. An easy analysis shows that the domain must be of the required shape. \square

3.4. Holomorphic triangles on grid diagrams.

Lemma 3.7. *Fix a snail-like domain D (as in Figure 4) for destabilization at one point. Then the count of holomorphic triangles in $\text{Sym}^n(T)$ with D as their shadow is one mod 2.*

Proof. The trivial domain D of type L corresponds to a homotopy class ϕ whose support is a disjoint union of triangles with 90° angles. Hence, the corresponding holomorphic count is one.

To establish the claim in general, note that the destabilization map at one point (given by counting holomorphic triangles) is a chain map (in fact, a quasi-isomorphism) from $C^-(G)$ to $EC^-(G)$, compare (20). By Proposition 3.6, the only counts involved in this map are the ones corresponding to snail-like domains. If to each snail-like domain we assign the coefficient one when counting it in the map, the result is the chain map (20). By induction on complexity, along the lines of [11, Lemma 3.5], we do indeed need to assign one to each snail-like domain in order for the result to be a chain map. \square

Proposition 3.6 and Lemma 3.7 imply that one can count combinatorially (mod 2) all the index zero holomorphic triangles in a grid diagram (with one point marked for destabilization), with fixed shadow. Indeed, if the shadow is a snail-like domain, then the count is one, and otherwise it is zero.

4. FORMAL COMPLEX STRUCTURES AND SURGERY

Let $\vec{L} \subset S^3$ be a link with framing Λ , and let \mathbf{u} be a Spin^c structure on the surgered manifold $S_\Lambda^3(L)$. Our goal in this section is to explain a combinatorial procedure for calculating the ranks of the groups $\mathbf{HF}^-(S_\Lambda^3(L), \mathbf{u})$. The procedure will be based on Theorem 2.10.

4.1. The complex of positive pairs. Let G be a grid diagram marked for destabilization at some of the O 's, as in Section 3.2. We defined there the (quasi-isomorphic) complexes $C^-(G)$ and $EC^-(G)$. Let us consider the Hom complex

$$\mathrm{Hom}_{\mathcal{R}}(C^-(G), EC^-(G)),$$

where $\mathcal{R} = \mathbb{F}[[U_1, \dots, U_n]]$. Since $C^-(G)$ is a free \mathcal{R} -module, we can naturally identify it with its dual using the basis given by $\mathbf{S}(G)$. Thus, if we view the Hom complex as an \mathbb{F} -vector space, its generators (in the sense of direct products) are pairs $[\mathbf{x}, \tilde{\mathbf{y}}]$, where $\tilde{\mathbf{y}}$ is an enhanced generator possibly multiplied by some powers of U , i.e., $\tilde{\mathbf{y}} = U_1^{N_1} \dots U_n^{N_n} \cdot (\mathbf{y}, \epsilon)$.

The homological degree of a generator in the Hom complex is $M(\tilde{\mathbf{y}}) - M(\mathbf{x})$. However, we would like to think of the generators as being enhanced domains (up to the addition of periodic domains), so in order to be consistent with the formula for the index of domains we set

$$I([\mathbf{x}, \tilde{\mathbf{y}}]) = M(\mathbf{x}) - M(\tilde{\mathbf{y}})$$

and view the Hom complex as a cochain complex, with a differential d that increases the grading. It has the structure of an \mathcal{R} -module, where multiplication by a variable U_i increases the grading by two: $U_i[\mathbf{x}, \tilde{\mathbf{y}}] = [\mathbf{x}, U_i\tilde{\mathbf{y}}]$. Note that the Hom complex is bounded from below with respect to the Maslov grading.

There is a differential on the complex given by

$$d[\mathbf{x}, \tilde{\mathbf{y}}] = [\partial^*\mathbf{x}, \tilde{\mathbf{y}}] + [\mathbf{x}, \partial\tilde{\mathbf{y}}].$$

Thus, taking the differential of a domain consists in summing over the ways of pre- and post-composing the domain with a rectangle.

If a pair $[\mathbf{x}, \tilde{\mathbf{y}}]$ is positive (as in Definition 3.5), then by definition it represents a positive domain; adding a rectangle to it keeps it positive. Indeed, note that if the rectangle crosses the O used for destabilization, then the real and total multiplicities increase by 1, but the fake multiplicity stays the same, so the inequalities (25) are still satisfied.

Therefore, the positive pairs $[\mathbf{x}, \tilde{\mathbf{y}}]$ generate a subcomplex $CP^*(G)$ of the Hom complex. For the moment, let us ignore its structure as an \mathcal{R} -module, and simply consider it as a cochain complex over \mathbb{F} . We denote its cohomology by $HP^*(G)$.

We make the following:

Conjecture 4.1. *For any toroidal grid diagram G with some O 's marked for destabilization, we have $HP^d(G) = 0$ for $d < 0$.*

Proposition 3.6 implies Conjecture 4.1 in the case when only one O is marked for destabilization. Indeed, in that case we have $CP^d(G) = 0$ for $d < 0$, so the homology is also zero.

In the case of several destabilization points, we can prove only a weaker form of the conjecture, namely Theorem 4.3 below. However, this will suffice for our application.

Definition 4.2. *Let G be a toroidal grid diagram with some O 's marked for destabilization. If none of the O 's marked for destabilization sit in adjacent rows or adjacent columns, we say that G is sparse.*

In particular, if we consider the sparse double $G_{\#}$ of G as in Section 2.11, with only some of the original markings used for destabilization (not the ones in the n additional unlinks), then $G_{\#}$ is clearly sparse.

Theorem 4.3. *If G is a sparse toroidal grid diagram with k O 's marked for destabilization, then $HP^d(G) = 0$ for $d < \min\{0, 2 - k\}$.*

Once we show this, it will also follow that, for sparse G , the homology of $CP^*(G) \otimes M$ is zero in degrees $< \min\{0, 2 - k\}$, where M is an \mathcal{R} -module (a quotient of \mathcal{R}) obtained by setting some of the U variables equal to each other.

The proof of Theorem 4.3 will be given in Section 5.

4.2. An extended complex of positive pairs. Let us now return to the set-up of Section 2, where \vec{L} is an oriented link with a grid presentation G .

When \mathcal{Z} is a consistent set of markings (X 's or O 's) on G , we set

$$J_\infty(\mathcal{Z}) = \{\mathbf{s} = (s_1, \dots, s_\ell) \in J(\mathcal{Z}) \mid s_i = \pm\infty \text{ for all } i\}.$$

Let \mathcal{Z}_0 and \mathcal{Z} be two disjoint sets of markings on G , such that $\mathcal{Z}_0 \cup \mathcal{Z}$ is consistent. For $\mathbf{s} \in J_\infty(\mathcal{Z}_0 \cup \mathcal{Z})$, we can define a cochain complex

$$CP^*(G; \mathcal{Z}_0, \mathcal{Z}, \mathbf{s}),$$

which is the subcomplex of

$$(30) \quad \text{Hom}_{\mathcal{R}}(\mathfrak{A}^-(\mathbb{T}_\alpha, \mathbb{T}_\beta^{\mathcal{Z}_0}, \mathbf{s}), \mathfrak{A}^-(\mathbb{T}_\alpha, \mathbb{T}_\beta^{\mathcal{Z}_0 \cup \mathcal{Z}}, \mathbf{s}))$$

consisting of positive pairs. Here positivity of pairs has the same meaning as before: it is defined in terms of enhanced domains, by looking at the grid diagram $G^{\mathcal{Z}_0}$ destabilized at the points of \mathcal{Z}_0 .

Lemma 4.4. *If \mathcal{Z} is a consistent set of markings on a grid G and $\mathbf{s}, \mathbf{s}' \in J_\infty(\mathcal{Z})$, then the complexes $CP^*(G; \mathcal{Z}_0, \mathcal{Z}, \mathbf{s})$ and $CP^*(G; \mathcal{Z}_0, \mathcal{Z}, \mathbf{s}')$ are canonically isomorphic.*

Proof. It suffices to consider the case when \mathbf{s} and \mathbf{s}' differ in only one place, say $s_i \neq s'_i$ for a component $L_i \notin L(\mathcal{Z})$, where $L(\mathcal{Z})$ is as in Section 2.5. Suppose $s_i = -\infty$ while $s'_i = +\infty$. Then the desired isomorphism is

$$(31) \quad CP^*(G; \mathcal{Z}_0, \mathcal{Z}, \mathbf{s}) \longrightarrow CP^*(G; \mathcal{Z}_0, \mathcal{Z}, \mathbf{s}'), \quad [\mathbf{x}, \tilde{\mathbf{y}}] \rightarrow U_i^{A_i(\mathbf{x}) - A_i(\mathbf{y})}[\mathbf{x}, \tilde{\mathbf{y}}],$$

where $\tilde{\mathbf{y}} = U_1^{N_1} \dots U_\ell^{N_\ell} \mathbf{y}$, and A_i is the component of the Alexander grading corresponding to L_i . If D is a positive domain representing the pair $[\mathbf{x}, \tilde{\mathbf{y}}]$, note that

$$A_i(\mathbf{x}) - A_i(\mathbf{y}) = \sum_{j \in \mathbb{X}_i} X_j(D) - \sum_{j \in \mathbb{O}_i} O_j(D)$$

and $N_i = \sum_{j \in \mathbb{O}_i} O_j(D)$, so

$$A_i(\mathbf{x}) - A_i(\mathbf{y}) + N_i \geq 0.$$

This implies that the right hand side generator has a nonnegative exponent of U_i , so it is a well-defined generator of the respective complex. It is then easy to check that the map from Equation (31) is a chain map that preserves positivity. Its inverse is given by multiplication with $U_i^{A_i(\mathbf{y}) - A_i(\mathbf{x})}$. \square

In view of Lemma 4.4, we can drop \mathbf{s} from the notation and refer to $CP^*(G; \mathcal{Z}_0, \mathcal{Z}, \mathbf{s})$ simply as $CP^*(G; \mathcal{Z}_0, \mathcal{Z})$.

Let $G^{\mathcal{Z}_0, \mathcal{Z}}$ be the grid diagram $G^{\mathcal{Z}_0}$, from which we delete all X markings on components $L_i \notin L(\mathcal{Z}_0 \cup \mathcal{Z})$, and relabel the remaining X markings as O . Further, we mark for destabilization the markings in \mathcal{Z} . Then $G^{\mathcal{Z}_0, \mathcal{Z}}$ is a grid diagram with only O 's as in Sections 3.2 and 4.1. As such, it has a complex of positive pairs $CP^*(G^{\mathcal{Z}_0, \mathcal{Z}})$, cf. Section 4.1.

In light of the isomorphisms (10), we see that

$$(32) \quad CP^*(G; \mathcal{Z}_0, \mathcal{Z}) \cong CP^*(G^{\mathcal{Z}_0, \mathcal{Z}}) \otimes M,$$

where M is an \mathcal{R} -module obtained by setting some U variables equal to each other.

Let $\mathcal{Z}_0, \mathcal{Z}, \mathcal{Z}'$ be disjoint sets of markings on G , such that $\mathcal{Z}_0 \cup \mathcal{Z} \cup \mathcal{Z}'$ is consistent. There are natural composition maps

$$(33) \quad \circ : CE^i(G; \mathcal{Z}_0 \cup \mathcal{Z}, \mathcal{Z}') \otimes CE^j(G; \mathcal{Z}_0, \mathcal{Z}) \rightarrow CE^{i+j}(G; \mathcal{Z}_0, \mathcal{Z} \cup \mathcal{Z}'),$$

obtained from the respective Hom complexes by restriction.

We define the *extended complex of positive pairs* associated to G to be

$$CE^*(G) = \bigoplus_{\mathcal{Z}_0, \mathcal{Z}} CP^*(G; \mathcal{Z}_0, \mathcal{Z}),$$

where the direct sum is over all collections $\mathcal{Z}_0, \mathcal{Z}$ such that $\mathcal{Z}_0 \cup \mathcal{Z}$ is consistent.

This breaks into a direct sum

$$CE^*(G) = \bigoplus_{k \geq 0} CE^*(G; k),$$

according to the cardinality k of \mathcal{Z} , i.e., the number of points marked for destabilization. Putting together the maps (33), we obtain global composition maps:

$$(34) \quad \circ : CE^i(G; k) \otimes CE^j(G; l) \rightarrow CE^{i+j}(G; k+l),$$

where the compositions are set to be zero when not a priori well-defined on the respective summands. These composition maps satisfy a Leibniz rule for the differential.

The complex $CE^*(G)$ was mentioned in the Introduction. There we stated Conjecture 1.1, which says that for any toroidal grid diagram G , we have

$$HE^d(G) = 0 \text{ when } d < 0.$$

Observe that Conjecture 1.1 would be a direct consequence of Conjecture 4.1, because of Equation (32).

We prove a weaker version of Conjecture 1.1, in terms of the sparse double $G_{\#}$ of G , cf. Section 2.11. We define a subcomplex of $CE^*(G_{\#})$, denoted

$$CE_{\#}^*(G) = \bigoplus_{k \geq 0} CE_{\#}^*(G; k),$$

to consist only of positive pairs coming from destabilization at the original markings of G , i.e., $\mathcal{Z}_0 \cup \mathcal{Z}$ should be disjoint from $\mathcal{Z}(\mathcal{U}_n)$, in the notation of Section 2.11. Observe that there is a natural composition map on $CE_{\#}^*(G)$, similar to (34).

Theorem 4.5. *Let G be a toroidal grid diagram representing a link L . Then $HE_{\#}^d(G; k) = 0$ whenever $d < \min\{0, 2 - k\}$.*

Proof. This follows from Theorem 4.3, using Equation (32). The key observation is that all destabilizations of the sparse double at markings away from \mathcal{U}_n are sparse. \square

4.3. Formal complex structures. Let G be a grid presentation for the link L . Our goal is to find an algorithm for computing $\mathbf{HF}_*^-(S_{\Lambda}^3(L), \mathbf{u})$, where Λ is a framing of L and \mathbf{u} a Spin^c structure on $S_{\Lambda}^3(L)$. We will first describe how to do so assuming that Conjecture 1.1 is true, and using Theorem 2.4.

By Theorem 2.4, we need to compute the homology of the complex $\mathcal{C}^-(G, \Lambda, \mathbf{u}) \subseteq \mathcal{C}^-(G\Lambda)$, with its differential \mathcal{D}^- . In the definition of \mathcal{D}^- we use the maps $\Phi_{\mathbf{s}}^{\vec{M}}$, which in turn are based on destabilization maps $\hat{D}_{\mathbf{s}}^{\vec{M}}$ of the kind constructed in Section 2.6. In turn, the maps $\hat{D}_{\mathbf{s}}^{\vec{M}}$, obtained by compression, are sums of compositions of maps of the type $D_{\mathbf{s}}^{\mathcal{Z}}$ (for unordered sets of markings \mathcal{Z}), as in Section 2.3.

Consider the maps

$$(35) \quad D_{\mathbf{s}}^{\mathcal{Z}} : \mathfrak{A}^-(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta}^{\mathcal{Z}_0}, \mathbf{s}) \rightarrow \mathfrak{A}^-(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta}^{\mathcal{Z}_0 \cup \mathcal{Z}}, \mathbf{s}),$$

for $\mathbf{s} \in J(\mathcal{Z}_0 \cup \mathcal{Z})$, compare Remark 2.2. Each such map is defined by counting holomorphic $(k+2)$ -gons of index $1 - k$ in $\text{Sym}^n(\mathcal{T})$, where k is the cardinality of \mathcal{Z} . The holomorphic polygon counts that appear are the same regardless of the value $\mathbf{s} \in J(\mathcal{Z}_0 \cup \mathcal{Z})$. Hence, if we fix \mathcal{Z}_0 and \mathcal{Z} , the map $D_{\mathbf{s}}^{\mathcal{Z}}$ for one value of \mathbf{s} determines those for all other \mathbf{s} .

Remark 4.6. One instance of this principle can be seen at the level of the maps $D_{\mathbf{s}}^{\vec{M}}$ defined in Equation (13). Fixing \vec{M} , the maps $D_{\mathbf{s}}^{\vec{M}}$ are determined by the map for one value of $\mathbf{s} \in J(\vec{M})$, using Equation (14). Indeed, in that situation the respective projection maps are invertible, essentially

because they satisfy Equation (14). Hence, taking $\vec{M}_1 = \vec{M}$ and $\vec{M}_2 = \vec{L} - M$, we find that the map $D_s^{\vec{M}}$ is determined by $D_{p^{\vec{L}-M}(s)}^{\vec{M}}$.

Pick some $s \in J_\infty(\mathcal{Z}_0 \cup \mathcal{Z}) \subset J(\mathcal{Z}_0 \cup \mathcal{Z})$. We will focus on the map (35) for that value s . We seek to understand the map $D_s^{\mathcal{Z}}$ combinatorially.

In the case $k = 0$, we know that holomorphic bigons are the same as empty rectangles on \mathcal{T} , cf. [10]. For $k = 1$, one can still count holomorphic triangles explicitly, cf. Section 3.4.

Unfortunately, for $k \geq 2$, the count of holomorphic $(k+2)$ -gons seems to depend on the almost complex structure on $\text{Sym}^n(\mathcal{T})$. The best we can hope for is not to calculate the maps $D^{\mathcal{Z}}$ explicitly, but to calculate them up to chain homotopy. In turn, this will give an algorithm for computing the chain complex $\mathcal{C}^-(G, \Lambda, u)$ up to chain homotopy equivalence, and this is enough for knowing its homology.

Recall that a complex structure j on the torus \mathcal{T} gives rise to a complex structure $\text{Sym}^n(j)$ on the symmetric product $\text{Sym}^n(\mathcal{T})$. In [19], in order to define Floer homology the authors used a certain class of perturbations of $\text{Sym}^n(j)$, which are (time dependent) almost complex structures on $\text{Sym}^n(\mathcal{T})$. For each almost complex structure J in this class, one can count J -holomorphic polygons for various destabilization maps. In particular, for any

$$\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta^{\mathcal{Z}_0}, \quad \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta^{\mathcal{Z}_0 \cup \mathcal{Z}}, \quad \tilde{\mathbf{y}} = U_1^{i_1} \dots U_\ell^{i_\ell} \mathbf{y}$$

there is a number

$$n_J^{\mathcal{Z}_0, \mathcal{Z}}(\mathbf{x}, \tilde{\mathbf{y}}) \in \mathbb{F}$$

such that the destabilization map (35) is given by

$$D_s^{\mathcal{Z}} \mathbf{x} = \sum_{\mathbf{y}} n_J^{\mathcal{Z}_0, \mathcal{Z}}(\mathbf{x}, \tilde{\mathbf{y}}) \cdot \tilde{\mathbf{y}}.$$

In other words, $n_J^{\mathcal{Z}_0, \mathcal{Z}}(\mathbf{x}, \tilde{\mathbf{y}})$ is the count of J -holomorphic $(k+2)$ -gons between \mathbf{x} and $\tilde{\mathbf{y}}$, in all possible homotopy classes ϕ with $\mu(\mathbf{x}, \mathbf{y}) = \mu(\phi) = 1 - k$, and coming from all possible orderings of the elements of \mathcal{Z} . (Recall that $D^{\mathcal{Z}}$ is a sum of maps of the form $D^{(Z_1, \dots, Z_k)}$, where (Z_1, \dots, Z_k) is an ordering of \mathcal{Z} .)

Observe that, in order for $n_J^{\mathcal{Z}_0, \mathcal{Z}}(\mathbf{x}, \tilde{\mathbf{y}})$ to be nonzero, the pair $[\mathbf{x}, \tilde{\mathbf{y}}]$ has to be positive. Hence, the set of values $n_J^{\mathcal{Z}_0, \mathcal{Z}}(\mathbf{x}, \tilde{\mathbf{y}})$ produces well-defined elements in the extended complex of positive pairs on G :

$$c_k(J) = \sum n_J^{\mathcal{Z}_0, \mathcal{Z}}(\mathbf{x}, \tilde{\mathbf{y}}) \cdot [\mathbf{x}, \tilde{\mathbf{y}}] \in CP^{1-k}(G; \mathcal{Z}_0, \mathcal{Z}) \subseteq CE^{1-k}(G; k), \quad k \geq 1.$$

Lemma 2.1 implies that the elements $c_k(J)$ satisfy the following compatibility conditions, with respect to the composition product (34):

$$dc_k(J) = \sum_{i=1}^{k-1} c_i(J) \circ c_{k-i}(J).$$

In particular, $dc_1(J) = 0$. Note that $c_1(J)$ is given by the count of snail-like domains, and therefore is independent of J . We denote it by

$$c_1^{\text{sn}} \in CE^0(G; 1).$$

Definition 4.7. A formal complex structure \mathfrak{c} on the grid diagram G consists of a family of elements

$$c_k \in CE^{1-k}(G; k), \quad k \geq 1,$$

satisfying $c_1 = c_1^{\text{sn}}$ and the compatibility conditions:

$$(36) \quad dc_k = \sum_{i=1}^{k-1} c_i \circ c_{k-i}.$$

In particular, an (admissible) almost complex structure J on $\text{Sym}^n(\mathcal{T})$ induces a formal complex structure $\mathfrak{c}(J)$ on G .

Remark 4.8. If we let $\mathfrak{c} = (c_1, c_2, \dots) \in CE^*(G)$, the relation (36) summarized by the equation

$$(37) \quad d\mathfrak{c} = \mathfrak{c} \circ \mathfrak{c}.$$

Definition 4.9. Two formal complex structures $\mathfrak{c} = (c_1, c_2, \dots)$, $\mathfrak{c}' = (c'_1, c'_2, \dots)$ on a grid diagram G are called homotopic if there exists a sequence of elements

$$h_k \in CE^{-k}(G; k), k \geq 1$$

satisfying $h_1 = 0$ and

$$(38) \quad c_k - c'_k = dh_k + \sum_{i=1}^{k-1} (c'_i \circ h_{k-i} + h_i \circ c_{k-i}).$$

Observe that, if J and J' are (admissible) almost complex structures on $\text{Sym}^n(\mathcal{T})$, one can interpolate between them by a family of almost complex structures. The resulting counts of holomorphic $(2+k)$ -gons of index $-k$ induce a homotopy between $\mathfrak{c}(J)$ and $\mathfrak{c}(J')$.

Lemma 4.10. Assume $HE^{1-k}(G; k) = 0$ for any $k \geq 2$. Then any two formal complex structures on G are homotopic.

Proof. Let $\mathfrak{c} = (c_1, c_2, \dots)$, $\mathfrak{c}' = (c'_1, c'_2, \dots)$ be two formal complex structures on G . We prove the existence of the elements h_k by induction on k . When $k = 1$, we have $c_1 = c'_1 = c_1^{\text{sn}}$ so we can take $h_1 = 0$.

Assume we constructed h_i for $i < k$ satisfying (38), and we need h_k . Since the cohomology group $CE(G; k)$ is zero in degree $1 - k$, it suffices to show that

$$c_k - c'_k - \sum_{i=1}^{k-1} (c'_i \circ h_{k-i} + h_i \circ c_{k-i})$$

is a cocycle. Indeed, we have

$$\begin{aligned} & d\left(c_k - c'_k - \sum_{i=1}^{k-1} (c'_i \circ h_{k-i} + h_i \circ c_{k-i})\right) \\ &= \sum_{i=1}^{k-1} c_i \circ c_{k-i} - \sum_{i=1}^{k-1} c'_i \circ c'_{k-i} - \sum_{i=1}^{k-1} (dc'_i \circ h_{k-i} + c'_i \circ dh_{k-i} + dh_i \circ c_{k-i} + h_i \circ dc_{k-i}) \\ &= \sum_{i=1}^{k-1} (c_i - c'_i - dh_i) \circ c_{k-i} + \sum_{i=1}^{k-1} c'_i \circ (c_{k-i} - c'_{k-i} - dh_{k-i}) - \sum_{i=1}^{k-1} dc'_i \circ h_{k-i} - \sum_{i=1}^{k-1} h_i \circ dc_{k-i} \\ &= \sum (c'_\alpha h_\beta c_\gamma + h_\alpha c_\beta c_\gamma) + \sum (c'_\alpha h_\beta c_\gamma + c'_\alpha c'_\beta h_\gamma) - \sum c'_\alpha c'_\beta h_\gamma - \sum h_\alpha c_\beta c_\gamma \\ &= 0. \end{aligned}$$

In the second-to-last line the summations are over $\alpha, \beta, \gamma \geq 1$ with $\alpha + \beta + \gamma = k$, and we suppressed the composition symbols for simplicity. \square

4.4. Combinatorial descriptions. Consider a formal complex structure \mathfrak{c} on a grid G , a framing Λ for L , and an equivalence class $\mathfrak{u} \in (\mathbb{H}(L) : H(L, \Lambda)) \cong \text{Spin}^c(S_\Lambda^3(L))$. We can define a complex $\mathcal{C}^-(G, \Lambda, \mathfrak{u}, \mathfrak{c})$ analogous to the complex $\mathcal{C}^-(G, \Lambda, \mathfrak{u})$ from Section 2.7, but defined using the elements c_k instead of the holomorphic polygon counts. In particular, if $\mathfrak{c} = \mathfrak{c}(J)$ for an actual almost complex structure J , then we recover the complex $\mathcal{C}^-(G, \Lambda, \mathfrak{u})$ from Section 2.7.

Taking into account the naturality properties of compression discussed at the end of Section 2.1, the proof of the following lemma is straightforward:

Lemma 4.11. *A homotopy between formal complex structures $\mathfrak{c}, \mathfrak{c}'$ on G induces a chain homotopy equivalence between the complexes $\mathcal{C}^-(G, \Lambda, \mathfrak{u}, \mathfrak{c})$ and $\mathcal{C}^-(G, \Lambda, \mathfrak{u}, \mathfrak{c}')$.*

If Conjecture 1.1 were true, we could then give the following combinatorial description of $\mathbf{HF}^-(S_\Lambda^3(L), \mathfrak{u})$: choose any formal complex structure on G , construct the complex $\mathcal{C}^-(G, \Lambda, \mathfrak{u}, \mathfrak{c})$, and take its homology. Indeed, note that $CE^*(G)$ is finite in each degree, and the number k of destabilization points is bounded above by the grid number, so the direct sum $\bigoplus_{k \geq 1} CE^{1-k}(G; k)$ is a finite set. Further, we know that a formal complex structure exists, because it could be induced by some almost complex structure J . Thus, we can find a formal complex structure \mathfrak{c} on G by a brute force approach: go over all the (necessarily finite) sequences $\mathfrak{c} = (c_1 = c_1^{\text{sn}}, c_2, c_3, \dots) \in \bigoplus_{k \geq 1} CE^{1-k}(G; k)$, and pick the first one that satisfies Equation (37). Then, by Lemma 4.10 we would know that all possible \mathfrak{c} 's are homotopic, and Lemma 4.11, together with Theorem 2.4, would tell us that the homology of $\mathcal{C}^-(G, \Lambda, \mathfrak{u}, \mathfrak{c})$ is indeed the right answer (after dividing out the factor corresponding to the homology of a torus).

However, since we cannot prove Conjecture 1.1 at the moment, we settle for a less efficient algorithm as follows. We use Theorem 2.10 instead of Theorem 2.4. Recall that the definition of the chain complex $\mathcal{C}_\#^-(G, \Lambda, \mathfrak{u})$ from Section 2.11 is based on counts of holomorphic $(k+2)$ -gons on the sparse double $G_\#$, but allowing destabilizations only at the original markings of G . These counts are based on the choice of an almost complex structure $J_\#$ on $\text{Sym}^{2n}(\mathcal{T}_\#)$. We can encode them into elements $c_k^\# \in CE_\#^{1-k}(G; k)$, where $CE_\#^*(G; k)$ are the complexes defined at the end of Section 4.2. When $k=1$ we get the count of snail-like domains $c_1^{\#, \text{sn}}$.

Definition 4.12. *A sparsed formal complex structure $\mathfrak{c}^\#$ on a grid diagram G consists of a family of elements*

$$c_k^\# \in CE_\#^{1-k}(G; k), \quad k \geq 1,$$

satisfying $c_1^\# = c_1^{\#, \text{sn}}$ and the compatibility conditions:

$$dc_k^\# = \sum_{i=1}^{k-1} c_i^\# \circ c_{k-i}^\#.$$

We can define the notion of homotopy of sparsed formal complex structures, by analogy with Definition 4.9. Theorem 4.5 then implies that any two sparsed formal complex structures on a grid are homotopic, by the same argument as in Lemma 4.10. A sparsed formal complex structure $\mathfrak{c}^\#$ produces a complex $\mathcal{C}_\#^-(G, \Lambda, \mathfrak{u}, \mathfrak{c}^\#)$, similar to the complex $\mathcal{C}_\#^-(G, \Lambda, \mathfrak{u})$ from Section 2.11, and chain homotopic to it.

Proof of Theorem 1.2. The algorithm to compute \mathbf{HF}^- of an integral surgery on a link \vec{L} goes as follows: start with a grid presentation G for \vec{L} , find a sparsed formal complex structure on G by a brute force approach, and compute the homology of the complex $\mathcal{C}_\#^-(G, \Lambda, \mathfrak{u}, \mathfrak{c}^\#)$. Theorem 2.10 and the observations above imply that this gives the correct answer.

Note that although this description is combinatorial in nature, the complex $\mathcal{C}_\#^-(G, \Lambda, \mathfrak{u}, \mathfrak{c}^\#)$ is still infinite dimensional, being an infinite direct product of modules over a ring of power series. However, we can replace it by a quasi-isomorphic, finite dimensional complex over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ using

the vertical and horizontal truncation procedures from [9, Section 6]. Taking the homology of the truncated complex is clearly an algorithmic task.

One can calculate the other versions of Heegaard Floer homology (\mathbf{HF}^∞, HF^+) in a similar way, using the sparse analog of Theorem 2.6 mentioned at the end of Section 2.11. \square

Proof of Theorem 1.3. One can calculate the maps induced by two-handle additions using the sparse analogs of Theorems 2.5 and 2.6, cf. Section 2.11. Building on this, we can calculate the mixed invariants of closed four-manifolds using the sparse analog of Proposition 2.9. In all cases, one proceeds by choosing an arbitrary sparsed formal complex structure $\mathfrak{c}^\#$ on the grid diagram G , and computing the respective groups or maps using polygon counts prescribed by $\mathfrak{c}^\#$. \square

Remark 4.13. Our present techniques do not give a combinatorial procedure for finding the rank (in a given degree) of an arbitrary cobordism map $F_{W,t}^-$. However, suppose W is a cobordism between connected three-manifolds Y_1 and Y_2 such that the induced maps $H_1(Y_1; \mathbb{Z})/\text{Tors} \rightarrow H_1(W; \mathbb{Z})/\text{Tors}$ and $H_1(Y_2; \mathbb{Z})/\text{Tors} \rightarrow H_1(W; \mathbb{Z})/\text{Tors}$ are surjective. Then the ranks of $F_{W,t}^-$ in fixed degrees can be computed using the same arguments as in [8, Section 4]. Indeed, they are determined by the ranks of the map induced by the two-handle additions which are part of the cobordism W .

5. SPARSE GRID DIAGRAMS

This section is devoted to the proof of Theorem 4.3. Let G be a sparse toroidal grid diagram, with some O 's marked for destabilization, as in Section 3.2. We define a filtration \mathcal{F} on $CP^*(G)$ as follows. Let us mark one X in each square of the grid with the property that neither its row nor its column contains an O marked for destabilization. (Note that these X 's have nothing to do with the original set of X 's from the link.) See Figure 10; and note also that in that figure, the squares marked by an X are in fact shaded.

Given a pair $[\mathbf{x}, \tilde{\mathbf{y}}]$, let (D, ϵ, ρ) be an enhanced domain from \mathbf{x} to $\tilde{\mathbf{y}}$. Let $X(D)$ be the number of X 's inside D , counted with multiplicity. Define

$$(39) \quad \mathcal{F}([\mathbf{x}, \tilde{\mathbf{y}}]) = -X(D) - \sum_{j=1}^k \rho_j.$$

It is easy to see that $X(D)$ does not change under addition of periodic domains. Since the same is true for the real multiplicities ρ_j , the value $\mathcal{F}([\mathbf{x}, \tilde{\mathbf{y}}])$ is well-defined. Furthermore, pre- or post-composing with a rectangle can only decrease \mathcal{F} , so \mathcal{F} is indeed a filtration on $CP^*(G)$.

To show that $HP^d(G) = 0$ (for $d > 0$) it suffices to check that the homology of the associated graded groups to \mathcal{F} are zero. In the associated graded complex $\text{gr}_{\mathcal{F}} CP^*(G)$, the differential only involves composing with rectangles r such that r is supported in a row or column going through some O_{i_j} marked for destabilization, but r does not contain O_{i_j} . We cannot post-compose with such a rectangle, because it would move the destabilization corners in \mathbf{y} , and that is not allowed. Thus, the differential of $\text{gr}_{\mathcal{F}} CP^*(G)$ only involves pre-composing with rectangles as above.

For each positive pair $\mathbf{p} = [\mathbf{x}, \tilde{\mathbf{y}}]$, we let $C\mathcal{F}^*(G, \mathbf{p})$ be the subcomplex of $\text{gr}_{\mathcal{F}} CP^*(G)$ generated by \mathbf{p} and those pairs related to \mathbf{p} by a sequence of nonzero differentials in $\text{gr}_{\mathcal{F}} CP^*(G)$. That is, for each complex over \mathbb{F} freely generated by a set S , we can form an associated graph whose set of vertices is S and with an edge from \mathbf{x} to \mathbf{y} ($\mathbf{x}, \mathbf{y} \in S$) whenever the coefficient of \mathbf{y} in $d\mathbf{x}$ is one. Then the graph of $C\mathcal{F}^*(G, \mathbf{p})$ is the connected component containing \mathbf{p} of the graph of $\text{gr}_{\mathcal{F}} CP^*(G)$ (with respect to the standard basis).

Definition 5.1. *Let G be a toroidal grid diagram with O_{i_j} marked for destabilization, $j = 1, \dots, k$. The four corners of the square containing O_{i_j} are called inner corners at O_{i_j} . An element $\mathbf{x} \in \mathbf{S}(G)$ is called inner if, for each $j = 1, \dots, k$, at least one of the inner corners at O_{i_j} is part of \mathbf{x} . The element \mathbf{x} is called outer otherwise.*

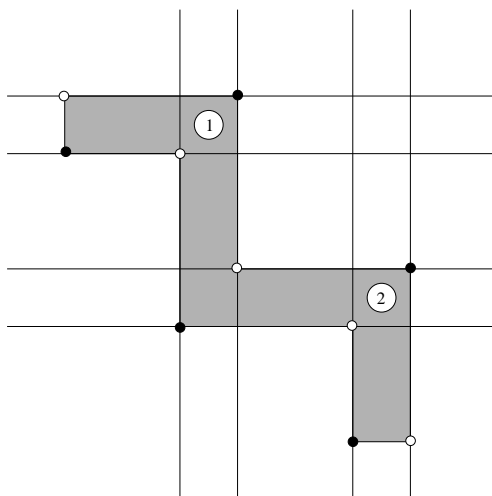


FIGURE 8. **An example of a positive domain with \mathbf{x} inner.** We have $k = 2$ and the two O 's marked for destabilization ($O_{i_j}, j = 1, 2$) are shown in the figure with the value of j written inside. We view this as a domain of type LL , meaning that $\epsilon_1 = \epsilon_2 = 0$. Both real multiplicities at the destabilization points are zero, so that $f_1 = f_2 = 1$. The domain has index 0, and the quantities n_1 and n_2 both equal $3 \cdot (1/4) + (3/4) + (1/4 + 3/4)/2 = 2$.

Lemma 5.2. *Let G be a sparse toroidal grid diagram with k O 's marked for destabilization, where $k \geq 2$. Let $\mathbf{p} = [\mathbf{x}, \tilde{\mathbf{y}}]$ an positive pair such that \mathbf{x} is inner. Then the index $I(\mathbf{p})$ is at least $2 - k$.*

Proof. Let (D, ϵ, ρ) be an enhanced domain going between \mathbf{x} and $\tilde{\mathbf{y}}$. Then

$$I(\mathbf{p}) = I(D, \epsilon, \rho) = I(D) - \sum_{j=1}^k (\epsilon_j + 2f_j)$$

by Equation (21).

Just as in the proof of Proposition 3.6, without loss of generality we can assume that every row or column contains at least one square where the multiplicity of D is zero; hence, as we move to an adjacent row (or column), the multiplicity of the domain D can only change by 0 or ± 1 .

According to Equation (22), the usual index $I(D)$ is given by the sum of the average multiplicities of D at the corners. For each $j = 1, \dots, k$, define n_j to be the sum of the average multiplicities at the corners situated on one of the following four lines: the two vertical lines bordering the column of O_{i_j} and the two horizontal lines bordering the row of O_{i_j} ; with the caveat that, if such a corner c (with average multiplicity a around it) appears on one of the four lines for O_{i_j} and also on one of the four lines for O_{i_l} , $l \neq j$, then we let c contribute $a/2$ to n_j and $a/2$ to n_l . For an example, see Figure 8. Note that, since the diagram is sparse, the average multiplicities at inner corners are only counted in one n_l .

We get

$$(40) \quad I(\mathbf{p}) \geq \sum_{j=1}^k (n_j - \epsilon_j - 2f_j).$$

We will prove that

$$(41) \quad n_j \geq 2f_j + \epsilon_j/2.$$

Indeed, the relations (25) imply that the average multiplicity at the destabilization point p_j for O_{i_j} (which is part of \mathbf{y}) is

$$\frac{a_j + b_j + c_j + d_j}{4} \geq f_j + \frac{2\epsilon_j - 1}{4}.$$

Since \mathbf{x} is inner, there is also one point of \mathbf{x} , call it x , in a corner of the square containing O_{i_j} . There are four cases, according to the position of x . We first consider the case when the marking at O_{i_j} is L (i.e., $\epsilon_j = 0$).

- (i) If x is the lower left corner (which is the same as the destabilization point), we have $a_j + d_j = b_j + c_j \geq 2f_j$ there, so the average multiplicity at x is at least f_j ; since the corner counts in both \mathbf{x} and \mathbf{y} , we get $n_j \geq 2f_j$, as desired.
- (ii) If x is the lower right corner, since $b_j, d_j \geq f_j$ and the multiplicity of D can change by at most ± 1 as we pass a column, we find that the average multiplicity at x is at least $f_j - 1/4$. Together with the contribution from $p_j \in \mathbf{y}$, this adds up to $2f_j - 1/2$. Suppose both the contributions from p_j and x are exactly $f_j - 1/4$. Then if r is a square other than O_j in the row through O_j , and s is the square directly below it, the local multiplicity at r is one greater than it is at s . This contradicts the positivity assumption combined with the assumption that there is at least one square with multiplicity 0 in the row containing O_j . Hence, the contribution from x and p_j is at least $2f_j$, as desired.
- (iii) The case when x is the upper left corner is similar to lower right, with the roles of the the row and the column through O_{i_j} swapped.
- (iv) Finally, if x is the upper right corner, then the average multiplicity there is at least $f_j - 3/4$. Together with the contribution from $p_j \in \mathbf{y}$, it adds up to $2f_j - 1$. There are two remaining corners on the vertical lines through p_j and x ; we call them $c_1 \in \mathbf{x}$ and $c_2 \in \mathbf{y}$, respectively. We claim that the contributions of the average multiplicities of c_1 and c_2 to n_j sum up to at least $1/2$. Indeed, if at least one of these average multiplicities is $\geq 3/4$, their sum is ≥ 1 , which might be halved (because the contribution is split with another n_l) to get $1/2$. If both of the average multiplicities are $1/4$ (i.e., both c_1 and c_2 are 90° corners), they must lie on the same horizontal line, and therefore their contributions are not shared with any of the other n_l 's; so they still add up to $1/2$. A similar argument gives an additional contribution of at least $1/2$ from the two remaining corners on the row through O_{i_j} . Adding it all up, we get $n_j \geq (2f_j - 1) + 1/2 + 1/2 = 2f_j$.

This completes the proof of Equation (41) when D is of type L at O_{i_j} . When D is of type R there (i.e., $\epsilon_j = 1$), the contribution of x to n_j is at least $f_j - \frac{3}{4}$. Studying the four possible positions of x , just as in the L case, gives additional contributions to n_j of at least 1, which proves Equation (41). In fact, the contributions are typically strictly greater than 1; the only situation in which we can have equality in (41) when D is of type R is when x is the upper right corner and the local multiplicities around x and p_j are exactly as in Figure 9.

Putting Equation (41) together with Inequality (40), we obtain

$$(42) \quad I(\mathbf{p}) \geq \sum_{j=1}^k (-\epsilon_j/2) \geq -k/2.$$

Our goal was to show that $I(\mathbf{p}) \geq 2 - k$. This follows directly from (42) for $k \geq 4$; it also follows when $k = 3$, by observing that $I(\mathbf{p})$ is an integer.

The only remaining case is $k = 2$, when we want to show $I(\mathbf{p}) \geq 0$, but Inequality (42) only gives $I(\mathbf{p}) \geq -1$. However, if $I(\mathbf{p}) = -1$ we would have equality in all inequalities that were used to arrive at Inequality (42). In particular, both destabilizations are of type R , the corresponding x 's are the upper right corners of the respective squares, and the local multiplicities there are as in Figure 9. Observe that as we move down from the row above O_{i_j} to the row containing O_{i_j} , the local multiplicity cannot decrease. The same is true as we move down from the row containing O_{i_j}

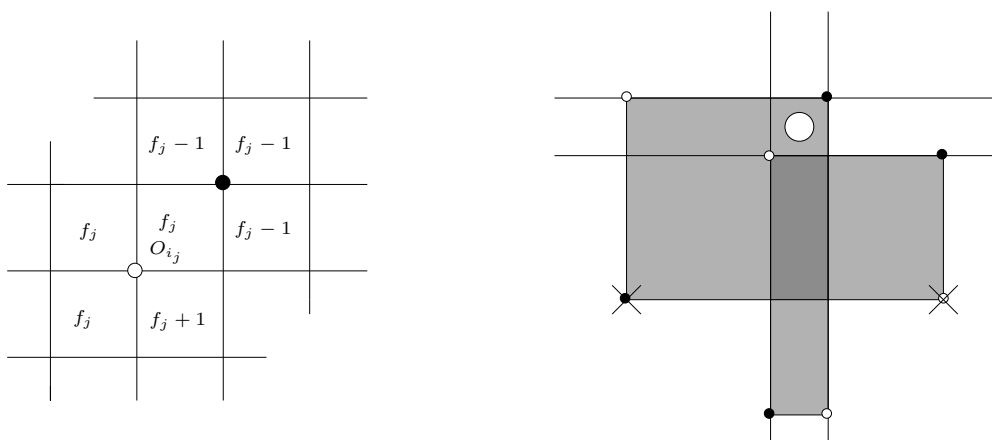


FIGURE 9. **Equality in Equation (41).** On the left, we show the local multiplicities around O_{i_j} in the case $\epsilon_j = 1, n_j = 2f_j + 1/2$. (Note that the real multiplicity at O_{i_j} is then zero.) On the right we picture a domain of this type, with $k = 1, f_1 = 1$. There are two corners, marked by \times , whose contributions are not counted in n_1 . As a consequence, the Inequality (40) is strict.

to the one just below. On the other hand, looking at the column of O_{i_1} , as we go down around the grid from the square just below O_{i_1} (where the multiplicity is $f_1 + 1$) to the one just above O_{i_1} (where the multiplicity is $f_j - 1$), we must encounter at least two horizontal circles where the multiplicity decreases. By our observation above, neither of these circles can be one of the two that bound the row through O_{i_2} . However, one or two of them could be the circles of the two remaining corners on the column through O_{i_1} . These corners only contribute to n_1 , not to n_2 , and since we had equality when we counted their contribution to be at least $1/2$, it must be the case that each of them is a 90° corner, with a contribution of $1/4$. This means that they must lie on the same horizontal circle. Hence, there must be one other horizontal circle along which local multiplicities of our domain decrease as we cross it from above. On this circle there are some additional corners, with a nontrivial contribution to $I(D)$ unaccounted for; compare the right hand side of Figure 9. (Note however that in that figure, we consider $k = 1$, rather than $k = 2$.) These vertices contribute extra to the vertex multiplicity, which means that $I(\mathbf{p}) > -1$. \square

Remark 5.3. It may be possible to improve the inequality (42) to $I(\mathbf{p}) \geq 0$ along the same lines, by doing a more careful analysis of the contributions to $I(D)$.

Lemma 5.4. *Let G be a sparse toroidal grid diagram, and $\mathbf{p} = [\mathbf{x}, \tilde{\mathbf{y}}]$ a positive pair such that \mathbf{x} is outer. Then the cohomology of $C\mathcal{F}^*(G, \mathbf{p})$ is zero.*

Proof. First, observe that, since the differential of $\text{gr}_{\mathcal{F}} CP^*(G)$ only involves pre-compositions, $\tilde{\mathbf{y}}$ is the same for all generators of $C\mathcal{F}^*(G, \mathbf{p})$. Furthermore, since \mathbf{x} is outer, all the other generators $[\mathbf{x}', \tilde{\mathbf{y}}]$ of $C\mathcal{F}^*(G, \mathbf{p})$ are outer.

Let j be such that no corner of O_{i_j} is in \mathbf{x} . Consider a new filtration \mathcal{G} on $C\mathcal{F}^*(G, \mathbf{p})$ given as follows. Let us mark one Y in each square of the grid that lies in a column or row through an O marked for destabilization, but does not lie in the column going through O_{i_j} (where j was chosen above), nor does it lie in the same square as one of the other O_{i_l} 's. Further, we mark $n - 1$ copies of Y in the square directly below the square of O_{i_j} . (Here n is the size of the grid.) Finally, we mark one extra Y in the square directly to the left of each O_{i_l} , $l \neq j$. See Figure 10 for an example. Observe that for every periodic domain equal to the row through some O_{i_l} minus the

O_1		Y Y		Y	Y Y Y Y
Y	O_2	Y	Y	Y	Y
	Y Y Y Y Y Y	O_3		Y	
		Y Y Y Y Y Y Y	O_4	Y	
Y		Y	Y Y	O_5	Y
				Y	O_6

FIGURE 10. **The markings defining the filtrations.** We show here a grid diagram of grid number 6, with two O 's marked for destabilization, namely $O_{i_1} = O_2$ and $O_{i_2} = O_5$. We draw small circles surrounding each of the two destabilization points. Each shaded square contains an X and this defines the filtration \mathcal{F} . One could also imagine an X marking in the upper right quadrant of each small destabilization disk, to account for the term ρ_j in Equation (39). Given an outer generator \mathbf{x} , we need to choose some $j \in \{1, 2\}$ such that no corner O_{i_j} is in \mathbf{x} . Suppose we chose $j = 1$. We then define a second filtration \mathcal{G} on the components of the associated graded of \mathcal{F} , using the Y markings as shown, plus five (invisible) Y markings in the lower left corner of each small destabilization disk. We first choose the Y markings in the unshaded squares, then mark the shaded squares so that every periodic domain has a total of zero markings, counted with multiplicities.

column through O_{i_l} (for some $l = 1, \dots, k$, including $l = j$), the signed count of Y 's in that domain is zero.

Consider now the squares in G that do not lie in any row or column that goes through an O marked for destabilization. We denote them by $s_{u,v}$, with $u, v \in \{1, \dots, n - k\}$, where the two indices u and v keep track of the (renumbered) row and column, respectively. Note that all these squares are already marked by an X , used to define the filtration \mathcal{F} . Additionally, we mark them with several Y 's, where the exact number $\alpha_{i,j}$ of Y 's in $s_{i,j}$ is to be specified soon.

Given one of the generators $\mathbf{p}' = [\mathbf{x}', \tilde{\mathbf{y}}]$ of $\mathcal{CF}^*(G, \mathbf{p})$, choose an enhanced domain (D, ϵ, ρ) from \mathbf{x}' to \mathbf{y} , and let $Y(D)$ be the number of Y 's inside D (counted with multiplicity). Set

$$(43) \quad \mathcal{G}(\mathbf{p}') = -Y(D) - (n - 1) \sum_{l=1}^k (c_l(D) - f_l(D) - \epsilon_l(D) + 1).$$

Here c_l, f_l, ϵ_l are as in Section 3.2. The second term in the formula above is chosen so that adding to D a column through some O_{i_l} does not change the value of \mathcal{G} ; this term can be interpreted as marking $n - 1$ additional Y 's in the lower left quadrant of the small destabilization disk around O_{i_l} .

We require that the quantity \mathcal{G} is well-defined, i.e. it should not depend on D , but only on the pair \mathbf{p}' . For this to be true, we need to ensure that the addition of a periodic domain to D does not

change \mathcal{G} . This can be arranged by a judicious choice of the quantities $\alpha_{u,v}$, for $u, v \in \{1, \dots, n-k\}$. Indeed, the only generators of the space of periodic domains that have not been already accounted for are those of the form: column minus row though some O not marked for destabilization. Note that there is a permutation σ of $\{1, \dots, n-k\}$ such that the unmarked O 's are in the squares $s_{u\sigma(u)}$. There are $n-k$ conditions that we need to impose on $\alpha_{u,v}$, namely

$$(44) \quad \sum_{v=1}^{n-k} \alpha_{uv} - \sum_{v=1}^{n-k} \alpha_{v\sigma(u)} = t_u,$$

for $u = 1, \dots, n-k$. Here t_u are determined by the number of Y 's that we already marked in the respective row and column (as specified above, in squares not marked by X), with an extra contribution in the case of the row just below some O_{i_l} and the column just to the left of O_{i_l} , to account for the term $c_l(D) - f_l(D) - \epsilon_l(D) + 1$ from Equation (43).

Note that $\sum_{u=1}^{n-k} t_u = 0$. Hence, there exists a solution (in rational numbers) to the linear system described in Equation (44). By multiplying all the values in the solution by a large integer, and then adding a sufficiently large constant, we can obtain a solution in nonnegative integers, which we take to be our definition of α_{uv} .

We have now arranged so that \mathcal{G} is an invariant of \mathbf{p}' . Moreover, pre-composing with a rectangle can only decrease \mathcal{G} , and it keeps \mathcal{G} the same only when the rectangle (which a priori has to be supported in one of the rows and columns through some O_{i_l}) is actually supported in the column through O_{i_j} , and does not contain the square right below O_{i_l} .

It follows that \mathcal{G} is indeed a filtration on $C\mathcal{F}^*(G, \mathbf{p})$. We denote the connected components of the associated graded by $C\mathcal{G}^*(G, j, \mathbf{p}')$; it suffices to show that these have zero cohomology. Without loss of generality, we will focus on $C\mathcal{G}^*(G, j, \mathbf{p})$.

The complex $C\mathcal{G}^*(G, j, \mathbf{p})$ can only contain pairs $[\mathbf{x}', \tilde{\mathbf{y}}]$ such that \mathbf{x}' differs from \mathbf{x} by either pre- or post-composition with a rectangle supported in the column through O_{i_j} . The condition that this rectangle does not contain the square right below O_{i_l} is automatic, because \mathbf{x} and \mathbf{x}' are outer.

We find that there can be at most two elements in $C\mathcal{G}^*(G, j, \mathbf{p})$, namely $\mathbf{p} = [\mathbf{x}, \tilde{\mathbf{y}}]$ and $\mathbf{p}' = [\mathbf{x}', \tilde{\mathbf{y}}]$, where \mathbf{x}' is obtained from \mathbf{x} by switching the horizontal coordinates of the components of \mathbf{x} in the two vertical circles bordering the column of O_{i_j} . Provided that \mathbf{p}' is positive, the pairs \mathbf{p} and \mathbf{p}' are related by a differential, so the cohomology of $C\mathcal{G}^*(G, j, \mathbf{p})$ would indeed be zero.

Therefore, the last thing to be checked is that \mathbf{p}' is positive. We know that \mathbf{p} is positive, so we can choose a positive enhanced domain D representing \mathbf{p} . Recall that the destabilization point near O_{i_j} is denoted p_j and is part of \mathbf{y} . Draw a vertical segment S going down the vertical circle from p_j to a point of \mathbf{x} . There are three cases:

- (i) There is no point of \mathbf{x}' on the segment S . Then there exists a rectangle going from \mathbf{x}' to \mathbf{x} , and \mathbf{p}' appears in $d\mathbf{p}$ by pre-composition. Adding the rectangle to D preserves positivity;
- (ii) There is a point of \mathbf{x}' on S , and the multiplicity of D just to the right of S is nonnegative. Then there is a rectangle, just to the right of S , going from \mathbf{x} to \mathbf{x}' . We get a positive representative for \mathbf{p}' by subtracting this rectangle from D ;
- (iii) There is a point of \mathbf{x}' on S , and the multiplicity of D is zero somewhere just to the right of S . Note that as we pass the segment S from left to right the difference in multiplicity is constant; since D is positive, this difference must be nonnegative. In particular, $c_j \geq d_j$. Relation (23) implies $a_j \geq b_j + 1$. Looking at the inequalities in (25), we see that we can use two of them (the ones involving b_j and d_j) to improve the other two:

$$(45) \quad a_j \geq b_j + 1 \geq f_j + 1, \quad c_j \geq d_j \geq f_j + \epsilon_j.$$

Let us add to D the periodic domain given by the column through O_{i_j} . This increases b_j, d_j , and f_j by 1 while keeping a_j and c_j constant. Nevertheless, Inequality (45) shows that the inequalities (25) are still satisfied for the new domain \tilde{D} . Thus \tilde{D} is positive, and

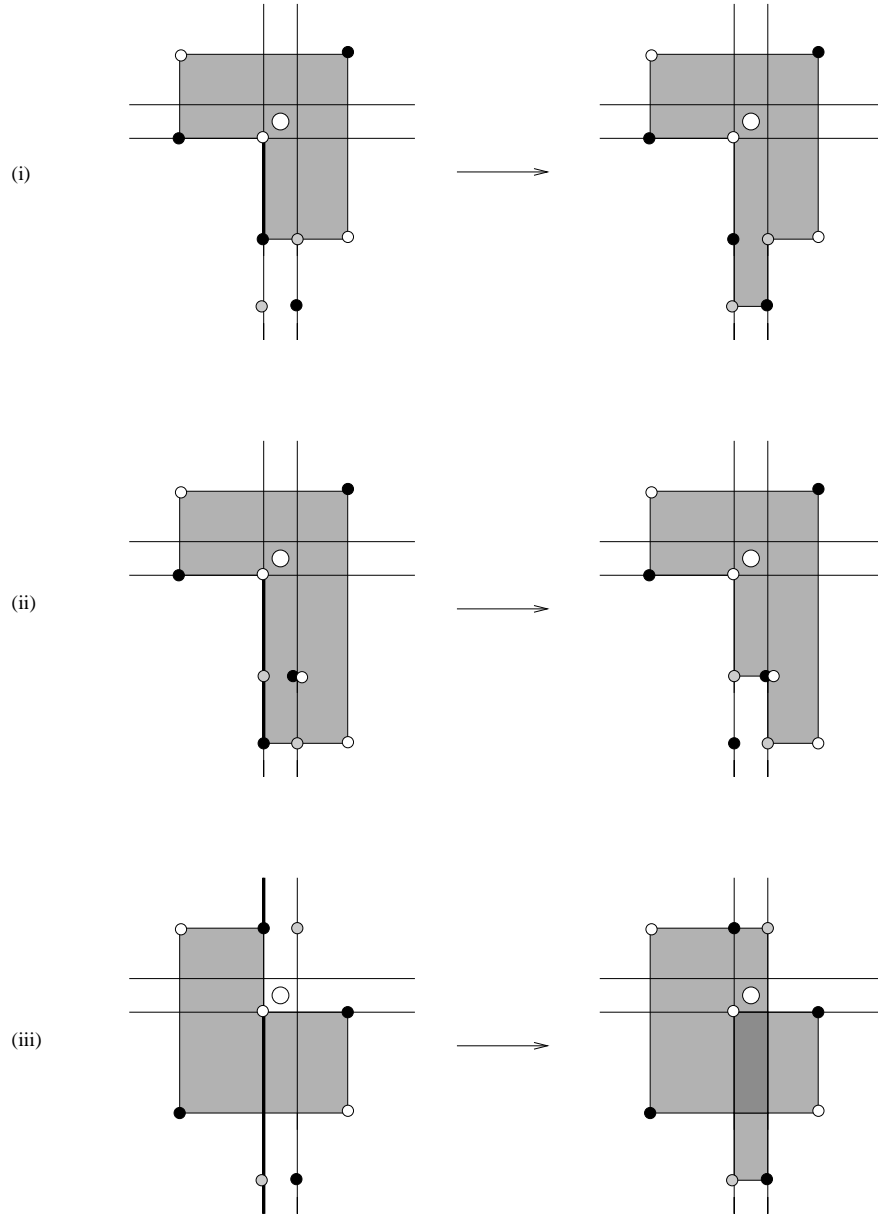


FIGURE 11. **The three cases in Lemma 5.4.** On the left of each picture we show a positive L domain representing $\mathbf{p} = [\mathbf{x}, \tilde{\mathbf{y}}]$. These domains have index 0, 2 and 2, respectively (assuming that all the real multiplicities are zero). On the right of each picture we show the corresponding positive domain representing $\mathbf{p}' = [\mathbf{x}', \tilde{\mathbf{y}}]$. These domains have all index 1. The components of \mathbf{x} are the black dots, the components of \mathbf{x}' the gray dots, and those of \mathbf{y} the white dots. The segment S is drawn thicker.

its multiplicity just to the right of S is everywhere nonnegative. We can then subtract a rectangle from \tilde{D} to obtain a positive representative for \mathbf{p}' .

The three cases are pictured in Figure 11. □

Proof of Theorem 4.3. The case when $k = 0$ is trivial, and the case $k = 1$ follows from Proposition 3.6. For $k \geq 2$, Lemma 5.2 says that for any pair $[\mathbf{x}, \tilde{\mathbf{y}}]$ of index $< 2 - k$, the generator \mathbf{x} is

outer. Lemma 5.4 then shows that the homology of $\text{gr}_{\mathcal{F}} CP^*(G)$ is zero in the given range, which implies the same for $HP^*(G)$. \square

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DEPARTMENT OF MATHEMATICS, UCLA, 520 PORTOLA PLAZA, LOS ANGELES, CA 90095
E-mail address: cm@math.ucla.edu

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, 2990 BROADWAY, NEW YORK, NY 10027
E-mail address: petero@math.columbia.edu

DEPARTMENT OF MATHEMATICS, BARNARD COLLEGE, COLUMBIA UNIVERSITY, NEW YORK, NY 10027
E-mail address: dthurston@barnard.edu