Math 225A: Practice Final

The final will consist of 7 problems, to be solved in 3 hours. No books or notes are allowed. Each problem is worth 10 points.

Some of the problems will be taken from the problem sets, and in some you will be asked to prove something that was proved in class. There will also be a few new problems.

Be wary of writing too much on any given problem: you may run out of time. Some of the problems require just a few lines for a complete solution. A sketch of the proof will produce partial or full credit, depending on how sketchy it is. I suggest going over all the problems first, and writing up short proofs. At the end, if you have time and you think some of your solutions are not detailed enough, you can go back and fill in the details.

Here is a practice exam:

1. Let M be the set of all lines in $\mathbb{R}^2$ (not just those which pass through the origin). Show that M is a smooth manifold and identify it with a well-known space.

2. Let X be a connected smooth manifold, and $x, y \in X$. Show that there exists a diffeomorphism $f : X \to X$ such that $f(x) = y$.

3. Let X be a k-dimensional submanifold of $\mathbb{R}^n$ for $n > 2k + 1$. Show that X admits an injective immersion into $\mathbb{R}^{n-1}$.

4. Let M be a smooth compact manifold of dimension n. Show that there is no immersion of M into $\mathbb{R}^n$.

5. A smooth n-dimensional manifold is called parallelizable if it admits n vector fields that are linearly independent at each point. Give an example of a manifold X such that X is not parallelizable, but $X \times S^1$ is. Justify your answer.

6. Show rigorously that the Möbius band is not orientable.

7. Show that if X is a smooth, compact, oriented manifold with $\chi(X) = 0$, then X admits a nonvanishing vector field. (For this problem, you can assume the truth of the Hopf Degree Theorem.)