Math 225A: Problem Set 3
due Wednesday, November 11

1. Guillemin and Pollack Ch. I: problems 2, 9, 10 from §5 p.32-33 ; problem 7 from §6 p.38 ; problems 7, 9, 10 from §8 p.55.

2. The manifold $S(X)$ from Guillemin and Pollack problem 9 in §8 is usually called the *unit tangent bundle* of X, and denoted $UT(X)$. Its definition depends on the embedding of X into a Euclidean space, so that the length of tangent vectors is well-defined. It turns out that different embeddings give rise to diffeomorphic unit tangent bundles, but you do not have to show that.

Show that $UT(S^2)$, the real projective space $\mathbb{RP}^3$, and the Lie group $SO(3) = \{A \in M_{3\times 3}(\mathbb{R}) | AA^t = I,\ \det(A) = 1\}$ are all homeomorphic. (In fact, they are diffeomorphic, but I don't want to make you check that the maps you'll be defining are smooth.)

3. Let Φ be a global flow on a smooth manifold X, and $\gamma : \mathbb{R} \to X$ a flow line of Φ. Show that one of the following possibilities holds true:

- γ is an injective immersion;
- γ is a periodic immersion, that is, it is an immersion and there exists $T > 0$ such that $\gamma(s) = \gamma(t)$ if and only if $s = t + kT$ for some $k \in \mathbb{Z}$;
- γ is constant.

If γ is a periodic immersion, its image is called a *closed orbit* of the flow.

4. Let X be a smooth manifold, and $Y \subset X$ a submanifold diffeomorphic to S^1. Use a partition of unity to show that Y is a closed orbit of a global flow on X.

5. Give an example of an injective immersion $\gamma : \mathbb{R} \to \mathbb{R}^2$ that cannot be a flow line of any global flow. (*Hint:* Arrange so that there is a sequence of real numbers $(t_n)_{n\geq 0}$ with $\gamma(t_n)$ converging to a point in $\mathbb{R}^2$, but $\gamma'(t_n)$ not converging to any vector.)

6. (a) Recall that there is a natural projection map $p : \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$. Show that its restriction to the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ produces a fiber bundle over $\mathbb{CP}^{n-1}$ with fiber S^1.

(b) Recall that $\mathbb{CP}^1$ and S^2 are diffeomorphic. Thus there is a fiber bundle $\pi : S^3 \to S^2$ with fiber S^1. This is called the *Hopf fibration*. Write down explicitly the formula for π, and draw two distinct fibers in $\mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$.

7. Construct a fiber bundle with total space the Lie group $O(n)$, base S^{n-1}, and fiber $O(n-1)$.

8. Let X be a compact manifold and $f : X \to \mathbb{R}$ a Morse function with only two critical points. Show that X is homeomorphic to a sphere. (*Hint:* Suppose 0 and 1 are the critical points. Use the Morse lemma to show that $f^{-1}(-\infty, \epsilon)$ and $f^{-1}(1-\epsilon, \infty)$ are diffeomorphic to balls, for small $\epsilon > 0$. Then use Ehresmann's theorem to understand what happens over the interval $(\epsilon, 1-\epsilon)$.)

9. Let $P(n, d)$ denote the vector space of all homogeneous polynomials in $\mathbb{C}[z_0, \ldots, z_n]$ of degree $d \geq 1$. For $f = f(z_1, \ldots, z_n) \in P(n, d)$, let

$$H_f = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{CP}^n | f(z_0, \ldots, z_n) = 0\}$$

be the zero set of f. Note that since the polynomial f is homogeneous, if an n-tuple $z = (z_0, \ldots, z_n)$ is a zero of the polynomial f, then any scalar multiple of z is also a zero, so H_f is well-defined.

(a) Calculate the dimension of $P(n,d)$ as a complex vector space.

(b) Consider the set

$$Z = \{(z,f) \in \mathbb{CP}^n \times P(n,d) \mid f(z) = 0\}.$$

Show that Z is a submanifold of $\mathbb{CP}^n \times P(n,d)$.

(c) Consider the map $p : Z \to P(n,d)$ obtained by composing the inclusion of Z in $\mathbb{CP}^n \times P(n,d)$ with the projection to the second factor. Show that $f \in P(n,d)$ is a regular value of p if and only if the complex valued functions

$$f, \frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}$$

have no common zero in $\mathbb{C}^{n+1} \setminus \{0\}$.

(d) Assume that the set $U(n,d) \subset P(n,d)$ of regular values of p is connected. (You do not have to prove this.) Deduce that for $f \in U(n,d), H_f$ is a smooth manifold whose diffeomorphism type is independent of f. We let $H(n,d)$ denote any manifold of the form $H_f, f \in U(n,d)$. This is called a *smooth hypersurface of degree d in* $\mathbb{CP}^n$, and plays an important role in algebraic geometry.

(e) Show that $H(n,1)$ is diffeomorphic to $\mathbb{CP}^{n-1}$.

(f) Show that $H(2,2)$ is diffeomorphic to S^2. (*Hint:* Consider the polynomial $z_0^2 - z_1 z_2$.)

(g) Show that $H(3,2)$ is diffeomorphic to $S^2 \times S^2$. (*Hint:* Consider the polynomial $z_0 z_1 - z_2 z_3$.)