Math 225A: Problem Set 2
due Wednesday, October 28

1. Let X be a smooth manifold, and $p \in X$. Recall that a *derivation at* p is a linear map $L : C^\infty(X) \to \mathbb{R}$ such that

$$L(fg) = L(f)g(p) + L(g)f(p),$$

for all $f, g \in C^\infty(X)$. Let $\mathcal{L}(X,p)$ be the vector space of derivations at p.

In this problem you will fill in the details of the proof that $\mathcal{L}(X,p)$ is isomorphic to the tangent space T_pX.

Choose a smooth bump function $\beta : X \to \mathbb{R}$ such that $\beta(x) = 1$ for all x in a neighborhood of p. For $i = 1, \ldots, n$, we let $x^i : \mathbb{R}^n \to \mathbb{R}$ be the ith coordinate function, that is, $x^i(x_1, \ldots, x_n) = x_i$. If $(U, \varphi : U \to \tilde{U} \subseteq \mathbb{R}^n)$ is a smooth chart near p, we defined linear maps

$$F : \mathcal{L}(X,p) \to T_pX, \;\; F(L) = (L(\beta \cdot (x^1 \circ \varphi)), \ldots, L(\beta \cdot (x^n \circ \varphi))$$

and

$$G : T_pX \to \mathcal{L}(X,p), \;\; G(U, \varphi, \vec{v})(f) = D_{\vec{v}}(f \circ \varphi^{-1})|_{\varphi(p)} \text{ for all } f \in C^\infty(X).$$

(a) Check that G is well-defined, that is, it does not depend on the chosen chart (U, φ).

(b) Check that $F \circ G$ is the identity.

(c) Let $g : V \to \mathbb{R}$ be a smooth function defined on a convex open neighborhood V of 0 in $\mathbb{R}^n$. Suppose $g(0) = 0$. Show that there exists smooth functions $h_1, \ldots, h_n : U \to \mathbb{R}$ such that

$$g = \sum_{i=1}^{n} x^i h_i.$$

(*Hint*: Choose $h_i(x) = \int_0^1 \frac{\partial}{\partial x_i} g(tx) dt$.)

(d) Show that $L(\beta f) = L(f)$, for any $L \in \mathcal{L}(X,p)$ and $f \in C^\infty(X)$.

(e) Check that $G \circ F$ is the identity. Conclude that F is well-defined, that is, it does not depend on the chosen chart.

2. Show that the boundary of a square

$$S = \{(x,y) \in \mathbb{R}^2 | \max(|x|, |y|) = 1\}$$

is not a smooth submanifold of $\mathbb{R}^2$.

3. Let $X \subseteq \mathbb{R}^n$ be a smooth submanifold. Show that a map $f : X \to \mathbb{R}^m$ is smooth if and only if for any $x \in X$, there exists a neighborhood $U \subseteq \mathbb{R}^n$ of x in $\mathbb{R}^n$ and a smooth map $g : U \to \mathbb{R}^m$ such that $g|_{X \cap U} = f|_{X \cap U}$. (This shows that the definition of smooth maps from class is the same as the one given in Guillemin and Pollack, Ch. I §1.)

4. Let $f : X \to Y$ be an embedding of manifolds. Show that f is proper if and only if the image $f(X)$ is closed.

5. If A and B are two disjoint, closed subsets of a smooth manifold X, show that there is a smooth function $\beta : X \to [0,1] \subset \mathbb{R}$ such that $\beta = 0$ on A and $\beta = 1$ on B. (*Hint:* Use partitions of unity.)

6. Problems 1, 2, 7, 10, 11, 13 from Guillemin and Pollack, Ch.1 §4, p. 25-27.