

Math 225A: Problem Set 1

due Wednesday, October 14

1. Show that a locally compact, Hausdorff topological space is regular.
2. Prove that the union of the two coordinate axes in \mathbb{R}^2 is not a manifold.
3. For $n \geq 1$, prove that the sphere S^n cannot admit an atlas composed of a single chart.
4. Let

$$X = \{(x, y) \in \mathbb{R}^2 \mid \max(|x|, |y|) = 1\}$$

be the boundary of a square. Show how to equip X with the structure of a smooth manifold.

5. Let $\mathbb{C}\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ be the complex projective space, with the smooth structure constructed in class.

- (a) Show that the natural projection $p: \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}\mathbb{P}^n$ is a smooth map.
- (b) Show that $\mathbb{C}\mathbb{P}^1$ is diffeomorphic to S^2 .

6. Let X be a (nonempty) smooth manifold of dimension $n \geq 1$. Show that X admits uncountably many distinct smooth structures. (Note that here we do not ask for “smooth structures up to diffeomorphism.” As mentioned in the first lecture, if X has dimension at most three, then all smooth structures on X are diffeomorphic.)

7. Let X be a connected smooth manifold, and $x, y \in X$. Show that there exists a diffeomorphism $f: X \rightarrow X$ such that $f(x) = y$. (*Hint:* It suffices to prove this when x and y are in the same coordinate chart.)

8. Let $B(k, n) \subset (\mathbb{R}^n)^k = \mathbb{R}^{nk}$ be the space of k -tuples of linearly independent vectors in \mathbb{R}^n , with the subspace topology induced from \mathbb{R}^{nk} . We define an equivalence relation \sim on $B(k, n)$ by saying that two k -tuples are equivalent if they span the same subspace of \mathbb{R}^n . The quotient $Gr(k, n) = B(k, n)/\sim$ (with the quotient topology) is called the *Grassmannian*. Its elements are the k -dimensional subspaces of \mathbb{R}^n .

- (a) Equip $Gr(k, n)$ with the structure of a smooth manifold.
- (b) Show that $Gr(k, n)$ is diffeomorphic to $Gr(n - k, n)$.

Extra credit: Give an example of a locally compact topological space X and point $x \in X$ with the property that x has no neighborhood whose closure is compact.