Inverse Strichartz estimates are a fundamental tool for studying nonlinear Schrödinger equations at critical regularity. For $L^2$ initial data, such estimates have been investigated in the context of the constant-coefficient equation, and exploit connections with Fourier restriction theory. The diminished utility of the Fourier transform presents a significant obstacle to the analysis for nontranslation-invariant equations.

Motivated by applications to mass-critical NLS, we prove an inverse Strichartz estimate given $L^2$ initial data for a class of Schrödinger operators in one space dimension with a class of potentials that includes the free particle (studied previously by Carles-Keraani via Fourier-analytic methods) and the harmonic oscillator. Consequently, we obtain a linear profile decomposition for such operators.

1. INTRODUCTION

The classical Strichartz inequality for the Schrödinger equation states that if $u(t, x)$ is a solution to the equation for a free particle

$$i\partial_t u = -\frac{1}{2} \Delta u, \quad u(0, \cdot) \in L^2(\mathbb{R}^d),$$

then

$$\left\| u \right\|_{L^{\frac{2(d+2)}{d+2}}(\mathbb{R} \times \mathbb{R}^d)} \leq C \left\| u(0) \right\|_{L^2(\mathbb{R}^d)}.$$

Whereas Strichartz inequalities are used to control the solution in terms of the initial data, inverse Strichartz inequalities do the opposite, and characterize initial data that give rise to solutions with large spacetime norm. Roughly speaking, an inverse Strichartz estimate says that the left side of (1.1) can be a nontrivial fraction of the right side only when the initial data $u(0)$ contains a “bubble of concentration”, by which we mean a function $\phi$ with a characteristic length scale, frequency, and location, which when removed from the initial data yields a remainder $u(0) - \phi$ with strictly smaller mass and linear evolution. One may remove such bubbles iteratively to decompose $u$ into essentially noninteracting “profiles”. Such decompositions are basic to the analysis of nonlinear Schrödinger equations.

Our interest in inverse estimates and profile decompositions arises from considering semilinear Schrödinger equations of the form

$$i\partial_t u = \left( -\frac{1}{2} \Delta + V \right) u + |u|^{4/d} u, \quad u(0, \cdot) \in L^2(\mathbb{R}^d),$$

for some real potential $V$. Solutions to the above formally conserve mass

$$M[u(t)] := \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx = M[u(0)].$$

When $V = 0$ the problem is called mass-critical because the scaling $u_\lambda(t, x) = \lambda^{-d/2} u(\lambda^{-2} t, \lambda^{-1} x)$ preserves the class of solutions and the mass. This problem was recently shown by Dodson to be globally wellposed \cite{4, 5, 6}. While discussing the PDE details would lead us too far astray, we mention that the proof uses the concentration compactness and rigidity approach and builds on the work of Tao-Visan-Zhang constructing minimal-mass blowup solutions \cite{17}, which in turn relies on inverse variants of the above Strichartz estimate \cite{4} developed by Carles-Keraani \cite{3} for $d = 1$, Moyua-Vargas-Vega \cite{15} and Merle-Vega \cite{14} for $d = 2$, and Begout-Vargas \cite{1} for $d \geq 3$. Regarding these estimates, see also the exposition in \cite{12}.

We wish to eventually investigate \cite{12} for more general potentials such as the harmonic oscillator. It turns out that the isotropic harmonic oscillator can be reduced to the free particle equation via the lens
Analogous relation holds for time-dependent harmonic oscillators of the form \( V = a(t)|x|^2 \); see \([2]\) and the references therein. However, this trick is quite fragile and breaks down for instance for an anisotropic harmonic oscillator \( V(x_1, x_2) = \omega_1 x_1^2 + \omega_2 x_2^2, \omega_1 \neq \omega_2 \). Studying \([1,2]\) in greater generality therefore requires a more robust line of attack, such as the induction on energy and concentration-compactness paradigm.

Implementing that strategy requires appropriate inverse Strichartz estimates and profile decompositions for the linear propagator. In this paper we obtain such estimates in the simplest case \( d = 1 \). The main new difficulty is that as the equation is not translation-invariant, the Fourier-analytic techniques (such as restriction estimates) underpinning the corresponding results for the constant coefficient equation do not carry over. We shall work instead in physical space.

Consider a family of Schrödinger operators on the real line

\[
H(t) = -\frac{1}{2} \partial_x^2 + V(t, x),
\]

where the potential conforms to the following hypotheses:

- \( V \) is subquadratic in the sense that there exists \( M_k < \infty \) for each \( k \geq 2 \) so that

\[
\|\partial_x^k V(t, x)\|_{L^\infty_x} + \|\partial_x^k \partial_t V(t, x)\|_{L^\infty_x} \leq M_k.
\]

- The third derivatives satisfy the decay condition that for some \( \varepsilon > 0 \),

\[
|x|^{1 + \varepsilon} \partial_x^2 V | + |x|^{1 + \varepsilon} \partial_x^3 V | \in L^\infty_t L^2_x.
\]

This implies by the fundamental theorem of calculus that the second derivative \( \partial_x^2 V(t, x) \) converges as \( x \to \pm \infty \). Here and throughout this paper we use the bracket notation \( \langle x \rangle = (1 + |x|^2)^{1/2} \).

Whereas the first condition is quite natural, we impose the artificial second condition to avoid possibly logarithmic growth in certain estimates (see the discussion surrounding Lemma 4.7). The two most basic examples of such potentials are \( V = 0 \) and \( V = \frac{1}{2} x^2 \).

The propagator \( U(t, s) \) for such Hamiltonians (which is a one-parameter unitary group \( U(t, s) = e^{-i(t-s)H} \) when \( V = V(x) \) is time-independent) is known (see Corollary 2.4 below) to obey Strichartz estimates at least locally in time:

\[
\|U(t, s)f\|_{L^6_t(x) \times \mathbb{R}} \lesssim I \|f\|_{L^2(\mathbb{R})}
\]

for any compact interval \( I \) and any fixed \( s \in \mathbb{R} \).

Our main result is an inverse form of this inequality which, as previously mentioned, asserts that nontriviality of the left side above implies concentration in the initial data. We shall detect such concentration by correlation with a suitably scaled, translated, and modulated test function.

To state the theorem, we need some notation. For \( \lambda > 0 \) and \((x_0, \xi_0) \in T^* \mathbb{R} \equiv \mathbb{R}_x \times \mathbb{R}_\xi \), define the scaling and phase space translation operators

\[
S_\lambda f(x) = \lambda^{-1/2} f(\lambda^{-1} x), \quad \pi(x_0, \xi_0) f(x) = e^{i(x-x_0)\xi_0} f(x - x_0).
\]

Throughout this paper we let \( \psi \) denote a fixed real, even Schwartz function \( \psi \in \mathcal{S}(\mathbb{R}) \) with \( \|\psi\|_{L^2} = (2\pi)^{-1/2} \). Then \( \pi(x_0, \xi_0) \psi \) is concentrated in space at \( x_0 \) and in frequency at \( \xi_0 \).

**Theorem 1.1.** There exists \( \beta > 0 \) such that if \( 0 < \varepsilon \leq \|U(t, 0)f\|_{L^6([\frac{1}{2}, \frac{3}{2}] \times \mathbb{R})} \) and \( \|f\|_{L^2} \leq A \), then

\[
\sup_{z \in T^* \mathbb{R}, 0 < \lambda \leq 1, |\xi| \leq 1/2} \|\pi(z)S_\lambda \psi, U(t, 0)f\|_{L^2(\mathbb{R})} \geq C\varepsilon (\frac{\xi_0}{\lambda})^\beta
\]

for some constant \( C \) depending on the seminorms in \([1.3]\) and \([1.4]\).

In Section 3 we use this to construct a linear profile decomposition, which will follow essentially from repeatedly applying the following corollary. For simplicity we state it assuming the potential is time-independent (so that \( U(t, 0) = e^{-itH} \)).
Corollary 1.2. Let \( \{f_n\} \subset L^2(\mathbb{R}) \) be a sequence such that \( 0 < \varepsilon \leq \|e^{-itH}f_n\|_{L^2_{\tau,s}([-\frac{1}{2}, \frac{1}{2}] \times \mathbb{R})} \) and \( \|f\|_{L^2} \leq A \) for some constants \( A, \varepsilon > 0 \). Then, after passing to a subsequence, there exist a sequence of parameters
\[
((\lambda_n, t_n, z_n))_n \subset (0, 1] \times [-1/2, 1/2] \times T^* \mathbb{R}
\]
and a function \( 0 \neq \phi \in L^2 \) such that,
\[
S_{\lambda_n}^{-1} \pi(z_n)^{-1} e^{-it_nH} f_n \to \phi \text{ in } L^2
\]
(1.6)
Further,
\[
\|\phi\|_{L^2} \gtrsim \varepsilon (\frac{\varepsilon}{A})^\beta
\]
(1.7)
Proof. By Theorem 1.1, there exist \( (\lambda_n, t_n, z_n) \) such that \( |\langle \pi(z_n)S_{\lambda_n} \psi, e^{-it_nH} f_n \rangle| \gtrsim \varepsilon (\frac{\varepsilon}{A})^\beta \). As the sequence \( S_{\lambda_n}^{-1} \pi(z_n)^{-1} e^{-it_nH} f_n \) is bounded in \( L^2 \), it has a weak subsequential limit \( \phi \in L^2 \). Passing to this subsequence, we have
\[
\|\phi\|_2 \gtrsim (\langle \psi, S_{\lambda_n}^{-1} \pi(z_n)^{-1} e^{-it_nH} f_n \rangle) \gtrsim \varepsilon (\frac{\varepsilon}{A})^\beta.
\]
To obtain (1.7), write the left side as
\[
2 \Re(\langle f_n - e^{it_nH} \pi(z_n)S_{\lambda_n} \phi, e^{-it_nH} \pi(z_n)S_{\lambda_n} \phi \rangle) = 2 \Re(\langle S_{\lambda_n}^{-1} \pi(z_n)^{-1} e^{-it_nH} f_n - \phi, \phi \rangle) \to 0,
\]
by the definition of \( \phi \).

The restriction to a compact time interval in the above statements is dictated by the generality of our hypotheses. For a generic subquadratic potential, the \( L^q_t \) norm of a solution need not be finite on \( \mathbb{R}_t \times \mathbb{R}_x \). For example, solutions to the harmonic oscillator \( V = x^2 \) are periodic in time. Nonetheless, the conclusions may be strengthened in some cases. In particular, our methods specialize to the case \( V = 0 \) to yield

Theorem 1.3. If \( 0 < \varepsilon \leq \|e^{i\mathcal{A} f}\|_{L^q_{\tau,s}(\mathbb{R} \times \mathbb{R})} \lesssim \|f\|_{L^2} = A \), then
\[
\sup_{z \in T^* \mathbb{R}, \lambda > 0, t \in \mathbb{R}} |\langle \pi(z)S_{\lambda} \psi, e^{i\mathcal{A} f} \rangle| \gtrsim \varepsilon (\frac{\varepsilon}{A})^\beta.
\]
This yields the analogue of Corollary 1.2, which can be upgraded to a linear profile decomposition for the 1d free particle as in the proof of Proposition 5.2. Such a profile decomposition was obtained originally by Carles-Keraani [3] using different methods.

Let us first make a few reductions. We shall assume in the sequel that the initial data \( f \) is Schwartz. This assumption will justify certain applications of Fubini’s theorem and may be removed a posteriori by an approximation argument. Further, note that it suffices to prove the theorem with the time interval \( [-\frac{1}{2}, \frac{1}{2}] \) replaced by \( [-\delta_0, \delta_0] \), where \( \delta_0 \) is to be chosen later (in Theorem 2.3) according to the seminorms \( M_k \) of the potential. Indeed, the interval \( [-\frac{1}{2}, \frac{1}{2}] \) can then be tiled by subintervals of length \( \delta_0 \).

With these preliminary remarks out of the way, let us describe the main ideas of the proof of Theorem 1.1.

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With these preliminary remarks out of the way, let us describe the main ideas of the proof of Theorem 1.1. We want to locate the parameters describing a bubble of concentration in the initial data. The relevant parameters in our setting are length scale \( \lambda_0 \), spatial center \( x_0 \), frequency center \( \xi_0 \), and a time parameter \( t_0 \) describing when the concentration occurs. Each of those parameters is associated with a noncompact symmetry or approximate symmetry of the Strichartz inequality. For instance, when \( V = 0 \) or \( V = \frac{1}{2} x^2 \), both sides of (1.5) are preserved by translations \( f \mapsto f(\cdot - x_0) \) and modulations \( f \mapsto e^{i\xi_0 f} \) of the initial data (see Lemma 2.5 below).

The existing proofs of inverse Strichartz inequalities for the free particle can be very roughly summarized as follows. First, one uses Fourier analysis to isolate a scale \( \lambda_0 \) and frequency center \( \xi_0 \). For example, Carles-Keraani proved in their Proposition 2.1 that for some \( 1 < p < 2 \),
\[
\|e^{i\mathcal{A} f}\|_{L^q_{\tau,s}(\mathbb{R} \times \mathbb{R})} \lesssim_p \left( \sup_J |J|^{\frac{1}{2} - \frac{1}{p}} \|\hat{f}\|_{L^p(J)} \right)^{1/3} \|f\|_{L^2(\mathbb{R})},
\]
where \( J \) ranges over all intervals and \( \hat{f} \) is the Fourier transform of \( f \). Then one employs a separate argument to determine \( x_0 \) and \( t_0 \). This strategy ultimately relies on the fact that the propagator for the free particle is diagonalized by the Fourier transform.
General Schrödinger operators do not enjoy that luxury as the momenta of particles may vary with time and in a position-dependent manner. Thus it is natural to consider the position and frequency parameters together. To this end, we use a wavepacket decomposition as a partial substitute for the Fourier transform.

Unlike the Fourier transform, however, the wavepacket transform requires that one first choose a length scale. This takes some work because the Strichartz inequality which we are trying to invert has no intrinsic length scale; the rescaling

$$f \mapsto \lambda^{-d/2}f(\lambda^{-1} \cdot), \ 0 < \lambda \ll 1$$

preserves both sides of the inequality exactly $V = 0$ and at least approximately for subquadratic $V$ such as the harmonic oscillator $V = |x|^2$.

The key ingredient that gets us started is a refinement of the Strichartz inequality in the time variable due to Killip-Visan. Using a direct physical-space argument, which we describe in Section 3, they show that if $u(t, x)$ is a solution with nontrivial $L^6_{t,x}$ norm, then there exists a time interval $J$ such that $u$ is large in $L^q_{t,x}(J \times \mathbb{R})$ for some $q < 6$. Unlike the $L^6_{t,x}$ norm, the $L^q_{t,x}$ norm of the solution is not scale-invariant. The width of the time interval then dictates a spatial length scale for the solution.

Armed with the temporal center $t_0$ and characteristic width $\lambda_0$, we then use an interpolation and rescaling argument to reduce matters to a refined $L^4_{t,x}$ estimate. This is then proved using a wavepacket decomposition, integration by parts, and consideration of the classical flow, revealing the parameters $x_0$ and $\xi_0$ simultaneously.

This paper is structured as follows. Section 2 collects some preliminary definitions and lemmas. The heart of the argument is presented in Sections 3 and 4. Finally, Section 5 discusses the linear profile decomposition.

As the identification of a time interval works in any number of spatial dimensions, Sections 2 and 3 are presented for a general subquadratic Schrödinger operator on $\mathbb{R}^d$. However, our subsequent reduction to $L^4$ relies on $d = 1$. A naive attempt to extend our argument to higher dimensions would have us to prove a refined $L^p$ estimate for some $2 < p < 4$, but our techniques currently exploit the fact that $4$ is an even integer.

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## 2. Preliminaries

### 2.1. Wavepackets

We recall the basic properties of the wavepacket transform; see for instance [7]. Fix a real, even Schwartz function $\psi \in S(\mathbb{R}^d)$ with $\|\psi\|_{L^2} = (2\pi)^{-d/2}$. For $f \in L^2(\mathbb{R}^d)$ and $z = (x, \xi) \in T^*\mathbb{R}^d = \mathbb{R}^d_+ \times \mathbb{R}^d_+$, define

$$Tf(z) = \int_{\mathbb{R}^d} e^{i(x-y)\xi} \psi(x-y) f(y) \, dy = \langle f, \psi_z \rangle_{L^2(\mathbb{R}^d)}.$$  

By taking the Fourier transform in the $x$ variable, we get

$$F_x Tf(\eta, \xi) = \int_{\mathbb{R}^d} e^{-iy\eta} \hat{\psi}(\eta - \xi) f(y) \, dy = \hat{\psi}(\eta - \xi) \hat{f}(\eta).$$

Thus $T$ maps $S(\mathbb{R}^d) \to S(\mathbb{R}^d \times \mathbb{R}^d)$ and is an isometry $L^2(\mathbb{R}^d) \to L^2(T^*\mathbb{R}^d)$. The hypothesis that $\psi$ is even implies the adjoint formula

$$T^*F(y) = \int_{T^*\mathbb{R}^d} F(z) \psi_z(y) \, dz$$

and the inversion formula

$$f = T^*Tf = \int_{T^*\mathbb{R}^d} \langle f, \psi_z \rangle_{L^2(\mathbb{R}^d)} \psi_z(dz).$$

### 2.2. Classical flow

We collect here some properties of the classical flow on $T^*\mathbb{R}^d$ for a subquadratic potential.

Let $V(t,x)$ satisfy $\partial_t^k V(t,\cdot) \in L^\infty(\mathbb{R}^d)$ for all $k \geq 2$, uniformly in $t$, and let $\Phi(t,s)$ denote the (time-dependent) hamiltonian flow on $T^*\mathbb{R}^d$ generated by the symbol $h = \frac{1}{2}|\xi|^2 + V$. Note that this is well-defined for all $s$ and $t$ since the hypothesis ([1,3]) implies that the Hamiltonian vector field $\xi \partial_x - (\partial_x V) \partial_\xi$ is uniformly Lipschitz. For $z = (x, \xi)$, write $z^t = (x^t(z), \xi^t(z)) = \Phi(t,0)(z)$.
Lemma 2.2. These computations immediately yield the following dynamical consequences:

\begin{equation}
0 \leq x_0^s - x_1^s < x_0^s - x_1^s + (t-s)(\xi_0^s - \xi_1^s) - \int_s^{t}(t-\tau)(\partial_{\tau}V(\tau, x_0^s) - \partial_{\tau}V(\tau, x_1^s)) d\tau.
\end{equation}

Bounding $|\partial_{\tau}V(\tau, x_0^s) - \partial_{\tau}V(\tau, x_1^s)| \leq \|\partial^2_{\tau}V\|_{L^\infty}|x_0^s - x_1^s|$, we obtain for $|t-s| \leq 1$

\begin{equation}
|0 \leq x_0^s - x_1^s| \leq \|\partial^2_{\tau}V\|_{L^\infty} |0 \leq x_0^s - x_1^s| + |t-s|\|\xi_0^s - \xi_1^s|.
\end{equation}

(2.1)

These computations immediately yield the following dynamical consequences:

Lemma 2.1. Assume the preceding setup.

- If $\delta \leq \min(1, \|\partial^2_{\tau}V\|_{L^\infty}^{-1})$, then

$$
|0 \leq x_0^s - x_1^s| \leq (t-s)(\xi_0^s - \xi_1^s) \leq \frac{1}{100}(0 \leq x_0^s - x_1^s| + |t-s|\|\xi_0^s - \xi_1^s|)
$$

whenever $|t-s| \leq \delta$. Hence if $|0 \leq x_1^s| \leq r$ and $C \geq 2$, then $|0 \leq x_1^s| \geq Cr$ for

$$
\frac{2Cr}{|0 \leq x_1^s|} \leq |t-s| \leq \delta.
$$

Informally, two particles colliding with sufficiently large relative velocity will interact only once during a length $\delta$ time interval.

- If $|0 \leq x_1^s| \leq r$, then

$$
|0 \leq x_1^s - (x_0^s - x_1^s)| \leq \min\left(\delta, \frac{2Cr}{|0 \leq x_1^s|}\right) Cr\|\partial^2_{\tau}V\|_{L^\infty}
$$

for all $t$ such that $|0 \leq x_1^s| \leq Cr$. That is, the relative velocity of two particles remains essentially constant during an interaction.

The following lemma will be used in Section 4.2.

Lemma 2.2. There exists a constant $C > 0$ so that if $Q_{\eta} = (0, \eta) + [-1, 1]^{2d}$ and $r \geq 1$, then

$$
|t-t_0| \leq \min(|\eta|^{-1}, 1)
$$

$$
\Phi(t, 0)^{-1}(z_0^t + rQ_\eta) \subset \Phi(t_0, 0)^{-1}(z_0^{t_0} + CrQ_\eta).
$$

In other words, if the trajectory $z^t$ of $z \in T^*R^d$ passes through the cube $z_0^t + rQ_\eta$ in phase space during some time window $|t-t_0| \leq |\eta|^{-1}$, then it must also pass through the twice-larger cube $z_0^{t_0} + 2rQ_\eta$ at time $t_0$.

Proof. If $z^t \in z_0^t + rQ_\eta$, then (2.1) and $|t-s| \leq \min(|\eta|^{-1}, 1)$ imply that

$$
|0 \leq x_1^t| \leq |0 \leq x_1^s| + \min(|\eta|^{-1}, 1)(|\eta| + r) \leq r,
$$

$$
|0 \leq x_1^t - (0 \leq x_1^s)| \leq r \min(|\eta|^{-1}, 1).
$$

\hfill \square

2.3. The Schrödinger propagator. In this section we establish some basic facts regarding the quantum propagator for subquadratic potentials.

Theorem 2.3 (Fujiwara \cite{8, 9}). Let $V(t, x)$ satisfy

$$
M_k = \|\partial^k_{\tau}V(t, x)\|_{L^\infty} < \infty
$$

for all $k \geq 2$. There exists a constant $\delta_0 > 0$ such that for all $0 < |t-s| \leq \delta_0$ the propagator $U(t, s)$ of $H = -\frac{1}{2}\Delta + V(t, x)$ has Schwartz kernel

$$
U(t, s)(x, y) = \left(\frac{1}{2\pi i(t-s)}\right)^{d/2} a(t, s, x, y)e^{is(t, s, x, y)}.
$$
For each $m > 0$ there is a constant $\gamma_m > 0$ such that
\[ |a(t, s, x, y) - 1|_{C^m(\mathbb{R}^2 \times \mathbb{R}^2)} \leq \gamma_m |t - s|^2. \]
Moreover
\[ S(t, s, x, y) = \frac{|x - y|^2}{2(t - s)} + (t - s) r(t, s, x, y), \]
with
\[ |\partial_x r| + |\partial_y r| \leq C(M_2)(1 + |x| + |y|), \]
and for each multiindex $\alpha$ with $|\alpha| \geq 2$, the quantity
\[ C_\alpha = \|\partial^\alpha_x y(t, s, \cdot, \cdot)\|_{L^\infty} \]
is finite. The map $U(t, s) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is a topological isomorphism, and all implicit constants depend on finitely many seminorms $M_k$.

**Definition 2.1.** A pair of exponents $(q, r)$ is (Schrödinger-)admissible if $(q, r, d) \neq (2, \infty, 2)$, $2 \leq q \leq \infty$, and $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$.

**Corollary 2.4** (Dispersive and Strichartz estimates). If $V$ satisfies the hypotheses of the previous theorem, then $U(t, s)$ admits the fixed-time bounds
\[ \|U(t, s)\|_{L^1_t(L^q_x(\mathbb{R}^d) \to L^r_x(\mathbb{R}^d))} \lesssim |t - s|^{-d/2} \]
whenever $|t - s| \leq \delta_0$. For any compact time interval $I$ and any admissible exponents $(q, r)$,
\[ \|U(t, s)f\|_{L^1_I L^r_x(\mathbb{R}^d)} \lesssim \|f\|_{L^q_x(\mathbb{R}^d)}. \]

**Proof.** It follows from the general machinery of Keel-Tao [11], the above pointwise bound for $U(t, s)$, and the unitarity of $U(t, s)$ on $L^2$ that for any fixed $s$,
\[ \|U(t, s)f\|_{L^1_I L^r_x([t-s] \leq \delta_0) \times \mathbb{R}^d)} \lesssim \|f\|_{L^2_x}. \]
If $I = [T_0, T_1]$ is a general time interval, partition it into subintervals $[t_{j-1}, t_j]$ of length at most $\delta_0$. For each such subinterval we can write $U(t, s) = U(t, t_{j-1})U(t_{j-1}, s)$, so
\[ \|U(t, s)f\|_{L^1_I L^r_x([t_{j-1}, t_j] \times \mathbb{R}^d)} \lesssim \|U(t, t_{j-1}) f\|_{L^2} = \|f\|_{L^2}. \]
The corollary follows from summing over the subintervals. \qed

Recall that solutions to the free particle equation $i \partial_t u = -\frac{1}{2} \Delta u$, $u(0) = \phi$ transform according to the following rule with respect to phase space translations of the initial data:
\[ e^{i \frac{\xi}{2} \cdot x} \pi(x, \xi_0) \phi(x) = e^{i[(x - x_0, \xi_0 - \frac{1}{4} t |\xi_0|^2)]} (e^{i \frac{\xi}{2} \cdot \phi}) (x - x_0 - t \xi_0). \]
Physically, $\pi(x_0, \xi_0) \phi$ represents the state of a quantum free particle with position $x_0$ and velocity $\xi_0$. The above relation states that the time evolution of $\pi(x_0, \xi_0) \phi$ in the absence of a potential oscillates in space and time at frequency $\xi_0$ and $-\frac{1}{4} |\xi_0|^2$, respectively, and tracks the classical trajectory $t \mapsto x_0 + t \xi_0$.

In the presence of a potential, the time evolution of such modified initial data admits a more complicated but structurally similar description:

**Lemma 2.5.** If $U(t, s)$ is the propagator for $H = -\frac{1}{2} \Delta + V(t, x)$, then
\[ U(t, s) \pi(z_0) \phi(x) = e^{i[(x - x_0, \xi_0 - \frac{1}{4} t |\xi_0|^2)]} U^{z_0}(t, s) \phi(x - x_0) \]
where
\[ L(t, x) = \frac{1}{2} |\xi|^2 - V(t, x), \]
is the classical Lagrangian, $U^{z_0}(t, s)$ is the propagator for $H^{z_0} = -\frac{1}{2} \Delta + V^{z_0}(t, x)$,
\[ V^{z_0}(t, x) = V(t, x_0 + x) - V(t, x_0) - x \partial_x V(t, x_0) = (x, Q x) \]
where
\[ Q = \int_0^1 (1 - \theta) \partial^2 V(t, x_0 + \theta x) d\theta. \]
and \( z_0^t = (x_0^t, \xi_0^t) \) is the trajectory of \( z_0 \) under the Hamiltonian flow of the symbol \( h = \frac{1}{2} |\xi|^2 + V(t, x) \). The propagator \( U^{z_0}(t, s) \) is continuous on \( S(R^d) \) uniformly in \( z_0 \) and \( |t-s| \leq \delta_0 \).

**Proof.** The formula for \( U(t, s)\pi(z_0^s)\phi \) is verified by direct computation. To obtain the last statement, we notice that \( \|\partial_x^k V z_0\|_{L^\infty} = \|\partial_x^k V\|_{L^\infty} \) for \( k \geq 2 \), and appeal to the last assertion of Theorem 2.3.

**Remarks.**
- This reduces to (2.3) when \( V = 0 \) and also yields analogous relations when \( V \) is a polynomial of degree at most 2. In particular, for a constant electric field \( V(x) = E x \) and \( z_0 = 0 \) (thus \( V z_0 = 0 \)) we recover the well-known Avron-Herbst formula.
- Direct computation shows that the above identity extends to semilinear equations of the form

\[
i\partial_t u = \left(-\frac{1}{2} \Delta + V\right) u + |u|^p u.
\]

That is, if \( u \) is the solution with \( u(0) = \pi(z_0)\psi \), then

\[
u(t) = e^{i \int_0^t L(\tau, z_0^\tau) d\tau} \pi(z_0^t) u(0)
\]

where \( u_{z_0} \) solves

\[
i\partial_t u_{z_0} = \left(-\frac{1}{2} \Delta + V_{z_0}\right) u_{z_0} + |u_{z_0}|^p u_{z_0}, \quad u_{z_0}(0) = \psi,
\]

with the potential \( V_{z_0} \) defined as above.
- One can combine this lemma with a wavepacket decomposition to represent a solution \( U(t, 0)f \) as a sum of wavepackets

\[
U(t, 0)f = \int_{z_0 \in T^* R^d} (f, \psi_{z_0}) U(t, 0)(\psi_{z_0}) dz_0,
\]

where the oscillation of each wavepacket \( U(t, 0)(\psi_{z_0}) \) is largely captured in the phase

\[
(x - x_0^t)\xi_0^t + \int_0^t \frac{1}{2} |\xi_0^\tau|^2 - V(\tau, x_0^\tau) d\tau.
\]

Our arguments later in this paper make essential use of this information. Analogous wavepacket representations have been constructed by Koch and Tataru [13, Theorem 4.3] for a broad class of pseudodifferential operators.

3. Locating a Length Scale

In this section we present an unpublished argument of Killip-Visan that identifies both a characteristic length scale and a temporal center for our sought-after bubble of concentration. Recall that the usual \( TT^* \) proof of the nonendpoint Strichartz inequality combines the dispersive estimate with the Hardy-Littlewood-Sobolev inequality in time. By using instead an inverse HLS inequality, one can locate a time interval on which the solution is large in a non-admissible spacetime norm.

**Proposition 3.1.** Let \((q, r)\) be admissible with \( 2 < q < \infty \), and suppose \( u = U(t, 0)f \) solves

\[
i\partial_t u = \left(-\frac{1}{2} \Delta + V\right) u, \quad u(0) = f \in L^2(R^d)
\]

with \( \|f\|_{L^2(R^d)} = 1 \) and \( \|u\|_{L^q_t L^r_x([-\delta_0, \delta_0] \times R^d)} \geq \varepsilon \), where \( \delta_0 \) is the constant from Theorem 2.3. Then there is a time interval \( J \subset [-\delta_0, \delta_0] \) such that

\[
\|u\|_{L^q_t L^r_x(J \times R^d)} \gtrsim |J|^{\frac{1}{q} - \frac{d}{2r}} e^{1 + \frac{d(q+2)}{4r^2}}
\]

**Remark.** That this estimate singles out a special length is easiest to see when \( V = 0 \). For \( \lambda > 0 \), let \( f_\lambda = \lambda^{-d/2} f(\lambda^{-1}) \) be a rescaling of some fixed \( f \in L^2 \), and let \( u_\lambda(t, x) = \lambda^{-d/2} u(\lambda^{-2} t, \lambda^{-1} x) = e^{it\lambda^2} (f_\lambda) \) be their linear evolutions (here \( u := u_1 \)). Both sides of the Strichartz inequality

\[
\|u_\lambda\|_{L^q_t L^r_x} \lesssim \|f_\lambda\|_{L^2}
\]

remain constant as \( \lambda \) varies.

We claim (supposing for example that \( J = [0, 1] \) in the lemma)

\[
\|u_\lambda\|_{L^q_t L^r_x([0,1] \times R^d)} \to 0 \quad \text{as} \quad \lambda \to 0
\]
Indeed, as \( \|u\|_{L^p_t L_q^s(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|f\|_{L^2} \), for each \( \eta > 0 \) there exists \( T > 0 \) so that (suppressing the region of integration in \( x \)) \( \|u\|_{L^p_t L^s_q((|x| > T)}) < \eta \). Then
\[
\|u\|_{L^p_t L^s_q([0,1])} \leq \|u\|_{L^p_t L^s_q([0,1]\lambda T')} + \|u\|_{L^p_t L^s_q(\lambda T',1)} \\
\leq (\lambda^2 T')^{\frac{1}{q'-1}} \|u\|_{L^p_t L^s_q([0,\lambda T])} + \|u\|_{L^p_t L^s_q(\lambda T',1)} \\
\leq (\lambda^2 T')^{\frac{1}{q'-1}} \|u\|_{L^p_t L^s_q(0,\lambda^2 T')},
\]
which yields the claim. Thus, a lower bound on \( \|u\|_{L^p_t L^s_q((|x| > T))} \) is compatible with the concentration of the solution at arbitrarily small scales. Similar considerations preclude \( \lambda \to \infty \).

To prove the proposition we shall use the following inverse Hardy-Littlewood-Sobolev inequality. For \( 0 < s < d \), denote by \( I_s f(x) = (|D|^{-s} f)(x) = c_{s,d} \int_{\mathbb{R}^d} \frac{f(x,y)}{|y|^{d-s}} dy \) the fractional integration operator.

**Lemma 3.2** (Inverse HLS). For \( 1 < p < \infty \) and \( 0 < s < d/p \),
\[
\|I_s f\|_{L^\frac{p}{p-s} (\mathbb{R}^d)} \lesssim \|f\|_{L^p}^{\frac{1}{2} + \frac{s(d-1)}{d (d-s)}},
\]
where the sup is taken over all balls.

**Proof.** We use a variant of the usual proof of the HLS inequality due to Hedberg [10]; see also [10] §VIII.4.2. Let
\[
\delta = \sup_B |B|^{\frac{1}{p}-1} \int_B f(y) dy \lesssim \|f\|_{L^p}.
\]

For \( r_1 < r_2 \) to be fixed shortly, decompose the integral as
\[
I_s f(x) = c_{s,d} \int_{\mathbb{R}^d} \frac{f(x,y)}{|y|^{d-s}} \lesssim \int_{|y| \leq r_1} + \int_{r_1 \leq |y| \leq r_2} + \int_{|y| > r_2} \\
\lesssim r_1^s Mf(x) + \delta r_1^{-d} r_2^d + \delta^\frac{d}{2} \|f\|_{L^p},
\]
where \( Mf \) is the Hardy-Littlewood maximal function and Hölder was used to estimate the second and third integrals. Choosing \( r_1 \) and \( r_2 \) to equate the terms, we find that
\[
(\frac{r_1}{r_2})^{d-s} \lesssim \|f\|_{L^p}, \quad r_2 = (\frac{\delta}{Mf})^{\frac{d}{2}} (\|f\|_{L^p})^{\frac{p}{p-s}}
\]
which yields the pointwise bound
\[
I_s f \lesssim \delta^{\frac{d}{2}} \frac{d}{d-s} Mf^{1-\frac{d}{2}} \|f\|_{L^p}^{\frac{p}{p-s}}.
\]
The conclusion follows. \( \square \)

**Proof of Proposition 3.1.** Define the map \( T : L^2_{s,d} \to L^q_{s,d} \) by \( Tf(t) = U(t,0)f \), which by Corollary 2.4 is continuous. By duality, \( \varepsilon \lesssim \|u\|_{L^1_t L^\infty_x} \) implies \( \varepsilon \lesssim \|T^* \psi\|_{L^{\infty}_x} \), where
\[
\psi = \frac{|u|^{r-1}}{\|u(t)|^{r-1}_{L^q_x}} \frac{\|u(t)||^{q-1}_{L^q_x}}{\|u||^{q-1}_{L^q_t L^q_x}}
\]
satisfies \( \|\psi\|_{L^q_t L^\infty_x} = 1 \), and
\[
T^* \psi = \int U(0,s) \psi(s) ds.
\]

By the dispersive estimate of Corollary 2.4,
\[
e^2 \lesssim (T^* \psi, T^* \psi) = (\psi, T T^* \psi)_{L^2} = \int \overline{\psi(t)} U(t,s) \psi(s) dx ds dt \lesssim \int \int G(t)G(s) \frac{ds dt}{|t-s|^{2/q}} \lesssim \int \int G(t)G(s) \frac{ds dt}{|t-s|^{2/q}},
\]
where \( G(t) = \|\psi(t)||^{q}_{L^q_x} \). Writing the last term as \( \|I_s G\|^{2}_{L^q_x} \), where \( s = \frac{1}{2} - \frac{1}{q} = \frac{1}{2} - \frac{1}{q} \), and appealing to the previous lemma with \( p = q' \), we can bound the above by
\[
(\sup_j |J|^{-\frac{1}{q'}} \|G\|^{q'}_{L^q(J)})^{1-(\frac{q'}{q} - \frac{1}{2})} = (\sup_j |J|^{-\frac{1}{q'}} \|u\|^{q'}_{L^q_t L^q_x} \|u^{q-1}_{L^q_t L^q_x(J \times \mathbb{R})})^{\frac{q(q-2)}{2q-1}}.
\]
Upon rearranging, we get
\[ \sup_J |J|^{-\frac{1}{q(r-1)}}\|u\|_{L_t^{q/r-1} L_x^r(\mathbb{R})} \gtrsim \varepsilon^{(1 + \frac{4(q+2)}{q(r-2)})}. \]

\[ \square \]

4. A refined \( L^4 \) estimate

4.1. Reduction to \( L^4 \). Now we specialize to the one-dimensional setting \( d = 1 \), and apply Proposition 3.1 to the Strichartz pair \( (q, r) = \left( \frac{7+\sqrt{33}}{2}, \frac{5+\sqrt{33}}{2} \right) \) determined by the conditions \( \frac{2}{q} + \frac{1}{r} = \frac{1}{2} \) and \( q - 1 = r \).

**Corollary 4.1.** With \( (q, r) \) as above, choose
\[ \frac{1}{q} - \frac{1}{r} < \theta < 1. \]

Suppose
\[ \varepsilon = \|U(t, 0)f\|_{L_t^{q}(\mathbb{R})} \lesssim \|f\|_{L^2} = A. \]

Then there exists a time interval \( J \) such that
\[ \|U(t, 0)f\|_{L_t^{q/r-1} L_x^r(\mathbb{R})} \gtrsim A|J|^{-\frac{1}{q(r-1)}} \left( \frac{\varepsilon}{A} \right)^{\frac{1}{2} + \frac{4(q+2)}{q(r-2)}}. \]

**Proof.** Let \((q_0, r_0)\) be any Strichartz pair with \( 4 < q < 6 \). Then with
\[ \theta = \frac{\frac{1}{q} - \frac{1}{r}}{q_0 - \frac{1}{q}}, \]
we have
\[ \varepsilon \leq \|U(t, 0)f\|_{L_t^{q_0} L_x^{r_0}} \leq \|U(t, 0)f\|_{L_t^{q/r-1} L_x^r(\mathbb{R})}^{1-\theta} \|U(t, 0)f\|_{L_t^{q_0} L_x^{r_0}}^\theta \lesssim A^{1-\theta}\|U(t, 0)f\|_{L_t^{q/r-1} L_x^r(\mathbb{R})}^\theta. \]

The claim now follows from the previous Lemma. \( \square \)

Let \( J = [t_0 - \lambda^2, t_0 + \lambda^2] \) be the interval from the above corollary, and set
\[ u(t, x) = \lambda^{-1/2} u_{\lambda} (\lambda^{-2} (t - t_0), \lambda^{-1} x), \]

where \( u_{\lambda} \) solves
\[ i\partial_t u_{\lambda} = (-\frac{1}{2}\partial_x^2 + V_{\lambda}) u_{\lambda} = 0, \quad u_{\lambda} (0, x) = \lambda^{1/2} u(t_0, \lambda x). \]

and \( V_{\lambda} (t, x) = \lambda^2 V(t_0 + \lambda^2 t, \lambda x) \) also satisfies the hypotheses [1.3] and [1.4] for all \( 0 < \lambda \leq 1 \). By the corollary and a change of variables,
\[ \|u\|_{L_t^{q/r-1} L_x^r(\mathbb{R})} \gtrsim A \left( \frac{\varepsilon}{A} \right)^{\frac{1}{2} + \frac{4(q+2)}{q(r-2)}}. \]

As \( 4 < q - 1 < 6 \), Theorem 1.1 will follow by interpolating between \( L_x^2 \to L_t^{q/r} \) Strichartz estimate and the following refined \( L_x^2 \to L_t^4 \) estimate:

**Proposition 4.2.** There exists \( \beta > 0 \) so that if \( \psi \in S(\mathbb{R}) \) is real, even, and \( L^2 \) normalized, and \( \eta(t) \) is a bump function vanishing when \( |t| \geq \delta_0 \), then
\[ \|U\psi(t, 0)f\|_{L_t^{q/r} L_x^r(\mathbb{R})} \lesssim \|f\|_{L^2}^{1-\beta} \sup_z \|\psi_z f\|^\beta \]

whenever \( U\psi \) is the propagator for a potential \( V \) satisfying the hypotheses [1.3] and [1.4].

**Remark.** The implicit constant will depend on the quantities \( \|\partial_t^k V\|_{L^\infty} + \|\partial_x^k V\|_{L^\infty}, \quad k \geq 2 \), as well as \( \sup_t |\partial_t V(t, 0)| \), which are all bounded with respect to the above rescaling and time translation.
4.2. **Proof of Proposition 4.2** We fix a potential $V$ and drop the subscript $V$ from the propagator. Assuming the setup of the proposition, we decompose $f$ into wavepackets $f = \int_{T^*\mathbb{R}}^{} f(x)\psi_x dz$, and expand the $L^4$ norm:

$$\|U(t,0)f\|_{L^4_t}^4 \leq \int_{(T^*\mathbb{R})^4} K(z_1, z_2, z_3, z_4) \prod_{j=1}^4 |\langle f, \psi_j \rangle| dz_1 dz_2 dz_3 dz_4,$$

where

$$K = |\langle U(t,0)(\psi_z)U(t,0)(\psi_{z2}), U(t,0)(\psi_z)U(t,0)(\psi_{z4}) \rangle|_{L^2_{x,t}(\eta(t)dxdt)}|.$$

There is no difficulty with interchanging the order of integration as $f$ was assumed to be Schwartz.

**Proposition 4.3.** For some $0 < \theta < 1$ the kernel

$$K(z_1, z_2, z_3, z_4) \max(|z_1 - z_2|^{\theta}, |z_3 - z_4|^{\theta})$$

is bounded as a map on $L^2(T^*\mathbb{R} \times T^*\mathbb{R})$.

We defer the proof for the moment and observe how it implies Proposition 4.2. Writing $a_z = |\langle f, \psi_z \rangle|$, we then have

$$\|U(t,0)f\|_{L^4_t}^4 \lesssim \left( \int_{(T^*\mathbb{R})^2} a_z^2 a_{z2}^2 \langle z_1 - z_2 \rangle^{-2\theta} dz_1 dz_2 \right)^{1/2} \left( \int_{(T^*\mathbb{R})^2} a_z^2 a_{z4}^2 \langle z_3 - z_4 \rangle^{-2\theta} dz_3 dz_4 \right)^{1/2}$$

By Young’s inequality, the convolution kernel $k(z_1, z_2) = \langle z_1 - z_2 \rangle^{-2\theta}$ is bounded from $L^p_z$ to $L^{p'}_z$ for some $p \in (1, 2)$, and the integral on the right is bounded by

$$\left( \int_{T^*\mathbb{R}} a_z^{2p} dz \right)^{2/p} \leq \|f\|_{L^2}^p \sup_z a_z^p.$$

Overall we obtain the desired bound

$$\|U(t,0)f\|_{L^4_t} \lesssim \|f\|_{L^2}^{\frac{1}{4} + \frac{\theta}{2}} \sup_z a_z^{\frac{1}{2p}}.$$ 

Thus it remains to prove Proposition 4.3. By Lemma 2.3

$$U(t,0)(\psi_{x_j})(x) = e^{it\psi} U_j(t,0) \psi(x - x_j^t),$$

where

$$\psi_j(t, x) = (x - x_j^t)\xi_j^t + \int_0^t \frac{1}{2}|\xi_j^\tau|^2 - V(\tau, x_j^\tau) d\tau$$

and $U_j$ is the propagator for $H_j = -\frac{1}{2}\partial_x^2 + V_j(t, x)$, where

$$V_j(t, x) = x^2 \int_0^1 (1 - s)\partial_x^2 V(t, x_j^s + sx) ds,$$

and the wavepackets $U_j(t,0)\psi(x - x_j^t)$ concentrate along the classical trajectories $t \mapsto x_j^t$:

$$|\partial_{x_j^t}^k U_j(t,0)\psi(x - x_j^t)| \lesssim_{k,N} \langle x - x_j^t \rangle^{-N}.$$

The kernel $K$ thus admits the crude bound

$$K(z) \lesssim_N \prod_{j=1}^4 \langle x - x_j^t \rangle^{-N} \eta(t)dxdt \lesssim \max(\langle z_1 - z_2 \rangle, \langle z_3 - z_4 \rangle)^{-1},$$

and Proposition 4.3 will follow from

**Proposition 4.4.** For $\delta > 0$ sufficiently small the kernel $K^{1-\delta}$ is bounded on $L^2(T^*\mathbb{R} \times T^*\mathbb{R})$. 

---
Proof. Define

\[ E_0 = \{ \vec{z} \in (T^* \mathbb{R})^4 : \min_{|t| \leq \delta_0} \max_{k, k'} |x_k^t - x_{k'}^t| \leq 1 \} \]

\[ E_m = \{ \vec{z} \in (T^* \mathbb{R})^4 : 2^{m-1} < \min_{|t| \leq \delta_0} \max_{k, k'} |x_k^t - x_{k'}^t| \leq 2^m \}, \ m \geq 1, \]

and decompose

\[ K = K_1 E_0 + \sum_{m \geq 1} K_1 E_m = K_0 + \sum_{m \geq 1} K_m. \]

Then

\[ K^{1-\delta} = K_0^{1-\delta} + \sum_{m \geq 1} K_m^{1-\delta}. \]

The \( K_0 \) term corresponds to the 4-tuples of particles that all collide at some time \( t \in [-\delta_0, \delta_0] \). As suggested by (4.3), this will be the dominant term. We shall show that for any \( N > 0 \),

\[ \| K_m^{1-\delta} \|_{L^2 \to L^2} \lesssim N 2^{-mN}, \]

which immediately implies the proposition upon summing in \( m \).

The proof of this relies on the following interaction estimate:

**Lemma 4.5.** For any \( N_1, N_2 > 0 \),

\[ |K_m(z)| \lesssim_{N_1, N_2} 2^{-mN_1} \min \left( \frac{\xi_1 - \xi_2}{1 + |\xi_2 - \xi_3|} - \frac{\xi_3 - \xi_4}{1 + |\xi_3 - \xi_4|} \right) \]

\[ \left( |\xi_1 - \xi_2| + |\xi_3 - \xi_4| \right)^2 \left( |\xi_1 - \xi_2|^2 - |\xi_3 - \xi_4|^2 \right)^2, \]

where \( t(z) \) is a time witnessing the minimum in the definition of \( E_m \).

This will be proved below. For the moment, let us use it with Schur’s test to deduce (4.4).

Fix \( (z_3, z_4) \) in the image of the projection \( E_m \subset (T^* \mathbb{R})^4 \to T^* \mathbb{R}_{z_3} \times T^* \mathbb{R}_{z_4} \), and let

\[ E_m(z_3, z_4) = \{ (z_1, z_2) \in (T^* \mathbb{R})^2 : (z_1, z_2, z_3, z_4) \in E_m \}. \]

Choose \( t_1 \) minimizing \( |x_3^{t_1} - x_4^{t_1}| \): the definition of \( E_m \) implies that \( |x_3^{t_1} - x_4^{t_1}| \leq 2^m \).

By Lemma 2.1 any collision time \( t(z) \) for \( (z_1, z_2) \in E_m(z_3, z_4) \) must belong to the interval

\[ I = \{ t \in [-\delta_0, \delta_0] : |t - t_1| \leq \min \left( 1, \frac{2^m}{|\xi_1^{t_1} - \xi_2^{t_1}|} \right) \}, \]

and for such \( t \),

\[ |\xi_1 - \xi_2 - (\xi_3^{t_1} - \xi_4^{t_1})| \lesssim \min \left( 2^m, \frac{2^{2m}}{|\xi_3^{t_1} - \xi_4^{t_1}|} \right). \]

Write \( Q_\xi = (0, \xi) + [-1, 1]^2 \subset T^* \mathbb{R} \), and denote by \( \Phi(t, s) \) the classical propagator for the Hamiltonian

\[ \hbar = \frac{1}{2} \xi^2 + V(t, x). \]

Using the shorthand \( z^t = \Phi(t, 0)(z) \), define for \( \mu_1, \mu_2 \)

\[ Z_{\mu_1, \mu_2} = \bigcup_{t \in I} (\Phi(t, 0) \circ \Phi(t, 0))^{-1} \left( z_3^{t_1} + z_4^{t_1} + 2^m Q_{\mu_1} \right) \times \left( z_3^{t_1} + z_4^{t_1} + 2^m Q_{\mu_2} \right), \]

where \( \Phi(t, 0) \circ \Phi(t, 0)(z_1, z_2) = (z_1^{t_1}, z_2^{t_1}) \) is the product flow on \( T^* \mathbb{R} \times T^* \mathbb{R} \). By definition

\[ E_m(z_3, z_4) \subset \bigcup_{\mu_1, \mu_2 \in \mathbb{Z}} Z_{\mu_1, \mu_2}. \]

**Lemma 4.6.** \( |Z_{\mu_1, \mu_2}| \lesssim 2^{4m} \max(|\mu_1|, |\mu_2|)|I| \), where \(| \cdot | \) on the left denotes Lebesgue measure on \((T^* \mathbb{R})^2\).
Proof. Without loss assume $|\mu_1| \geq |\mu_2|$. Partition the interval $I$ into subintervals of width $|\mu_1|^{-1}$. For each $t'$ in the partition, Lemma 4.5 implies that
\[ \bigcup_{|t-t'| \leq |\mu_1|^{-1}} \Phi(t,0)^{-1} \left( \frac{\zeta_3 + \zeta_4}{2} + 2mQ_{\mu_1} \right) \subset \Phi(t',0)^{-1} \left( \frac{\zeta_3' + \zeta_4'}{2} + C2^mQ_{\mu_2} \right) \]
\[ \bigcup_{|t-t'| \leq |\mu_1|^{-1}} \Phi(t,0)^{-1} \left( \frac{\zeta_3 + \zeta_4}{2} + 2mQ_{\mu_2} \right) \subset \Phi(t',0)^{-1} \left( \frac{\zeta_3' + \zeta_4'}{2} + C2^mQ_{\mu_1} \right), \]

hence
\[ \bigcup_{|t-t'| \leq |\mu_1|^{-1}} \left( \Phi(t,0) \otimes \Phi(t,0) \right)^{-1} \left( \frac{\zeta_3 + \zeta_4}{2} + 2mQ_{\mu_1} \right) \times \left( \frac{\zeta_3' + \zeta_4'}{2} + 2mQ_{\mu_2} \right) \subset \left( \Phi(t',0) \otimes \Phi(t',0) \right)^{-1} \left( \frac{\zeta_3' + \zeta_4'}{2} + C2^mQ_{\mu_1} \right) \times \left( \frac{\zeta_3' + \zeta_4'}{2} + C2^mQ_{\mu_2} \right). \]

By Liouville’s theorem, the right side has measure $O(2^{4m})$ in $(T^*\mathbb{R})^2$. The claim follows by summing over the partition. \qed

For each $(z_1, z_2) \in E_m(z_3, z_4) \cap Z_{\mu_1, \mu_2}$, choose $t$ in $I$ such that $z_j' \in \frac{\zeta_3 + \zeta_4}{2} + 2mQ_{\mu_j}$. As
\[ \xi_j' = \frac{\xi_3 + \xi_4}{2} + \mu_j + O(2^m), \quad j = 1, 2, \]
the second assertion of Lemma 4.1 implies that
\[ \xi_1(\bar{\xi}) + \xi_2(\bar{\xi}) - \xi_3(\bar{\xi}) - \xi_4(\bar{\xi}) = \mu_1 + \mu_2 + O(2^m) \]
\[ \xi_1(\bar{\xi}) - \xi_2(\bar{\xi}) = \mu_1 - \mu_2 + O(2^m), \]

hence by Lemma 4.5
\[ |K_m| \lesssim N 2^{-3mN} \min \left( \frac{(\mu_1 + \mu_2 + O(2^m))^N}{1 + |\mu_1 - \mu_2| + |\xi_3(\bar{\xi}) - \xi_4(\bar{\xi})| + O(2^m)} \right) \frac{1 + |\mu_1 - \mu_2| + |\xi_3(\bar{\xi}) - \xi_4(\bar{\xi})| + O(2^m)}{\left( (\mu_1 - \mu_2)^2 - (\xi_3(\bar{\xi}) - \xi_4(\bar{\xi}))^2 \right)^{\delta}}. \]

Applying Lemma 4.6 writing $\max(|\mu_1|, |\mu_2|) \leq |\mu_1 + \mu_2| + |\mu_1 - \mu_2|$, and absorbing $|\mu_1 + \mu_2|$ into the factor $(\mu_1 + \mu_2)^{-N}$,
\[ \int K_m(z_1, z_2, z_3, z_4)1^{-\delta} \, dz_1 dz_2 \leq \sum_{\mu_1, \mu_2 \in \mathbb{Z}} \int K_m(z_1, z_2, z_3, z_4)1^{-\delta} 1_{Z_{\mu_1, \mu_2}}(z_1, z_2) \, dz_1 dz_2 \]
\[ \lesssim \sum_{\mu_1, \mu_2 \in \mathbb{Z}} 2^{-mN} \min \left( \frac{(\mu_1 + \mu_2)^{-N}}{1 + |\mu_1 - \mu_2| + |\xi_3(\bar{\xi}) - \xi_4(\bar{\xi})| + O(2^m)} \right) \frac{1 + |\mu_1 - \mu_2| + |\xi_3(\bar{\xi}) - \xi_4(\bar{\xi})| + O(2^m)}{\left( (\mu_1 - \mu_2)^2 - (\xi_3(\bar{\xi}) - \xi_4(\bar{\xi}))^2 \right)^{\delta}}. \]

As the portion of the sum where $|\mu_1 - \mu_2| \leq 1$ is of size $2^{-mN}$, we may restrict attention to the terms where $|\mu_1 - \mu_2| \geq 1$.

When $|\mu_1 - \mu_2| \geq 2|\xi_3(\bar{\xi}) - \xi_4(\bar{\xi})|$, the above expression is bounded by
\[ \sum_{\mu_1, \mu_2 \in \mathbb{Z}} 2^{-mN} \min \left( \frac{(\mu_1 + \mu_2)^{-N}}{1 + |\mu_1 - \mu_2| + |\xi_3(\bar{\xi}) - \xi_4(\bar{\xi})| + O(2^m)} \right)^{1-\delta} \lesssim N 2^{-mN}. \]

Otherwise, one has the bound
\[ \sum_{\mu_1, \mu_2 \in \mathbb{Z}} 2^{-mN} \min \left( \frac{(\mu_1 + \mu_2)^{-N}}{1 + |\mu_1 - \mu_2|, \mu_1 - \mu_2| - |\xi_3(\bar{\xi}) - \xi_4(\bar{\xi})|} \right)^{1-\delta} \lesssim N 2^{-mN}. \]

Therefore
\[ \int K_m(z_1, z_2, z_3, z_4)1^{-\delta} \, dz_1 dz_2 \lesssim N 2^{-mN}. \]
and the same considerations apply with the roles of \((z_1, z_2)\) and \((z_3, z_4)\) reversed. The bound \((4.4)\) now follows from Schur’s test. Modulo Lemma 4.5 this completes the proof of Proposition 4.4. \(\square\)

The remainder of this section is devoted to the interaction estimate.

**Proof of Lemma 4.5.** From \((4.3)\) and the definition of \(E_m\) we immediately get the cheap bound
\[
|K_m(\tilde{z})| \lesssim 2^{-mN},
\]
but we can usually do better integrating by parts. As the argument is essentially the same for all \(m\), we shall for simplicity take \(m = 0\) in the sequel.

Suppose that \(t(\tilde{z}) = 0\). Then by Lemma 2.5
\[
K_0(\tilde{z}) = \left| \int e^{i\Phi} \prod_{j=1}^{4} U_j(t, 0) \psi(x - x_j^t) \eta(t) dx dt \right|
\]
where \(\sigma = (+, +, -, -), \prod_{j=1}^{4} c_j = c_1 c_2 r_3 r_4\), and
\[
\Phi = \sum_j \sigma_j ((x - x_j^t) \xi_j^t + \int_0^t \frac{1}{2} |\xi_j^\tau|^2 - V(t, x_j^\tau) \, d\tau).
\]

Let \(1 = \theta_0 + \sum_{\ell \geq 1} \theta_\ell\) be a partition of unity such that \(\theta_0\) is supported in the unit ball and \(\theta_\ell\) is supported in the annulus \(\{2^{\ell-1} < |x| < 2^{\ell+1}\}\). Also choose \(\chi \in C^\infty_0\) equal to 1 on \(|x| \leq 8\). Further decompose
\[
K_0 \leq \sum_{\ell} K_0^\ell,
\]
where
\[
K_0^\ell = \left| \int e^{i\Phi} \prod_{j} U_j \psi(x - x_j^t) \theta_\ell(x - x_j^t) \eta(t) dx dt \right|
\]
Fix \(\ell\), and write \(\ell^* = \max \ell_j\). By Lemma 2.1 the integrand is supported on
\[
\{(t, x) : |t| \leq \min(1, \frac{2^{\ell^*}}{\max |\xi_j^t - \xi_k^t|}), |x - x_j^t| \leq 2^{\ell_j}\},
\]
and for all \(t\) subject to the above restriction
\[
|x_j^t - x_k^t| \lesssim 2^{\ell^*}, \quad |\xi_j^t - \xi_j| \lesssim \min(2^{\ell^*}, \frac{2^{2\ell^*}}{\max |\xi_j^t - \xi_k^t|}).
\]

We estimate \(K_0^\ell\) using integration by parts. The relevant derivatives of the phase function are
\[
\partial_x \Phi = \sum_j \sigma_j \xi_j^t, \quad \partial_x^2 \Phi = 0,
\]
\[
-\partial_t \Phi = \sum_j \sigma_j h(t, z_j^t) + \sum_j \sigma_j (x - x_j^t) \partial_x V(t, x_j^t).
\]

Integrating by parts repeatedly in \(x\) yields, for any \(N \geq 0\),
\[
|K_0^\ell| \lesssim_N \int |\xi_1^t + \xi_2^t - \xi_3^t - \xi_4^t|^{-N} |\partial_x^N \prod_{j} U_j \psi(x - x_j^t) \theta_\ell(x - x_j^t) | \eta(t) dx dt
\]
\[
\lesssim_N 2^{-\ell^* N} \frac{\xi_1^t + \xi_2^t - \xi_3^t - \xi_4^t}{1 + |\xi_1 - \xi_2| + |\xi_3 - \xi_4|} -N
\]
where we have used \((4.6)\) to replace \(\xi_1^t + \xi_2^t - \xi_3^t - \xi_4^t\) with \(\xi_1 + \xi_2 - \xi_3 - \xi_4 + O(2^{\ell^*})\).

We can also integrate by parts using a vector field adapted to the average trajectory of the four particles. Define
\[
\overline{x^t} = \frac{1}{4} \sum_j x_j^t, \quad \overline{\xi^t} = \frac{1}{4} \sum_j \xi_j^t,
\]
\[
x_j^t = \overline{x^t} + \overline{x_j^t}, \quad \xi_j^t = \overline{\xi^t} + \overline{\xi_j^t}.
\]

Note that
\[
\max_j |\overline{x^t}_j| \sim \max_{j, k} |x_j^t - x_k^t|, \quad \max_j |\overline{\xi^t}_j| \sim \max_{j, k} |\xi_j^t - \xi_k^t|.
\]
Define the vector field 
\[ D = \partial_t + \xi^j \partial_x. \]
Then
\[
-D\Phi = \sum \sigma_j h(t, z_j^i) + \sum \sigma_j [(x - x_j^i)\partial_x V(t, x_j^i) - \xi^j] \\
= \frac{1}{2} \sum \sigma_j [\xi^j]^2 + \sum \sigma_j \{V(t, x_j^i) + (x - x_j^i)\partial_x V(t, x_j^i)\}.
\]
This is better expressed in the “Lagrangian” variables \( \pi_j \) and \( \xi_j \). Each term in the second sum can be written as
\[
V(t, \pi^j + x) + (x - x_j^i)\partial_x V(t, \pi^j + x) \\
= V(t, \pi^j + x) - V(t, \pi^j) - \xi^j \partial_x V(t, \pi^j) \\
+ (x - x_j^i)(\partial_x V(t, \pi^j + x) - \partial_x V(t, \pi^j)) \\
= V^\pi(t, \pi^j + x) + (x - x_j^i)\partial_x V^\pi(\pi^j + x) + (x - x_j^i)\partial_x V(t, \pi^j),
\]
where
\[
(4.10) \quad V^\pi(t, x) = V(t, \pi^j + x) - V(t, \pi^j) - x\partial_x V(t, \pi^j) = x^2 \int_0^1 (1 - s)\partial_x^2 V(t, \pi^j + sx) \, ds.
\]
The terms without the subscript \( j \) cancel upon summing, and we obtain
\[
(4.11) \quad -D\Phi = \sum \sigma_j \frac{1}{2} [\xi^j]^2 + \sum \sigma_j [V^\pi(t, \pi^j + x) + (x - x_j^i)\partial_x V^\pi(t, \pi^j)].
\]
Thus the contribution to \( D\Phi \) from the potential depends essentially only on the relative positions \( x_j^i - x_k^i \); by (4.5), (4.6), and (4.9), that contribution is at most \( O(2^\gamma) \).

Note also that
\[
(\xi^j)^2 = (\xi_j^i)^2 + O(2^\gamma),
\]
as can be seen via
\[
\frac{d}{dt} [\frac{1}{2} (\xi^j)^2] = -\xi^j_\partial_x V^\pi(t, \pi^j) = (-\xi_j^i + O(2^\gamma))\partial_x V^\pi(t, \pi^j),
\]
the fundamental theorem of calculus, (4.5), and (4.9). It follows that if
\[
(4.12) \quad \left| \sum \sigma_j (\xi^j)^2 \right| \geq C \cdot 2^{2^\gamma}
\]
for some large constant \( C > 0 \), then on the support of the integrand
\[
(4.13) \quad \left| D\Phi \right| \geq \left| \sum \sigma_j (\xi^j)^2 \right| = \frac{1}{2} \left| (\xi_1 + \xi_2)^2 - (\xi_3 + \xi_4)^2 + (\xi_1 - \xi_2)^2 - (\xi_3 - \xi_4)^2 \right|
\]
\[
\geq \left| \xi_1 - \xi_2 \right|^2 - \left| \xi_3 - \xi_4 \right|^2,
\]
where the last line follows from the fact that \( \xi_1 + \xi_2 + \xi_3 + \xi_4 = 0 \).

The second vector field derivative is
\[
-D^2\Phi = \sum \sigma_j \xi^j \left( \frac{1}{4} \sum_k \partial_x V(t, x_k^i) - \partial_x V(t, x_j^i) \right) + \sum (x - x_j^i)\xi^j \partial_x^2 V(t, x_j^i) + \xi^j \sum \sigma_j \partial_x V(t, x_j^i)
\]
\[
+ \sum \sigma_j [\partial_t V(t, x_j^i) + (x - x_j^i)\partial_x V(t, x_j^i)]
\]
\[
= \sum \sigma_j \xi^j \left( \frac{1}{4} \sum_k \partial_x V(t, x_k^i) - \partial_x V(t, x_j^i) \right) + \sum \sigma_j (x - x_j^i)\xi^j \partial_x^2 V(t, x_j^i)
\]
\[
+ \sum \sigma_j [\partial_t V(t, x_j^i) + (x - x_j^i)\partial_x V(t, x_j^i)] + \xi^j \sum \sigma_j [\partial_x V(t, x_j^i) + (x - x_j^i)\partial_x^2 V(t, x_j^i)].
\]
As before, the last two sums may be rewritten to yield

\[-D^2\Phi = \sum_{j} \sigma_j \sum_{k} \frac{1}{4} \partial_x V(t, x_k^j) \partial_x V(t, x_j^x) + \sum_{j} \frac{1}{4} \sum_{k} \frac{1}{4} \partial_x V(t, x_k^j) \partial_x V(t, x_j^x) + \sum_{j} \sigma_j (x - x_j^x) \partial_x V(t, x_j^x)\]

(4.14)

+ \sum_{j} \sigma_j [(\partial_x V)^2(t, x_j^x) + (x - x_j^x) \partial_x (\partial_x V)^2(t, x_j^x)]

+ \sum_{j} \sigma_j [(\partial_x V)^2(t, x_j^x) + (x - x_j^x) \partial_x (\partial_x V)^2(t, x_j^x)],

where

\[(\partial_x V)^2(t, x) = x^2 \int_0^1 (1 - s) \partial_x V(t, x + sx) \, ds\]

\[(\partial_x V)^2(t, x) = x^2 \int_0^1 (1 - s) \partial_x V(t, x + sx) \, ds.\]

Assuming (4.12), integrate by parts using the vector field \(D\) to get

\[K^2_0 \leq \int e^{i\phi} \frac{D^2\Phi}{(D\Phi)^2} \prod U_j \psi(x - x_j^x) \theta_{\ell_j}(x - x_j^x) \eta(t) \, dx \, dt\]

+ \int e^{i\phi} \frac{2D^2\Phi}{(D\Phi)^3} \prod U_j \psi(x - x_j^x) \theta_{\ell_j}(x - x_j^x) \eta(t) \, dx \, dt\]

+ \int e^{i\phi} \frac{1}{(D\Phi)^2} D^2 \prod U_j \psi(x - x_j^x) \theta_{\ell_j}(x - x_j^x) \eta(t) \, dx \, dt\]

= I + II + III.

Consider first the term \(I\). Write \(I \leq I_a + I_b + I_c\), where \(I_a, I_b, I_c\) correspond respectively to the first, second, and third lines in the expression \((4.14)\) for \(D^2\Phi\).

In view of (4.3), (4.6), (4.13), and the fact that

\[\frac{1}{4} \sum_{k} \partial_x V(t, x_k^x) - \partial_x V(t, x_j^x) = \frac{1}{4} \sum_{k} (x_k^x - x_j^x) \int_0^1 \partial_x^2 V(t, x_j^x + s(x_k^x - x_j^x)) \, ds = O(2^{\epsilon'})

we have

\[I_a \lesssim_N \int 2^{\epsilon'} \frac{\sum_j |\xi_j|}{|D\Phi|^2} \prod 2^{-\epsilon_j} N \chi \left(\frac{x - x_j^x}{2^\epsilon_j}\right) \eta(t) \, dx \, dt\]

\[\lesssim 2^{\epsilon'} (1 + \sum_j |\xi_j|) \int \prod 2^{-\epsilon_j} N \chi \left(\frac{x - x_j^x}{2^\epsilon_j}\right) \eta(t) \, dx \, dt\]

\[\lesssim_N 2^{-\epsilon N} \frac{\xi_1 + \xi_2 - \xi_3 - \xi_4 + |\xi_1 - \xi_2| + |\xi_3 - \xi_4|}{|\xi_1 - \xi_2|^2 - (\xi_3 - \xi_4)^2 |} \cdot \frac{1}{1 + |\xi_1 - \xi_2| + |\xi_3 - \xi_4|},\]

where we have observed that

\[\sum_j |\xi_j| \sim (\sum_j |\xi_j|^2)^{1/2} \sim (|\xi_1 + \xi_2|^2 + |\xi_1 - \xi_2|^2 + |\xi_3 + \xi_4|^2 + |\xi_3 - \xi_4|^2)^{1/2}\]

\[\lesssim |\xi_1 + \xi_2 - \xi_3 - \xi_4| + |\xi_1 - \xi_2| + |\xi_3 - \xi_4|].\]

Similarly,

\[I_b \lesssim \int 2^{\epsilon'} \frac{\sum_j |\xi_j|}{|D\Phi|^2} \prod 2^{-\epsilon_j} N \chi \left(\frac{x - x_j^x}{2^\epsilon_j}\right) \eta(t) \, dx \, dt\]

\[\lesssim_N 2^{-\epsilon N} \frac{2^{-\epsilon N}}{|(\xi_1 - \xi_2)^2 - (\xi_3 - \xi_4)^2|} \cdot \frac{1}{1 + |\xi_1 - \xi_2| + |\xi_3 - \xi_4|}.\]
To estimate $I_c$, use the decay hypothesis $|\partial_t^2 V(x)| \lesssim (x)^{-1-\epsilon}$ to obtain

\[ I_c \lesssim \int_0^1 \left[ \frac{2^{2\epsilon^*}}{|\partial_t \Phi|^2} \left( \int_0^1 \sum_j (\bar{x}^j + s\bar{x}^j)^{1-\epsilon} ds \right) \prod_j 2^{-\epsilon_j} N \chi(x^j - x^j) \eta(t) dx dt \right] \lesssim \int_0^1 \sum_j \int_{|t| \leq \delta_0} \sum_j \int_0^1 \frac{2^{-\epsilon_j} N \chi(\bar{x}^j + s\bar{x}^j)^{1-\epsilon}}{|(\xi_1 - \xi_2)^2 - (\xi_3 - \xi_4)^2|^2} dt ds. \]

The integral on the right is estimated in the following technical lemma.

**Lemma 4.7.**

\[ \int_0^1 \sum_j \int_{|t| \leq \delta_0} |\bar{x}^j + s\bar{x}^j|^{1-\epsilon} dt ds = O(2^{(2+\epsilon)^*}). \]

**Proof.** It will be convenient to replace the average trajectory $\langle \bar{x}^j, \bar{\xi}^j \rangle$ with the trajectory of the average initial data $\overline{\langle x^j, \xi^j \rangle}$. The accuracy of this approximation remains acceptable for the relevant $t$, for Hamilton’s equations imply that

\[ \bar{x}^j - x^j = -\int_0^t (t - \tau) \left( \frac{1}{4} \sum_k \partial_x V(\tau, x^j_k) - \partial_x V(\tau, \bar{x}^j) \right) d\tau \]

\[ = -\int_0^t (t - \tau) \left( \frac{1}{4} \sum_k (\bar{x}^j_k + \bar{x}^j - \bar{x}^j) \int_0^1 \partial_x^2 V(\tau, \bar{x}^j + s(x^j_k - \bar{x}^j)) ds \right) d\tau \]

\[ = -\int_0^t (t - \tau)(\bar{x}^j - \bar{x}^j) \left( \int_0^1 \frac{1}{4} \sum_k \partial_x^2 V(\tau, \bar{x}^j + s(x^j_k - \bar{x}^j)) ds \right) + O(2^\epsilon^* t^2), \]

so by Gronwall

\[ |\bar{x}^j - x^j| = O(2^\epsilon^*). \]

Similar considerations yield

\[ |\bar{\xi}^j - \xi^j| = O(2^\epsilon^*). \]

As also $\bar{x}^j = O(2^\epsilon^*)$, we are reduced to showing

\[ \int_{|t| \leq \delta_0} |\bar{x}^j| |(\bar{x}^j)^{1-\epsilon} dt = O(1). \]

Integrating the ODE

\[ \frac{d}{dt} x^j = \xi^j, \quad \frac{d}{dt} \xi^j = -\partial_x V(t, x^j), \]

yields the estimates

\[ |x^j - x - t\xi| \leq C|t|^2(1 + |x| + |t\xi|), \]

\[ |\xi^j - \xi| \leq C|t|(1 + |x| + |\xi|) \]

for some constant $C$ depending on $\sup_{|t|} |\partial_x V(t, 0)|$. By subdividing the time interval $[-\delta_0, \delta_0]$ and repeating the argument on each subinterval, we may assume that $C|t| \leq 1/10$.

Consider separately the cases $|x| \leq |\xi|$ and $|x| \geq |\xi|$. When $|x| \leq |\xi|$, $2|\xi| \geq |\xi^j| \geq |\xi| - \frac{1}{10} (1 + 2|\xi|) \geq \frac{1}{2} |\xi|$ (assuming as we may that $|\xi| \geq 1$), the bound follows from the change of variables $y = x^j$. If $|x| \geq |\xi|$, then $|x^j| \geq \frac{1}{2} |x|$, $|\xi^j| \geq 2|x|$, which also yields the desired bound.

Returning to $I_c$, we conclude that

\[ I_c \lesssim N \frac{2^{-\epsilon_j} N}{|(\xi_1 - \xi_2)^2 - (\xi_3 - \xi_4)^2|^2}. \]

Overall

\[ I \leq I_a + I_b + I_c \lesssim N 2^{-\epsilon_j} N \frac{(\xi_1 + \xi_2 - \xi_3 - \xi_4)}{|(\xi_1 - \xi_2)^2 - (\xi_3 - \xi_4)^2|^2}. \]
For \( II \), we have
\[
(4.16) \quad D[U_j \psi(x - x_j')] = -iH_j U_j \psi(x - x_j') - \xi_j \partial_x U_j \psi(x - x_j')
\]
and estimating as in \( I \),
\[
II \lesssim_N \frac{1 + \sum_j |\xi_j|}{|\xi_1 - \xi_2|^2 - |\xi_3 - \xi_4|^2} \int |D^2 \Phi|^2 \prod 2^{-\varepsilon_j N} \chi\left(\frac{x - x_j'}{2\varepsilon_j}\right) \eta dx dt
\]
\[
\lesssim_N 2^{-\varepsilon N} \left(\frac{(\xi_1 + \xi_2 - \xi_3 - \xi_4) + |\xi_1 - \xi_2| + |\xi_3 - \xi_4|}{|\xi_1 - \xi_2|^2 - (\xi_3 - \xi_4)^2|}\right) \frac{\xi_1 + \xi_2 - \xi_3 - \xi_4}{|\xi_1 - \xi_2|^2 - (\xi_3 - \xi_4)^2|2}.
\]
It remains to consider \( III \). The derivatives can distribute in various ways:
\[
(4.17) \quad III \lesssim \frac{1}{|D\Phi|^2} \left( \int |D^2[U_1 \psi(x - x_1')]|^2 \prod \theta_{t, k}(x - x_k') \right) dx dt
\]
\[
\lesssim \int |D[U_1 \psi(x - x_1')]| |D[U_2 \psi(x - x_2')]| \prod \theta_{t, k}(x - x_k') \eta dx dt
\]
\[
\lesssim \int \prod \theta_{t, k}(x - x_k') \eta dx dt
\]
where the first two terms represent sums over the appropriate permutations of indices.

We focus on the terms involving double derivatives of \( U_j \) as the other terms can be dealt with as in the estimate for \( II \). From \( 4.16 \),
\[
(4.18) \quad D^2[U_j \psi(x - x_j')] = -i\partial_t V_j(t, x - x_j') U_j(t, x - x_j') + (H_j)^2 U_j \psi(x - x_j')
\]
\[
+ 2i\xi_j \partial_x H_j U_j \psi(x - x_j') + \frac{1}{4} \sum_k \partial_x V(t, x_k') \partial_x V(t, x_j') \partial_x U_j \psi(x - x_j') + (\xi_j)^2 \partial^2_x U_j \psi(x - x_j').
\]
Recalling from \( 4.2 \) that
\[
\partial_t V_j(t, x) = x^2 \left[ \xi_j \int_0^1 (1 - s) \partial^2_x V(x_j' + sx) ds + \int_0^1 (1 - s) \partial_x \partial^2_x V(t, x_j' + sx) ds \right],
\]
it follows that
\[
\int |\partial_t V_1(t, x - x_1') U_1 \psi(x - x_1') \prod \theta_{t, k}(x - x_k') \eta(t) dx dt
\]
\[
\lesssim 2^{2\xi_1} \left[ \int_0^1 |\xi_j \partial^3 V_j(t, x_1' + s(x - x_1'))| ds \right]
\]
\[
+ \int_0^1 |\partial_t \partial^2_x V(t, x_j' + s(x - x_2'))| ds \prod \theta_{t, k}(x - x_k') \eta dx dt
\]
\[
\lesssim N 2^{-\varepsilon N},
\]
where the terms involving \( \partial^2_x V \) are handled as in \( I_c \) above. Also, from \( 4.3 \) and \( 4.6 \),
\[
\int |(\xi_1)^2 \partial^2_x U_1 \psi(x - x_1') \prod \theta_{t, k}(x - x_k') \eta(t) dx dt
\]
\[
\lesssim N \frac{2 - \varepsilon N (1 + |\xi_1|^2)}{1 + |\xi_1 - \xi_2| + |\xi_3 - \xi_4|}. 
\]
The intermediate terms in (4.18) and the other terms in the the expansion (4.17) yield similar upper bounds. We conclude overall that

\[
III \lesssim N 2^{-εrN} \left( \frac{1}{|ξ_1 - ξ_2|^2 - (ξ_3 - ξ_4)^2 |^2} + \left| \frac{1}{|ξ_1 - ξ_2|^2 - (ξ_3 - ξ_4)^2 |^2} \right|^2 \cdot (1 + |ξ_1 - ξ_2| + |ξ_3 - ξ_4|) \right)
\]

Note also that in each of the integrals I, II, and III we may integrate by parts in x to obtain arbitrarily many prefactors of $|ξ_1 + ξ_2 - ξ_3 - ξ_4|^{-1}$. All instances of $|ξ_1 + ξ_2 - ξ_3 - ξ_4|$ in the above estimates may therefore be replaced by 1.

Combining I, II, and III, we obtain

\[
|K^r_{0}| \lesssim N 2^{-εrN} \left( \frac{1 + |ξ_1 - ξ_2| + |ξ_3 - ξ_4|}{|ξ_1 - ξ_2|^2 - (ξ_3 - ξ_4)^2 |^2} \right)
\]

under the hypothesis (4.12); in general

\[
|K^r_{0}| \lesssim N 2^{-εrN} \min \left(1, \frac{1 + |ξ_1 - ξ_2| + |ξ_3 - ξ_4|}{|ξ_1 - ξ_2|^2 - (ξ_3 - ξ_4)^2 |^2} \right).
\]

Combining this with (4.8),

\[
|K^r_{0}| \lesssim N_{1}, N_{2} 2^{-εrN_{1}} \min \left( |ξ_1 + ξ_2 - ξ_3 - ξ_4|^{-N_{2}}, \frac{1 + |ξ_1 - ξ_2| + |ξ_3 - ξ_4|}{|ξ_1 - ξ_2|^2 - (ξ_3 - ξ_4)^2 |^2} \right)
\]

for any $N_{1}, N_{2} > 0$. Lemma 4.5 now follows from summing in $t$, at least when $t(\vec{z}) = 0$.

For general $t(\vec{z})$, use Lemma 2.5 to write $U(t, 0)\psi_{z_j} = U(t, t(\vec{z}))U(t(t(\vec{z})), 0)(\psi_{z_j}) = U(t, t(\vec{z}))(\psi_{z_j(t(\vec{z}))})$, where

\[
\psi = e^{-σt(\vec{z})ξ_{i}^{(\sigma)} + f_{0}(\vec{z})x_{z_j}^{(\sigma)} - \frac{1}{2}(ξ_{z_j}^{(\sigma)})^{2} - V(\vec{z}, t\vec{z}^{(\sigma)})} dτ U^{z_j}(t(\vec{z}), 0)\psi
\]

is bounded in $S(R)$ uniformly in $z_j$ and $t(\vec{z})$, and argue as before.

5. A LINEAR PROFILE DECOMPOSITION

In this discussion we assume for simplicity that $V = V(x)$ is time-independent and satisfies hypotheses (1.3) and (1.4). The propagator is then a one-parameter unitary group $e^{-itH}$. Let $δ_0$ be the constant from Theorem 2.3 so that the dispersive estimate

\[
\|e^{-itH}\|_{L^1(R) \to L^∞(R)} \lesssim |t|^{-1/2}
\]

holds for all $|t| \leq δ_0$. All spacetime norms in this section will be taken over the time interval $[-δ_0, δ_0]$.

Given a bounded sequence $\{f_n\} \subset L^2$, we can apply Corollary 1.2 inductively to obtain a full profile decomposition. But first we introduce some systematic (and standard) notation and terminology.

- A frame is a sequence $\{(λ_n, t_{n}, z_n)\} \subset (0, 1] \times [-δ_0, δ_0] \times T^{*}R$.
- Two frames $\{(λ_{n}^{L}, t_{n}^{L}, z_{n}^{L})\}$ and $\{(λ_{n}^{R}, t_{n}^{R}, z_{n}^{R})\}$ are orthogonal if

\[
\frac{λ_{n}^{L}}{λ_{n}^{R}} + \frac{λ_{n}^{R}}{λ_{n}^{L}} + \left( (λ_{n}^{L})^{-2} + (λ_{n}^{R})^{-2} \right) |t_{n}^{L} - t_{n}^{R}| + |s_{λ_{n}^{L}}((z_{n}^{L})^{t_{n}^{L} - t_{n}^{L}} - z_{n}^{L})| + |s_{λ_{n}^{R}}((z_{n}^{R})^{t_{n}^{R} - t_{n}^{R}} - z_{n}^{R})| \to \infty
\]

Here $s_{λ}(x, ξ) = (λ^{-1}x, λξ)$, and $t \mapsto z^t$ is the trajectory of $z$ under the Hamiltonian flow on $T^{*}R$ generated by $h = \frac{1}{2}|ξ|^2 + V(x)$.

- Two frames $\{(λ_{n}^{L}, t_{n}^{L}, z_{n}^{L})\}$ and $\{(λ_{n}^{R}, t_{n}^{R}, z_{n}^{R})\}$ are equivalent if the following limits exist as $n \to ∞$:

\[
\frac{λ_{n}^{L}}{λ_{n}^{R}} \to λ_{∞} \in (0, ∞), \ (λ_{n}^{L})^{-2}(t_{n}^{L} - t_{n}^{R}) \to t_{∞}
\]

\[
|s_{λ_{n}^{L}}((z_{n}^{L})^{t_{n}^{L} - t_{n}^{L}} - z_{n}^{L})| \to z_{∞}
\]

\[
|s_{λ_{n}^{R}}((z_{n}^{R})^{t_{n}^{R} - t_{n}^{R}} - z_{n}^{R})| \to z'_{∞}.
\]
Lemma 5.1. Let $\Gamma^j = (\lambda^j, t^j, z^j)$ and $\Gamma^k = (\lambda^k, t^k, z^k)$ be two frames, and denote by

$$g^j_n = \pi(z^j_n)S_{\lambda^j_n}, g^k_n = \pi(z^k_n)S_{\lambda^k_n}.$$ 

the associated symmetry operators on $L^2$.

(a) If $\Gamma^j$ and $\Gamma^k$ are equivalent, then

$$(g^k_n)^{-1}e^{it^k_n - t^j_n}Hg^j_n$$

converges strongly in $L^2$.

(b) If $\Gamma^j$ and $\Gamma^k$ are orthogonal, then

$$(g^k_n)^{-1}e^{it^k_n - t^j_n}g^j_n \phi, \psi \rightarrow 0$$

for all $\phi, \psi$ in $L^2$.

Proof. Write $t_n = t^j_n - t^k_n$. Then by Lemma 2.5

$$(g^k_n)^{-1}e^{it_n^k}Hg^j_n = S_{(\lambda^j_n)^{-1}\lambda^k_n}(z^j_n - z^k_n)U^\langle_0((\lambda^j_n)^{-2}t_n, 0) = S_n\pi_nU_n.$$ 

(a) Equivalence of the frames implies that $S_n$, $\pi_n$, and $U_n$ all converge strongly.

(b) By continuity, it suffices to prove the claim with $\phi$ and $\psi$ Schwartz. Suppose first that both $(\lambda^k_n)^{-2}t_n$ and $(\lambda^j_n)^{-2}t_n$ diverge to infinity. Assuming without loss that $\lambda^k_n \geq \lambda^j_n$, the dispersive estimate yields

$$|e^{it_n^k}Hg^j_n \phi, g^j_n \phi| \lesssim |t_n|^{-1/2}(\lambda^k_n)^{1/2}(\lambda^j_n)^{1/2}||\phi||_1||\psi||_1 \leq \lambda^j_n|t_n|^{-1/2}||\phi||_1||\psi||_1 \rightarrow 0.$$ 

Suppose now that $(\lambda^k_n)^{-2}t_n$ stays bounded; if $\sup_n |(\lambda^k_n)^{-2}t_n| < \infty$, the same following considerations apply after taking adjoints. By Lemma 2.5, the operators $U_n$ are uniformly continuous on $S(R)$. For fixed Schwartz $\phi$ and any $N > 0$, the functions $\langle x \rangle^NU_n\phi$ and $\langle x \rangle^N\partial_xU_n\phi$ are bounded uniformly in $n$, so $\{U_n\}_{n}\in L^2$. If $N\lambda^k_n^{-1}\lambda^j_n \rightarrow 0$, then $\pi_n S_n^{-1} \phi$ converges weakly to zero as it becomes increasingly concentrated or dispersed. If on the other hand $(\lambda^k_n)^{-2}t_n \rightarrow \lambda_{\infty}$ in $(0, \infty)$, then also $(\lambda^k_n)^{-2}t_n \rightarrow t_{\infty}$, and inequivalence implies that, after interchanging $j$ and $k$ if necessary,

$$|S_{\lambda^j_n}(z^j_n t^k_n - t^j_n)| \rightarrow \infty,$$

Hence $\pi_n \rightarrow 0$ weakly on $L^2$, and

$$(g^k_n)^{-1}e^{it_n^k}g^j_n \phi, \psi = (\pi_n U_n\phi, S_n^{-1}\psi) \rightarrow 0$$

since $S_n^{-1}$ converges strongly and $\{U_n\}_{n}$ is precompact.

□

Proposition 5.2. Suppose $\{f_n\} \subset L^2(R)$ is bounded. Then, after passing to a subsequence, there exist $J^* \in \{0, 1, \ldots, \} \cup \{\infty\}$, functions $\phi^j \in L^2(R)$, mutually orthogonal frames $\Gamma^j = \{(\lambda^j, t^j, z^j)\}$, and for every finite $J \leq J^*$ a sequence $r_n^j$, which obey the following properties:

For each finite $J \leq J^*$,

$$f_n = \sum_{j=1}^J e^{it_n^j}\pi(z^j_n)S_{\lambda^j_n} \phi^j + r_n^J = \sum_{j=1}^J e^{it_n^j}Hg^j_n \phi^j + r_n^J.$$ 

(5.1)

$$\lim_{n \rightarrow \infty} \|f_n\|^2 - \sum_{j=1}^J \|g^j_n \phi^j\|^2 - \|r_n^J\|^2 = 0$$

(5.2)

$$(g^j_n)^{-1}e^{-it_n^j}Hr_n^j \rightarrow 0 \text{ as } n \rightarrow \infty.$$ 

(5.3)
Proof. Let \( r_n^0 = f_n \), and define inductively
\[
\varepsilon_j = \limsup_{n \to \infty} \| e^{-itH} r_n^j \|_{L^2(t,x)}, \quad A_j = \limsup_{n \to \infty} \| r_n^j \|_{L^2},
\]
where the \( \limsup \) for the \( A_j \) is evaluated along a subsequence that realizes the \( \limsup \) for \( \varepsilon_j \). After passing to a subsequence in \( n \), the \( \limsup \)s may be replaced by genuine limits. If \( \varepsilon_j > 0 \), apply Corollary 1.2 to obtain a frame \( \Gamma^{j+1} = \{ (\lambda_n^{j+1}, t_n^{j+1}, z_n^{j+1}) \}_n \) and a profile
\[
\phi^{j+1} = \lim_n g_n^{j+1} r_n^j.
\]
where the limit is taken in the \( L^2 \) sense. Set
\[
r_n^{j+1} = r_n^j - e^{it_n^{j+1}H} g_n^{j+1} \phi^{j+1}.
\]
Continue until either \( \limsup_{n \to \infty} \| e^{-itH} r_n^j \|_{L^2(t,x)} = 0 \) (in which case set \( J^* = J \)) or forever \( (J^* = \infty) \). The decoupling (5.1) of \( L^2 \) norms follows from applying the corresponding assertion (1.7) in Corollary 1.2 at each step of the construction.

To see (5.3) in the case that \( J^* = \infty \), note that by \( L^2 \) decoupling and the lower bound (1.6) for the \( L^2 \) norm of each profile,
\[
A_{j+1}^2 \leq A_j^2 - C\varepsilon_j^\alpha A_j^{-2\beta} - A_j^2 (1 - C\varepsilon_j^\alpha A_j^{-2\beta-2}),
\]
which, together with the Strichartz estimate \( \varepsilon_j \lesssim A_j \) and the boundedness of \( f_n \) in \( L^2 \), implies that \( \lim_{j \to \infty} \varepsilon_j = 0 \).

To prove the mutual inequivalence of frames, suppose on the contrary that two frames are equivalent (after possibly passing to a subsequence). Choose \( k \) minimal so that \( \Gamma^j \) and \( \Gamma^k \) are equivalent for some \( j < k \). By definition,
\[
r_n^{j-1} = e^{it_n^jH} g_n^j \phi^j + e^{it_n^jH} g_n^k \phi^k + \sum_{j<\ell<k} e^{it_n^\ellH} g_n^\ell \phi^\ell + r_n^k,
\]
so
\[
(g_n^k)^{-1} e^{i(t_n^j - t_n^k)H} g_n^j (g_n^j)^{-1} e^{-it_n^jH} r_n^{j-1} - \phi^j = \phi^k + \sum_{j<\ell<k} (g_n^k)^{-1} e^{i(t_n^j - t_n^\ell)H} g_n^\ell \phi^\ell + (g_n^k)^{-1} e^{-it_n^kH} r_n^k.
\]
Taking \( n \to \infty \), recalling the definition of \( \phi^j \), and invoking the previous Lemma, we deduce that \( \phi^k = 0 \). But each \( \phi^k \) is nonzero by construction. \( \square \)

References
