DESCENT PROPERTIES OF HOMOTOPY K-THEORY

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Abstract

In this paper, we show that the widely held expectation that Weibel's homotopy K-theory satisfies cdh-descent is indeed fulfilled for schemes over a field of characteristic zero. The main ingredient in the proof is a certain factorization of the resolution of hypersurface singularities. Some consequences are derived. Finally, some evidence for a conjecture of Weibel concerning negative K-theory is given.

1. Introduction

The behavior of K-theory under blow-ups has been studied for a long time, starting with Grothendieck and his coworkers [2] for K_0 and continued for higher (and lower) K-theory by Thomason [18]. These results concern blow-ups with regularly embedded centers, in which case it is shown that K-theory satisfies *descent*. It is also known that negative K-theory satisfies a descent condition for finite blow-ups (i.e., essentially, normalization or closed covers), a property that is implied by excision for ideals and invariance under infinitesimal extensions. It is easy to see that algebraic K-theory as defined in [19] cannot satisfy descent for arbitrary blow-ups since it is not homotopy invariant in characteristic zero—as would be implied by homotopy invariance for smooth varieties and resolution of singularities. However, there is a variant of K-theory introduced by Weibel [24] that is homotopy invariant and widely expected to satisfy descent with respect to arbitrary abstract blow-ups.

In this paper, we show that that is in fact the case for schemes over a field of characteristic zero; more precisely, we prove the following.

THEOREM 1.1
Let X be a scheme essentially of finite type over a field of characteristic 0. *Then there is a strongly convergent spectral sequence*

$$H^p_{cdh}(X, a_{cdh}K_{-q}) \Longrightarrow KH_{-p-q}(X).$$

Here $a_{cdh}K_{-q}$ is the cdh-sheafification of the K-theory presheaf.

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Our approach is roughly as follows.

(1) Reformulate the problem as follows: show that for any scheme X, essentially of finite type over a field F of characteristic 0, the natural map $\mathscr{K}\mathscr{H}(X) \to \mathscr{K}^{cdh}(X)$ from homotopy K-theory of X to cdh-fibrant K-theory of X is a weak equivalence. The advantage of this formulation is that it can (in principle) be verified for any given scheme.

(2) Solve the reformulated problem for hypersurfaces. This is done by factoring their resolution of singularities into morphisms of two types: blow-ups with regularly embedded center and finite abstract blow-ups. For both these types, we understand the behavior of $\mathscr{K}\mathscr{H}$.

(3) Reduce the general case to the case of hypersurfaces. This is surprisingly easy and needs only descent for closed covers and invariance under infinitesimal extensions.

Using the same methods, we also give some evidence for a conjecture of Weibel regarding vanishing of K-theory in sufficiently negative degrees.

We give a short overview of the paper. In Section 2, we fix notation and recall some assorted technical results needed. In Section 3, we define what we mean by descent and recall what is known about it. We also give the reformulation of the basic problem we are going to use. In Section 4, we briefly recall the statement of resolution of singularities as proven by Hironaka [10]. In Sections 5 and 6, we prove some facts about reduction ideals and finally state and prove the main theorem of this paper. In Section 7, we derive some of the consequences of our main theorem, in particular, computing some of the negative KH-groups of singular schemes. Finally, in Section 8, we apply our methods to obtain some results on the negative K-theory (as opposed to KH-theory) of singular schemes.

2. Recollections and notation

Throughout this paper, k is a field, and all k-schemes are separated and essentially of finite type over k. We reserve F for fields of characteristic 0. We let $\mathscr{K}$ denote the presheaf of spectra of nonconnective K-theory as defined in [19, Section 6.4], and $\mathscr{K}\mathscr{H}$ denotes Weibel's homotopy invariant K-theory of [24], as formulated in [19, Section 9.11]. For a scheme X, we write $K_n(X)$, respectively, $KH_n(X)$, for the nth homotopy group of $\mathscr{K}(X)$, respectively, $\mathscr{K}\mathscr{H}(X)$.

We quickly recall the definitions of abstract blow-up and the cdh-topology on the category Sch/k of k-schemes.

Definition 2.1
An *abstract blow-up square* of k-schemes is a Cartesian square of k-schemes

$$\begin{array}{ccc} E & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

where $X' \to X$ is a proper morphism and $Z \to X$ is a closed embedding such that the induced morphism $(X' - E)^{\mathrm{red}} \to (X - Z)^{\mathrm{red}}$ is an isomorphism. We also say (abusing language) that $X' \to X$ is an abstract blow-up with center Z. Clearly, an actual blow-up is an abstract blow-up in this sense.

Recall that the Nisnevich topology on Sch/k is the topology generated by covers of the form $\{U \to X, V \to X\}$, where $U \to X$ is an open embedding and $V \to X$ is an étale morphism that is an isomorphism over $X - U$. If we have such a cover, the square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

is called an elementary distinguished Nisnevich square.

Definition 2.2
The *cdh-topology* on Sch/k is the topology generated by the Nisnevich topology and covers of the form $\{Z \to X, X' \to X\}$ for abstract blow-ups $X' \to X$ with center Z.

Remark 2.3
Suslin and Voevodsky prove in [17, Theorem 5.13] that a d-dimensional scheme has cdh-cohomological dimension at most d.

In this paper, we are concerned with certain presheaves of spectra on the category of schemes over a field and their *descent properties*.

A map of presheaves of spectra $E \to F$ on a site $\mathscr{C}$ is called a *local weak equivalence* if it induces an isomorphism

$$a\pi_*(E) \to a\pi_*(F)$$

on sheaves of stable homotopy groups. We need the following result about presheaves of spectra on Sch/k.

THEOREM 2.4

*Let t be the Nisnevich (resp., cdh) topology on Sch/k. There is a model structure (actually, several) on the category of presheaves of spectra on Sch/k such that weak equivalences are t-local weak equivalences and any fibrant presheaf of spectra E satisfies the following property (called t-*excision*):*

If

$$\begin{array}{ccc} Z' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

is an elementary distinguished Nisnevich square (resp., an elementary distinguished Nisnevich or abstract blow-up square), then the square of spectra

$$\begin{array}{ccc} E(Z') & \longleftarrow & E(X') \\ \uparrow & & \uparrow \\ E(Z) & \longleftarrow & E(X) \end{array}$$

is homotopy Cartesian.

Proof

One possible model structure is the one of [11]. The Nisnevich excision property is proven, for example, in [13, Corollary 1.4]. The same proof applies to show that the cdh-excision property follows from the corresponding unstable statement for presheaves of simplicial sets. This in turn follows from the same statement for simplicial *sheaves* (argue as in the proof of [13, Theorem 1.3]). Finally, the excision property for fibrant simplicial sheaves follows from [21, Lemma 4.3], the fact that the "combined cd-structure," which defines the cdh-topology, is bounded and regular (cf. [22, Proposition 2.12 and Lemma 2.13]), and the fact that sheaves that are fibrant in Jardine's model structure are also fibrant in the local projective (or "Brown-Gersten" in the terminology of [21]) model structure (cf. also [3, Lemma 4.1]). □

Remark 2.5

Note that all the model structures satisfying the above theorem have the property that any local weak equivalence of fibrant presheaves $E \to F$ is an objectwise weak equivalence. For any U, the induced map on sections $E(U) \to F(U)$ is a (stable) weak equivalence. For Jardine's model structure, this fact can be found in [11].

Definition 2.6

Let $\mathscr{L}$ be a presheaf of spectra on Sch/k. Then $\mathscr{L}^{cdh}$ denotes a (functorial) fibrant replacement of $\mathscr{L}$ in the model category structure of Theorem 2.4; that is, $\mathscr{L}^{cdh}$ is fibrant and there is a map $\mathscr{L} \to \mathscr{L}^{cdh}$ that is a local weak equivalence.

Remark 2.7

If resolution of singularities holds over k, then any cdh-cover of k-schemes admits a refinement consisting of smooth schemes. Hence, under the assumption of resolution of singularities, a natural transformation $\mathscr{L}_1 \to \mathscr{L}_2$ of presheaves inducing a weak equivalence on smooth schemes induces a weak equivalence $\mathscr{L}_1^{cdh} \to \mathscr{L}_2^{cdh}$ on all schemes.

In particular, the natural transformation $\mathscr{K}^{cdh} \to \mathscr{K}\mathscr{H}^{cdh}$ is a weak equivalence, and we are justified in just writing $\mathscr{K}^{cdh}$ in place of $\mathscr{K}\mathscr{H}^{cdh}$, which we do from here on.

We also need the following well-known result.

THEOREM 2.8

Let $\mathscr{L}$ be a presheaf of spectra on Sch/k such that the sheaf $a_{cdh}\pi_q\mathscr{L}$ of homotopy groups vanishes for q small enough. For any scheme X/k (recall our schemes are always essentially of finite type and in particular Noetherian and finite dimensional), there is a strongly convergent spectral sequence

$$E_2^{p,q} = H^p_{cdh}\big(X, a_{cdh}\pi_{-q}(\mathscr{L})\big) \Longrightarrow \pi_{-p-q}\big(\mathscr{L}^{cdh}(X)\big).$$

The same holds if one replaces the cdh-topology by the Zariski or Nisnevich ones.

Proof

The proof given by Jardine in [12, Section 6.1] carries over to our setting; just observe that the cdh-cohomological dimension (Zariski, Nisnevich cohomological dimension) is locally bounded in the following sense: for any object X of our site, there exists $d \geq 0$ (namely, the dimension of X—see Remark 2.3) such that any covering of X has a refinement consisting of objects with cohomological dimension at most d. (In other words, our site is "of finite type" in the sense of [14, Definition 2.1.31].) □

To finish this section, we state an obvious fact that is used repeatedly.

LEMMA 2.9

Let $f : X \to Y$ be a morphism of schemes. Assume $D \subset D' \subset Y$ are closed subschemes such that $D^{\mathrm{red}} = D'^{\mathrm{red}}$. (In this case we say that D' is an infinitesimal extension *of D.) Then* $f^{-1}(D) \to f^{-1}(D')$ *is an infinitesimal extension. (Here* $f^{-1}(D)$ *denotes the scheme-theoretic fiber.)*

3. Descent properties of K-theory

We define the descent properties investigated in this paper and recall some results regarding descent for $\mathscr{K}\mathscr{H}$.

We say a spectrum X is k-connected if $\pi_i(X) = 0$ for $i \leq k$. A map of spectra is a $(k+1)$-equivalence if its homotopy fiber is k-connected. Thus, a $(k+1)$-equivalence induces an isomorphism in stable homotopy groups up to degree k and an epimorphism in degree $k+1$.

Definition 3.1
Let

$$\begin{array}{ccc} E & \longrightarrow & X \\ \downarrow & & \downarrow \\ D & \longrightarrow & Y \end{array}$$

be a Cartesian square of schemes, and let $\mathscr{L}$ be a presheaf of spectra on Sch.

(1) We say that $\mathscr{L}$ satisfies descent with respect to this square if the induced square

$$\begin{array}{ccc} \mathscr{L}(E) & \longleftarrow & \mathscr{L}(X) \\ \uparrow & & \uparrow \\ \mathscr{L}(D) & \longleftarrow & \mathscr{L}(Y) \end{array}$$

is homotopy Cartesian. If $X \to Y$ is the blow-up along D, we also say that $\mathscr{L}$ satisfies descent with respect to the blow-up of Y along D. Note that $\mathscr{L}$ satisfies descent if and only if the natural map

$$\mathrm{hofib}\big(\mathscr{L}(Y) \to \mathscr{L}(D)\big) \longrightarrow \mathrm{hofib}\big(\mathscr{L}(X) \to \mathscr{L}(E)\big) \tag{3.2}$$

is a weak equivalence. Equivalently, $\mathscr{L}$ satisfies descent if and only if the map

$$\mathrm{hofib}\big(\mathscr{L}(Y) \to \mathscr{L}(X)\big) \longrightarrow \mathrm{hofib}\big(\mathscr{L}(D) \to \mathscr{L}(E)\big) \tag{3.3}$$

is a weak equivalence.

(2) We say that $\mathscr{L}$ satisfies k-descent for some integer k with respect to the square if the natural map 3.2 (or, equivalently, the map 3.3) is a $(k+1)$-equivalence. Sometimes, in particular if $\mathscr{L} = \mathscr{K}$, we write $\mathscr{L}(Y, X, D)$ for the spectrum $\mathrm{hofib}\big(\mathrm{hofib}(\mathscr{L}(Y) \to \mathscr{L}(D)) \to \mathrm{hofib}(\mathscr{L}(X) \to \mathscr{L}(E))\big)$. (If $\mathscr{L} = \mathscr{K}$ and the square is a conductor square, this is known as *triple relative K-theory*.)

With this notation, $\mathscr{L}$ satisfies k-descent if and only if $\mathscr{L}(Y, X, D)$ is k-connected.

If $\mathscr{L}$ satisfies k-descent, there is an induced long exact sequence, for $i \le k$,

$$\cdots \to \pi_i\big(\mathscr{L}(Y)\big) \to \pi_i\big(\mathscr{L}(D)\big) \oplus \pi_i\big(\mathscr{L}(X)\big) \to \pi_i\big(\mathscr{L}(E)\big) \\ \to \pi_{i-1}\big(\mathscr{L}(Y)\big) \to \cdots .$$

Again, if $X \to Y$ is the blow-up along D, then we say that $\mathscr{L}$ satisfies k-descent with respect to this blow-up.

(3) We say that negative K-theory satisfies descent with respect to a square if $\mathscr{K}$ satisfies 0-descent with respect to the same square.

Remark 3.4

By Theorem 2.4, for any presheaf $\mathscr{L}$ of spectra on Sch/k, $\mathscr{L}^{cdh}$ satisfies descent with respect to any abstract blow-up or elementary Nisnevich square.

In Section 6, we prove the following theorem.

THEOREM 3.5

Let F be a field of characteristic 0. Then the presheaf of spectra $\mathscr{K}\mathscr{H}$ on Sch/F satisfies descent with respect to any abstract blow-up square.

The following result is a consequence of Thomason's theorem [18, Theorem 2.1].

THEOREM 3.6

Let $D \to X$ be a regular closed embedding of schemes. Then $\mathscr{K}\mathscr{H}$ satisfies descent with respect to the blow-up of X along D.

Proof

First, we observe that for any n, $D \times \Delta^n \to X \times \Delta^n$ is a regular embedding and $Bl_{D\times\Delta^n}(X \times \Delta^n) = Bl_D X \times \Delta^n$. Let $E_n = E_0 \times \Delta^n$ denote the exceptional divisor of the blow-up of $X \times \Delta^n$ along $D \times \Delta^n$. Thomason's result (in the form of [8, Theorem 5]) for the blow-up square

$$\begin{array}{ccc} E_n & \longrightarrow & Bl_D X \times \Delta^n \\ \downarrow & & \downarrow \\ D \times \Delta^n & \longrightarrow & X \times \Delta^n \end{array}$$

is equivalent to the assertion that the square

$$\begin{array}{ccc} \mathscr{K}(E_n) & \longleftarrow & \mathscr{K}(Bl_D X \times \Delta^n) \\ \uparrow & & \uparrow \\ \mathscr{K}(D \times \Delta^n) & \longleftarrow & \mathscr{K}(X \times \Delta^n) \end{array}$$

is homotopy co-Cartesian. (Note that a square of spectra is homotopy Cartesian if and only if it is homotopy co-Cartesian.) That is, the square

$$\begin{array}{ccc} \mathscr{K}\mathscr{H}(E_0) & \longleftarrow & \mathscr{K}\mathscr{H}(Bl_D X) \\ \uparrow & & \uparrow \\ \mathscr{K}\mathscr{H}(D) & \longleftarrow & \mathscr{K}\mathscr{H}(X) \end{array}$$

is a homotopy colimit (over Δ^{op}) of homotopy co-Cartesian squares and hence is homotopy co-Cartesian itself. This is equivalent to the assertion of the theorem. □

Unfortunately, we have no idea how to prove descent for blow-ups along centers that are not regularly embedded directly. Therefore we change our viewpoint: instead of trying to prove that $\mathscr{K}\mathscr{H}$ satisfies descent for all kinds of abstract blow-up squares, we prove that $\mathscr{K}\mathscr{H}$ and $\mathscr{K}^{cdh}$ coincide on as large a class of schemes as possible; for any abstract blow-up square with all schemes in this class, descent is then automatic.

It is well known that $\mathscr{K}\mathscr{H}$ and $\mathscr{K}^{cdh}$ are equivalent for smooth schemes.

PROPOSITION 3.7
Let X be a smooth scheme over a field k such that resolution of singularities holds over k. Then $\mathscr{K}\mathscr{H}(X) \cong \mathscr{K}^{cdh}(X)$.

Proof
For a smooth k-scheme U, we have $\mathscr{K}(U) \simeq \mathscr{K}\mathscr{H}(U)$. For each n, the presheaf $U \mapsto K_n(U)$ is a homotopy invariant pseudo-pretheory on Sm/k (see [6, Corollary 11.4]). Consequently, the associated Nisnevich sheaf $a_{Nis}K_n$ is a homotopy invariant sheaf with transfers on Sm/k (as first observed by Voevodsky in [20, Section 3.4]). We have descent spectral sequences (cf. Theorem 2.8)

$$H^p(X, a_{Nis}K_{-q}) \Longrightarrow K_{-p-q}(X) = KH_{-p-q}(X) \tag{3.8}$$

and

$$H^p(X, a_{cdh}K_{-q}) \Longrightarrow K^{cdh}_{-p-q}(X) \tag{3.9}$$

and a natural transformation of spectral sequences 3.8 $\longrightarrow$ 3.9. By [17, Corollary 5.12.3], this map is an isomorphism on E_2-terms, and the result follows. □

It is immediate from Proposition 3.7 and descent for closed covers that normal crossing schemes also have the property that $\mathscr{K}\mathscr{H} \cong \mathscr{K}^{cdh}$. We include a proof of this fact because similar proofs are used several times later in the paper.

Definition 3.10

A reduced k-scheme D is called a *normal crossing scheme* (over k) if it can be embedded as a reduced strict normal crossing divisor into some smooth k-scheme X. That is, it can be embedded into a smooth scheme X in such a way that D is locally defined by a squarefree monomial in a regular system of parameters and all the irreducible components are smooth.

Example 3.11

The boundary of the affine n-simplex,

$$T_{n-1} = \operatorname{Spec}\big(k[X_0, \ldots, X_n]/(X_0 + \cdots + X_n - 1, X_0 \cdots X_n)\big)$$

is an example of a normal crossing scheme. An irreducible nodal curve is not. (Its only component is not smooth.)

PROPOSITION 3.12

Let V be a normal crossing scheme over a field k for which we have resolution of singularities. Then $\mathscr{K}\mathscr{H}(V) \cong \mathscr{K}^{cdh}(V)$.

Proof

We can assume that V is connected and hence equidimensional. Let V_i, $i = 1, \ldots, s$, be the irreducible components of V. We recall that

(1) V_i is smooth for all i;

(2) $\bigcup_{j \neq i} V_j$ is again a normal crossing scheme;

(3) $V_i \cap \bigcup_{j \neq i} V_j$ is a normal crossing divisor on V_i.

Now the statement of the proposition follows by induction on the dimension of V and on the number of components s, using the fact that $\mathscr{K}\mathscr{H}$ satisfies descent for closed covers (cf. [24, Corollary 4.10]) and Proposition 3.7. Indeed, the claim is true in dimension 0 or for $s = 1$ because a normal crossing scheme (as defined above) with only one component is smooth. Now let $s > 1$, let the dimension d of V be bigger than 0, and assume that the assertion is proven for normal crossing schemes of dimension at most $d - 1$ or with fewer than s components. Write $V = V_1 \cup \big(\bigcup_{j \neq 1} V_j\big)$. Then V_1 is smooth, $\bigcup_{j \neq 1} V_j$ has $s - 1$ components, and the intersection is a normal crossing scheme of dimension $d - 1$. By induction and descent for closed covers, the assertion follows. □

4. Hironaka's resolution of singularities

We recall the notion of normal flatness and Hironaka's resolution of singularities (which we need in its explicit form). Recall that all our schemes are Noetherian.

Definition 4.1

Let X be a Noetherian scheme, and let $D \subset X$ be a closed subscheme. Let $x \in D$, and denote by $J \subset \mathcal{O}_{X,x}$ the ideal defining D at x. We say that X is normally flat along D at x if the graded algebra $gr_J(\mathcal{O}_{X,x})$ is flat over $\mathcal{O}_{X,x}/J$. If X is normally flat along D at every point, then we also say that X is normally flat along D (or D is normally flat in X).

We recall the main result of Hironaka (see [10, Theorem 1*]).

THEOREM 4.2

Let F be a field of characteristic zero, and let X/F be an equidimensional scheme of finite type, which need not be reduced or irreducible. Then there exists a sequence of monoidal transformations

$$X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = X$$

such that the following hold:

(1) *the reduced subscheme X_r^{red} is smooth over F;*

(2) *the center D_i of the monoidal transformation $X_{i+1} \to X_i$ is smooth and connected;*

(3) *X_i is normally flat along D_i;*

(4) *D_i is nowhere dense in X_i.*

Proof

The only statement not explicitly made in [10, Theorem 1*] is that all the centers are nowhere dense. Instead, Hironaka makes a more precise statement: All the points in D_i are either singular points of X_i^{red} or points of X_i^{red} where X_i is not normally flat along X_i^{red}. Both sets of points are nowhere dense (by generic smoothness and generic flatness, resp.). □

Actually, Hironaka proved an even stronger statement known as *embedded resolution of singularities*. This in turn implies the following result, used in Sections 7 and 8. Here we say that a (strict) normal crossing divisor D in a smooth scheme U meets a closed subscheme $X \subset U$ normally (or has normal crossing with X) if for any point $x \in X \cap D$, there is a regular system of parameters $(f_1, \ldots, f_n)$ of U at x such that (at x) D is defined by a (squarefree!) monomial in the f_i and X is defined by a subset of the f_i. In particular, if X intersects D properly, then $X \cap D$ is a (strict) normal crossing divisor in X. (Note that X is clearly smooth along $X \cap D$.)

THEOREM 4.3

Let X be an irreducible variety over a field F of characteristic 0, embedded as subvariety in some smooth F-variety U. Then there exists a subscheme D of U, supported in the singular set of X, such that if $\tilde{U} \to U$ denotes the blow-up along D and $\tilde{X}$ denotes the strict transform of X, then $\tilde{U}$ and $\tilde{X}$ are smooth varieties and the reduced subscheme of the exceptional divisor $E \subset \tilde{U}$ of this blow-up is a divisor with strict normal crossings meeting $\tilde{X}$ normally. (In particular, the reduced subscheme of the exceptional divisor $E \times_{\tilde{U}} \tilde{X}$ of the blow-up of X along D is a strict normal crossing divisor.)

5. Reduction ideals and blow-up of normally flat subschemes

The plan to prove that a given F-scheme X satisfies $\mathscr{K}\mathscr{H}(X) \cong \mathscr{K}^{cdh}(X)$ is now to choose a resolution of singularities of X and factor it into morphisms for which descent is already known, that is, blow-ups along a regularly embedded subscheme (cf. Theorem 3.6) and finite abstract blow-ups (cf. [24, Proposition 4.9]). To realize that plan, we need some ideas from commutative algebra. We begin by recalling the definition of reduction ideals and the main result from [15].

Definition 5.1

Let A be a ring, and let I be an ideal of A. An ideal $J \subset I$ is called a *reduction* if there is a natural number $n > 0$ such that $JI^{n-1} = I^n$ (in this case the same holds for all bigger n). A reduction is called *minimal* if it does not contain any other reduction of I. If S is a scheme and D a closed subscheme defined by an ideal sheaf $\mathscr{I}$, then we call a closed subscheme $\tilde{D}$ containing D a *reduction* of D if it is defined by an ideal sheaf that is locally a reduction of $\mathscr{I}$.

Northcott and Rees prove the following result (see [15, Theorems 1 in Sections 2, 3, 4]).

THEOREM 5.2

Let A be a Noetherian local ring with infinite residue field $A/\mathfrak{m}$, let $I \subset A$ be an ideal, and let $l(I) = \dim(A/\mathfrak{m} \otimes_A gr_I A)$ be the analytic spread of I. (Here "dim" is the Krull-dimension of the ring.) Then there exists a minimal reduction J of I that can be generated by $l(I)$ elements.

In the case where an ideal is normally flat in a ring, more can be said about the analytic spread. The following result is a consequence of [9, Theorem 23.12], in light of the fact that finitely generated flat modules over local rings are free.

LEMMA 5.3
Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$, and let $\mathfrak{p}$ be a prime ideal in A, such that A is normally flat along $\mathfrak{p}$. Then we have an equality

$$l(\mathfrak{p}) = \dim(A/\mathfrak{m} \otimes_A gr_{\mathfrak{p}} A) = ht(\mathfrak{p}).$$

Now we can easily conclude the following.

PROPOSITION 5.4
Let A be a Noetherian local Cohen-Macaulay ring with maximal ideal $\mathfrak{m}$ and infinite residue field, and let $\mathfrak{p} \subset A$ be a prime ideal along which A is normally flat. Then there is a (minimal) reduction J of $\mathfrak{p}$ that is generated by a regular sequence.

Proof
Using Lemma 5.3 and Theorem 5.2, we see that there is a reduction J of $\mathfrak{p}$ generated by $r = ht(\mathfrak{p})$ elements. This is in turn equal to the codimension of J (because the radical of J is $\mathfrak{p}$, which is therefore the only minimal prime of J). Therefore J is generated by a regular sequence (see [5, Corollary 17.7]). □

In what follows, we write $A[It]$ for the graded (blow-up) algebra $A \oplus I \oplus I^2 \dots$ for an ideal I in a ring A.

We next quote a result of Weibel.

LEMMA 5.5 (Weibel)
Let A be a Noetherian ring, let $I \subset A$ be an ideal, and let J be a reduction of I. Then the natural morphism of blow-up schemes $\mathrm{Proj}(A[It]) \to \mathrm{Proj}(A[Jt])$ *is finite.*

Proof
This is proven in [25, Theorem 1.5]. Note that the implication we want does not need the condition that A be a domain. □

We are ready to prove the necessary descent properties for $\mathcal{KH}$. We first prove a local result. All schemes are Noetherian.

PROPOSITION 5.6
Assume that S is a local Cohen-Macaulay scheme with infinite residue field and D is

an integral subscheme along which S is normally flat. Denote by S_D the blow-up of S along D and by D' the exceptional divisor. Then homotopy K-theory satisfies descent with respect to this blow-up; that is,

$$\begin{array}{ccc} \mathscr{K}\mathscr{H}(S) & \longrightarrow & \mathscr{K}\mathscr{H}(D) \\ \downarrow & & \downarrow \\ \mathscr{K}\mathscr{H}(S_D) & \longrightarrow & \mathscr{K}\mathscr{H}(D') \end{array}$$

is homotopy Cartesian.

Proof
Using Proposition 5.4, we can choose a reduction $\tilde{D}$ of D which is regularly embedded in S. Denote by $S_{\tilde{D}}$ the blow-up along $\tilde{D}$ and by $\tilde{D}'$ the corresponding exceptional divisor. The blow-up along D factors as $S_D \to S_{\tilde{D}} \to S$. Homotopy K-theory satisfies descent with respect to the blow-up along $\tilde{D}$ by Theorem 3.6. Moreover, the pullback square

$$\begin{array}{ccc} \tilde{D}'' & \longrightarrow & S_D \\ \downarrow & & \downarrow \\ \tilde{D}' & \longrightarrow & S_{\tilde{D}} \end{array}$$

is an abstract blow-up with all the morphisms finite; hence, homotopy K-theory satisfies descent with respect to this square (cf. [24, Proposition 4.9]). Finally, the natural morphism $D' \to \tilde{D}''$ (induced by pulling back the embedding $D \to \tilde{D}$) is an infinitesimal extension (since $D \to \tilde{D}$ is) and so induces an equivalence upon applying $\mathscr{K}\mathscr{H}$ by [24, Theorem 2.3]. The result follows. □

THEOREM 5.7
Assume that X is a Cohen-Macaulay scheme with all residue fields infinite and that $D \subset X$ is an integral subscheme along which X is normally flat. Then $\mathscr{K}\mathscr{H}$ satisfies descent with respect to the blow-up of X along D.

Proof
Let E be the exceptional divisor of the blow-up along D, and set $X' = Bl_D X$. Let $\mathscr{F}_1$ be the presheaf of spectra on X_{Zar} defined as

$$\mathscr{F}_1(U) = \mathrm{hofib}\big(\mathscr{K}\mathscr{H}(U) \longrightarrow \mathscr{K}\mathscr{H}(X' \times_X U)\big),$$

and let $\mathscr{F}_2$ be the presheaf

$$\mathscr{F}_2(U) = \mathrm{hofib}\big(\mathscr{K}\mathscr{H}(D \times_X U) \longrightarrow \mathscr{K}\mathscr{H}(E \times_X U)\big).$$

The presheaves $\mathscr{F}_i$ satisfy Zariski descent (because $\mathscr{KH}$ does), and the natural transformation $\mathscr{F}_1 \to \mathscr{F}_2$ is a local weak equivalence by Proposition 5.6. Now an application of the descent spectral sequence shows that $\mathscr{F}_1(X) \to \mathscr{F}_2(X)$ is a weak equivalence, as asserted. □

6. Main results

In this section, we state and prove the main results.

The results of the previous section, namely, Theorem 5.7, enable us to carry out our plan of showing that a scheme X satisfies $\mathscr{KH}(X) \cong \mathscr{K}^{cdh}(X)$ using a resolution of singularities only for those schemes X that are Cohen-Macaulay and have the property that all the intermediate schemes in their resolution of singularities are Cohen-Macaulay too. A good class of schemes satisfying these conditions are hypersurfaces (by which we mean effective Cartier divisors on smooth schemes.)

THEOREM 6.1

Let F be a field of characteristic 0, and let $X \subset U$ be a hypersurface (i.e., an effective Cartier divisor) in some smooth F-variety U. Then $\mathscr{KH}(X) \cong \mathscr{K}^{cdh}(X)$.

Proof

We proceed by induction on the dimension of X. The statement is clearly true for $\dim(X) = 0$. So assume that the claim holds true for all hypersurfaces of dimension at most $n-1$, and assume that $n = \dim(X) > 0$. For any scheme Y, denote by $c(Y)$ the minimal length of a sequence of monoidal transformations as in Theorem 4.2 needed to resolve the singularities of Y. We prove that $\mathscr{KH}(X) \cong \mathscr{K}^{cdh}(X)$ by induction on $c(X)$. If $c(X) = 0$, then X^{red} is smooth and thus the assertion is true (cf. [24, Theorem 2.3] and Proposition 3.7).

Assume $c(X) > 0$. Choose a sequence of monoidal transformations (as in Theorem 4.2) $X_r \to \cdots \to X_0 = X$ resolving singularities of X such that $r = c(X)$; write D_i for the center of the blow-up $X_{i+1} \to X_i$. Then X_1 is again a hypersurface (in $Bl_{D_0}(U)$, a smooth variety) that is clearly of dimension n and $c(X_1) < c(X)$. By induction on $c(X)$, $\mathscr{KH}(X_1) \cong \mathscr{K}^{cdh}(X_1)$.

The blow-up of X along D_0 satisfies the conditions of Theorem 5.7 with $D = D_0$. (Note that any hypersurface is Cohen-Macaulay.) Moreover, the exceptional divisor D' of that blow-up is a Cartier divisor on the exceptional divisor E of the blow-up of U along D_0; but E, being the exceptional divisor of a blow-up of a smooth scheme along a smooth center, is itself smooth. Therefore D' is a hypersurface of dimension $n-1$, so that $\mathscr{KH}(D') \cong \mathscr{K}^{cdh}(D')$ by induction on the dimension. Finally, D_0 is smooth, and therefore $\mathscr{KH}(D_0) \cong \mathscr{K}^{cdh}(D_0)$. Now Theorem 5.7 implies that the assertion holds for X. □

It is conceivable that a similar kind of argument might work for more general Cohen-Macaulay schemes, but the behavior of the Cohen-Macaulay property under blow-ups is not well understood. As a next step we therefore generalize from hypersurfaces to local complete intersections.

COROLLARY 6.2

Assume that F is a field of characteristic 0. Let X/F be a local complete intersection inside some smooth F-scheme U. Then $\mathcal{K}\mathcal{H}(X) \cong \mathcal{K}^{cdh}(X)$.

Proof

By Zariski descent, the assertion is local. Hence we can assume that $X \subset U$ is actually a complete intersection inside an affine scheme, defined by a regular sequence $(f_1, \dots, f_c)$. We proceed by induction on the embedding codimension c. For $c = 1$, we are in the hypersurface case of Theorem 6.1. So let $c > 1$, and assume that the assertion is true for codimension less than c. The subscheme $V(f_1 f_2, f_3, \dots, f_c)$ defined by the ideal $(f_1 f_2, f_3, \dots, f_c)$ is a complete intersection in U of codimension $c - 1$; it has a closed cover

$$V(f_1 f_2, f_3, \dots, f_c) = V(f_1, f_3, \dots, f_c) \cup V(f_2, f_3, \dots, f_c)$$

by complete intersections of codimension $c - 1$, and the intersection

$$V(f_1, f_3, \dots, f_c) \cap V(f_2, f_3, \dots, f_c)$$

is equal to X. Now the inductive hypothesis and descent for closed covers (cf. [24, Corollary 4.10]) complete the proof. □

For a general scheme, the method of proof used for hypersurfaces very probably does not work, as the following example shows.

Example 6.3

Let X/F be an isolated, normal, non-Cohen-Macaulay singularity. Then there is no finite birational map to X and there is no subscheme $D \subset X$ supported in the singular point that is regularly embedded. Consequently, no resolution of singularities of X can be factored into blow-ups with regularly embedded center and finite abstract blow-ups.

Surprisingly, though, the general case easily follows from what we already know, using only descent for closed covers.

THEOREM 6.4

Assume that F is a field of characteristic 0. Let X be any F-scheme. Then $\mathcal{K}\mathcal{H}(X) \cong \mathcal{K}^{cdh}(X)$.

Proof

By descent for closed covers and invariance for infinitesimal extensions (see [24]), we can assume that X is irreducible and reduced. (To be exact, if we prove the assertion for all integral schemes of dimension at most d, then it follows for all schemes of dimension at most d.) Using Zariski descent, we also can assume that X is local. We proceed by induction on the dimension d of X. The result is true for schemes of dimension 0. So assume that $d > 0$, and assume that the assertion is known for schemes of dimension less than d. Embed X into some smooth connected local F-scheme U; there is a complete intersection $\tilde{X} \subset U$ such that X is one of its (isolated!) components. (In fact, X is defined by a prime $\mathfrak{p}$ in the regular local ring $\mathscr{O}(U)$. Hence $\mathfrak{p}$ contains a regular sequence of length $ht(\mathfrak{p})$ over which $\mathfrak{p}$ is automatically minimal.) Write X' for the union of the other components (i.e., the reduced scheme on the closure of $\tilde{X} - X$) and Y for the intersection $X \cap X'$. Note that the dimension of Y is less than d, and so $\mathscr{K}\mathscr{H}(Y) \cong \mathscr{K}^{cdh}(Y)$ by our inductive hypothesis. The natural transformation $\mathscr{K}\mathscr{H} \to \mathscr{K}^{cdh}$ induces a map of long exact (closed Mayer-Vietoris) sequences

$$\begin{array}{ccccccccc}
KH_{n+1}(Y) & \longrightarrow & KH_n(\tilde{X}) & \longrightarrow & KH_n(X) \oplus KH_n(X') & \longrightarrow & KH_n(Y) & \longrightarrow & KH_{n-1}(\tilde{X}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K^{cdh}_{n+1}(Y) & \longrightarrow & K^{cdh}_n(\tilde{X}) & \longrightarrow & K^{cdh}_n(X) \oplus K^{cdh}_n(X') & \longrightarrow & K^{cdh}_n(Y) & \longrightarrow & K^{cdh}_{n-1}(\tilde{X})
\end{array}$$

(note that the union of X and X' is actually the reduced subscheme of $\tilde{X}$, but both $\mathscr{K}\mathscr{H}$ and $\mathscr{K}^{cdh}$ are invariant under infinitesimal extensions), and the first two and last two vertical arrows are isomorphisms because Y is of lower dimension and $\tilde{X}$ is a local complete intersection. By the five lemma and additivity, we conclude that the map $KH_n(X) \to K^{cdh}_n(X)$ is an isomorphism for all n. This completes the induction step and proves our assertion. □

We are finally able to prove Theorem 3.5.

COROLLARY 6.5

Let F be a field of characteristic 0. Then the presheaf of spectra $\mathscr{K}\mathscr{H}$ on Sch/F satisfies descent with respect to any abstract blow-up square.

Proof

This is true for the presheaf $\mathscr{K}^{cdh}$ by Theorem 2.4. Hence the assertion is immediately implied by Theorem 6.4. □

Another immediate consequence of our main theorem is Theorem 1.1.

COROLLARY 6.6

Let X be a scheme essentially of finite type over a field of characteristic 0. Then there is a strongly convergent spectral sequence

$$H^p_{cdh}(X, a_{cdh}K_{-q}) \Longrightarrow KH_{-p-q}(X).$$

Proof

This is immediate from Theorem 6.4 and Theorem 2.8. □

7. Computations and consequences

In this section, we give some computations regarding the negative KH-groups of singular schemes.

THEOREM 7.1

Let X be any F-scheme. Then $KH_n(X) = 0$ for $n < -d$ and $KH_{-d}(X) = H^d_{cdh}(X, \mathbb{Z})$, where $d = \dim(X)$.

Proof

By the Corollary 6.6, we have a descent spectral sequence

$$E_2^{p,q} = H^p_{cdh}(X, aK_{-q}) \Longrightarrow KH_{-p-q}(X),$$

where aK_n is the cdh-sheafification of the presheaf $U \mapsto K_n(U)$. By [17, Theorem 5.13], the cdh-cohomological dimension of X is at most d; and by resolution of singularities, $aK_n = 0$ for $n < 0$. Moreover, $aK_0 = \mathbb{Z}$. The result follows. □

In the case of an isolated singularity, the lowest-degree nonvanishing KH-group can actually be computed from an embedded resolution of that singularity. To do that, we first need to compute the respective group for a normal crossing scheme; that can be done from the incidence set associated with it.

Let $\mathcal{N}_F$ be the category whose objects are normal crossing schemes (cf. Definition 3.10) over F and such that a morphism (in $\mathcal{N}_F$) $V \to W$ is a morphism of schemes that is, on each connected component of V, an embedding of a union of irreducible components of W.

Definition 7.2

The *resolution functor* $\mathbf{R} : \mathcal{N}_F \to \mathcal{N}_F$ is defined by setting $\mathbf{R}(V)$ to be the disjoint union of the irreducible components of V. (This is a functor because of the definition of the morphisms in $\mathcal{N}_F$.) So $\mathbf{R}(V)$ is always smooth. (Recall that "normal crossing scheme" for us means in particular that all the components are smooth schemes.)

We have the following obvious lemma.

LEMMA 7.3
Let V be a normal crossing scheme. Consider the simplicial scheme

$$\mathscr{R}(V) = \big(n \mapsto \mathbf{R}(V) \times_V \cdots \times_V \mathbf{R}(V)\big)$$

(with $n+1$ factors in degree n). Then $\mathscr{R}(V)_n$ is smooth for each n, and $\mathscr{R}$ defines a functor on the category $\mathscr{N}_F$.

Remark 7.4
Note that the nondegenerate part of $\mathscr{R}(V)_n$ is the disjoint union of intersections of n irreducible components $V^{(n)}$.

LEMMA 7.5
Let E be a presheaf of spectra on Sch/k, fibrant in Jardine's model structure (cf. Theorem 2.4). If V/k is a normal crossing scheme, the natural map of spectra

$$E(V) \longrightarrow \operatorname{holim}_\Delta E\big(\mathscr{R}(V)\big)$$

is a weak equivalence.

Proof
This follows easily from Theorem 2.4 using induction on the number of components of V. (Note that we can discard the degenerate part of $\mathscr{R}(V)$ and are then left with an iterated pullback.) □

COROLLARY 7.6
Let V be a normal crossing scheme. Then there is a natural (on $\mathscr{N}_F$) weak equivalence (induced by augmentation)

$$\mathscr{K}\mathscr{H}(V) \simeq \operatorname{holim}_\Delta \mathscr{K}\mathscr{H}\big(\mathscr{R}(V)\big).$$

Proof
This follows readily from Lemma 7.3, Proposition 3.12, and the fact that $\mathscr{K}^{cdh}$ satisfies descent for closed covers (cf. Corollary 7.5). □

The following is a standard definition.

Definition 7.7
Let V be a normal crossing scheme over F. The *incidence set* of V is the simplicial

set $S(V) = (n \mapsto \pi_0 \mathscr{R}(V)_n)$. (So, e.g., $S(V)_0$ is the set of irreducible components of V.)

Remark 7.8
If V is of dimension 1, then $S(V)$ is homotopy equivalent to the *graph* of V as defined, for example, in [25, Definition 2.1].

PROPOSITION 7.9
Let V be a normal crossing scheme of dimension d. Then $S(V)$ is a d-dimensional complex and $KH_{-d}(V) \cong H^d(S(V))$.

Proof
By Corollary 7.6, we have a Bousfield-Kan spectral sequence

$$H^p\big(KH_{-q}(\mathscr{R}(V))\big) \Longrightarrow KH_{-q-p}(V).$$

Let $\tilde{\mathscr{R}}(V)$ be the nondegenerate part of $\mathscr{R}(V)$. (That is, we forget all the intersections where at least one component of V occurs multiple times; as a price, we do not have any degeneracies anymore, only face maps.) For each q, the map (induced by inclusion) of cochain complexes (obtained by taking alternating sums of boundary maps) $KH_q(\mathscr{R}(V)) \to KH_q(\tilde{\mathscr{R}}(V))$ is a quasi-isomorphism. In other words, we get a spectral sequence

$$E_2^{p,q} = H^p\big(KH_{-q}(\tilde{\mathscr{R}}(V))\big) \Longrightarrow KH_{-q-p}(V).$$

Since V is a normal crossing scheme, we have that $\dim(\tilde{\mathscr{R}}(V)_n) = d - n$; in particular, $\tilde{\mathscr{R}}(V)_n$ is empty for $n > d$. First of all, this shows that $S(V)$ is a d-dimensional complex. Recall that all the schemes $\tilde{\mathscr{R}}(V)_n$ are smooth, so that we can as well replace KH by K in the E_2-term, and $E_2^{p,q} = 0$ for $q > 0$. From this we conclude that

$$KH_{-d}(V) = H^d\big(K_0(\tilde{\mathscr{R}}(V))\big) = K_0\big(\tilde{\mathscr{R}}(V)_d\big)/\partial\big(K_0(\tilde{\mathscr{R}}(V)_{d-1})\big),$$

where ∂ denotes the differential in the complex computing the E_2-terms. Now ∂ is an alternating sum of maps induced by inclusions of points into smooth curves; but for any such inclusion $x \to C$, the induced map $K_0(C) \to K_0(x) \cong \mathbb{Z}$ is isomorphic to the rank map. That is, the image of ∂ does not change if we replace $K_0(\tilde{\mathscr{R}}(V)_{d-1})$ by $\mathrm{Maps}(\tilde{\mathscr{R}}(V)_{d-1}, \mathbb{Z})$, but this is just the group of $(d-1)$-cochains on $S(V)$, and $K_0(\tilde{\mathscr{R}}(V)_d) \cong \mathrm{Maps}(\tilde{\mathscr{R}}(V)_d, \mathbb{Z})$ anyway (because $\tilde{\mathscr{R}}(V)_d$ has dimension 0). This shows that $KH_{-d}(V) \cong H^d(S(V))$, as asserted. □

THEOREM 7.10
Let X be a scheme over a field F of characteristic zero with a single isolated singularity in the closed point $x \in X$. Assume that $d = \dim(X) \geq 2$. Further, let $X' \to X$

be an embedded resolution of X with exceptional divisor E. Then

$$KH_{-d}(X) \cong H^{d-1}\big(S(E)\big).$$

Proof
By Theorem 3.5, $\mathscr{K}\mathscr{H}$ satisfies descent with respect to the square

$$\begin{array}{ccc} E & \longrightarrow & X' \\ \downarrow & & \downarrow \\ D & \longrightarrow & X \end{array}$$

for some $D \subset X$ with $D^{\text{red}} = x$. Since we assumed that $\dim(X) \geq 2$ and because X' and x are smooth, we obtain an isomorphism $KH_{1-d}(E) \cong KH_{-d}(X)$. Now we apply Proposition 7.9. □

Remark 7.11
By Remark 7.8, this reproduces Weibel's result (see [25]) for normal surfaces.

In particular, we observe that KH_{-d} is of a rigid nature.

COROLLARY 7.12
Let X be a scheme with only isolated singularities over an algebraically closed field F of characteristic zero. Let $d = \dim(X)$, and assume that E/F is a field extension. Then the natural map $KH_{-d}(X) \to KH_{-d}(X \times E)$ is an isomorphism.

Proof
Choose an embedded resolution $X' \to X$ with exceptional divisor V. Now observe that $S(V) = S(V \times E)$ since F is algebraically closed. □

Another method to compute KH-groups which is made possible by our result is the motivic spectral sequence, extended to singular schemes by means of cdh-fibrant replacements.

THEOREM 7.13
Assume that F is a field of characteristic 0 and X is an F-scheme. There is a natural strongly convergent spectral sequence

$$H^{p-q}_{\mathscr{M}}\big(X, \mathbb{Z}(-q)\big) \Longrightarrow KH_{-p-q}(X).$$

Proof
Given Theorem 6.4, this is purely formal from the main result of [6]. Choose a fibrant

replacement functor (in the local projective model structure for presheaves of spectra on $(Sch/F)_{cdh}$, say), and apply it to the Friedlander-Suslin tower defining the motivic spectral sequence. The resulting tower defines a spectral sequence. The E_2-terms are as asserted by the very definition of motivic cohomology for singular schemes, strong convergence follows because cdh-cohomological dimension of X is finite, and the abutment is K^{cdh}, which is isomorphic to KH by Theorem 6.4. □

8. Negative K-theory

We derive some results about and make some computations of the negative K-theory of hypersurfaces and isolated singularities. These results are weaker than we would like because we know descent for negative K-theory with respect to finite morphisms only in the affine case.

In particular, we give some evidence for the following conjecture of Weibel (originally posed in [23, Questions 2.9]). Here a scheme Y is K_s-regular for an integer s if the natural map $K_s(Y) \to K_s(Y \times \mathbb{A}^r)$ is an isomorphism for all r. By [4, Corollary 4.4], K_s-regularity implies K_{s-1}-regularity.

CONJECTURE 8.1

Let X be a Noetherian scheme of dimension d. Then $K_n(X) = 0$ for $n < -d$ and X is K_{-d}-regular.

This conjecture is known to be true in dimension at most 2 (see [23] for the case of dimension at most 1 and [25] for the case of excellent surfaces).

It is also known (see [4]) that the affine tetrahedron (cf. Example 3.11) satisfies $K_{-n}(T_n) = \mathbb{Z}$. Hence the vanishing part of the conjecture is sharp.

In light of the fact that $\mathscr{K}\mathscr{H}$ and $\mathscr{K}$ are equivalent with finite coefficients (see [24]), Theorem 6.4 immediately implies the following version of Weibel's conjecture with finite coefficients.

THEOREM 8.2

Let F be a field of characteristic 0, and let X be an F-scheme of dimension d. Let $n > 1$. Then $K_m(X, \mathbb{Z}/n) = 0$ for $m < -d$.

To get integral results, we first need to study the descent properties of K-theory. The following technical results are certainly well known to the experts; we collect them here for ease of reference.

The following lemma is a straightforward generalization of part of [25, Lemma 2.5], with identical proof.

LEMMA 8.3

Suppose S is an affine scheme. Let $X \to X'$ be an infinitesimal extension (i.e., the induced morphism of reduced subschemes $X^{\mathrm{red}} \to X'^{\mathrm{red}}$ is an isomorphism) of S-schemes, with X' of dimension d, and let T be an affine S-scheme. Then $\mathscr{K}$ satisfies $(-d)$-descent with respect to the square

$$\begin{array}{ccc} \varnothing & \longrightarrow & X \times_S T \\ \downarrow & & \downarrow \\ \varnothing & \longrightarrow & X' \times_S T \end{array}$$

that is, the natural map $\mathscr{K}(X' \times_S T) \to \mathscr{K}(X \times_S T)$ is a $(-d+1)$-equivalence.

Proof

Swan proved that for X' affine, $\mathscr{K}(X') \to \mathscr{K}(X)$ is a 1-equivalence (cf. [1, Chapter IX, Theorem 1.3]). In the general case, if $\mathscr{F}$ denotes the presheaf of spectra on X'_{Zar} defined by $\mathscr{F}(U) = \mathrm{hofib}(\mathscr{K}(U \times_S T) \to \mathscr{K}(X \times_{X'} U \times_S T))$, then $\mathscr{F}$ is locally 0-connected by the result for affine schemes. Now $\mathscr{F}$ satisfies Zariski descent, so the descent spectral sequence $H^p_{Zar}(X', a_{Zar}\pi_q(\mathscr{F})) \Rightarrow \pi_{q-p}(\mathscr{F}(X'))$ shows that $\mathscr{F}(X')$ is $(-d)$-connected, as asserted. □

COROLLARY 8.4

Let $X \to X'$ be an infinitesimal extension of d-dimensional schemes, and let $n \le -d$ be an integer. Then X is K_n-regular if and only if X' is K_n-regular.

PROPOSITION 8.5

Let

$$\begin{array}{ccc} E & \longrightarrow & X \\ \downarrow & & \downarrow \\ D & \longrightarrow & Y \end{array}$$

be a finite abstract blow-up with Y affine. Then $\mathscr{K}$ satisfies (-1)-descent with respect to this square.

Proof

We follow Weibel's proof that $\mathscr{K}\mathscr{H}$ satisfies descent with respect to finite abstract blow-ups (cf. [24, Proposition 4.9]). Suppose that $Y = \mathrm{Spec}(A)$ and D is defined by an ideal $J \subset A$. Since finite morphisms are affine, X is affine, say $X = \mathrm{Spec}(B)$; let f denote the morphism $A \to B$. By hypothesis, $A[s^{-1}] \cong B[s^{-1}]$ for any $s \in J$. Consequently, the map $J \to B$ is injective and $s^n B \subset f(A)$ for any $s \in J$ and n big enough. In other words, $I = \{s \in J \mid sB \subset f(A)\}$ is an ideal in A with the same radical as J such that $f(I)$ is an ideal in B, and $f : I \to f(I)$ is an isomorphism. By excision for negative K-theory (see [1, Chapter IX, Theorem 5.3]), the natural map

$$\mathrm{hofib}\big(\mathscr{K}(A) \to \mathscr{K}(A/I)\big) \longrightarrow \mathrm{hofib}\big(\mathscr{K}(B) \to \mathscr{K}(B/I)\big)$$

is a 1-equivalence. Now Lemma 8.3 implies that

$$\mathrm{hofib}\big(\mathscr{K}(A) \to \mathscr{K}(A/J)\big) \longrightarrow \mathrm{hofib}\big(\mathscr{K}(B) \to \mathscr{K}(B/JB)\big)$$

is a 0-equivalence, as asserted. □

Unfortunately, we cannot say as much in the nonaffine case. But at least a weaker result still holds.

COROLLARY 8.6
Let

$$\begin{array}{ccc} E & \longrightarrow & X \\ \downarrow & & \downarrow \\ D & \longrightarrow & Y \end{array}$$

be a finite abstract blow-up such that the dimension of Y *is* d. *Then* $\mathscr{K}(- \times \mathbb{A}^r)$ *satisfies* $(-d-1)$-*descent with respect to this square for any natural number* r.

Proof
Let $\mathscr{F}_1$ denote the functor on open subschemes of Y given as $U \mapsto \mathrm{hofib}(\mathscr{K}(U \times \mathbb{A}^r) \to \mathscr{K}(X \times_Y U \times \mathbb{A}^r))$, and let $\mathscr{F}_2$ be the functor on open subschemes of Y defined by $U \mapsto \mathrm{hofib}(\mathscr{K}(D \times_Y U \times \mathbb{A}^r) \to \mathscr{K}(E \times_Y U \times \mathbb{A}^r))$. Then $\mathscr{F}_1$ and $\mathscr{F}_2$ are presheaves of spectra on the small Zariski site of Y and clearly both satisfy Zariski descent. By Proposition 8.5, the natural transformation $\mathscr{F}_1 \to \mathscr{F}_2$ is locally a 0-equivalence. Now the descent spectral sequence for Zariski topology and the fact that the cohomological dimension of Y_{Zar} is at most d show that $\mathscr{F}_1(Y) \to \mathscr{F}_2(Y)$ is a $(-d)$-equivalence, as asserted. (Note that the descent spectral sequence applies for the homotopy fiber of $\mathscr{F}_1 \to \mathscr{F}_2$ since its homotopy groups are locally bounded below.) □

For the following, we refer to [16, Appendix A].

Remark 8.7
Note that $\mathcal{K}(- \times \mathbb{A}^r)$ actually satisfies $(1-d)$-descent for finite blow-up squares

$$\begin{array}{ccc} E & \longrightarrow & X \\ \downarrow & & \downarrow \\ D & \longrightarrow & Y \end{array}$$

satisfying the following two properties:

(1) the square is a conductor square; that is, if $\mathcal{J} \subset \mathcal{O}_Y$ is the ideal sheaf defining D, then $\mathcal{J}$ is also an ideal sheaf in $p_*(\mathcal{O}_X)$, with p the morphism from X to Y, defining

$$E = \operatorname{Spec}\big(p_*(\mathcal{O}_X)/\mathcal{J}\big) \subset \operatorname{Spec}\big(p_*(\mathcal{O}_X)\big) = X;$$

(2) $\dim(D) \leq d - 1$.

This follows from [1, Chapter IX, Theorem 5.3], the descent spectral sequence, and the fact that the triple relative K-theory presheaves (see Definition 3.1 for the meaning of that term) $U \mapsto \pi_n \mathcal{K}(U, X \times_Y U, D \times_Y U)$ have Zariski sheafification supported in D.

For closed covers, the following holds.

LEMMA 8.8
Let X be a reduced scheme of dimension d, and let $X = A \cup B$ be a closed covering of that scheme by reduced subschemes. Then, for all r, $\mathcal{K}$ satisfies $(1-d)$-descent with respect to the square

$$\begin{array}{ccc} A \times_X B \times \mathbb{A}^r & \longrightarrow & A \times \mathbb{A}^r \\ \downarrow & & \downarrow \\ B \times \mathbb{A}^r & \longrightarrow & X \times \mathbb{A}^r \end{array}$$

Proof
First, assume that $X = \operatorname{Spec}(R)$ is affine, $A = \operatorname{Spec}(R/I)$, and $B = \operatorname{Spec}(R/J)$. In that case, the square of rings

$$\begin{array}{ccc} R[X_1, \ldots, X_r] & \longrightarrow & R/I[X_1, \ldots, X_r] \\ \downarrow & & \downarrow \\ R/J[X_1, \ldots, X_r] & \longrightarrow & R/(I+J)[X_1, \ldots, X_r] \end{array}$$

is actually a Milnor square (i.e., it is Cartesian, and one of the maps to the base is surjective). By [1, Chapter IX, Theorem 5.3] and the main result of [7], this implies that

$\mathscr{K}$ satisfies 1-descent with respect to this closed cover. Now the general case follows again by an application of the descent spectral sequence for the Zariski topology. □

Now we can proceed to attack Weibel's conjecture. As a warm-up, we prove Weibel's conjecture for normal crossing schemes. For general hyperplane arrangements, this (and much more) was already proven in [4].

PROPOSITION 8.9

Let V/F be a normal crossing scheme (cf. Definition 3.10) of dimension d. Then $K_n(V) = 0$ for $n < -d$ and $K_{-d}(V)$ is finitely generated.

Proof

As for Proposition 3.12, we prove this by induction on the dimension d of V and the number s of components; we can assume V is connected. The assertion is true for $d = 0$ (a reduced 0-dimensional ring is regular) and for $s = 1$ because negative K-theory vanishes for regular schemes. So assume $d > 0$, $s > 1$; our assertion is then proven for dimension less than d or dimension equal to d and fewer than s components. Write $V = \bigcup_{i=1}^{s} V_i$. Set $V' = V_1 \cap \big(\bigcup_{j\neq 1} V_j\big)$ and $V'' = \bigcup_{j\neq 1} V_j$. Note that both V' and V'' are normal crossing schemes (in particular, they are reduced). Since $\dim(V') < d$, we have that $K_n(V') = 0$ for $n \le -d$ and $K_{1-d}(V')$ is finitely generated, by induction on d. Because V'' has $s-1$ components, we conclude that $K_n(V'') = 0$ for $n < -d$ and $K_{-d}(V'')$ is finitely generated, by induction on s. Finally, $K_n(V_1) = 0$ for $n < 0$ (in particular, $K_{-d}(V_1) = 0$). Now the inductive step is completed by applying Lemma 8.8 to the closed cover $V = V_1 \cup V''$ with $V_1 \cap V'' = V'$. Indeed, set $F_1 = \operatorname{hofib}(\mathscr{K}(V) \to \mathscr{K}(V''))$ and $F_2 = \operatorname{hofib}(\mathscr{K}(V_1) \to \mathscr{K}(V'))$. The natural map $F_1 \to F_2$ is a $(-d+2)$-equivalence, by Lemma 8.8. Thus, we obtain short exact (in the middle) sequences

$$\pi_n(F_2) \to K_n(V) \to K_n(V'') \tag{8.10}$$

for $n \le -d$.

The long exact homotopy sequence for the fibration $F_2 \to \mathscr{K}(V_1) \to \mathscr{K}(V')$ shows that $\pi_n(F_2) = 0$ for $n < -d$ and that there is a surjection $K_{1-d}(V') \to \pi_{-d}(F_2)$, which implies that $\pi_{-d}(F_2)$ is finitely generated. Now the exactness of (8.10) shows that $K_n(V) = 0$ for $n < -d$ and $K_{-d}(V)$ is finitely generated, as asserted. □

PROPOSITION 8.11

Let V/F be a normal crossing scheme of dimension d. Then V is K_{1-d}-regular.

Proof

As usual, we use induction on d and the number s of irreducible components. Since regular schemes are K_n-regular for any n, the assertion holds true if $d = 0$ or $s = 1$. Suppose that $d > 0$, $s > 1$, and the claim has been proven for normal crossing schemes of dimension at most $d - 1$ or of dimension d and with less than s components. Writing once again $V = \bigcup_{i=1}^{s} V_i$ and defining $V' = V_1 \cap \bigcup_{j\neq 1} V_j$ and $V'' = \bigcup_{j\neq 1} V_j$, the inductive hypotheses yield the following:

(1) $K_n(V') \to K_n(V' \times \mathbb{A}^r)$ is an isomorphism for $n \leq 2 - d$ and any r;

(2) $K_n(V'') \to K_n(V'' \times \mathbb{A}^r)$ is an isomorphism for $n \leq 1 - d$ and any r.

Lemma 8.8 implies that $\mathscr{K}$ satisfies $(1-d)$-descent with respect to each of the squares

$$\begin{array}{ccc} V' \times \mathbb{A}^r & \longrightarrow & V'' \times \mathbb{A}^r \\ \downarrow & & \downarrow \\ V_1 \times \mathbb{A}^r & \longrightarrow & V \times \mathbb{A}^r \end{array}$$

Since V_1 and V' are K_{2-d}-regular, and writing

$$F'(\mathbb{A}^r) = \mathrm{hofib}\big(\mathscr{K}(V_1 \times \mathbb{A}^r) \to \mathscr{K}(V' \times \mathbb{A}^r)\big),$$

we obtain isomorphisms

$$\pi_n\big(F'(\mathbb{A}^0)\big) \cong \pi_n\big(F'(\mathbb{A}^r)\big)$$

for all r and all $n \leq 1 - d$. Now setting

$$F(\mathbb{A}^r) = \mathrm{hofib}\big(\mathscr{K}(V \times \mathbb{A}^r) \to \mathscr{K}(V'' \times \mathbb{A}^r)\big),$$

we obtain isomorphisms $\pi_{1-d}(F'(\mathbb{A}^0)) \cong \pi_{1-d}(F'(\mathbb{A}^r)) \cong \pi_{1-d}(F(\mathbb{A}^r))$ for all r. This implies that the natural map $\pi_{1-d}(F(\mathbb{A}^0)) \to \pi_{1-d}(F(\mathbb{A}^r))$ is an isomorphism for all r. That is, for any r, we get the following diagram of abelian groups with exact rows:

$$\begin{array}{ccccccc} \pi_{1-d}\big(F(\mathbb{A}^0)\big) & \longrightarrow & K_{1-d}(V) & \longrightarrow & K_{1-d}(V'') & \longrightarrow & \pi_{-d}\big(F(\mathbb{A}^0)\big) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_{1-d}\big(F(\mathbb{A}^r)\big) & \longrightarrow & K_{1-d}(V \times \mathbb{A}^r) & \longrightarrow & K_{1-d}(V'' \times \mathbb{A}^r) & \longrightarrow & \pi_{-d}\big(F(\mathbb{A}^r)\big) \end{array}$$

where the first, third, and fourth vertical arrows are isomorphisms. Now the five lemma implies that $K_{1-d}(V) \to K_{1-d}(V \times \mathbb{A}^r)$ is an epimorphism. Since it is also a split injection, it is an isomorphism, showing that V is K_{1-d}-regular. □

COROLLARY 8.12

Let V/F be a scheme such that V^{red} is a normal crossing scheme of dimension d. Then V is K_{-d}-regular and $K_n(V) = K_n(V^{\mathrm{red}})$ for $n \leq -d$.

Proof
The assertion follows from Proposition 8.11 and Corollary 8.4. □

COROLLARY 8.13
Let V/F be a scheme such that V^{red} is a normal crossing scheme of dimension d. Then there is an isomorphism

$$K_{-d}(V) \cong H^d\big(S(V^{\mathrm{red}})\big).$$

Proof
This follows immediately from Corollary 8.12, Proposition 7.9, and [24, Proposition 1.5]. □

Next, we prove Weibel's conjecture for isolated Cohen-Macaulay singularities.

THEOREM 8.14
Let X be a d-dimensional Cohen-Macaulay variety with only isolated singularities over a field F of characteristic zero. Then $K_n(X) = 0$ for $n < -d$ and X is K_{-d}-regular.

Proof
Using Zariski descent and the vanishing of negative K-theory for smooth schemes, we can assume that $X = \mathrm{Spec}(A)$ is local with isolated singular point given by the maximal ideal $\mathfrak{m}$. We can also assume that the dimension of X is at least 2 and X is integral. Using embedded resolution of singularities (cf. Theorem 4.3), we can find an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ such that the blow-up $\tilde{X}$ of X along $\mathfrak{q}$ is smooth and the reduced subscheme of the exceptional divisor is a normal crossing scheme (of dimension $d-1$).

By [15, Section 6], there is a reduction J of $\mathfrak{q}$ generated by a regular sequence. Using Lemma 5.5, we see that we can factor the blow-up along $\mathfrak{q}$ into two steps: first, the blow-up $X' \to X$ along J, and then a finite abstract blow-up $\tilde{X} \to X'$.

Now for the blow-up $X' \to X$, the center has reduced subscheme a point and hence is K_0-regular by Corollary 8.4. Therefore Thomason's theorem (see [18, Theorem 2.1]) implies that X is K_{-d}-regular and $K_n(X) = 0$ for $n < -d$ if and only if the same holds for X'.

The finite abstract blow-up looks like

$$\begin{array}{ccc} E & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ D & \longrightarrow & X' \end{array}$$

where D is the exceptional divisor of $X' \to X$ and E is an infinitesimal extension of the exceptional divisor of the blow-up $\tilde{X} \to X$. We can assume as well that D is reduced. Using the method of the proof of Proposition 8.5 (which commutes with localization), we can find an infinitesimal extension D' of D such that the resulting square

$$\begin{array}{ccc} E' & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ D' & \longrightarrow & X' \end{array}$$

is a conductor square. By Remark 8.7, $\mathscr{K}(- \times \mathbb{A}^r)$ satisfies $(1-d)$-descent for this square for all $r \geq 0$.

Let $F_1(r) = \mathrm{hofib}(\mathscr{K}(X' \times \mathbb{A}^r) \to \mathscr{K}(\tilde{X} \times \mathbb{A}^r))$ and $F_2(r) = \mathrm{hofib}(\mathscr{K}(D' \times \mathbb{A}^r) \to \mathscr{K}(E' \times \mathbb{A}^r))$. Since J is generated by a regular sequence, $D'^{\mathrm{red}} \cong \mathbb{P}^{d-1}$ is regular and hence D' is K_{1-d}-regular. Moreover, E'^{red} is a normal crossing scheme, so that E is K_{1-d}-regular by Corollary 8.12.

The five lemma now implies that the natural map $\pi_{-d} F_2(0) \to \pi_{-d} F_2(r)$ is an isomorphism for all r and that $\pi_s F_2(r) = 0$ for all $s < -d$ and all r. Moreover, we have $(1-d)$-descent for the conductor square, so $\pi_s F_1(r) \to \pi_s F_2(r)$ is also an isomorphism for all r and all $s \leq 1-d$. Since $\tilde{X}$ is actually regular, we obtain isomorphisms

$$\pi_s F_2(0) \cong \pi_s F_1(r) \cong K_s(X' \times \mathbb{A}^r)$$

for all r and $s \leq -d$. This shows that X', and hence X, satisfies Weibel's conjecture, as asserted. □

As in the case of normal crossing schemes (cf. Corollary 8.13), we conclude the following.

COROLLARY 8.15

Let X be an isolated Cohen-Macaulay singularity in characteristic 0, of dimension $d \geq 2$. Assume there is an embedded resolution $X' \to X$ with exceptional divisor E (a normal crossing divisor of dimension $d-1$). Then

$$K_{-d}(X) \cong KH_{-d}(X) \cong H^d_{cdh}(X, \mathbb{Z}) \cong H^{d-1}\big(S(E)\big).$$

Proof

This follows from Theorems 7.1 and 7.10 and [24, Proposition 1.5]. □

Remark 8.16

Again, this reproduces Weibel's result in [25] for $d = 2$.

We obtain a considerably weaker result for hypersurfaces.

PROPOSITION 8.17
Assume that X is a Cohen-Macaulay scheme of dimension d with all residue fields infinite, and assume that $D \subset X$ is an integral closed subscheme along which X is normally flat. Then, for all r, $\mathscr{K}(- \times \mathbb{A}^r)$ satisfies $(-2d-2)$-descent with respect to the blow-up of X along D.

Proof
First, assume that X is local. Using Proposition 5.4 and Lemma 5.5, we can factor the blow-up along D into a blow-up along a regular sequence $X' \to X$ and a finite abstract blow-up $\tilde{X} \to X'$. For the former, $\mathscr{K}(- \times \mathbb{A}^r)$ satisfies descent by Thomason's theorem (see [18, Theorem 2.1]). For the latter, the resulting Cartesian square of schemes is an infinitesimal extension of one for which $\mathscr{K}(- \times \mathbb{A}^r)$ satisfies $(-d-1)$-descent by Corollary 8.6.

Since infinitesimal extensions induce $(-d+1)$-equivalences on applying $\mathscr{K}$ (cf. Lemma 8.3), we conclude that $\mathscr{K}(- \times \mathbb{A}^r)$ satisfies $(-d-2)$-descent for the finite abstract blow-up $\tilde{X} \to X'$. Thus, $\mathscr{K}(- \times \mathbb{A}^r)$ satisfies $(-d-2)$-descent for $\tilde{X} \to X$ (the blow-up along D). The general case (with X not local) follows from this once again by applying the descent spectral sequence for the Zariski topology. □

THEOREM 8.18
Let $X \subset U/F$ be a hypersurface in some smooth F-variety (or localization thereof). Set $d = \dim(X)$. Then $K_n(X) = 0$ for $n < -2d-1$ and X is K_{-2d-2}-regular.

Proof
We proceed as in the proof of Theorem 6.1. The result follows from Lemma 8.3 if $d = \dim(X) = 0$. We use induction on d. So assume $d > 0$. Choose a sequence of blow-ups $X_n \to \cdots \to X_0 = X$ resolving the singularities of X as in Theorem 4.2, of minimal length $n = c(X)$. If $c(X) = 0$, then X^{red} is smooth and the assertion follows again from Lemma 8.3. Proceed by induction on c. Let $r \geq 0$. By Proposition 8.17, $\mathscr{K}(- \times \mathbb{A}^r)$ satisfies $(-2d-2)$-descent with respect to the blow-up $X_1 \to X$ with center D_0. The exceptional divisor E of that blow-up is a hypersurface of dimension $d-1$, so $K_n(E \times \mathbb{A}^r) = 0$ for $n < 1-2d$. Moreover, $K_n(D_0 \times \mathbb{A}^r) = 0$ for $n < 0$ since D_0 is smooth. Finally, $c(X_1) < c(X)$, so $K_n(X_1 \times \mathbb{A}^r) = 0$ for $n < -2d-1$ by induction on c. Now the fact that $\mathscr{K}(- \times \mathbb{A}^r)$ satisfies $(-2d-2)$-descent for the blow-up along D finishes the induction step and completes the proof. □

Finally, in the general case, we can at least still obtain the following result.

THEOREM 8.19
Let X be any F-scheme. Then there is an integer $N(X)$ such that $K_n(X) = 0$ and X is K_n-regular for all $n < N(X)$.

Proof
First, if X is a local complete intersection, the claim follows from Theorem 8.18 using Lemma 8.8 by the same method used in the proof of Corollary 6.2. The general case follows from this using, once again, Lemmas 8.8 and 8.3 and applying the method of proof of Theorem 6.4. □

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