

1. Let A^\bullet be a cochain complex of abelian groups and assume A^n is free for all $n \in \mathbb{Z}$. Show that A^\bullet is acyclic if and only if the identity $\mathbf{1}_{A^\bullet}$ is null homotopic.

2. Let X be a non-empty set. Define a chain complex of abelian groups $R(X)_\bullet$ by setting $R(X)_n = \mathbb{Z}(X^{\times n})$ to be the free abelian group on the n -th power of X (note that $X^{\times 0}$ is by definition the set with one element $\{e\}$, and $R(X)_n = 0$ for $n < 0$), and the differential $\partial_n : R(X)_n \rightarrow R(X)_{n-1}$ is given on generators by the formula

$$\partial_n(x_1, \dots, x_n) = \sum_{i=1}^n (-1)^i (x_1, \dots, \hat{x}_i, \dots, x_n)$$

if $n \geq 2$ and $\partial_1(x) = e$. (Here the hat means "remove that component"). Compute $H_*(R(X)_\bullet)$.

3. Let \mathcal{A} be an arbitrary abelian category, and let

$$0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$$

be an exact sequence of complexes in $\mathbf{C}(\mathcal{A})$. Using only the universal properties of kernels and cokernels, construct the connecting homomorphism in the long exact sequence in cohomology, and prove that the resulting long sequence is, in fact, exact.

4. Let \mathcal{V}_k be the abelian category of vector spaces over a field k . Show that, for any complex A^\bullet of k -vector spaces, there is a quasiisomorphism $A^\bullet \rightarrow H^*(A^\bullet)$ where the cohomology is considered as a complex with zero differentials.

5. Prove that chain homotopy is an additive equivalence relation of the category of cochain complexes over an abelian category.