1. Give examples of the following: a ring $R$ and module $M$ such that $M$ contains no simple submodule. And a ring $R$ such that every module over $R$ contains a simple submodule, and module $M$ that is not semisimple.

2. Let $k$ be a field, and $R = k[x]/(x^n)$ for some integer $n > 1$. Prove that the category of $R$-modules is equivalent to the category whose objects are pairs $(V, A)$ with $V$ a $k$-vector space and $A \in \text{End}_k(V)$ a linear transformation such that $A^n = 0$ and whose morphisms are linear transformations $T : V \to V'$ commuting with the nilpotent operators in the obvious manner, that is, such that $T \circ A = A' \circ T$. Further prove that if $(V, A)$ and $(V, B)$ are two such modules (with same underlying vector space) then they are isomorphic if and only if $A$ and $B$ are conjugate linear transformations.

3. Now let $k$ be an algebraically closed field of positive characteristic $p$, and $R = k[x]/(x^p)$. List all simple $R$-modules up to isomorphism.

4. An $R$-module $M$ is called indecomposable if, whenever we have an isomorphism $M \cong N \oplus Q$, then either $N = 0$ or $Q = 0$. Show that simple modules are indecomposable, and give an example of a ring $R$ and an indecomposable $R$-module $M$ that is not simple.

5. For the ring $R$ of problem 3., list all indecomposable $R$-modules up to isomorphism.

6. Continuing with the same ring as in problems 3. and 5., show that every finitely generated $R$-module $M$ is a direct sum of indecomposable modules, and that both these indecomposable modules up to isomorphism and their multiplicities are uniquely determined by $M$.

7. Consider the $\mathbb{R}$-algebra $\mathcal{D} = \mathbb{R}\{x, \partial\}$ with two generators $x$ and $\partial$ subject to the relation $\partial x - x \partial = 1$. Show that the vector space $\mathbb{C}^\infty$ of infinitely differentiable functions on the real line is a module over $\mathcal{D}$ with $x$ acting as multiplication with the function $f(x) = x$ and $\partial$ acting as derivative.

8. Continuing from problem 7., let $P \in \mathcal{D}$ be a linear algebraic differential operator, and set $M(P) = \mathcal{D}/\mathcal{D}P$. Show that the space of infinitely differentiable solutions of the differential equation $Pf = 0$ is isomorphic to the vector space $\text{Hom}_\mathcal{D}(M(P), \mathbb{C}^\infty)$.